An Exact Test of the Improvement of the Minimum Variance Portfolio[1]

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Abstract

I propose an exact finite sample test of the risk reduction of the global minimum variance (GMV) portfolio. The GMV test statistic has a straightforward geometric and portfolio interpretation and complements the celebrated GRS test in Gibbons, Ross and Shanken (1989). In practical applications, the GMV test leads to a rejection of the null hypothesis of no improvement in the GMV portfolio more often than the GRS test rejects the null hypothesis of no improvement in the risk-return profile of the tangent portfolio. The power of the GMV test increases with the variance reduction of the global minimum variance portfolio. Using test asset returns scaled by pre-determined instrumental variables is equivalent to increasing the overall number of test assets and leads to substantial power gains.

Key Words: Tangent portfolio, global minimum variance portfolio, mean-variance spanning.

JEL Classification: G11, G12.

1 Introduction

Following the pioneering work on the mean-variance trade-off in Markowitz (1952) the question of mean-variance efficiency of an asset or a set of proposed asset-pricing factors has been of great importance to investment practitioners as well as finance researchers. Early theoretical work on the first two moments of risky asset returns focused on preference-free methods of ranking like first and second-order stochastic dominance as well as necessary and sufficient parametric restrictions on investors' preferences or the data-generating process of risky asset returns that would lead to a meaningful equilibrium trade-off between the first two return moments. Early empirical tests relied mostly on asymptotic econometric tests on relatively short historical records of a small set of portfolios and individual stock returns (see Gibbons (1982), Jobson and Korkie (1982, 1985, 1989) and MacKinlay (1987), for example). It was not until the ground-breaking work in Gibbons, Ross and Shanken (1989) (GRS) that presented a finite sample test of mean-variance efficiency of one or a set of assets with respect to another set of basis securities, facilitating reliable inferences in empirical work using only a limited historical time series of returns. Even more importantly, GRS presented the finite sample distribution of their test under both the null hypothesis of mean-variance efficiency *and* the alternative hypothesis of certain level of inefficiency. Their test is based on the proportional improvement in one plus the squared Sharpe ratio of the tangent portfolio following the addition of the test asset or assets to the mean-variance frontier consisting of the base securities. The power of the GRS test increases in the relative improvement in the risk-return trade-off offered by the inclusion of the test assets. Recent work extending the classic GRS test statistic includes Chou and Zhou (2006), Barilla and Shanken (2015) and Hwang and Satchell (2015) among others.

In this paper, I propose a new exact test of the reduction of the variance of the global minimum variance (GMV) portfolio based on the proportional improvement in the variance of the GMV portfolio following the addition of the test assets to the mean-variance frontier constructed with the base assets. The GMV test has an identical finite sample distribution with the GRS test under both the null *and* the alternative hypothesis. Furthermore, the GMV test often produces higher values for the test statistic than the GRS test. The reason for this is the fact that the proportional change in the squared value of a given variable is always higher than the proportional change in one plus the same squared variable. I present a graphical illustration to illustrate the intuition behind this straightforward mathematical fact. Furthermore, I present a substantial amount of empirical evidence documenting that superior value of the GMV test over the GRS test using various sets of US and international stock portfolios in both an unconditional and several conditional forms.

The contributions of this paper are as follows. First, I derive a finite-sample GMV improvement test based on the risk reduction in the GMV portfolio following the addition of the test assets to the set of base assets. Second, I offer a geometric and a portfolio intuition behind the GMV test and present an intuitive explanation as to the reason it will typically lead to larger values than the GRS test. Thirdly, I present the power function of the GMV test and offer thoughts on experimental design cases where the test will be more powerful. Finally, I apply both the GRS and the GMV tests to a large set of US and international portfolios sorted by various characteristics and past returns as well as by industry. I perform both unconditional and conditional tests using several pre-determined instrumental variables. The empirical evidence overwhelmingly support the theoretical intuition behind the derivation of the GMV test. In practice, this raises suspicion over the validity of many unconditional versions of the market model of Sharpe (1964), the three-factor model of Fama and French (1992) and the four-factor model of Carhart (1997) while many conditional models are either marginally rejected or fail to be rejected when using the GMV test.

The paper proceeds as follows. Section 2 presents the derivation of the GMV test. Section 3 discusses the geometric interpretation while Section 4 demonstrates the logic of the GMV statistic in a mean-variance optimal portfolio framework. Section 5 briefly discusses issues relating to the power of the GMV test. Section 6 presents the empirical findings of applying both the GMV and the GRS tests with several robustness checks discussed in Section 7. Finally, Section 8 offers a few concluding thoughts.

2 Test Statistic Derivation

Let R_1 be the $T \times K$ matrix of realized excess returns on the K test assets over T periods. Similarly, let R_2 be the $T \times N$ matrix of realized excess returns on the N basis assets. Consider the following linear regression of the basis assets returns on the test assets returns

$$R_{2t} = \alpha + R_{1t}\beta + \epsilon_t, \tag{1}$$

where β is a $K \times N$ matrix of loadings of the test assets on the basis assets and $\Sigma = var(\epsilon_t)$. Expressing the above regression in matrix form we have

$$R_2 = XB + E, \tag{2}$$

where X is a $T \times (K+1)$ matrix with a typical row of $[1, R_{1t}^{'}]$, $B = [\alpha, \beta]'$, and E is a $T \times N$ matrix with $\epsilon_t^{'}$ as a typical row. The maximum likelihood estimate of B and Σ are

$$\hat{B} = (X'X)^{-1}(X'R_2), \tag{3}$$

$$\hat{\Sigma} = \frac{1}{T}(R_2 - X\hat{B})'(R_2 - X\hat{B}). \tag{4}$$

Huberman and Kandel (1987) show that testing for spanning of the test assets by the benchmark assets involves checking the following parameter restrictions

$$\alpha = 0_N, \qquad 1_K\beta = 1_N, \tag{5}$$

where 1_M is an M-element column-vector of ones. Both of these restrictions make good economic sense. Intuitively, they amount to an exact factor pricing test. If the returns of the additional assets are exact linear combinations of the returns of the benchmark assets then there would be no improvement in the mean-variance frontier generated only by the benchmark assets after the additional assets are included in the set of choices available to the investor. Alternatively, one may think of this result as a case in which security markets are complete with respect to both the set of benchmark assets *and* the expanded set. The additional assets will, thus, be redundant and, hence, their returns will be spanned completely by the returns of the benchmark assets. Gibbons, Ross and Shanken (1989) focus on testing the first component in (5) while the focus of this paper is on testing the second part.

Focusing on the test of the second restriction we need to consider the distribution of $\hat{\delta}$ which is just a linear function of $\hat{\beta}$:

$$\hat{\delta} = 1_N - 1_K \hat{\beta}. \tag{6}$$

It is straightforward to show that

$$\sqrt{T/\left(1_K' \hat{V}_{11} 1_K\right)}\hat{\delta} \sim N\left(\sqrt{T/\left(1_K' \hat{V}_{11} 1_K\right)}\delta, \Sigma\right), \tag{7}$$

where T is the number of time series observations, $\hat{\delta}' \equiv \left(\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_N\right)$, $\hat{V}_{11}$ is the variance-covariance matrix of the benchmark assets and Σ the variance-covariance matrix of the residuals. $\hat{\delta}$ and $\hat{\Sigma}$ are independent where $(T-2)\hat{\Sigma}$ has a Wishart distribution with parameters $(T-2)$ and Σ. This leads to $(T(T-N-K)/N(T-K-1))W_u$ having a noncentral F distribution with degrees of freedom N and $T-N-K$, where

$$W_u \equiv \frac{\hat{\delta}' \hat{\Sigma}^{-1} \hat{\delta}}{1_K' \hat{V}_{11}^{-1} 1_K}, \tag{8}$$

and a non-centrality parameter, λ, given by

$$\lambda \equiv T \frac{\hat{\delta}' \hat{\Sigma}^{-1} \hat{\delta}}{1_K' \hat{V}_{11}^{-1} 1_K}. \tag{9}$$

Finally, the GMV statistic can be expressed as follows:

$$GMV = \left(\frac{T}{N}\right)\left(\frac{T-N-K}{T-K-1}\right)\frac{\hat{\delta}' \hat{\Sigma}^{-1} \hat{\delta}}{1_K' \hat{V}_{11}^{-1} 1_K}. \tag{10}$$

Note that under the null hypothesis W_u and GMV has a central F distribution unconditionalally (see Rao (1951, 1972) as well as Stewart (1997)) and the actual returns of the reference assets do not improve upon the investment opportunity set offered by the test assets. Unfortunately, we do not know the unconditional distribution of W_u and GMV under the alternative hypothesis. Nevertheless, the assumption of conditional multivariate normality of the reference asset returns conditional on the test asset returns is a good starting point for any statistical analysis. Under this assumption the distribution of W_u and GMV under the alternative hypothesis is non-central F.

3 Geometric Interpretation

Consider a straightforward example with $K = 2$ test assets and $N = 2$ basis assets. There are several possibilities of how the minimum-variance frontier will improve following the addition of the test assets to the basis assets. Figure 1 plots the two mean-variance frontiers, the tangent portfolios as well as the global minimum variance portfolio in the case where the Sharpe ratio improves considerably while the improvement in the risk of the global minimum variance portfolio is quite small. Naturally, this leads to a large value of the GRS statistic and a small value of the GMV statistic. It is quite likely that for a sufficiently large value of T the GRS test will reject the null hypothesis of mean-variance efficiency of the test assets while the GMV test will fail to reject the null hypothesis of no improvement in the GMV portfolio.

Insert Figure 1 about here.

Figure 2 plots the opposite possibility where the improvement in the Sharpe ratio of the tangent portfolio is quite small while the reduction in risk of the global minimum variance portfolio is substantial.

Insert Figure 2 about here.

The final possibility is illustrated in Figure 3. In this case, both the GRS test statistic as well as the GMV test statistic have low values as neither the tangent portfolio nor the minimum variance portfolio improves considerably. For an appropriate range of values of

T, N and K we are likely to find that both tests fail to reject the null hypothesis of no improvement in the tangent and minimum variance portfolios, respectively.

Insert Figure 3 about here.

To gain some further intuition as to why the GRS test statistic tends to produce lower values than the GMV test statistic, it will be useful to express both as follows:

$$GRS = \left(\frac{T}{N}\right)\left(\frac{T-N-K}{T-K-1}\right)\frac{\hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha}}{1+\hat{\mu}_K'\hat{V}_{11}^{-1}\hat{\mu}_K} \propto \frac{(1+\hat{s}_{tg,2}^2)-(1+\hat{s}_{tg,1}^2)}{(1+\hat{s}_{tg,1}^2)}, \tag{11}$$

and

$$GMV = \left(\frac{T}{N}\right)\left(\frac{T-N-K}{T-K-1}\right)\frac{\hat{\delta}'\hat{\Sigma}^{-1}\hat{\delta}}{1_K'\hat{V}_{11}^{-1}1_K} \propto \frac{\hat{\sigma}_{mv,1}^2-\hat{\sigma}_{mv,2}^2}{\hat{\sigma}_{mv,2}^2}, \tag{12}$$

where $\hat{\sigma}_{mv,1}^2$ is the estimate of the variance of return of the global minimum variance portfolio consisting of the benchmark assets only and $\hat{\sigma}_{mv,2}^2$ is the estimated variance of return of the global minimum variance portfolio consisting of both the benchmark and the test assets. The coefficient of proportionality is the same across both tests and depends only on T, N and K.

Note that the GRS test is proportional to the change in the one plus the squared Sharpe ratio of the tangent portfolio between the two mean-variance frontiers. Similarly, the GMV test is proportional to the change in the variance of return of the global minimum variance portfolio following the addition of the test assets to the set of basis assets. For a given set of $\hat{\sigma}_{mv,1}^2$ and $\hat{\sigma}_{mv,2}^2$ as well as $\hat{s}_{tg,1}^2$ and $\hat{s}_{tg,2}^2$ the GMV test will almost always produce a much larger value than the GRS test. Figure 4 plots the surfaces of possible values for both test statistics for a range of values for the Sharpe ratio of the tangent portfolio and the standard deviation of return of the global minimum variance portfolio before and after the addition of the test assets. Note that the GMV values are almost always larger than the GRS values. The intuition behind this finding is that the proportional change in a squared quantity is larger than the proportional change in one plus the squared quantity. Of course, it is conceivable that occasionally the values of the Sharpe ratios and the standard deviations of the global minimum variance portfolios are such that the GRS test will instead exceed the GMV test. Figure 4 illustrates this possibility but also highlights the fact that it will be unlikely in practical applications.

Insert Figure 4 about here.

4 Portfolio Interpretation

It is convenient to relate the regression estimates in (1) to the weights of the tangent and minimum variance portfolios on the extended frontier. Using the partitioned matrix inverse formula it can be shown that

$$w_{mv,a} = \frac{1}{\left(1_K'\hat{V}_{11}^{-1}1_K + \hat{\delta}'\hat{\Sigma}^{-1}\hat{\delta}\right)}\begin{bmatrix}\hat{V}_{11}^{-1}1_K - \hat{\beta}'\hat{\Sigma}^{-1}\hat{\delta} \\ \hat{\Sigma}^{-1}\hat{\delta}\end{bmatrix}, \tag{13}$$

and

$$w_{tg,a} = \frac{1}{\left(1_K\hat{V}_{11}^{-1}\hat{\mu}_1 + \hat{\delta}'\hat{\Sigma}^{-1}\hat{\alpha}\right)}\begin{bmatrix}\hat{V}_{11}^{-1}\hat{\mu}_1 - \hat{\beta}'\hat{\Sigma}^{-1}\hat{\alpha} \\ \hat{\Sigma}^{-1}\hat{\alpha}\end{bmatrix}. \tag{14}$$

It is readily observed that if $\hat{\delta} = 0$ then the minimum variance portfolios on both frontiers coincide at that point. In addition, the expanded minimum variance portfolio does not include any of the test assets, i.e., $w_{mv,a}' = [w_{mv,b}', 0_N']$. Similarly, if $\hat{\alpha} = 0$ the tangent portfolios on both frontiers coincide and the expanded tangent portfolio does not include any holdings of the test assets $w_{tg,a}' = [w_{tg,b}', 0_N']$. If both conditions hold then both mean-variance frontiers are identical and we have that the benchmark assets span the extended mean-variance frontier.

5 Power of the Test Statistic

Knowing the finite sample distribution of a test statistic under both the null hypothesis as well as the alternative hypothesis allows us to investigate the power of the test. The power of a test is given by the probability of rejecting a false null hypothesis. In this case, the distribution of the GMV test exactly mirrors the distribution of the GRS test and so both tests will have identical power. Figures 5 and 6 plot the power of both tests for several combinations of sample sizes given by T, base assets N and test assets K. The plots are slightly different from Gibbons, Ross and Shanken (1989) since a high value of GRS is driven by an increase of the tangent portfolio Sharpe ratio s, while a high value of the GMV test is caused by a reduction in the variance of the global minimum variance portfolio following the addition of the test assets to the base assets. Combinations of values below the 45 degree line in the horizontal plane are meaningless as the Sharpe ratio always weakly increases and the risk of the global minimum variance portfolio always weakly decreases following the addition of new assets to an existing mean-variance frontier.

Insert Figure 5 about here.

Insert Figure 6 about here.

6 Empirical Implementation

Panel A in Table 1 presents the Hotelling T^2 statistic, the GRS and the GMV test statistics along with the ex post Sharpe ratios of the tangent portfolio and the standard deviation of return of the global minimum variance portfolio for sets of 10 basis portfolios sorted by market capitalization, book-to-market, momentum, short-term reversal, long-term reversal, and industry. The data comes from Ken French Online Data Library and covers the period 1960:01 until 2013:12 with monthly value-weighted returns.[1]

I find that the GMV test rejects the null hypothesis of no improvement of the GMV portfolio following the addition of the test assets for all sets of portfolios I use while the GRS test fails to reject the null hypothesis of no improvement of the tangent portfolio fairly often. For example, when I use the ten deciles sorted on market capitalization, the GRS fails to reject the improvement of the tangent portfolio for all three different models under consideration as well as the SMB as a stand alone test asset. Similarly, the GRS test fails to reject the improvement of the tangent portfolio for all three sets of factors when I use

[1] I am grateful to Ken French for providing the portfolio and factor return data.

book-to-market decile portfolios and long-term reversal decile portfolios. In the meantime, the GMV test statistic delivers overwhelming rejections for all sets of basis portfolios and sets of test assets used. This confirms the intuition from prior sections that the GMV test presents a higher bar for the test assets to jump over leading to a more stringent test of the inward shift of the mean-variance frontier.

Panel B of Table 1 presents the same quantities of interest as Panel A and for the same sets of basis and test asset but using daily value-weighted returns during the period of January 4, 1960 until December 31, 2013. Specifically, the GRS test fails to reject the null hypothesis of no improvement of the tangent portfolio for the FF3 and C4 factors with respect to ten portfolio sorted by market capitalization, book-to-market ratios, and long-term reversal. The GRS test also fails to reject the null hypothesis of no improvemnt of the tangent portfolio with respect to ten industry portfolios. At the same time, the GMV test rejects the same test assets quite strongly. The empirical findings and conclusions are unchanged. In fact, the larger number of degrees of freedom serves to strengthen the values of both tests and leads to even stronger rejections of the null hypotheses of no improvement in the mean-variance frontier in some cases.

Insert Table 1 about here.

Next, I turn to the empirical test of GRS and GMV with a wider cross-section of base assets. Specifically, I use portfolios double-sorted on various characteristics as well as past returns. Panel A of Table 2 presents the results from the GRS and GMV tests using monthly value-weighted returns double-sorted on size/book-to-market, size/momentum, size/short-term reversal, size/long-term reversal as well as 30 industry portfolios as the base assets. I test whether the addition of the market portfolio, the FF3 and the C4 factors improve the mean-variance frontier with respect to each set of base assets. The data covers the period between January 1960 until December 2013. The GRS test rejects the null of no improvement of all test assets with the exception of C4 with respect to the 25 size/long-term reversal portfolios as well as the market portfolio and the C4 factors with respect to the 30 industry portfolios. At the same time, the GMV test rejects the null of no improvement in the GMV portfolio for all three sets of factors with respect to all 5 sets of base portfolios.

Panel B of Table 2 repeats the analysis using daily portfolio and factor returns between January 4, 1960 until December 31, 2013 with very similar results. This time the only factor for which we fail to reject the null of no improvement in the mean-variance frontier is the market factor with respect to the 30 industry portfolios. The C4 factors are rejected at a 10% level of significance with respect to the same industry portfolios. Once again, the GMV test uniformly rejects the null hypothesis of no improvement in the GMV portfolio for all sets of factors with respect to all 5 sets of portfolios when using daily returns over the same time period as in Panel A.

Insert Table 2 about here.

As a further application of the GRS and GMV tests, I turn to international portfolio returns single-sorted on dividend yield, earnings yield, cash earnings yield and book-to-market as well as double-sorted on size/book-to-market and size/momentum. I use three sets of factors with two versions of each. Panel A of Table 3 tests the null hypothesis of no improvement in the mean-variance frontier following the addition of the market portfolio to a set of 9 eequity portfolios in Australia, France, Germany, Italy, Japan and the United Kingdom using monthly value-weighted returns for the period between January 1975 and December 2013 and Canada between January 1977 and December 2013. The GRS test rejects the null hypothesis of no improvement of the slope of the tangent portfolio following the addition of the market portfolio in all countries with the exception of Italy and a marginal rejection at the 10% significance level for France. At the same time, the GMV test strongly rejects the null hypothesis of no improvement in the GMV portfolio using the market portfolio in all 7 countries.

Panel B of Table 3 reports the findings for the tests of the null of no improvement in the mean-variance frontier following the addition of the market portfolio, the FF3 factors and the C4 factors in five regions of the world using regional as well global versions of the factors and a two-by-three sort of base portfolio sorted on size and book-to-market using monthly value-weighted returns covering the period between November, 1990 and December, 2013. Interestingly, the GRS test fails to reject the null of no improvement for all three sets of factors both local as well as global with the exception of a minor rejection at the 10%

significance level for the global market portfolio. Similarly, the GRS test fails to reject the local and global FF3 and C4 factors in Europe. At the same time, the GMV test overwhelmingly rejects the null of no improvement of the GMV portfolio for all sets of factors in all five regions.

Panel C of Table 3 reports the findings of the tests of the null hypotheses of no improvement in the mean-variance frontier for the same local and global factors using double-sorted portfolio on size and momentum in all five regions of the world. Once again, the GRS test statistic failed to reject both the local and global market portfolio, the FF3 factors and the C4 factors in Japan and marginally rejected the global C4 factors at a 10% significance level in North America. Once again, the GMV test overwhelmingly rejects the null hypothesis of no improvement in the mean-variance frontier around the GMV portfolio for both the local and global factors in all five regions of the world.

Insert Table 3 about here.

Next, I consider a wider set of base assets in all five regions of the world double-sorted on size and book-to-market as well as size and momentum with monthly value-weighted returns covering the period between November 1990 and December 2013. Panel A of Table 4 presents the findings for the 25 size/book-to-market base portfolios. These findings are qualitatively the same as those reported in Panel B of Table 3. The GRS test fails to reject the null of no improvement in the tangent portfolio for all sets of local and global factors in Japan as well as the local and global FF3 and C4 factors in Europe. The GMV test rejects the null of no improvement of the GMV portfolio for both the local and global sets of all factors in all five regions of the world.

Panel B of Table 4 reports the findings of the GRS and GMV tests in all five regions of the world with both local and global factors on 25 portfolios sorted on size and momentum with monthly value-weighted returns between November, 1990 and December, 2013. Using these base assets, the GRS test statistic only fails to reject the local and global factor sets in Japan. The GRS test uniformly rejects both the local and the global factor sets in all the other four regions of the world. At the same time, the GMV test rejects both versions of the three sets of factors in all five regions of the world.

Insert Table 4 about here.

Finally, I investigate the performance of the two test statistics in a time-varying framework with pre-specified conditioning instrumental variables. I interpret these instrumental variables and their cross-product with the factors as scaled dynamic portfolios. I use the twelve-month dividend yield on the US stock market, the book-to-market ratio of the US stock market, the one-month Treasury bill rate, the credit spread of Moody's BAA-rated index less the AAA-rated index and US inflation rate as instrumental variables that potentially influence the dynamics investment opportunity set from Goyal and Welch (2008).[2]

Panel A of Table 5 presents the findings of both test statistics using ten portfolios sorted on market capitalization with one instrumental variable at a time as well as all five instrumental variables included simultaneously. The portfolio returns are monthly and value-weighted and cover the period between January 1960 and December 2013. It is notable that the GRS test fails to reject the null hypothesis of no improvement in the tangent portfolio for all sets of factors except SMB as a stand-alone factor. Even then, the GMV test produces larger test statistic values and leads to a stronger rejection. The only exception is when I use the inflation rate as an instrumental variable. In this case, the GMV test fails to reject while the GRS test does reject the null hypothesis of no improvement in the mean-variance frontier following addition of the SMB factor to the size decile portfolios. In all the other tests of the market portfolio, the FF3 factors and the C4 factors the GRS test fails to reject the null hypothesis of no improvement in the tangent portfolio while the GMV test rejects the null hypothesis of no improvement in the GMV portfolio more often than not with the notable exception of the market portfolio with the T-bill rate as an instrument and the C4 factors with the T-bill rate as an instrument.

Panel B of Table 5 reports the findings from both test statistics using ten portfolios sorted on book to market and the same instrumental variables as in Panel A. The overall performance of the tests is largely similar to that reported in Panel A for size deciles. The GMV test tends to reject more often than not and the GRS fails to reject very often with the exception of testing the null of no improvement in the tangent portfolio following the

[2] I am grateful to Amit Goyal for updating the instrumental variables through to the end of 2013 and sharing them on his website.

addition of the HML factor to the book-to-market decile portfolios using the inflation rate as an instrumental variable. There is also a marginal rejection of the null hypothesis of no improvement in the tangent portfolio following the addition of the market portfolio to the book-to-market decile portfolio using the credit spread as the instrumental variable. Note that researchers who use the GRS test might be tempted to conclude that the book-to-market portfolio decile returns can span the mean-variance frontier generated by the conditional set of factors scaled with the instrumental variables. However, the GMV test statistic of the same factors with the given instrumental variables point strongly towards a rejection of the null hypothesis of no improvement in the GMV portfolio.

Turning to the findings for momentum decile portfolios reported in Panel C of Table 5, we note that the GRS test rejects the null hypothesis of no improvement in the tangent portfolio for most factor sets, including the momentum factor as a stand-alone factor, with most instrumental variables with the notable exception of the C4 factors. Even then, using the inflation rate as an instrumental variable leads to a rejection by the GRS test but a failure to reject with the GMV test. In most other cases the previously reported pattern is repeated with the GMV test rejecting more strongly than the GRS test. Other decile portfolios sorted on past returns that I consider include the short-term reversal deciles and long-term reversal deciles. The findings from the conditional versions of both tests are reported in Panel D and Panel E of Table 5, respectively. For the most part, the GRS test rejects the null of no improvement of the tangent portfolio of most sets of factors with most instrumental variables and short-term reversal portfolios as the base assets with two notable exceptions. First, all conditional versions that use either the T-bill rate or the credit spread lead to a failure to reject or a marginal rejection at a 10% significance level. Second, all conditional versions of the GRS test of the efficiency of the C4 factors lead to a failure to reject regardless of whether we use one instrumental variable at a time or all five instrumental variables simultaneously. When long-term reversal decile portfolios are used as the base assets the results are even more dramatic. All the tests performed with the GRS statistic fail to reject with the only exception of the long-term reversal factor as a stand-alone asset and using the inflation rate as the instrumental variable. Interestingly, in this specific instance the GMV test fails to reject. Nevertheless, the overall picture that emerges from both tests can lead to very different conclusions from the results of a stand-alone GRS test compared to those from the GMV test.

Finally, I report the findings for both tests using ten industry portfolio in Panel F of Table 5 and thirty industry portfolios in Panel G of Table 5. The findings for both tests lead to very different conclusions. The GRS test fails to reject any of the factors sets tested with all five instrumental variables either stand-alone or jointly included in the regression. At the same time, the GMV test rejects the null of no improvement in the GMV portfolio for all sets of factors with the exception of the C4 factors with respect to ten industry portfolios and all instrumental variables included. This is largely an artifact of using ten base portfolios. Once I extend the base portfolio set to include all thirty industry portfolios, the same test leads to a strong rejection of the null hypothesis.

Insert Table 5 about here.

7 Robustness Checks

In this section I perform several robustness checks in order to determine how sensitive the proposed test is to portfolio constraints, finite sample sizes and non-Gaussian error distributions.

7.1 Short-Selling Constraints

The first robustness check involves limiting negative positions in the tangent and minimum variance portfolios on the two mean-variance frontiers and follows closely De Roon et al (2001).[3] It is straightforward to show that the short-sale constrained test statistic will follow the same functional form as the unconstrained test statistics with a slight modification of

[3]Note that the short-selling constraints can only be applied to the reference or base assets. Applying the short-selling constraints to the test assets can lead to a deterioration of the expanded frontier. To see how this counterintuitive situation can arise consider a portfolio on the frontier consisting of the test assets only which happens to have only positive weights. However, the unconstrained corresponding portfolio on the expanded frontier may involve selling short some or all of the test assets. Imposing the short-selling constraint will then potentially lead to a lower Sharpe ratio of the tangent portfolio and/or higher variance of the minimum variance portfolio.

the regression parameters:

$$GRS^{+} = \left(\frac{T}{N}\right)\left(\frac{T-N-K}{T-K-1}\right)\frac{(1+(\hat{s}^{+}_{tg,2})^2)-(1+(\hat{s}^{+}_{tg,1})^2)}{(1+(\hat{s}^{+}_{tg,1})^2)}, \quad (15)$$

$$GMV^{+} = \left(\frac{T}{N}\right)\left(\frac{T-N-K}{T-K-1}\right)\frac{\left(\hat{\sigma}^{+}_{mv,1}\right)^2-\left(\hat{\sigma}^{+}_{mv,2}\right)^2}{\left(\hat{\sigma}^{+}_{mv,2}\right)^2}, \quad (16)$$

where $\hat{s}^{+}_{tg,2}$ and $\hat{s}^{+}_{tg,1}$ are the Sharpe ratios of the short-sales constrained tangent portfolios on the extended and test assets mean-variance frontiers, while $\left(\hat{\sigma}^{+}_{mv,1}\right)^2$ and $\left(\hat{\sigma}^{+}_{mv,2}\right)^2$ are the variances of the short-sales constrained minimum variance portfolios on the extended and test assets mean-variance frontiers, respectively.

Table 6 presents the key findings for the short-sales constrained GRS^{+} and GMV^{+} tests using various sets of portfolios. Panel A presents the results using monthly portfolio returns. The values of the constrained statistics are lower than the values of the unconstrained statistics but the general finding still holds that the GMV^{+} test tends to reject the null hypothesis much more often than the GRS^{+}. The only set of decile portfolios for which both tests decisively reject the null hypothesis is the set of portfolios sorted by standard deviation of portfolio return. Hence, researches could be misled into failing to reject the null hypothesis when performing only the GRS test and not performing the GMV test. Panel B of Table 6 presents the findings of the constrained versions of both tests using daily portfolio returns. The results for daily portfolio returns are largely the same as for monthly portfolio returns. The short-sales constrained GMV^{+} test tends to reject the null hypothesis of no improvement in the GMV portfolio much more often than the short-sales constrained GRS^{+} test rejects the null of no improvement in the tangent portfolio. This suggests that the imposition of short sales constraints does not materially change the relative performance of the GRS and GMV tests.

Insert Table 6 about here.

Table 7 presents similar findings using sets of 25 and 30 portfolios as baseline or reference assets with the same test assets as before. Panel A reports the results for the case of monthly portfolio returns while Panel B reports the results for the case of daily portfolio returns. The rejection frequency of the GRS^{+} test relative to the GMV^{+} test is somewhat higher compared to the case when $N = 10$. Note that when $N = 25$ and we use portfolios double-sorted on size/short-term reversal, size/momentum and size/long-term reversal, both tests strongly reject the null hypotheses of no improvement in the tangent and GMV portfolios, respectively.

Insert Table 7 about here.

7.2 Size of GRS Test in Finite Samples

The next robustness check involves checking the actual size, in the sense of probability of rejection, of both the GRS and the GMV using simulated portfolio returns under the corresponding null hypothesis. Knowing that the null hypothesis is true in this simulation allows us the evaluate that actual frequency of rejecting a true null. For each simulation, I generate 10,000 random samples of portfolio returns for various values of N, K, T, as well as nominal sizes of the test. I present my findings from this simulation exercise in Table 8. Panel A uses $N = 10$ and I tried to match the first two moments of the simulated portfolio returns to the corresponding moments of the decile portfolio sorted by market capitalization.[4] Panel B uses $N = 25$ where the simulated moments are designed to match the first two moments of the 25 portfolios sorted by size and book-to-market. Finally, Panel C uses $N = 30$ and the simulated portfolio returns match the first two moments of the 30 industry portfolios. Several key findings emerge from the table. First, both tests tend to under-reject relative to the nominal size. This is in some sense not so bad as the tests are rejecting a null hypothesis that was true by construction. Hence, researchers can be assured that if the null hypothesis of mean-variance efficiency is true in reality then both the GRS test and the GMV test will have a slightly lower probability of a false rejection relative to the nominal size used.

Insert Table 8 about here.

Note that for $K = 1$ and several values of T both tests reject equally often. In fact, the larger the set of base assets, N, the closer the actual rejection rates get to the nominal

[4]Note that an exact match under the null hypothesis will generically be impossible as first moment will not match exactly when we require $\alpha = 0_N$ and the second moments will not match exactly when we impose $\delta = 0_N$.

rejection rate of the null hypothesis test. As soon as we move to cases where more than one asset's mean-variance efficiency is tested ($K > 1$) the GMV test rejects considerably less often a true null hypothesis. The relative under-rejection becomes larger for smaller nominal sizes of the test. This could be an artifact of the simulation sampling the extreme tails of the distribution of the test statistics. The only mitigating factor to this effect is increasing the sample size in which case the actual rejection rates of both tests are very close to each other.

7.3 Effect of Fat-Tailed Error Distributions

The final robustness check I perform involves simulating portfolio returns under the null hypothesis as in the previous sub-section but with errors that follow a Student-t distribution. I choose to simulate errors with 5 degrees of freedom. This is partly driven by prior research investigating the best fit of portfolio returns to a Student-t distribution (Tu and Zhou (2004), Glabadanidis (2014)) as well as by the intention to have an experiment where the returns have a really fat-tailed distribution.

Insert Table 9 about here.

Table 9 presents the findings from this simulation exercise. First, it is apparent that both the GRS and the GMV tests under-reject when $K > 1$. Second, and most important, the GRS test under-rejects for $K = 1$ while the GMV test tends to over-reject. Furthermore, this over-rejection of the GMV for the case of $K = 1$ becomes more severe for larger sample sizes although this is somewhat mitigated as N increases. This is a notable difference compared to the finite sample actual rates of rejection by the GMV test of the null hypothesis of no improvement in the GMV portfolio where return innovations follow a Gaussian distribution. Finally, it should be noted that when $K > 1$ and large sample size the GMV tests gets closer to the nominal size of the test of the null hypothesis than the GRS test.

8 Concluding Comments

In this paper, I have presented the GMV test of the null hypothesis of no improvement of the GMV portfolio in finite samples. I show that it has a very nice interpretation as a linear function of the proportional improvement in the variance of the global minimum variance portfolio. I also provide some intuition as to why the GMV test will tend to produce larger values than the GRS test. In addition, I offer a graphical and a portfolio interpretation as well as a few thoughts and ideas regarding the power of the GMV test in practical applications. I apply both tests in an unconditional as well as several conditional frameworks to illustrate and compare the performance of the GMV test to the performance of the GRS test. The empirical findings are that very often the conclusions drawn from the GRS test are dramatically different from the conclusions drawn from the GMV test. In particular, testing the null hypothesis of no improvement in the tangent portfolio following the addition of the market portfolio, the FF3 factors and the C4 factors, especially in their conditional versions, lead to a failure to reject. At the same time, the test of the same null hypothesis using unconditional versions of these factors very often reject using the GMV test. The only models for which the GMV test fails to reject are conditional models with one or more scaling state variable.

The conditional framework I adopt is *ad hoc* and leaves out a few potentially interesting questions. An important avenue of future research would be to explicitly model the dynamics of the joint set of base and test returns with the instrumental variable returns. Furthermore, imposing and testing any cross-sectional restrictions would allow researchers to explicitly test the statistical significance of test assets' risk premia as well as hedging risk premia associated with the instrumental variables. This is a natural next step to take as it would once again raise the bar in terms of the conditions we expect a priced factor or a relevant instrumental variable to satisfy both in the time series as well as in the cross-sectional implications of an empirical asset pricing test of the models in Chen, Roll and Ross (1998) and Ross (1976). I leave these and other interesting questions to future research.

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Table 1. GRS and GMV Empirical Results.

This table reports the findings of testing the null hypotheses of no improvement of the tangent and minimum variance portfolio, respectively, using the GRS and the GMV test with several sets of decile portfolios as the base assets and three separate sets of factors.

Panel A: Decile portfolios monthly value-weighted returns between 1960:01 and 2013:12.

Base	Test	T^2_{GRS}	GRS	T^2_{GMV}	GMV	s_1	s_2	$\sigma_{mv,1}$	$\sigma_{mv,2}$
Size	MKT	0.014	0.900	0.259	16.534***	0.111	0.163	0.045	0.040
Size	SMB	0.022	1.389	1.487	95.044***	0.074	0.165	0.030	0.019
Size	FF3	0.008	0.539	0.504	32.176***	0.219	0.239	0.017	0.014
Size	C4	0.011	0.723	0.353	22.532***	0.319	0.338	0.014	0.012
BM	MKT	0.019	1.232	0.194	12.380***	0.111	0.179	0.045	0.041
BM	HML	0.029	1.826*	1.477	94.369***	0.135	0.217	0.028	0.018
BM	FF3	0.007	0.430	0.657	42.006***	0.219	0.235	0.017	0.013
BM	C4	0.007	0.439	0.433	27.648***	0.319	0.330	0.014	0.012
CP	MKT	0.029	1.831*	0.220	14.064***	0.111	0.204	0.045	0.040
CP	FF3	0.005	0.351	0.354	22.626***	0.219	0.232	0.017	0.014
CP	C4	0.004	0.264	0.299	19.102***	0.319	0.326	0.014	0.012
DP	MKT	0.023	1.478	0.332	21.211***	0.111	0.189	0.045	0.039
DP	FF3	0.017	1.111	0.128	8.177***	0.219	0.258	0.017	0.016
DP	C4	0.018	1.164	0.068	4.360***	0.319	0.349	0.014	0.014
EP	MKT	0.038	2.447***	0.220	14.085***	0.111	0.226	0.045	0.040
EP	FF3	0.011	0.716	0.446	28.511***	0.219	0.245	0.017	0.014
EP	C4	0.009	0.549	0.338	21.607***	0.319	0.333	0.014	0.012
Mom	MKT	0.070	4.495***	0.373	23.839***	0.111	0.289	0.045	0.038
Mom	UMD	0.064	4.094***	2.076	132.643***	0.172	0.309	0.042	0.024
Mom	FF3	0.080	5.112***	0.102	6.524***	0.219	0.363	0.017	0.016
Mom	C4	0.040	2.584***	0.151	9.630***	0.319	0.382	0.014	0.013
Str	MKT	0.036	2.283**	0.377	24.099***	0.111	0.220	0.045	0.038
Str	STR	0.040	2.560***	1.545	98.731***	0.168	0.264	0.031	0.019
Str	FF3	0.029	1.874**	0.151	9.640***	0.219	0.281	0.017	0.016
Str	C4	0.031	1.963**	0.071	4.543***	0.319	0.368	0.014	0.014
Ltr	MKT	0.020	1.293	0.373	23.832***	0.111	0.181	0.045	0.038
Ltr	LTR	0.026	1.693*	1.271	81.223***	0.121	0.204	0.025	0.017
Ltr	FF3	0.008	0.490	0.367	23.423***	0.219	0.237	0.017	0.014
Ltr	C4	0.006	0.410	0.197	12.584***	0.319	0.330	0.014	0.013
Ind	MKT	0.022	1.422	0.639	40.810***	0.111	0.187	0.045	0.035
Ind	FF3	0.044	2.822***	0.440	28.105***	0.219	0.307	0.017	0.014
Ind	C4	0.033	2.091**	0.252	16.122***	0.319	0.371	0.014	0.013
Std	MKT	0.262	16.724***	1.747	111.636***	0.111	0.527	0.045	0.027
Std	FF3	0.224	14.311***	0.487	31.123***	0.219	0.532	0.017	0.014
Std	C4	0.212	13.571***	0.282	17.989***	0.319	0.579	0.014	0.012

Table 1 Continued:

Panel B: Decile portfolios daily value-weighted returns between 1960/01/04 and 2013/12/31.

Base	Test	T^2_{GRS}	GRS	T^2_{GMV}	GMV	s_1	s_2	$\sigma_{mv,1}$	$\sigma_{mv,2}$
Size	MKT	0.001	1.637*	0.740	1004.662***	0.024	0.042	0.010	0.007
Size	SMB	0.002	2.265**	1.360	1846.854***	0.012	0.042	0.005	0.003
Size	FF3	0.001	1.559	0.533	723.750***	0.057	0.066	0.003	0.002
Size	C4	0.001	1.455	0.379	514.202***	0.086	0.092	0.002	0.002
BM	MKT	0.001	1.887**	0.143	194.581***	0.024	0.044	0.010	0.009
BM	HML	0.002	2.400***	1.243	1688.858***	0.038	0.057	0.005	0.003
BM	FF3	0.001	0.926	0.551	748.200***	0.057	0.063	0.003	0.002
BM	C4	0.000	0.638	0.389	529.020***	0.086	0.089	0.002	0.002
Mom	MKT	0.004	4.981***	0.275	373.826***	0.024	0.065	0.010	0.009
Mom	UMD	0.003	4.586***	1.553	2108.871***	0.048	0.076	0.007	0.004
Mom	FF3	0.005	7.084***	0.133	180.124***	0.057	0.092	0.003	0.003
Mom	C4	0.003	3.712***	0.137	186.244***	0.086	0.101	0.002	0.002
Str	MKT	0.039	52.679***	0.260	353.689***	0.024	0.198	0.010	0.009
Str	STR	0.029	39.039***	1.457	1978.684***	0.162	0.236	0.007	0.004
Str	FF3	0.039	53.569***	0.071	97.005***	0.057	0.207	0.003	0.003
Str	C4	0.042	57.032***	0.026	35.392***	0.086	0.223	0.002	0.002
Ltr	MKT	0.002	2.176**	0.346	469.626***	0.024	0.047	0.010	0.008
Ltr	LTR	0.002	2.356***	1.030	1398.505***	0.029	0.050	0.004	0.003
Ltr	FF3	0.001	1.362	0.265	360.141***	0.057	0.065	0.003	0.002
Ltr	C4	0.001	1.144	0.202	273.838***	0.086	0.091	0.002	0.002
Ind	MKT	0.001	1.456	0.728	988.667***	0.024	0.041	0.010	0.007
Ind	FF3	0.003	3.838***	0.392	531.865***	0.057	0.078	0.003	0.002
Ind	C4	0.002	2.554***	0.203	276.184***	0.086	0.097	0.002	0.002
Std	MKT	0.027	36.922***	4.656	6324.450***	0.024	0.167	0.010	0.004
Std	FF3	0.026	35.294***	0.757	1028.647***	0.057	0.171	0.003	0.002
Std	C4	0.026	34.962***	0.432	586.175***	0.086	0.183	0.002	0.002

Table 2. GRS and GMV Empirical Results.

This table reports the findings of testing the null hypotheses of no improvement in the tangent and minimum variance portfolio, respectively, using the GRS and the GMV test using several sets of 25 and 30 portfolios as the base assets and three separate sets of factors.

Panel A: 25/30 portfolios monthly value-weighted returns between 1960:01 and 2013:12.

Base	Test	T^2_{GRS}	GRS	T^2_{GMV}	GMV	s_1	s_2	$\sigma_{mv,1}$	$\sigma_{mv,2}$
25 Size/BM	MKT	0.159	3.958***	0.710	17.710***	0.111	0.416	0.045	0.034
25 Size/BM	FF3	0.119	2.958***	0.882	22.007***	0.219	0.415	0.017	0.012
25 Size/BM	C4	0.093	2.313***	0.616	15.375***	0.319	0.451	0.014	0.011
25 Size/Str	MKT	0.196	4.884***	0.694	17.331***	0.111	0.459	0.045	0.034
25 Size/Str	FF3	0.168	4.197***	0.545	13.592***	0.219	0.474	0.017	0.013
25 Size/Str	C4	0.195	4.871***	0.354	8.836***	0.319	0.563	0.014	0.012
25 Size/Mom	MKT	0.210	5.248***	0.826	20.620***	0.111	0.475	0.045	0.033
25 Size/Mom	FF3	0.190	4.748***	0.523	13.051***	0.219	0.498	0.017	0.013
25 Size/Mom	C4	0.127	3.177***	0.485	12.110***	0.319	0.492	0.014	0.012
25 Size/Ltr	MKT	0.116	2.896***	0.815	20.342***	0.111	0.360	0.045	0.033
25 Size/Ltr	FF3	0.080	1.998***	0.659	16.453***	0.219	0.363	0.017	0.013
25 Size/Ltr	C4	0.043	1.079	0.424	10.575***	0.319	0.386	0.014	0.012
30 Industry	MKT	0.048	0.990	0.924	19.066***	0.111	0.247	0.045	0.032
30 Industry	FF3	0.073	1.501**	0.658	13.580***	0.219	0.353	0.017	0.013
30 Industry	C4	0.064	1.320	0.409	8.445***	0.319	0.415	0.014	0.012

Panel B: 25/30 portfolios daily value-weighted returns between 1960/01/04 and 2013/12/31.

Base	Test	T^2_{GRS}	GRS	T^2_{GMV}	GMV	s_1	s_2	$\sigma_{mv,1}$	$\sigma_{mv,2}$
25 Size/BM	MKT	0.011	6.210***	0.933	506.431***	0.024	0.110	0.010	0.007
25 Size/BM	FF3	0.009	4.726***	0.776	420.958***	0.057	0.109	0.003	0.002
25 Size/BM	C4	0.008	4.411***	0.557	302.444***	0.086	0.125	0.002	0.002
25 Size/Str	MKT	0.356	193.395***	0.984	534.010***	0.024	0.598	0.010	0.007
25 Size/Str	FF3	0.359	194.597***	0.437	237.341***	0.057	0.602	0.003	0.002
25 Size/Str	C4	0.363	196.748***	0.255	138.135***	0.086	0.610	0.002	0.002
25 Size/Mom	MKT	0.017	9.153***	1.257	682.118***	0.024	0.132	0.010	0.007
25 Size/Mom	FF3	0.015	7.982***	0.451	244.741***	0.057	0.134	0.003	0.002
25 Size/Mom	C4	0.010	5.509***	0.411	223.101***	0.086	0.133	0.002	0.002
25 Size/Ltr	MKT	0.007	3.754***	1.087	590.022***	0.024	0.087	0.010	0.007
25 Size/Ltr	FF3	0.004	2.255***	0.588	319.261***	0.057	0.086	0.003	0.002
25 Size/Ltr	C4	0.003	1.848***	0.401	217.473***	0.086	0.104	0.002	0.002
30 Industry	MKT	0.002	0.968	0.947	428.360***	0.024	0.052	0.010	0.007
30 Industry	FF3	0.004	1.910***	0.565	255.359***	0.057	0.086	0.003	0.002
30 Industry	C4	0.003	1.393*	0.327	147.837***	0.086	0.103	0.002	0.002

Table 3. International Evidence using GRS and GMV.

This table reports the findings of testing the null hypotheses of no improvement in the tangent and minimum variance portfolio, respectively, using the GRS and the GMV test using several sets of 9 portfolios sorted on B/M, E/P, CE/P and D/P as well as sets of 6 portfolios double-sorted on size/book-to-market and size/momentum as the base assets and the respective regional (Reg) and global (Glob) market portfolio, three-factor portfolios and four-factor portfolios as the test assets. Characteristics-sorted portfolio returns are value-weighted at the monthly frequency between 1975:01 and 2013:12 and denominated in USD. Size/BM and size/momentum portfolios returns are value-weighted at the monthly frequency between 1990:11 and 2013:12. The risk-free asset used is the monthly US risk-free rate of return.

Panel A: Characteristics-sorted international portfolios.

Country	T^2_{GRS}	GRS	T^2_{GMV}	GMV	s_1	s_2	$\sigma_{mv,1}$	$\sigma_{mv,2}$
Australia	0.058	2.952***	0.160	8.171***	0.176	0.301	0.068	0.063
Canada	0.055	2.652***	0.290	14.041***	0.179	0.297	0.057	0.050
France	0.037	1.872*	0.229	11.687***	0.180	0.265	0.066	0.059
Germany	0.067	3.413***	0.127	6.512***	0.179	0.317	0.061	0.058
Italy	0.027	1.388	0.137	6.981***	0.126	0.208	0.075	0.071
Japan	0.059	3.025***	0.122	6.253***	0.144	0.285	0.061	0.058
UK	0.109	5.590***	0.126	6.435***	0.209	0.397	0.063	0.059

Table 3 Continued:

Panel B: Size/BM-sorted international portfolios.

Region	Test	T^2_{GRS}	GRS	T^2_{GMV}	GMV	s_1	s_2	$\sigma_{mv,1}$	$\sigma_{mv,2}$
AP	Reg MKT	0.123	5.580***	0.152	6.910***	0.134	0.378	0.061	0.057
AP	Reg FF3	0.084	3.814***	0.969	44.087***	0.233	0.378	0.020	0.014
AP	Reg C4	0.051	2.332**	0.624	28.376***	0.369	0.441	0.016	0.013
AP	Glob MKT	0.130	5.899***	0.058	2.658***	0.115	0.380	0.043	0.042
AP	Glob FF3	0.111	5.065***	0.198	9.014***	0.229	0.412	0.013	0.012
AP	Glob C4	0.075	3.425***	0.166	7.559***	0.337	0.444	0.012	0.011
EU	Reg MKT	0.055	2.522**	0.394	17.936***	0.118	0.265	0.050	0.042
EU	Reg FF3	0.031	1.402	0.943	42.901***	0.196	0.265	0.015	0.011
EU	Reg C4	0.010	0.437	0.664	30.196***	0.377	0.391	0.013	0.010
EU	Glob MKT	0.058	2.647**	0.304	13.837***	0.115	0.269	0.043	0.038
EU	Glob FF3	0.034	1.531	0.602	27.405***	0.229	0.296	0.013	0.010
EU	Glob C4	0.020	0.908	0.439	19.977***	0.337	0.369	0.012	0.010
JP	Reg MKT	0.030	1.368	0.176	8.019***	0.002	0.173	0.058	0.054
JP	Reg FF3	0.004	0.172	0.856	38.940***	0.162	0.174	0.020	0.015
JP	Reg C4	0.002	0.105	0.570	25.927***	0.175	0.182	0.016	0.013
JP	Glob MKT	0.041	1.862*	0.135	6.128***	0.115	0.234	0.043	0.041
JP	Glob FF3	0.020	0.905	0.200	9.100***	0.229	0.271	0.013	0.012
JP	Glob C4	0.021	0.934	0.150	6.817***	0.337	0.369	0.012	0.011
GL	Glob MKT	0.171	7.792***	0.693	31.546***	0.115	0.432	0.043	0.033
GL	Glob FF3	0.128	5.802***	0.909	41.326***	0.229	0.432	0.013	0.009
GL	Glob C4	0.093	4.252***	0.692	31.495***	0.337	0.467	0.012	0.009
NA	Reg MKT	0.114	5.196***	0.465	21.174***	0.166	0.380	0.043	0.036
NA	Reg FF3	0.090	4.089***	0.891	40.518***	0.224	0.380	0.017	0.012
NA	Reg C4	0.068	3.105***	0.712	32.398***	0.289	0.397	0.015	0.012
NA	Glob MKT	0.116	5.287***	0.415	18.890***	0.115	0.362	0.043	0.037
NA	Glob FF3	0.093	4.234***	0.716	32.568***	0.229	0.388	0.013	0.010
NA	Glob C4	0.070	3.171***	0.605	27.519***	0.337	0.437	0.012	0.009

Table 3 Continued:

Panel C: Size/Momentum-sorted international portfolios.

Region	Test	T^2_{GRS}	GRS	T^2_{GMV}	GMV	s_1	s_2	$\sigma_{mv,1}$	$\sigma_{mv,2}$
AP	Reg MKT	0.147	6.709***	0.475	21.618***	0.134	0.410	0.061	0.050
AP	Reg FF3	0.159	7.250***	0.669	30.444***	0.233	0.472	0.020	0.016
AP	Reg C4	0.079	3.581***	0.530	24.097***	0.369	0.475	0.016	0.013
AP	Glob MKT	0.151	6.857***	0.269	12.229***	0.115	0.407	0.043	0.039
AP	Glob FF3	0.150	6.841***	0.198	9.019***	0.229	0.459	0.013	0.012
AP	Glob C4	0.094	4.273***	0.147	6.700***	0.337	0.467	0.012	0.011
EU	Reg MKT	0.258	11.753***	0.456	20.724***	0.118	0.525	0.050	0.042
EU	Reg FF3	0.256	11.661***	0.560	25.474***	0.196	0.552	0.015	0.012
EU	Reg C4	0.143	6.512***	0.485	22.059***	0.377	0.553	0.013	0.011
EU	Glob MKT	0.263	11.976***	0.454	20.665***	0.115	0.529	0.043	0.036
EU	Glob FF3	0.261	11.880***	0.293	13.349***	0.229	0.572	0.013	0.011
EU	Glob C4	0.173	7.886***	0.232	10.557***	0.337	0.554	0.012	0.010
JP	Reg MKT	0.013	0.569	0.403	18.323***	0.002	0.112	0.058	0.049
JP	Reg FF3	0.021	0.942	0.525	23.865***	0.162	0.218	0.020	0.016
JP	Reg C4	0.017	0.760	0.501	22.810***	0.175	0.219	0.016	0.013
JP	Glob MKT	0.024	1.109	0.348	15.846***	0.115	0.195	0.043	0.037
JP	Glob FF3	0.032	1.437	0.171	7.773***	0.229	0.293	0.013	0.012
JP	Glob C4	0.035	1.585	0.111	5.045***	0.337	0.390	0.012	0.011
GL	Glob MKT	0.135	6.136***	0.845	38.440***	0.115	0.387	0.043	0.032
GL	Glob FF3	0.122	5.530***	0.581	26.410***	0.229	0.425	0.013	0.010
GL	Glob C4	0.070	3.186***	0.492	22.362***	0.337	0.438	0.012	0.009
NA	Reg MKT	0.111	5.049***	0.725	32.985***	0.166	0.376	0.043	0.033
NA	Reg FF3	0.080	3.622***	0.551	25.046***	0.224	0.366	0.017	0.013
NA	Reg C4	0.048	2.162**	0.467	21.233***	0.289	0.367	0.015	0.012
NA	Glob MKT	0.115	5.237***	0.767	34.916***	0.115	0.360	0.043	0.033
NA	Glob FF3	0.087	3.943***	0.470	21.385***	0.229	0.379	0.013	0.011
NA	Glob C4	0.045	2.041*	0.426	19.382***	0.337	0.404	0.012	0.010

Table 4. More International Evidence using GRS and GMV.

This table reports the findings of testing the null hypotheses of no improvement in the tangent and minimum variance portfolio, respectively, using the GRS and the GMV test using several sets of 25 portfolios double-sorted on size/book-to-market and size/momentum as the base assets and the respective regional (Reg) and global (Glob) market portfolio, three-factor portfolios and four-factor portfolios as the test assets. Size/BM and size/momentum portfolios returns are value-weighted at the monthly frequency between 1990:11 and 2013:12 and denominated in USD. The risk-free asset used is the monthly US risk-free rate of return.

Panel A: 25 Size/BM-sorted international regional portfolios.

Region	Test	T^2_{GRS}	GRS	T^2_{GMV}	GMV	s_1	s_2	$\sigma_{mv,1}$	$\sigma_{mv,2}$
AP	Reg MKT	0.314	3.188***	0.477	4.848***	0.134	0.581	0.061	0.050
AP	Reg FF3	0.271	2.747***	0.987	10.013***	0.233	0.583	0.020	0.014
AP	Reg C4	0.212	2.146***	0.652	6.617***	0.369	0.613	0.016	0.013
AP	Glob MKT	0.320	3.247***	0.221	2.249***	0.115	0.581	0.043	0.039
AP	Glob FF3	0.296	3.008***	0.351	3.564***	0.229	0.604	0.013	0.011
AP	Glob C4	0.234	2.376***	0.289	2.926***	0.337	0.612	0.012	0.010
EU	Reg MKT	0.153	1.549*	0.706	7.169***	0.118	0.411	0.050	0.038
EU	Reg FF3	0.126	1.280	0.961	9.747***	0.196	0.411	0.015	0.011
EU	Reg C4	0.111	1.128	0.709	7.196***	0.377	0.519	0.013	0.010
EU	Glob MKT	0.154	1.561**	0.580	5.891***	0.115	0.411	0.043	0.035
EU	Glob FF3	0.127	1.292	0.665	6.744***	0.229	0.432	0.013	0.010
EU	Glob C4	0.122	1.232	0.541	5.488***	0.337	0.499	0.012	0.009
JP	Reg MKT	0.116	1.180	0.613	6.220***	0.002	0.341	0.058	0.046
JP	Reg FF3	0.094	0.952	0.882	8.944***	0.162	0.350	0.020	0.015
JP	Reg C4	0.087	0.878	0.656	6.652***	0.175	0.347	0.016	0.012
JP	Glob MKT	0.129	1.311	0.426	4.325***	0.115	0.379	0.043	0.036
JP	Glob FF3	0.111	1.125	0.333	3.382***	0.229	0.411	0.013	0.011
JP	Glob C4	0.099	1.008	0.257	2.610***	0.337	0.474	0.012	0.010
GL	Glob MKT	0.423	4.299***	1.094	11.104***	0.115	0.665	0.043	0.030
GL	Glob FF3	0.367	3.728***	0.945	9.592***	0.229	0.663	0.013	0.009
GL	Glob C4	0.319	3.235***	0.748	7.588***	0.337	0.685	0.012	0.009
NA	Reg MKT	0.321	3.258***	0.906	9.198***	0.166	0.598	0.043	0.031
NA	Reg FF3	0.288	2.925***	0.953	9.672***	0.224	0.594	0.017	0.012
NA	Reg C4	0.230	2.329***	0.800	8.114***	0.289	0.577	0.015	0.011
NA	Glob MKT	0.327	3.322***	0.863	8.758***	0.115	0.587	0.043	0.032
NA	Glob FF3	0.288	2.918***	0.785	7.960***	0.229	0.596	0.013	0.010
NA	Glob C4	0.213	2.156***	0.678	6.872***	0.337	0.592	0.012	0.009

Table 4 Continued:

Panel B: Size/Momentum-sorted international regional portfolios.

Region	Test	T^2_{GRS}	GRS	T^2_{GMV}	GMV	s_1	s_2	$\sigma_{mv,1}$	$\sigma_{mv,2}$
AP	Reg MKT	0.416	4.227***	1.083	11.000***	0.134	0.665	0.061	0.042
AP	Reg FF3	0.427	4.330***	0.784	7.950***	0.233	0.710	0.020	0.015
AP	Reg C4	0.336	3.410***	0.613	6.213***	0.369	0.720	0.016	0.013
AP	Reg MKT	0.418	4.242***	0.596	6.047***	0.115	0.661	0.043	0.034
AP	Glob FF3	0.419	4.248***	0.321	3.261***	0.229	0.702	0.013	0.011
AP	Glob C4	0.335	3.397***	0.231	2.339***	0.337	0.698	0.012	0.010
EU	Reg MKT	0.483	4.902***	0.885	8.988***	0.118	0.710	0.050	0.037
EU	Reg FF3	0.450	4.566***	0.648	6.576***	0.196	0.711	0.015	0.012
EU	Reg C4	0.308	3.122***	0.553	5.611***	0.377	0.702	0.013	0.010
EU	Glob MKT	0.487	4.947***	0.802	8.145***	0.115	0.712	0.043	0.032
EU	Glob FF3	0.447	4.539***	0.445	4.514***	0.229	0.723	0.013	0.011
EU	Glob C4	0.349	3.541***	0.365	3.703***	0.337	0.709	0.012	0.010
JP	Reg MKT	0.084	0.856	0.549	5.575***	0.002	0.290	0.058	0.047
JP	Reg FF3	0.083	0.838	0.562	5.704***	0.162	0.333	0.020	0.016
JP	Reg C4	0.079	0.801	0.540	5.476***	0.175	0.335	0.016	0.013
JP	Glob MKT	0.095	0.960	0.466	4.736***	0.115	0.330	0.043	0.036
JP	Glob FF3	0.096	0.976	0.292	2.963***	0.229	0.392	0.013	0.011
JP	Glob C4	0.104	1.054	0.219	2.220***	0.337	0.479	0.012	0.010
GL	Glob MKT	0.502	5.096***	1.373	13.942***	0.115	0.722	0.043	0.028
GL	Glob FF3	0.471	4.774***	0.639	6.479***	0.229	0.740	0.013	0.010
GL	Glob C4	0.397	4.028***	0.546	5.539***	0.337	0.746	0.012	0.009
NA	Reg MKT	0.382	3.874***	1.095	11.113***	0.166	0.648	0.043	0.030
NA	Reg FF3	0.328	3.330***	0.601	6.097***	0.224	0.628	0.017	0.013
NA	Reg C4	0.280	2.840***	0.520	5.273***	0.289	0.622	0.015	0.012
NA	Glob MKT	0.388	3.937***	1.125	11.421***	0.115	0.637	0.043	0.030
NA	Glob FF3	0.329	3.339***	0.549	5.570***	0.229	0.631	0.013	0.010
NA	Glob C4	0.270	2.738***	0.509	5.167***	0.337	0.644	0.012	0.009

Table 5. GRS and GMV with Instrumental Variables.

This table reports the findings of testing the null hypotheses of no improvement in the tangent and minimum variance portfolio, respectively, using the GRS and the GMV test using several sets of ten portfolios as the base assets and three separate sets of factors where several pre-determined instrumental variables are used to model a time-varying intercept and time-varying slopes for each factor. The instrumental variables are the twelve-month dividend yield on the stock market (DY), the book-to-market ratio of the stock market (BM), the 30-day Treasury bill rate (TB), the Moody's BAA minus AAA credit spread (CS), and the inflation rate (INF) from Goyal and Welch (2008). Portfolio returns are value-weighted and cover the period between 1960:01 and 2013:12.

Panel A: Size Deciles.

Base	Test	Instr	T^2_{GRS}	GRS	T^2_{GMV}	GMV
Size	MKT	DY	0.017	1.059	0.041	2.594***
Size	MKT	BM	0.015	0.931	0.106	6.802***
Size	MKT	TB	0.024	1.516	0.012	0.771
Size	MKT	CS	0.020	1.271	0.039	2.493***
Size	MKT	INF	0.012	0.751	0.016	1.052
Size	MKT	All	0.015	0.941	0.018	1.161
Size	SMB	DY	0.031	1.992**	0.133	8.492***
Size	SMB	BM	0.032	2.027**	0.362	23.150***
Size	SMB	TB	0.039	2.505***	0.047	3.024***
Size	SMB	CS	0.020	1.285	0.035	2.209**
Size	SMB	INF	0.035	2.218**	0.022	1.383
Size	SMB	All	0.021	1.343	0.064	4.109***
Size	FF3	DY	0.016	1.029	0.037	2.391***
Size	FF3	BM	0.015	0.987	0.143	9.120***
Size	FF3	TB	0.024	1.550	0.039	2.469***
Size	FF3	CS	0.018	1.130	0.041	2.641***
Size	FF3	INF	0.010	0.628	0.035	2.213**
Size	FF3	All	0.014	0.906	0.076	4.851***
Size	C4	DY	0.015	0.927	0.043	2.767***
Size	C4	BM	0.015	0.937	0.107	6.808***
Size	C4	TB	0.024	1.537	0.023	1.448
Size	C4	CS	0.020	1.268	0.033	2.081**
Size	C4	INF	0.010	0.669	0.017	1.090
Size	C4	All	0.017	1.072	0.065	4.169***

Table 5 Continued:

Panel B: Book-to-market Deciles.

Base	Test	Instr	T^2_{GRS}	GRS	T^2_{GMV}	GMV
BM	MKT	DY	0.005	0.351	0.051	3.283***
BM	MKT	BM	0.005	0.300	0.073	4.661***
BM	MKT	TB	0.008	0.495	0.121	7.760***
BM	MKT	CS	0.026	1.632*	0.177	11.312***
BM	MKT	INF	0.019	1.194	0.102	6.534***
BM	MKT	All	0.016	1.047	0.058	3.732***
BM	HML	DY	0.016	0.991	0.106	6.754***
BM	HML	BM	0.017	1.080	0.591	37.761***
BM	HML	TB	0.020	1.263	0.178	11.368***
BM	HML	CS	0.017	1.061	0.144	9.213***
BM	HML	INF	0.053	3.415***	0.071	4.549***
BM	HML	All	0.015	0.971	0.010	0.652
BM	FF3	DY	0.008	0.505	0.044	2.832***
BM	FF3	BM	0.009	0.553	0.230	14.713***
BM	FF3	TB	0.009	0.602	0.081	5.184***
BM	FF3	CS	0.019	1.215	0.108	6.928***
BM	FF3	INF	0.018	1.157	0.082	5.258***
BM	FF3	All	0.013	0.822	0.021	1.356
BM	C4	DY	0.006	0.380	0.025	1.572
BM	C4	BM	0.009	0.575	0.163	10.418***
BM	C4	TB	0.008	0.524	0.025	1.620*
BM	C4	CS	0.016	1.033	0.018	1.162
BM	C4	INF	0.014	0.901	0.048	3.048***
BM	C4	All	0.009	0.581	0.013	0.815

Table 5 Continued:

Panel C: Momentum Deciles.

Base	Test	Instr	T^2_{GRS}	GRS	T^2_{GMV}	GMV
MOM	MKT	DY	0.035	2.237**	0.064	4.106***
MOM	MKT	BM	0.039	2.487***	0.036	2.317**
MOM	MKT	TB	0.018	1.179	0.081	5.151***
MOM	MKT	CS	0.046	2.931***	0.060	3.820***
MOM	MKT	INF	0.050	3.164***	0.038	2.399***
MOM	MKT	All	0.037	2.341**	0.028	1.802*
MOM	UMD	DY	0.037	2.393***	0.098	6.258***
MOM	UMD	BM	0.042	2.691***	0.462	29.515***
MOM	UMD	TB	0.037	2.385***	0.079	5.040***
MOM	UMD	CS	0.020	1.283	0.070	4.502***
MOM	UMD	INF	0.084	5.347***	0.057	3.638***
MOM	UMD	All	0.031	1.981**	0.024	1.533
MOM	FF3	DY	0.029	1.833*	0.017	1.065
MOM	FF3	BM	0.032	2.062**	0.037	2.361***
MOM	FF3	TB	0.018	1.165	0.009	0.584
MOM	FF3	CS	0.049	3.153***	0.055	3.495***
MOM	FF3	INF	0.054	3.453***	0.012	0.761
MOM	FF3	All	0.030	1.901**	0.023	1.497
MOM	C4	DY	0.017	1.062	0.026	1.656*
MOM	C4	BM	0.018	1.139	0.092	5.849***
MOM	C4	TB	0.012	0.751	0.006	0.362
MOM	C4	CS	0.023	1.490	0.013	0.851
MOM	C4	INF	0.036	2.293**	0.017	1.114
MOM	C4	All	0.015	0.983	0.004	0.254

Table 5 Continued:

Panel D: Short-term Reversal Deciles.

Base	Test	Instr	T^2_{GRS}	GRS	T^2_{GMV}	GMV
STR	MKT	DY	0.038	2.418***	0.039	2.483***
STR	MKT	BM	0.041	2.641***	0.060	3.804***
STR	MKT	TB	0.017	1.093	0.026	1.641*
STR	MKT	CS	0.018	1.175	0.042	2.669***
STR	MKT	INF	0.028	1.760*	0.035	2.208**
STR	MKT	All	0.030	1.929**	0.080	5.123***
STR	STR	DY	0.057	3.623***	0.054	3.465***
STR	STR	BM	0.058	3.684***	0.359	22.921***
STR	STR	TB	0.026	1.649*	0.029	1.849**
STR	STR	CS	0.019	1.239	0.066	4.194***
STR	STR	INF	0.049	3.116***	0.036	2.297**
STR	STR	All	0.035	2.207**	0.037	2.347**
STR	FF3	DY	0.030	1.903**	0.006	0.392
STR	FF3	BM	0.034	2.191**	0.029	1.853**
STR	FF3	TB	0.018	1.157	0.017	1.105
STR	FF3	CS	0.019	1.188	0.034	2.160**
STR	FF3	INF	0.026	1.677*	0.020	1.250
STR	FF3	All	0.027	1.701*	0.034	2.147**
STR	C4	DY	0.021	1.345	0.009	0.599
STR	C4	BM	0.023	1.460	0.032	2.059**
STR	C4	TB	0.012	0.737	0.026	1.668*
STR	C4	CS	0.019	1.236	0.032	2.043**
STR	C4	INF	0.016	1.002	0.020	1.295
STR	C4	All	0.016	1.007	0.015	0.984

Table 5 Continued:

Panel E: Long-term Reversal Deciles.

Base	Test	Instr	T^2_{GRS}	GRS	T^2_{GMV}	GMV
LTR	MKT	DY	0.011	0.732	0.055	3.500***
LTR	MKT	BM	0.011	0.727	0.060	3.860***
LTR	MKT	TB	0.013	0.827	0.097	6.184***
LTR	MKT	CS	0.009	0.553	0.031	1.977**
LTR	MKT	INF	0.010	0.618	0.056	3.570***
LTR	MKT	All	0.012	0.749	0.045	2.843***
LTR	LTR	DY	0.019	1.238	0.029	1.882**
LTR	LTR	BM	0.021	1.313	0.428	27.373***
LTR	LTR	TB	0.018	1.138	0.117	7.485***
LTR	LTR	CS	0.010	0.626	0.035	2.222**
LTR	LTR	INF	0.033	2.116**	0.021	1.349
LTR	LTR	All	0.016	1.046	0.026	1.687*
LTR	FF3	DY	0.009	0.550	0.013	0.841
LTR	FF3	BM	0.009	0.571	0.101	6.437***
LTR	FF3	TB	0.007	0.432	0.015	0.929
LTR	FF3	CS	0.010	0.669	0.038	2.411***
LTR	FF3	INF	0.002	0.148	0.030	1.928**
LTR	FF3	All	0.009	0.544	0.028	1.782*
LTR	C4	DY	0.008	0.520	0.017	1.112
LTR	C4	BM	0.009	0.583	0.038	2.428***
LTR	C4	TB	0.008	0.524	0.018	1.177
LTR	C4	CS	0.010	0.639	0.029	1.849**
LTR	C4	INF	0.002	0.117	0.033	2.129**
LTR	C4	All	0.008	0.521	0.047	3.032***

Table 5 Continued:

Panel F: Ten Industry Portfolios.

Base	Test	Instr	T^2_{GRS}	GRS	T^2_{GMV}	GMV
IND10	MKT	DY	0.021	1.366	0.190	12.159***
IND10	MKT	BM	0.018	1.160	0.212	13.569***
IND10	MKT	TB	0.010	0.660	0.137	8.774***
IND10	MKT	CS	0.023	1.467	0.066	4.211***
IND10	MKT	INF	0.016	1.002	0.075	4.800***
IND10	MKT	All	0.023	1.501	0.044	2.782***
IND10	FF3	DY	0.017	1.088	0.059	3.787***
IND10	FF3	BM	0.021	1.341	0.140	8.939***
IND10	FF3	TB	0.015	0.958	0.091	5.831***
IND10	FF3	CS	0.023	1.439	0.060	3.856***
IND10	FF3	INF	0.025	1.576	0.046	2.963***
IND10	FF3	All	0.018	1.120	0.026	1.642*
IND10	C4	DY	0.012	0.756	0.046	2.947***
IND10	C4	BM	0.014	0.903	0.099	6.345***
IND10	C4	TB	0.012	0.783	0.034	2.198**
IND10	C4	CS	0.017	1.110	0.045	2.856***
IND10	C4	INF	0.017	1.067	0.028	1.773*
IND10	C4	All	0.012	0.790	0.007	0.443

Panel G: Thirty Industry Portfolios.

Base	Test	Instr	T^2_{GRS}	GRS	T^2_{GMV}	GMV
IND30	MKT	DY	0.048	0.996	0.310	6.387***
IND30	MKT	BM	0.044	0.901	0.335	6.915***
IND30	MKT	TB	0.042	0.874	0.273	5.640***
IND30	MKT	CS	0.046	0.951	0.175	3.604***
IND30	MKT	INF	0.056	1.161	0.194	3.994***
IND30	MKT	All	0.055	1.144	0.116	2.384***
IND30	FF3	DY	0.041	0.846	0.174	3.583***
IND30	FF3	BM	0.047	0.966	0.276	5.687***
IND30	FF3	TB	0.047	0.975	0.201	4.139***
IND30	FF3	CS	0.049	1.011	0.184	3.801***
IND30	FF3	INF	0.065	1.338	0.159	3.271***
IND30	FF3	All	0.048	0.998	0.120	2.473***
IND30	C4	DY	0.044	0.910	0.159	3.279***
IND30	C4	BM	0.046	0.942	0.213	4.397***
IND30	C4	TB	0.041	0.849	0.123	2.526***
IND30	C4	CS	0.045	0.924	0.106	2.179***
IND30	C4	INF	0.055	1.143	0.130	2.683***
IND30	C4	All	0.041	0.846	0.089	1.825***

Table 6. GRS^+ and GMV^+ Empirical Results with Short Sales Constraints.

This table reports the findings of testing the null hypotheses of no improvement in the tangent and minimum variance portfolio, respectively, using the GRS^+ and the GMV^+ test when short sales constraints are imposed using several sets of ten portfolios as the base assets and three separate sets of factors.

Panel A: Decile portfolios monthly value-weighted returns between 1960:01 and 2013:12.

Base	Test	$T^2_{GRS^+}$	GRS^+	$T^2_{GMV^+}$	GMV^+	s_1^+	s_2^+	$\sigma^+_{mv,1}$	$\sigma^+_{mv,2}$
Size	MKT	0.013	0.805	0.042	2.673***	0.111	0.159	0.045	0.044
Size	SMB	0.022	1.389	1.394	89.078***	0.074	0.165	0.030	0.020
Size	FF3	0.001	0.085	0.155	9.900***	0.219	0.223	0.017	0.015
Size	C4	0.001	0.065	0.115	7.377***	0.319	0.320	0.014	0.013
BM	MKT	0.016	1.017	0.125	7.965***	0.111	0.169	0.045	0.042
BM	HML	0.029	1.826*	1.477	94.369***	0.135	0.217	0.028	0.018
BM	FF3	0.002	0.118	0.157	10.030***	0.219	0.224	0.017	0.015
BM	C4	0.003	0.164	0.109	6.987***	0.319	0.323	0.014	0.013
CP	MKT	0.023	1.490	0.145	9.235***	0.111	0.190	0.045	0.042
CP	FF3	0.004	0.279	0.074	4.709***	0.219	0.230	0.017	0.016
CP	C4	0.003	0.206	0.063	4.012***	0.319	0.324	0.014	0.014
DP	MKT	0.020	1.249	0.244	15.565***	0.111	0.179	0.045	0.040
DP	FF3	0.005	0.326	0.023	1.458	0.219	0.231	0.017	0.016
DP	C4	0.010	0.648	0.015	0.979	0.319	0.336	0.014	0.014
EP	MKT	0.029	1.830*	0.115	7.340***	0.111	0.203	0.045	0.042
EP	FF3	0.009	0.585	0.050	3.185***	0.219	0.240	0.017	0.016
EP	C4	0.006	0.392	0.043	2.776***	0.319	0.329	0.014	0.014
Mom	MKT	0.019	1.196	0.080	5.103***	0.111	0.177	0.045	0.043
Mom	UMD	0.064	4.094***	2.076	132.643***	0.172	0.309	0.042	0.024
Mom	FF3	0.026	1.664*	0.002	0.102	0.219	0.275	0.017	0.017
Mom	C4	0.021	1.370	0.065	4.181***	0.319	0.354	0.014	0.014
Str	MKT	0.015	0.973	0.073	4.654***	0.111	0.167	0.045	0.043
Str	STR	0.029	1.829*	1.244	79.482***	0.168	0.240	0.031	0.021
Str	FF3	0.007	0.452	0.007	0.476	0.219	0.236	0.017	0.017
Str	C4	0.015	0.948	0.005	0.332	0.319	0.343	0.014	0.014
Ltr	MKT	0.010	0.667	0.111	7.074***	0.111	0.152	0.045	0.042
Ltr	LTR	0.026	1.693*	1.237	79.024***	0.121	0.204	0.025	0.017
Ltr	FF3	0.005	0.329	0.027	1.741*	0.219	0.231	0.017	0.016
Ltr	C4	0.001	0.065	0.018	1.133	0.319	0.320	0.014	0.014
Ind	MKT	0.022	1.432	0.480	30.684***	0.111	0.187	0.045	0.037
Ind	FF3	0.024	1.506	0.152	9.736***	0.219	0.270	0.017	0.015
Ind	C4	0.018	1.181	0.100	6.400***	0.319	0.349	0.014	0.013
Std	MKT	0.262	16.724***	1.637	104.600***	0.111	0.527	0.045	0.028
Std	FF3	0.214	13.686***	0.091	5.825***	0.219	0.522	0.017	0.016
Std	C4	0.212	13.571***	0.077	4.891***	0.319	0.579	0.014	0.014

Table 6 Continued:

Panel B: Decile portfolios daily value-weighted returns between 1960/01/04 and 2013/12/31.

Base	Test	$T^2_{GRS^+}$	GRS^+	$T^2_{GMV^+}$	GMV^+	s_1^+	s_2^+	$\sigma^+_{mv,1}$	$\sigma^+_{mv,2}$
Size	MKT	0.001	1.432	0.742	1007.608***	0.0241	0.0405	0.0098	0.0074
Size	SMB	0.002	2.265**	1.360	1846.854***	0.0115	0.0424	0.0050	0.0032
Size	FF3	0.000	0.067	0.105	141.990***	0.0568	0.0572	0.0027	0.0026
Size	C4	0.000	0.061	0.079	107.793***	0.0862	0.0864	0.0023	0.0023
BM	MKT	0.001	1.409	0.101	137.289***	0.0241	0.0402	0.0098	0.0093
BM	HML	0.002	2.400***	1.243	1688.858***	0.0384	0.0570	0.0047	0.0032
BM	FF3	0.000	0.278	0.095	128.406***	0.0568	0.0586	0.0027	0.0026
BM	C4	0.000	0.202	0.072	97.916***	0.0862	0.0870	0.0023	0.0023
Mom	MKT	0.001	1.849**	0.060	80.836***	0.0241	0.0441	0.0098	0.0095
Mom	UMD	0.003	4.586***	1.553	2108.871***	0.0482	0.0756	0.0068	0.0043
Mom	FF3	0.002	2.516***	0.000	0.218	0.0568	0.0713	0.0027	0.0027
Mom	C4	0.001	1.280	0.038	51.137***	0.0862	0.0915	0.0023	0.0023
Str	MKT	0.018	24.074***	0.041	55.660***	0.0241	0.1353	0.0098	0.0096
Str	STR	0.000	0.197	1.201	1631.528***	0.1624	0.1629	0.0070	0.0047
Str	FF3	0.018	24.056***	0.000	0.000	0.0568	0.1449	0.0027	0.0027
Str	C4	0.021	28.187***	0.000	0.000	0.0862	0.1683	0.0023	0.0023
Ltr	MKT	0.001	1.313	0.113	152.929***	0.0241	0.0394	0.0098	0.0093
Ltr	LTR	0.002	2.356***	1.030	1398.505***	0.0285	0.0505	0.0043	0.0030
Ltr	FF3	0.001	0.819	0.012	15.740***	0.0568	0.0619	0.0027	0.0027
Ltr	C4	0.000	0.529	0.006	8.689***	0.0862	0.0884	0.0023	0.0023
Ind	MKT	0.001	1.385	0.645	876.255***	0.0241	0.0400	0.0098	0.0076
Ind	FF3	0.002	2.120**	0.123	166.718***	0.0568	0.0692	0.0027	0.0026
Ind	C4	0.001	1.423	0.066	90.201***	0.0862	0.0921	0.0023	0.0023
Std	MKT	0.027	36.922***	4.588	6231.967***	0.0241	0.1667	0.0098	0.0041
Std	FF3	0.026	35.294***	0.480	651.749***	0.0568	0.1712	0.0027	0.0023
Std	C4	0.026	34.962***	0.283	384.101***	0.0862	0.1826	0.0023	0.0021

Table 7. GRS^+ and GMV^+ Further Empirical Results with Short Sales Constraints.

This table reports the findings of testing the null hypotheses of no improvement in the tangent and minimum variance portfolio, respectively, using the GRS^+ and the GMV^+ test when short sales constraints are imposed using several sets of 25 and 30 portfolios as the base assets and three separate sets of factors.

Panel A: 25/30 portfolios monthly value-weighted returns between 1960:01 and 2013:12.

Base	Test	$T^2_{GRS^+}$	GRS^+	$T^2_{GMV^+}$	GMV^+	s_1^+	s_2^+	$\sigma^+_{mv,1}$	$\sigma^+_{mv,2}$
25 Size/BM	MKT	0.087	2.181***	0.209	5.212***	0.1114	0.3176	0.0447	0.0406
25 Size/BM	FF3	0.034	0.857	0.307	7.664***	0.2194	0.2901	0.0166	0.0145
25 Size/BM	C4	0.037	0.920	0.237	5.917***	0.3187	0.3770	0.0141	0.0127
25 Size/Str	MKT	0.097	2.418***	0.091	2.264***	0.1114	0.3324	0.0447	0.0428
25 Size/Str	FF3	0.043	1.064	0.089	2.215***	0.2194	0.3047	0.0166	0.0159
25 Size/Str	C4	0.083	2.073***	0.048	1.194	0.3187	0.4394	0.0141	0.0138
25 Size/Mom	MKT	0.110	2.755***	0.122	3.055***	0.1114	0.3524	0.0447	0.0422
25 Size/Mom	FF3	0.058	1.450*	0.058	1.445*	0.2194	0.3302	0.0166	0.0162
25 Size/Mom	C4	0.064	1.600**	0.232	5.785***	0.3187	0.4150	0.0141	0.0127
25 Size/Ltr	MKT	0.100	2.500***	0.202	5.033***	0.1114	0.3374	0.0447	0.0408
25 Size/Ltr	FF3	0.029	0.721	0.202	5.052***	0.2194	0.2801	0.0166	0.0152
25 Size/Ltr	C4	0.035	0.869	0.101	2.520***	0.3187	0.3741	0.0141	0.0135
30 Industry	MKT	0.031	0.644	0.592	12.211***	0.1114	0.2098	0.0447	0.0354
30 Industry	FF3	0.030	0.620	0.165	3.401***	0.2194	0.2822	0.0166	0.0154
30 Industry	C4	0.027	0.548	0.116	2.402***	0.3187	0.3617	0.0141	0.0134

Panel B: 25/30 portfolios daily value-weighted returns between 1960/01/04 and 2013/12/31.

Base	Test	$T^2_{GRS^+}$	GRS^+	$T^2_{GMV^+}$	GMV^+	s_1^+	s_2^+	$\sigma^+_{mv,1}$	$\sigma^+_{mv,2}$
25 Size/BM	MKT	0.007	3.853***	0.814	441.751***	0.0241	0.0877	0.0098	0.0073
25 Size/BM	FF3	0.002	1.231	0.230	124.936***	0.0568	0.0742	0.0027	0.0025
25 Size/BM	C4	0.002	1.146	0.169	91.918***	0.0862	0.0977	0.0023	0.0022
25 Size/Str	MKT	0.227	123.004***	0.901	489.135***	0.0241	0.4768	0.0098	0.0071
25 Size/Str	FF3	0.224	121.529***	0.020	11.087***	0.0568	0.4774	0.0027	0.0027
25 Size/Str	C4	0.227	123.396***	0.010	5.345***	0.0862	0.4863	0.0023	0.0023
25 Size/Mom	MKT	0.011	6.022***	0.957	519.374***	0.0241	0.1081	0.0098	0.0070
25 Size/Mom	FF3	0.007	3.601***	0.032	17.538***	0.0568	0.0994	0.0027	0.0027
25 Size/Mom	C4	0.006	3.038***	0.089	48.564***	0.0862	0.1143	0.0023	0.0022
25 Size/Ltr	MKT	0.007	3.742***	0.673	365.498***	0.0241	0.0865	0.0098	0.0076
25 Size/Ltr	FF3	0.003	1.746**	0.099	53.841***	0.0568	0.0803	0.0027	0.0026
25 Size/Ltr	C4	0.003	1.772**	0.064	34.774***	0.0862	0.1035	0.0023	0.0023
30 Industry	MKT	0.001	0.638	0.691	312.205***	0.0241	0.0447	0.0098	0.0075
30 Industry	FF3	0.002	0.832	0.168	76.085***	0.0568	0.0712	0.0027	0.0025
30 Industry	C4	0.001	0.645	0.101	45.653***	0.0862	0.0941	0.0023	0.0022

Table 8. Finite Sample Size of the GRS and GMV Tests.

This table reports the findings on the actual size of the GRS and GMV tests based on 10,000 randomly generated samples of returns for a combination of sample sizes T, number of test assets K and number of base assets N where the portfolio returns follow a Gaussian distribution. In each simulation the null hypothesis of the respective test is imposed on the first two moments of portfolio returns. All reported values are in percent with the nominal size of the test reported in the first column. The values are the actual percentage of falsely rejecting a true null hypothesis based on the 10,000 simulations.

Panel A: $N = 10$.

		$K=1$		$K=3$		$K=4$	
Size	T	GRS	GMV	GRS	GMV	GRS	GMV
10	60	7.1	7.2	3.9	1.2	2.9	0.9
5	60	3.4	3.7	1.7	0.4	1.2	0.3
1	60	0.6	0.8	0.3	0.0	0.1	0.0
10	120	6.6	7.5	4.1	2.8	4.0	3.0
5	120	3.4	3.5	1.8	0.9	1.8	0.9
1	120	0.5	0.7	0.3	0.1	0.3	0.1
10	600	7.1	6.7	3.8	4.1	4.6	4.0
5	600	3.5	3.4	1.9	1.8	2.0	1.5
1	600	0.7	0.6	0.3	0.2	0.4	0.3

Panel B: $N = 25$.

		$K=1$		$K=3$		$K=4$	
Size	T	GRS	GMV	GRS	GMV	GRS	GMV
10	60	7.8	8.7	4.5	0.1	3.8	0.1
5	60	3.6	4.3	2.0	0.0	1.6	0.0
1	60	0.8	0.7	0.3	0.0	0.2	0.0
10	120	8.4	8.3	4.9	0.9	4.3	0.6
5	120	4.0	4.1	2.2	0.3	1.8	0.2
1	120	0.9	0.9	0.4	0.0	0.3	0.0
10	600	7.5	7.6	4.7	4.1	4.3	3.6
5	600	3.8	3.6	2.2	1.8	2.0	1.7
1	600	0.7	0.7	0.4	0.2	0.2	0.2

Panel C: $N = 30$.

		$K=1$		$K=3$		$K=4$	
Size	T	GRS	GMV	GRS	GMV	GRS	GMV
10	60	7.5	8.5	4.3	0.3	3.3	0.1
5	60	3.7	4.2	1.7	0.1	1.2	0.0
1	60	0.6	0.7	0.3	0.0	0.1	0.0
10	120	8.5	8.5	5.9	0.8	5.0	0.4
5	120	4.1	4.1	2.7	0.2	2.2	0.1
1	120	0.8	0.7	0.4	0.0	0.3	0.0
10	600	8.3	8.1	6.0	4.3	5.5	3.7
5	600	3.9	3.9	2.6	1.9	2.5	1.5
1	600	0.9	0.7	0.4	0.3	0.3	0.2

Table 9. The Effect of Fat-Tailed Error Distribution on the Size of the GRS and GMV Tests.

This table reports the findings on the actual size of the GRS and GMV tests based on 10,000 randomly generated samples of returns for a combination of sample sizes T, number of test assets K and number of base assets N where the portfolio returns follow a Student-t distribution with $\nu = 5$ degrees of freedom. In each simulation the null hypothesis of the respective test is imposed on the first two moments of portfolio returns. All reported values are in percent with the nominal size of the test reported in the first column. The values are the actual percentage of falsely rejecting a true null hypothesis based on the 10,000 simulations.

Panel A: $N = 10$.

		$K = 1$		$K = 3$		$K = 4$	
Size	T	GRS	GMV	GRS	GMV	GRS	GMV
10	60	6.5	11.9	4.2	1.9	2.4	0.9
5	60	3.2	6.2	2.3	0.4	0.7	0.4
1	60	0.8	1.6	1.0	0.0	0.0	0.0
10	120	6.3	13.9	4.7	2.6	4.2	4.5
5	120	3.4	8.2	2.3	1.3	1.4	1.7
1	120	0.5	2.0	0.1	0.2	0.1	0.4
10	600	5.5	21.1	3.9	7.3	4.1	6.3
5	600	2.9	13.3	1.6	3.6	1.3	2.8
1	600	0.4	4.7	0.2	0.4	0.1	0.6

Panel B: $N = 25$.

		$K = 1$		$K = 3$		$K = 4$	
Size	T	GRS	GMV	GRS	GMV	GRS	GMV
10	60	7.1	12.5	4.1	0.3	3.1	0.2
5	60	3.4	6.4	2.2	0.0	1.1	0.1
1	60	0.6	1.4	0.2	0.0	0.1	0.0
10	120	7.3	13.3	5.5	2.1	3.6	0.9
5	120	3.3	7.4	2.0	0.6	1.6	0.1
1	120	0.1	2.0	0.4	0.0	0.3	0.0
10	600	7.0	18.6	4.9	6.7	4.1	6.2
5	600	3.6	10.5	2.7	3.4	2.4	2.4
1	600	0.8	3.5	0.6	0.4	0.2	0.7

Panel C: $N = 30$.

		$K = 1$		$K = 3$		$K = 4$	
Size	T	GRS	GMV	GRS	GMV	GRS	GMV
10	60	8.7	10.2	5.4	1.1	2.8	0.4
5	60	4.4	5.9	2.3	0.3	1.4	0.2
1	60	0.7	1.5	0.8	0.0	0.4	0.1
10	120	8.4	12.6	6.0	1.2	4.6	1.2
5	120	3.8	7.2	2.2	0.5	1.7	0.7
1	120	1.0	1.8	0.3	0.2	0.1	0.2
10	600	10.0	14.7	6.7	6.2	6.6	6.4
5	600	4.5	8.2	3.0	2.7	2.1	1.7
1	600	0.8	2.1	0.5	0.1	0.1	0.4

Figure 1. Large GRS, small GMV

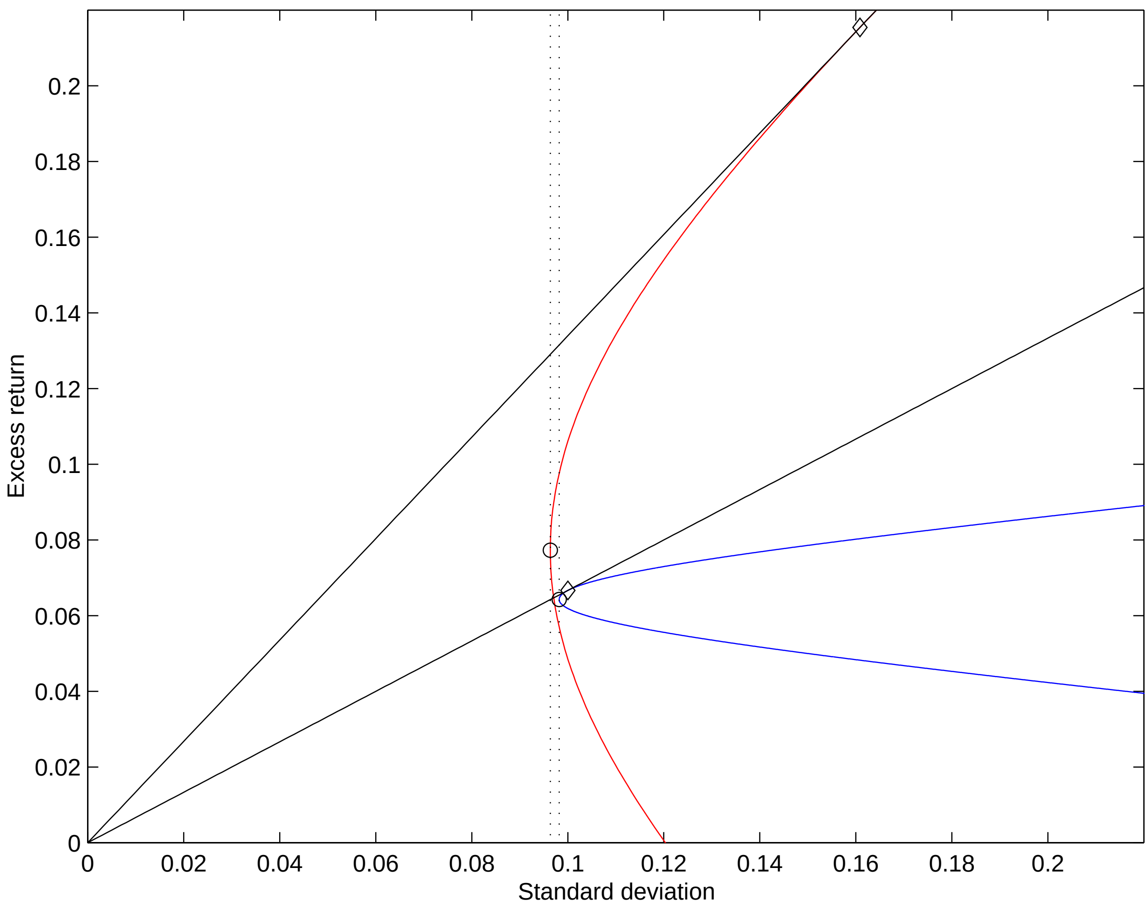

Figure 1 presents the mean-variance frontiers, the tangent portfolios (diamonds) and the global minimum variance portfolios (circles) with $N = 2$ basis assets and $K = 2$ test assets with pre-specified excess returns, variances and correlations. The percentage change in one plus the squared Sharpe ratio is 93.41% while the percentage change in the variance of the minimum variance portfolio is 3.86%.

Figure 2. Small GRS, large GMV

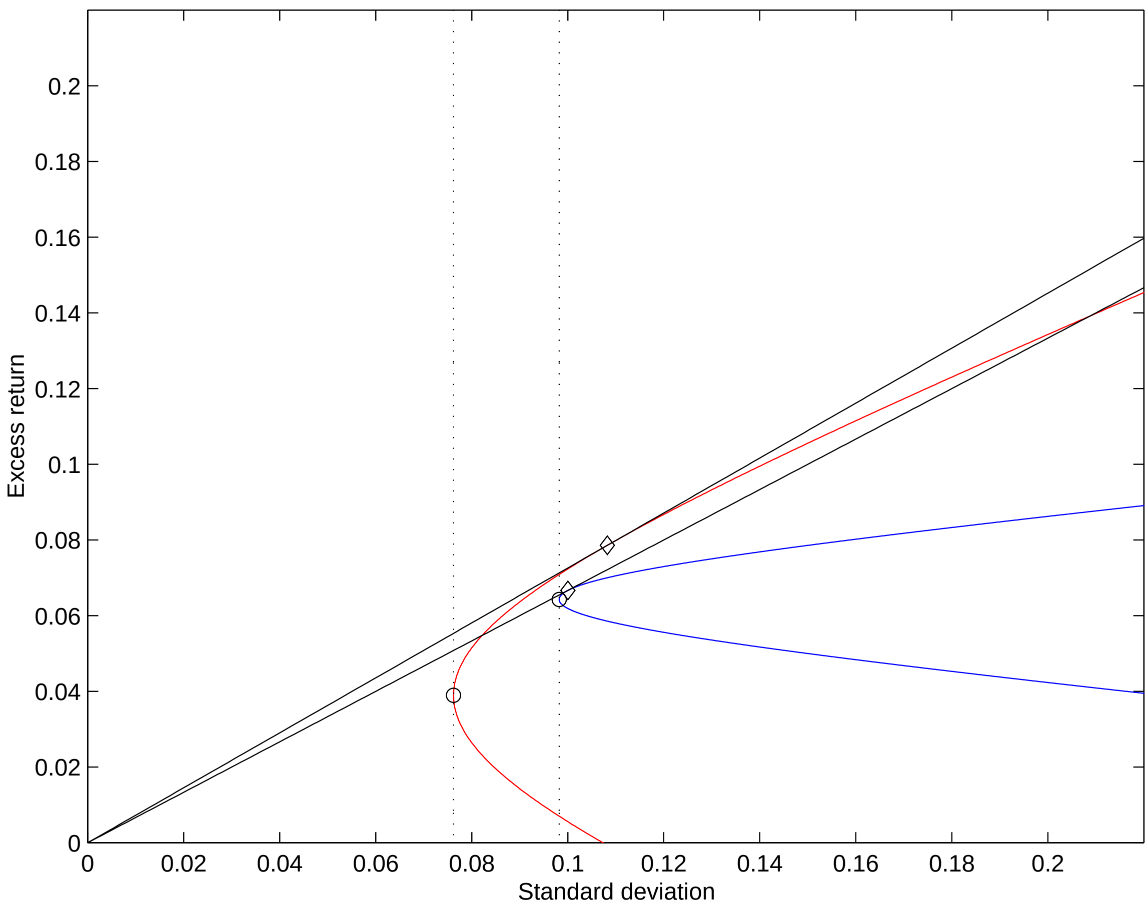

Figure 2 presents the mean-variance frontiers, the tangent portfolios (diamonds) and the global minimum variance portfolios (circles) with $N = 2$ basis assets and $K = 2$ test assets with pre-specified excess returns, variances and correlations. The percentage change in one plus the squared Sharpe ratio is 5.73% while the percentage change in the variance of the minimum variance portfolio is 66.02%.

Figure 3. Small GRS, small GMV

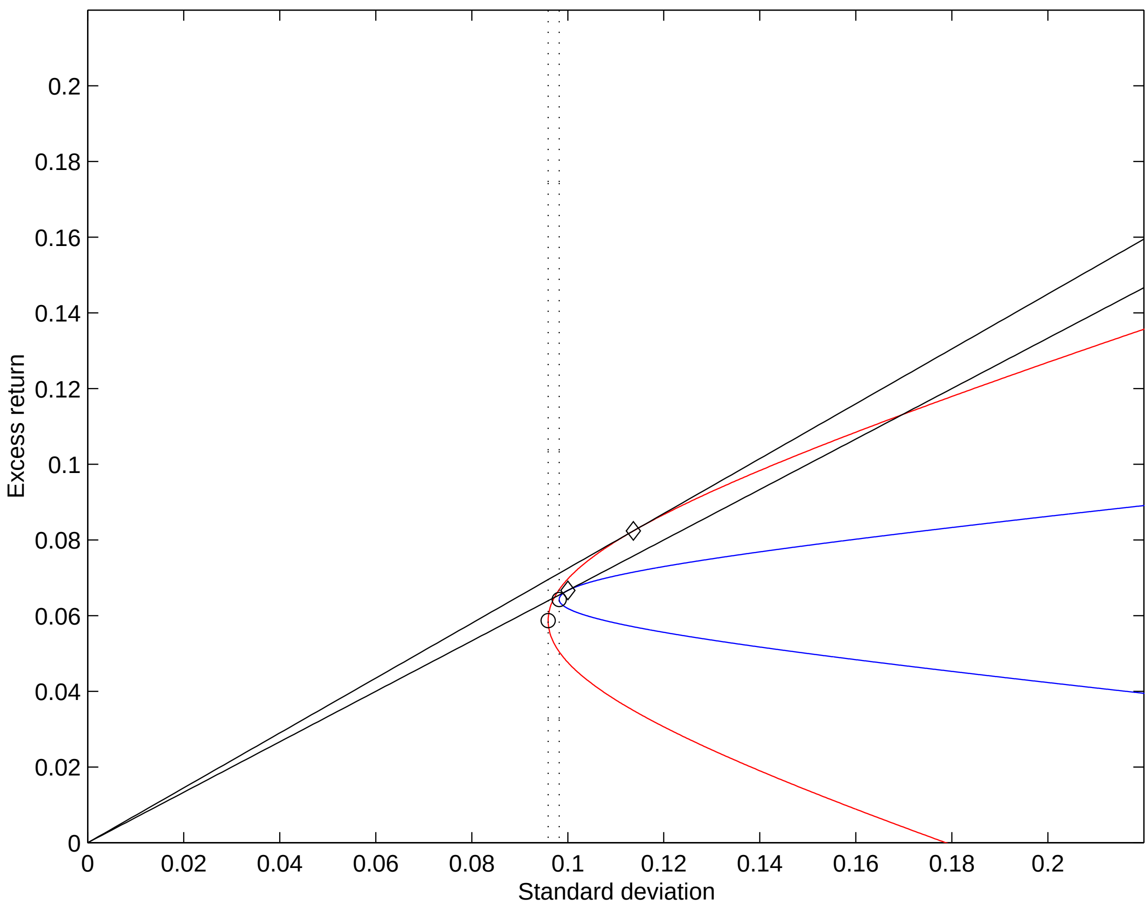

Figure 3 presents the mean-variance frontiers, the tangent portfolios (diamonds) and the global minimum variance portfolios (circles) with $N = 2$ basis assets and $K = 2$ test assets with pre-specified excess returns, variances and correlations. The percentage change in one plus the squared Sharpe ratio is 5.62% while the percentage change in the variance of the minimum variance portfolio is 4.84%.

Figure 4. Hotelling T^2 Statistic for GRS and GMV

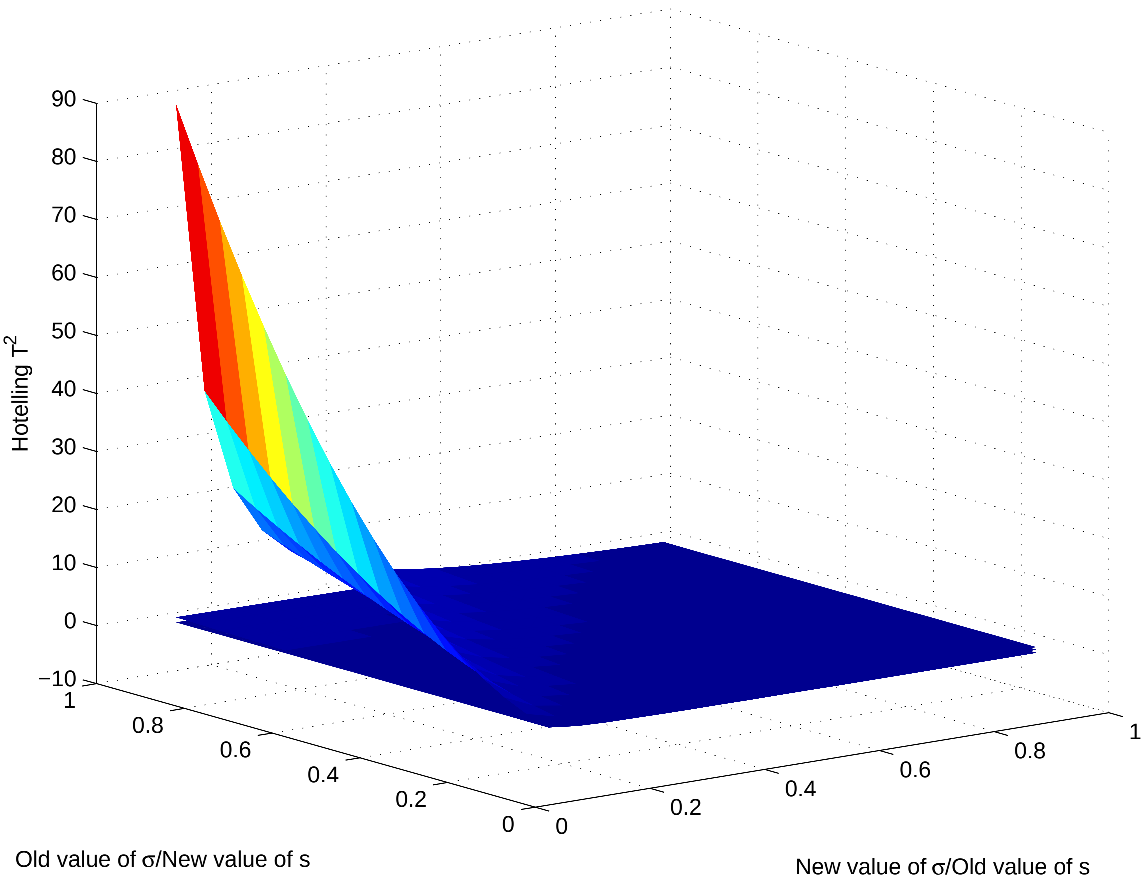

Figure 4 presents the Hotelling T^2 statistic for combinations of values for the Sharpe ratio of the tangent portfolio and the standard deviation of the global minimum variance portfolio before and after the addition of the test assets to the set of basis assets. The values are plotted for $N = 10$, $K = 2$, and $T = 120$. In order to convert the Hotelling T^2 statistic into the GRS (GMV) statistic it needs to be multiplied by a factor of $(T/N)((T - N - K)/(T - K - 1))$. The critical values for the GRS (GMV) statistic are 2.49, 1.92, and 1.66, respectively, at 1%, 5%, and 10% level of significance.

Figure 5. Power of GRS and GMV Tests with $K = 1$

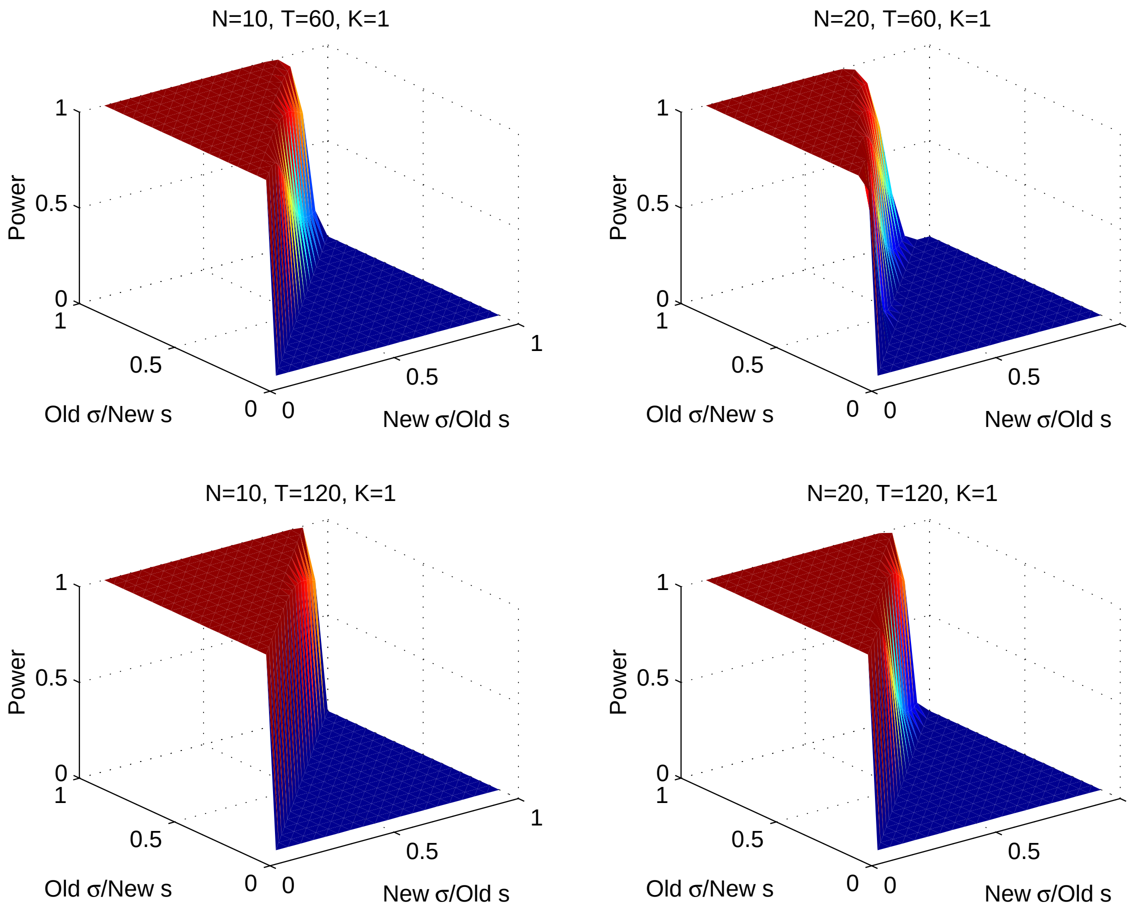

Figure 5 presents the power of the GRS and GMV tests for various combinations of T and N for $K = 1$.

Figure 6. Power of GRS and GMV Tests with $K = 4$

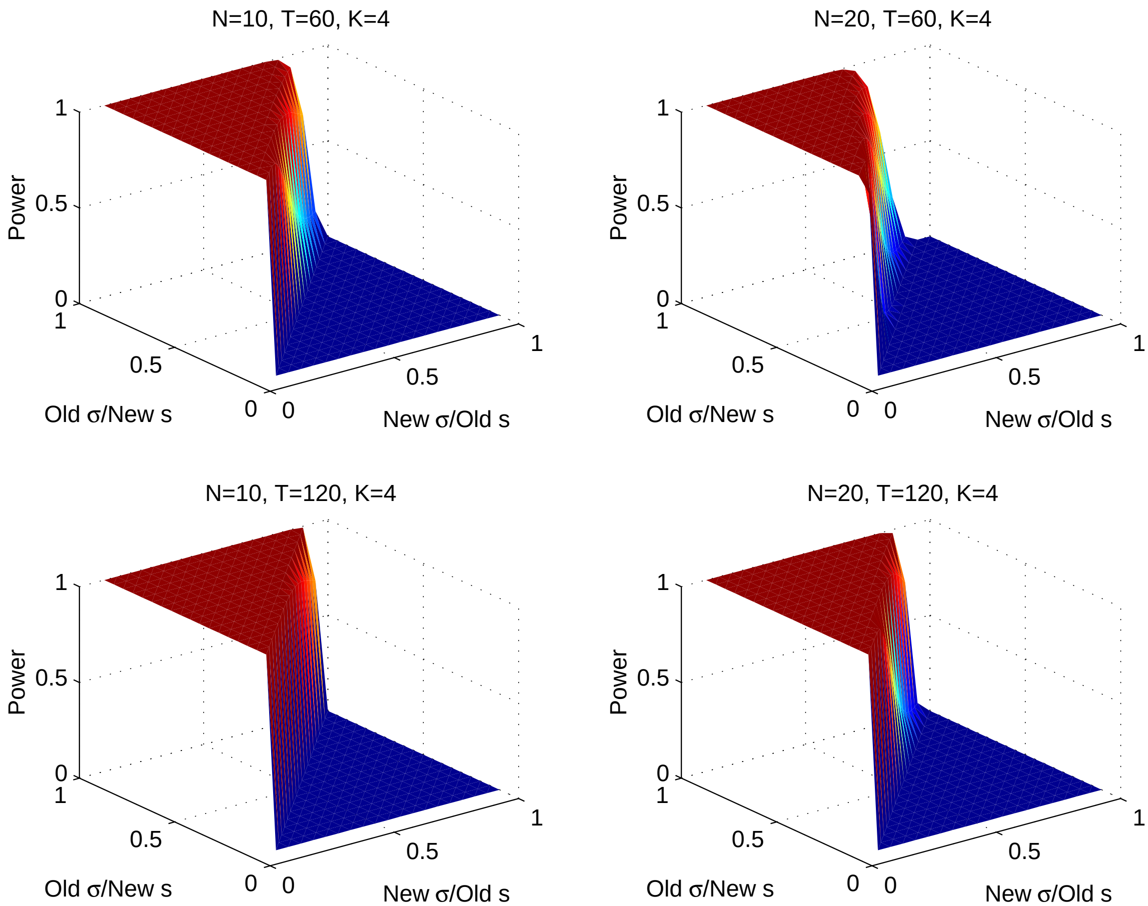

Figure 6 presents the power of the GRS and GMV tests for various combinations of T and N for $K = 4$.