Modeling default correlation in a US retail loan portfolio

DENNIS BAMS, MAGDALENA PISA and CHRISTIAN WOLFF*

October 18, 2012

ABSTRACT

This paper generalizes the existing asymptotic single-factor model to address issues related to industry heterogeneity, default clustering and capital requirement's parameter uncertainty in US retail loan portfolios. We argue that the Basel II capital requirement overstates the riskiness of small businesses even with prudential adjustments.

Moreover, our estimates show that both location and spread of loss distribution bare uncertainty. Their shifts over the course of the recent crisis have important risk management implications. The results are based on a unique representative dataset of US small businesses from 2005 to 2011 and give fundamental insights into the US economy.

PORTFOLIO LOSS DISTRIBUTION CAN be subject to changes in dynamic macroeconomic conditions. Typically, the expected loss associated with provisions level is considered not to bare uncertainty which in turn is associated with the unexpected losses. The expectation about the loss may however shift when new information becomes available thus changing the expectation about loss level. We build on the existing single-factor literature to include aspects of portfolio diversification, dynamics of risk and capital requirements. Our interest

*Bams is at the Maastricht University and De Lage Landen (DLL), Pisa is at the Maastricht University and Luxembourg School of Finance and Wolff is at Luxembourg School of Finance. The authors thank Lamia Bekkour and the participants at the Mathematical Finance Days at HEC Montreal for useful comments. Alla Kramskaia provided research assistance with the Dun & Bradstreet database. Special thanks to Dimitris Pongas for detailed comments and suggestions. Financial assistance is acknowledged from the Fonds National de la Recherche (Luxembourg).

lies in empirical study of credit risk in US retail loan portfolios that consist of loans granted to small businesses.

The principal aim of this paper is to provide empirical insights into risk management of US retail loan portfolios. Our contribution to the existing literature is threefold. First, we focus our attention on privately held firms. With small businesses being an engine of economic growth, job creation and a majority of active firms in the US our findings give fundamental insights into risk sources and segmentation of the US economy. The study conducts the analysis on a unique panel of exposures to US private firms from 2005 to 2011 that captures the evolution of small business risk during the turmoil of 2007 to 2009. The importance of those results becomes apparent due to the fact that although small businesses underpin the US economy, they remain an opaque research subject due to lack of financial statements and market trading. Second, this paper discusses whether the regulatory formula captures accurately the underlying small business credit risk or rather distorts the risk management practices in financial institutions which hold such portfolios. We verify existence of capital allocation inefficiencies in US retail loan portfolios arising from the Basel II formula for asset correlation. Lastly, we overcome the limited information availability in small businesses by deriving a simple yet effective estimation technique of joint default risk in retail loan portfolios. Importantly for corporate debt portfolios our estimation technique yields results which are coherent with Basel II capital requirements. Thus the results for retail loan portfolios can be positioned next to the regulatory ones.

Small businesses in the US are very peculiar by nature. Although they contribute about 50% to the US GDP and employ 50% of private workforce their financial picture is rather limited. In practice this ambiguity stems from the absence of financial statements and market trading. Also, oddly, until recently most of the available knowledge on this significant segment of the US economy was based on estimates rather than hard data. While some efforts were undertaken to shed light in the area of default dependency in the US retail portfolios, they are limited to aggregated measures of small business credit risk (Lee, Wang, and Zhang (2009)) or to loans originated under the SBA guarantee program (Glennon and

Nigro (2005)). Unlike those earlier works, our study performs an empirical analysis on a new and comprehensive default data of US private firms covering a period of seven years from 2005 to 2011. The panel contains quarterly observations on small and medium sized firms across all the credit ratings and industries in the US with an average of nearly 240,000 obligors per time period. It provides a sole opportunity to analyze the credit risk in retail loans in the pre-, during and post-crisis phases. Non-US studies of small businesses comprise Carling, Rönnegård, and Roszbach (2004) who analyze Swedish retail loan market and Düllmann and Scheule (2003) with their study on German small and large firms. Additionally we pay attention to evolution of portfolio risk during recession times that is changes to location and spread of loss distribution.

Moreover, credit risk in small businesses is of special interest to US financial institutions. As the FDIC reports, the US commercial banks' exposure to loans granted to small businesses is significant and amounts in June 2011 to 24.9% of the commercial and industrial loans. Large size of the retail loan portfolios and limited information on borrower creditworthiness makes small businesses an object of particular relevance for Basel II capital requirement. A discussion of the Basel II capital requirement can be found also in Botha and van Vuuren (2010) and Lopez (2004). The former asks how the asset correlation derived from loss data relates to the one specified in Basel II and its corresponding capital charges. The latter outlines empirical asset correlation for US, Japanese, and European corporate and private firms. Both studies however neglect a possible parameter uncertainty in the asset correlation and capital requirement estimates, which however in our study provides a basis for a prudential approach to capital requirement.

Driven by the objective to verify the validity of the Basel II minimum capital formulas for the US retail loan portfolios we chose a general multi-factor model as a setting for our analysis. This choice positions our estimates next to the outcomes of the Basel II single-factor model and at the same time allows the economy to have a more advanced structure. In such a multi-factor economy the risks can be industry or firm size specific as opposed to a single global risk factor in the Basel II model. It becomes unnecessary to impose a strong assumption of single-factor economy. Instead multi-factor setting incorporates obligors' and risk factors' heterogeneity and allows to test for it. To the sizable family of single-factor models belongs also research by Dietsch and Petey (2002, 2004). The former study investigates the capital requirements in the context of probit and gamma models and its deviation from the Basel II Accord but focuses on French small businesses. The latter study delves on the nature of asset correlation in small businesses but is confined to French and German market. Although very relevant, the study of Dietsch and Petey (2004) only uses a single-factor model which is improved on in the subsequent work from 2009 by considering multiple common risk factors. However, their generalized linear mixed model assumes that financial institutions possess a considerable set of information on their borrowers often not available for small business.

As Jorion and Zhang (2009) observe, calibration of the portfolio credit risk models from single-factor family is notoriously difficult. However, we propose a simple estimation technique in which we demonstrate that the observed default frequencies per homogenous obligor class are sufficient to estimate the joint default risk in a retail loan portfolio. To model and estimate the default dependencies we begin with the Vasicek (1987) firm value model elaborated in Gordy (2003) that shows its applicability to banks' capital requirement. This type of models finds its roots in work of Merton (1974) and is applied in practice by Credit Metrics (Gupton, Finger, and Bhatia (1997)) and KMV (Crosbie and Bohn (2003)). The advantage of the estimator proposed however lies in the minimal information required to assess the joint default risk in a retail loan portfolio. In fact our model is of an *incomplete information* type as described by Giesecke (2006) in which the investors observe a default barrier and obtain a noisy report on firm's asset value. And although there exist more sophisticated empirical models of joint default risk which include Duffie, Saita, and Wang (2007), McNeil and Wendin (2007), Duffie et al. (2009) and Azizpour, Giesecke, and Schwenkler (2012), the restricted data availability hinders their use in informationally opaque small businesses loans.

In the empirical analysis we address some fundamental questions which investigate how the common risk factors are distributed across the economy and which firm characteristics are relevant for diversification. In accordance to that we first select the dimensions to partition obligors into homogenous classes. Here the industry and credit rating play important role in portfolio segmentation. In general, we find that sensitivity to obligor class-specific common risk factors remains low and varies between 0.00-18.41% with only 0.00-3.39% of the asset variability explained by the common risk factors. The remaining 96.61-100.00% of small business risk is due to changes in the firm specific characteristics. During the whole period analyzed the implied asset correlation averages around 0.41%. Also, regardless of the small business' riskiness, industry or firm size our estimates are significantly lower than any available estimates for corporate firms. Second, we find empirical support that a single factor model assumed by Basel II capital calculations is too simplistic to summarize the entire structure of the US economy. In fact the US economy shows more complexity and has more sources of risk than a nation-wide single factor. Next, we analyze how the riskiness of US small business evolved over the course of the financial crisis. It boils down to analysis of two important elements of default risk in a portfolio of loans: location and spread of defaults. We find empirical support that the firms which withstood the crisis showed less sensitivity to economic conditions, a substantial reliance on the firm characteristics and lower default clustering from macroeconomic exposures. The importance of firm-specific risk as a source of default risk was also discussed in Jarrow and Yu (2001) who link it to the individual business connections of a firm. Our analysis recovers a delay in response of sensitivity parameter to the recession which we read as a sign that in the recent crisis small businesses were suffering its consequences rather than inducing it.

Lastly, we position our results with Basel II capital requirement calculations which imply a substantially larger exposure of retail loan portfolios to common risk factors. What we observe is a sizable overstatement of retail debt risk as perceived by the Basel II vis-à-vis our method. In our view it is heavily driven by an overly-simplistic way in which Basel II models and estimates the asset correlations in retail loan portfolios. In fact our results show that from a credit risk perspective retail exposures are safer investment than the regulator would suggests.We summarize the empirical results by discussing the parameter uncertainty of our estimates. A prudential adjustment of the capital requirement can be done by accounting for the parameter uncertainty but also by allowing for fat-tail distributed risk factors. The measured parameter uncertainty aims to provide a better understanding of the presented results for risk management practices.

The paper is organized as follows. The next section introduces the probabilistic model of joint default risk and the proposed estimators. Section III outlines the D&B dataset of small US businesses. The empirical results for the pre-, during and post-crisis phases are presented in section IV which also summarizes the implications of our findings for risk management and capital requirements in financial institutions. Finally, section 5 concludes.

II Methodology

We generalize the existing asymptotic single-factor model to a multi-factor one which includes aspects of diversification and segmentation. The model used departs from the Basel II asymptotic single risk factor in a sense of allowing for more flexibility in choice of the number of risk factors in the economy (i.e. a common factor per obligor class) as opposed to a single global risk factor. This general framework finds an empirical support in the next section in form of industry related heterogeneity and multiple common risk factors in the US economy. Despite the generalization, our model is equivalent to the regulatory one if we observe perfectly correlated common risk factors and yields estimates consistent with the regulatory ones.

Consider a portfolio of N small obligors which are ordered into homogenous obligor classes $k \in \{1, \dots, K\}$ categorized with respect to creditworthiness, industry, etc. Let a latent variable A_{it} denote the asset value of obligor i in obligor class k at time t which without loss of generality is standardized and centered on zero. The asset value is driven by two independent components: a common risk factor x_{kt} per obligor class k and an idiosyncratic risk factor ϵ_{it} per obligor i:

$$A_{it} = w_k x_{kt} + \sqrt{1 - w_k^2}\epsilon_{it} \qquad i \in k \qquad t = 1, \ldots, T \tag{1}$$

where $\mathrm{E}[x_{kt}\epsilon_{it}] = 0$. The class specific common risk factor x_{kt} represents changes in the economic conditions common to all obligors in obligor class k and the idiosyncratic risk factor ϵ_{it} stands for firm specific risk attributed to each obligor. The weight w_k of the common risk factor measures the sensitivity of obligor i to its economic conditions. Given that any two firms classified into the same obligor class are sufficiently homogenous it is typical to assume that the class specific factor has an identical effect on their asset value (McNeil and Wendin (2007), Gordy (2000)). It follows that the weight w_k is the same for obligors in one obligor class. Credit portfolio concentration risk depends heavily upon the magnitude with which obligors' asset value responds to the common risk factor. The higher the firm's sensitivity to its common risk factor the more responsive the asset value to unanticipated changes in the economic environment. In fact the default dependency in a loan portfolio arises from co-movements in asset value that is induced by those common risk factors correlated across obligor classes with a correlation matrix Ω, where:

$$\Omega_{kl} = \mathrm{Corr}[x_k, x_l] \tag{2}$$

Also, although not explicitly modeled in the methodology section a time variation in the sensitivity w_k can be achieved by applying a moving-time-window technique. As a result of shifting the time windows between pre-, during- and post-crisis phases we are able to investigate changes in the sensitivity w_k over time.

The state of obligor i depends on a relative distance of its asset value to a threshold that defines the default event. We assume that the risk factors x_{kt} and ϵ_{it}, and hence the asset value, are standard normally distributed. The default threshold is equal to $\Phi^{-1}(\bar{p}_k)$ where $\Phi^{-1}(\cdot)$ denotes the standard normal CDF and $\bar{p}_k$ stands for the unconditional probability of default in obligor class k. Our model shares the definition of default event with the structural models that date back to work of Merton (1974) and Black and Cox (1976). In this framework an obligor i defaults at time t if the following condition is met:

$$w_k x_{kt} + \sqrt{1 - w_k^2}\epsilon_{it} < \Phi^{-1}(\bar{p}_k) \Leftrightarrow D_{it} = 1 \tag{3}$$

where D_{it} denotes a default indicator of firm i. By definition D_{it} takes value 1 if firm i defaults at time t and 0 otherwise. From (3) it follows that if the economic conditions x_{kt} are good, a firm defaults only if the realization of the idiosyncratic risk factor ϵ_{it} is worse. Also, the asset correlation between two obligors i and j is derived to be:

$$\rho_{ij} = \mathrm{Corr}[A_{it}, A_{jt}] = w_k w_l \Omega_{kl} \qquad i \in k, j \in l \tag{4}$$

From the above relationship one can see that, holding Ω constant, with increase in the sensitivity parameter w the obligors become more correlated but with decrease in w it is the idiosyncratic risk that dominates.

In this setup we derive the theoretical moment for joint probability of default to be equal to the probability of two obligors being simultaneously below the default threshold (for the derivation please refer to Appendix A). Hence, the joint probability of default of obligor i and j follows as:

$$\begin{aligned} p_{kl} &\equiv P[D_{it} = 1, D_{jt} = 1] \\ &= \int_{-\infty}^{\Phi^{-1}(\bar{p}_l)} \Phi\left(\frac{\Phi^{-1}(\bar{p}_k) - \Omega_{kl} w_k w_l y}{\sqrt{1 - \Omega_{kl}^2 w_k^2 w_l^2}}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy \end{aligned} \tag{5}$$

where obligor i belongs to obligor class k and obligor j belongs to obligor class l. The empirical analysis focuses on estimation of the parameter set $\theta \equiv (w, \Omega)$ where $w \equiv (w_1, \ldots, w_K)$ denotes the vector of common risk factors sensitivities and Ω represents the matrix of common risk factors correlations. The proposed method of moments for credit risk is compatible with a statistical analysis of obligors clustered into obligor classes.

Equation (5) is at the center of the estimation procedure. The left hand side of the equation gives to theoretical moment for joint probability of default that is caused by the aggregate behavior of obligors in an obligor class. Next, we minimize the distance between this theoretical moment and its sample counterpart. Denote an observed default frequency in obligor class k at time t by ODF_{kt}. It follows that the observed default frequency is equal to a ratio of all default events in obligor class k to the total number of obligors in this class $ODF_{kt} = \sum_{i \in k}^{N_{kt}} D_{it}/N_{kt}$. It can be shown (see Appendix A) that for two obligor classes k and l, the sample joint probability of default corresponds to a historical average of products of their observed default frequencies. As a result, the following relationship holds for joint probability of default for two obligors i and j in obligor classes k and l respectively:

$$\hat{p}_{kl} = \frac{1}{T} \sum_{t=1}^{T} (ODF_{kt} \cdot ODF_{lt}) \tag{6}$$

We refer to expression in (6) as the *between obligor class* sample moment since it depicts the joint probability of default for obligors in two different obligor classes. By analogy, the *within obligor class* sample moment for joint probability of default for two obligors in the same obligor class k follows as:

$$\hat{p}_{kk} = \frac{1}{T} \sum_{t=1}^{T} (ODF_{kt} \cdot ODF_{kt}) \tag{7}$$

Estimate of w follows from method of moments applied to equation (5) using the sample moments in (7). The obtained vector w is used in the next step to estimate Ω from equations (5) and (6) [1]. Importantly, only minimum information on the obligor class level is required to estimate the relevant parameter vector θ, that is the observed default frequencies per obligor classes. Moreover, this information is usually readily available within a financial institution which facilitates an easy application of the approach proposed in small business finance providers. The advantage of the multi-factor model over a single-factor one is a more realistic modeling of portfolio risk which estimates and incorporates into the analysis the

[1] Essentially it is a numerical optimization which minimizes the sum of squared errors between the population and sample moments over a domain of θ.

dependencies between different risk factors. Thus by estimating Ω from the *between obligor class* moments one obtains a more comprehensive view of the portfolio risk, its diversification possibilities and a more informed segmentation of exposures. Interestingly, the single-factor model is estimated based solely on sub-portfolios composed of homogenous obligors which is equivalent to estimation of the *within obligor class* moments (see Gordy (2000), Dietsch and Petey (2002, 2004)).

Furthermore, this multi-factor model collapses to a single-factor in case of perfectly correlated common risk factors x's. It follows that the common risk factor x is one-dimensional (as assumed in Gordy (2003)). In other words, the perfect correlation imposes a single common risk factor as the sole external source of default correlation. The above property can be used to test the single factor assumption and homogeneity of obligors with obligor class.

III Data

In this section we outline the main characteristics of a unique dataset provided by Dun & Bradstreet. We conduct an extensive analysis of nearly 240,000 US small businesses per time period from the D&B dataset that contains rich quarterly information on firms' actual borrowing and payment behavior, public detrimental information such as county court judgments, legal pre-failure events (receivership, bankruptcy, etc.), credit ratings but also legal form, age, industry and firm's location. The sample covers annually \$19 billion of small business financial activity, providing a representative outlook on the economy. The average credit outstanding per firm is \$31,860.33 with 24.49% of the exposures below \$1 thousand and 99.75% below \$1 million.

The dataset spans a period of seven years from 2005 to 2011 and covers the pre-, during and post-crisis phases. During this time the study looks at payment behavior of thousands of small businesses across all the credit ratings, industries and firm sizes in the US which compose a representative cross section of the US economy. In this sample firms represent all the major US industries with a high concentration in services (40.78%), retail trade (14.82%) and construction (13.61%). Aside from the non-classified firms it is manufacturing that experiences the highest default rate of 17.48%, also illustrated in Figure 1. In the context of recession this high default rate is explained by the fact that consumers obstruct from new purchases and repair the equipment they already own (consistent with lower default rate in services).

In the empirical analysis of small businesses we turn our attention to the information we believe is the most reliable, namely the loan and trade records stored by financial institutions, suppliers and vendors. Informational coverage of the US economy is substantial with about 6,000 major firms reporting to D&B. Also, we adopt the Basel Accords view in which a default takes place if a payment is 90 days past due or unlikely to be paid which accounts for events like bad debt, suit-filed, non-sufficient funds, and credit refused, placed for collection or repossession.

A review of the business size reveals that 56.59% of firms have fewer than 5 and 98.29% fewer than 100 employees. Surprisingly the very small firms seem to perform on average better than small or medium firms. Also Table I reports that the annual default rate increases with firm size from 9.67% for very small firms (up to 5 employees) to 35.98% for those which employ more than 100 people. Similar result can be found in Glennon and Nigro (2005) who also report higher default rates for larger firms. The observed regularity can be due to higher cash holdings in very small businesses which create a buffer for financial distress (Steijvers and Niskanen (2009)).

With the vast majority of records containing information on privately held firms (99.97%) this study sheds light on the private small business economy. The firms analyzed are located in all major US regions with a higher concentration in California in the West, Texas in the Southwest and New York in the Northeast representing respectively 12.09%, 6.74% and 6.56% of the population.

The homogeneous obligor classes are differentiated with respect to three criteria: credit rating, industry and firm size. For purpose of our study we adopt the D&B credit evaluation points (CPOINTS) as an indicator of firm's creditworthiness. On their basis we construct the credit ratings as percentiles of the whole distribution such that the credit rating "1"

Table I

Small businesses in the US

Descriptive statistics for US small businesses in the D&B dataset covering period from 2005 to 2011. The values: number of firms, % total and default rate (%) represent a historical average. Geographic regions are defined as: Central: IA, KS, MN, MO, NE, ND, SD; West: AK, AZ, CA, CO, HI, ID, MT, NV, NM, OR, UT, WA, WY; Northeast: CT, ME, MA, NH, NJ, NY, PA, RI, VT; Midwest: IL, IN, MI, OH, WI; Southeast: DE, DC, FL, GA, MD, VA, NC, SC, WV, AL, KY, MS, TN; Southwest: AR, LA, TX, OK.

	# firms	% total	min	max	defaults (%)
1. SIC industry division					
Agriculture, Forestry, Fishing (A)	9,902	4.19	9,340	10,188	8.39
Mining (B)	825	0.35	758	872	12.55
Construction (C)	32,180	13.61	27,048	36,275	13.13
Manufacturing (D)	16,382	6.93	14,155	18,278	17.48
Transportation, Communications, Electric, Gas, and Sanitary Services (E)	8,123	3.44	6,963	9,046	14.12
Wholesale Trade (F)	16,048	6.79	14,063	17,836	16.02
Retail Trade (G)	35,032	14.82	29,552	39,993	14.19
Finance, Insurance, and Real Estate (H)	20,020	8.47	17,170	22,310	11.34
Services (I)	96,379	40.78	85,672	104,065	11.19
Public Administration and non-classified (J)	1,467	0.62	1,358	1,831	23.88
2. Firm size					
1-5	133,755	56.59	115,434	147,547	9.67
6-10	44,125	18.67	38,308	49,158	12.89
11-20	28,244	11.95	24,731	31,174	15.82
21-30	10,890	4.61	9,778	11,867	18.53
31-50	9,150	3.87	8,344	9,904	21.16
51-100	6,149	2.60	5,670	6,751	26.42
>100	4,043	1.71	3,700	4,446	35.98
3. $ outstanding					
$0-500	38,530	16.30	29,436	48,676	5.78
$501-1,000	18,648	7.89	15,510	24,119	7.57
$1,001-2,000	22,880	9.68	19,990	27,531	9.15
$2,001-5,000	32,174	13.61	29,208	35,538	10.94
$5,000-15,000	48,536	20.54	42,366	52,458	12.67
$15,001-30,000	28,001	11.85	24,930	31,288	14.73
>$30,001	47,589	20.13	38,951	53,303	22.23
4. Region					
Central	17,512	7.41	16,135	18,876	10.65
West	53,754	22.74	45,590	59,743	12.84
Northeast	49,437	20.92	43,212	54,240	12.37
Midwest	36,319	15.37	32,368	39,741	12.31
Southeast	55,219	23.36	47,174	61,552	14.00
Southwest	24,118	10.20	21,533	26,437	12.62
5. Private					
Yes	236,284	99.97	206,140	260,471	12.74
No	74	0.03	50	117	42.58
Total	236,358	100.00	206,196	260,590	12.74

contains the 10% most creditworthy obligors and credit rating "10" the 10% least creditworthy obligors. Accuracy ratio of the credit rating is 19.2% and the power curve of this credit rating

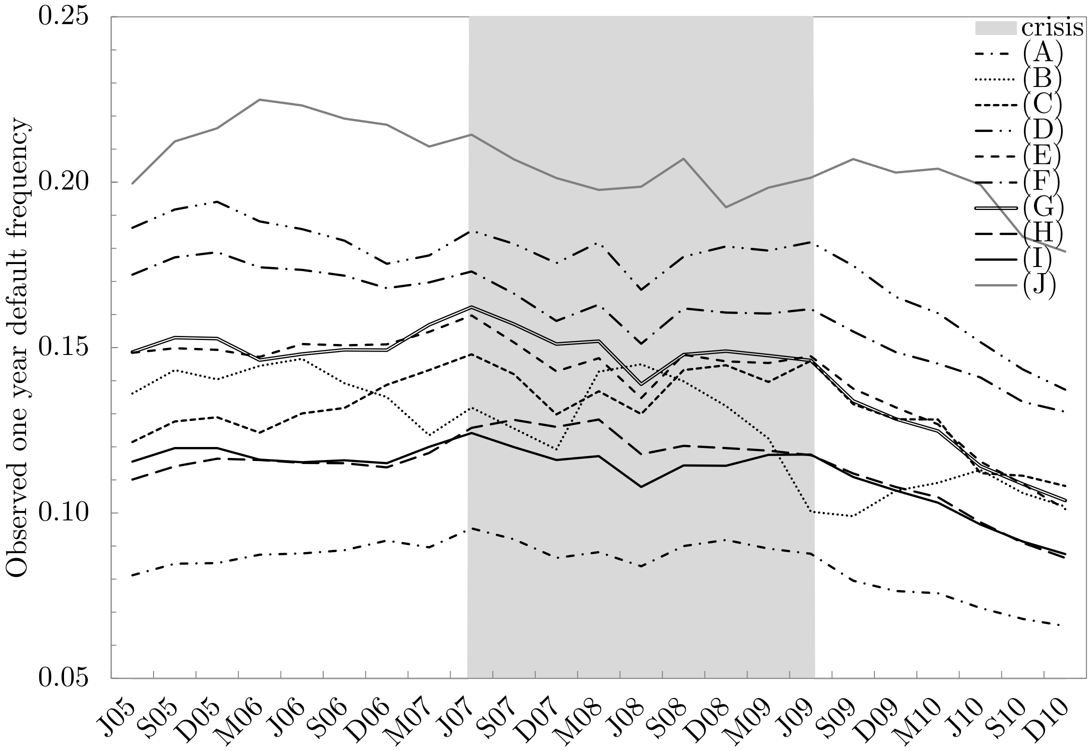

Figure 1: Observed default frequencies per industry classes. The shadowed area illustrates the crisis phase as defined by NBER business cycle reference dates. The pre-crisis phase covers June 2005 till September 2007; crisis is from December 2007 until June 2009 and post-crisis phase covers September 2009 until December 2010.

is illustrated in Figure 2. The discriminatory power of this rating is highly significant which confirm the Kolmogorov-Smirnov test and the Mann-Whitney U test (see Table II).

First, we categorize the firms into sets of homogenous obligor classes based on their credit rating and ten major SIC industry divisions. But in the absence of industry classification, financial institutions may turn to other available information to classify their exposures. Hence we conduct the analysis for credit ratig and seven firm size classes which are differentiated with respect to number of employees. Those seven firm size classes include very small firms with less than 5 employees, or those which employ between 6 and 10, 11 and 20, 21 and 30, 31 and 50, 51 and 100 or more than 100 people.

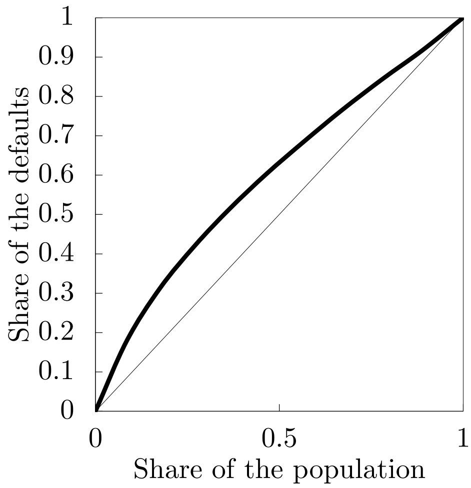

Figure 2: Power curve of the credit rating. It is based on percentiles of the small business population classified according to D&B credit evaluation points (CPOINTS) in the entire sample period. Power curve is denoted by the bold solid line and the 45° line by thin solid line. Accuracy ratio of the credit rating is 19.2%

Table II
Discriminatory power of the credit rating

Mann-Whitney U and Kolmogorov-Smirnov test for discriminatory power of the credit rating based on percentiles of population classified according to D&B credit evaluation CPOINTS. Based on random samples of 111,769 observations.

	Mann-Whitney U test	Kolmogorov-Smirnov test
Number	111,769	111,769
Defaults	16,666	16,714
Non-defaults	95,103	95,055
Test statistics	651,760,305.50	16.79
p-value	0.00	0.00

IV Results

In this section the estimator proposed is applied to the dataset described in the previous section. Particular interest is paid to industry and firm size heterogeneity among obligors and validity of the single-factor assumption in retail portfolios. Also we discuss different elements of risk in such portfolios as location and spread of defaults, default clustering and lastly comparing the obtained results to the minimum capital requirements imposed by Basel II including parameter uncertainty.

The empirical analysis begins by addressing the issues related to obligors' homogeneity in the US economy. What is meant by obligor class homogeneity in this paper is (1) a degree to which obligors react in similar fashion to a change in external common risk factors and (2) an exposure to the same or perfectly correlated common risk factors. The first relates to sharing by obligors the same sensitivity to external environment although being essentially different with respect to other firm's characteristics. In the multi-factor setting homogeneity is defined as a situation in which firms from same obligor class have equal sensitivity parameters w even if further segmented into smaller subclasses. The second translates into receiving equal stimulus from the external environment by all obligors classified as homogenous. In multi-factor model it is equivalent to perfectly correlated common risk factors within one homogenous class that is the correlation matrix of those subclasses Ω being equal to a matrix of ones. If both conditions are satisfied then the obligor classes can be considered as homogenous. Thus if we consider all obligors from one industry which may have different credit ratings, the obligors in that industry are homogenous only if their sensitivities w_k are equal and credit rating common risk factors are perfectly correlated.

In practice credit rating often serves to identify the homogenous obligor classes. It is a procedure which separates the firms according to their distribution with respect to risk. Hence for purpose of the study we select credit rating as the primary dimension of the analysis which is then subdivided into industry or firm size categories. Dietsch and Petey (2009) and McNeil and Wendin (2007) underline the relevance of other than credit rating sources of heterogeneity such as industry. Their claim is that a specialization in financing to specific industry may question the capital requirement requirements based solely on credit rating hence should include industry characteristic as well. Results of their study are based on corporate exposures. In small businesses we find some support for this hypothesis which can be seen in Table III. The table reports point estimates of the sensitivities to the common risk factors w_k for firms classified with respect to both: credit rating and industry, but also estimates only for credit ratings and only for industries. Indeed, the sensitivities to common risk factors per credit rating are not affected by the industry related heterogeneity. Inverse holds true as well. Thus, all credit ratings in one industry react in a similar fashion to a change in common risk factors. On the other hand, the industry related heterogeneity in credit ratings is revealed as different common risk factors per obligor class. Table IV presents Jennrich (1970) test for equality of correlation matrices where the reference matrix is equal to a matrix of ones hence a perfect correlation matrix. Panel A shows significant evidence of industry heterogeneity in the common risk factors. Only credit rating 4 and 9 remain robust to the industry related heterogeneity. It is good news for portfolio risk management as the industry related heterogeneity is followed by diversification benefits that

Table III
Sensitivity to the credit rating & industry common risk factors

The credit rating is constructed to represent deciles of the firms' risk distribution where 1 represents the lowest and 10 the highest credit risk. Industry is defined as in Table I. Sensitivity w_k without the industry/credit rating related heterogeneity is reported at the bottom of the table. Significant difference to w_k without the credit rating related heterogeneity (Panel C) is denoted by * at the 90% level, ** at the 95% level and *** at 99% level. Significant difference to w_k without the industry related heterogeneity (Panel B) is denoted by † at the 90% level, †† at the 95% level and ††† at 99% level. Bootstrapped S.E. in parenthesis ($\times 10^{-2}$).

	Sensitivity w_k (%)									
Industry	Credit rating 1	2	3	4	5	6	7	8	9	10
Panel A: w_k to the credit rating & industry common risk factors										
(A)	3.70	0.00	5.54	0.00	7.74	8.39	7.61	6.86	6.34	6.14
	(4.05)	(3.73)	(3.96)	(3.9)	(4.27)	(4.04)	(4.53)	(4.52)	(4.54)	(4.80)
(B)	0.00	18.41	2.76	0.00	6.58	0.00	0.00	0.00	0.00	16.07
	(7.81)	(12.86)	(11.23)	(11.87)	(13.11)	(12.78)	(13.53)	(15.53)	(15.59)	(18.79)
(C)	5.49	5.59	4.84	5.00	6.44	5.94	6.14	6.37	7.82*	8.72*
	(2.92)	(2.23)	(2.16)	(2.2)	(1.79)	(1.83)	(1.71)	(1.61)	(1.47)	(1.53)
(D)	5.85	4.46	5.43	6.15	4.59	4.68	6.69	6.10	7.06	6.31
	(2.76)	(3.07)	(2.76)	(2.6)	(2.84)	(2.7)	(2.7)	(2.37)	(2.25)	(2.64)
(E)	0.00	2.33	6.52	8.11	0.00	5.61	5.15	7.41	7.78	8.97
	(4.21)	(4.67)	(4.57)	(4.15)	(3.94)	(4.04)	(3.88)	(3.51)	(3.82)	(3.36)
(F)	4.34	6.45	2.19	5.74	6.43	2.57	6.49	4.61	7.17	6.29
	(2.7)	(2.71)	(2.66)	(2.98)	(2.77)	(2.78)	(2.73)	(2.87)	(2.54)	(3.00)
(G)	3.81	6.71	6.30	7.10	7.70	6.77	7.14	6.37	6.59	8.85
	(2.62)	(2.33)	(1.95)	(1.69)	(1.67)	(1.67)	(1.57)	(1.41)	(1.33)	(1.45)
(H)	3.77	4.75	2.62	6.72	4.63	4.94	8.68	6.92	7.63	7.58
	(2.67)	(2.73)	(2.47)	(2.22)	(2.73)	(2.73)	(2.43)	(2.52)	(2.89)	(3.01)
(I)	3.68	3.68	4.19	5.03	4.99	4.96	5.40	5.40	5.68	7.22**
	(1.23)	(1.17)	(1.03)	(1.01)	(0.99)	(0.94)	(0.91)	(0.93)	(0.94)	(1.00)
(J)	7.36	0.00	0.00	5.77	5.56	5.93	11.83	12.00	6.18	9.69
	(4.77)	(5.65)	(6.59)	(7.77)	(8.59)	(8.58)	(8.39)	(8.68)	(8.45)	(7.46)
Panel B: w_k assuming no industry related heterogeneity										
Industries	4.31	4.73	4.51	5.41	5.20	5.02	5.93	5.82	6.14	7.07
	(0.64)	(0.66)	(0.60)	(0.56)	(0.55)	(0.55)	(0.53)	(0.53)	(0.55)	(0.62)
Panel C: w_k assuming no credit rating related heterogeneity										
	Industry (A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)	(I)	(J)
Credit ratings	4.71	5.28	5.02	5.45	6.33	5.32	6.82	5.34	4.81	2.52
	(1.45)	(4.81)	(0.58)	(0.81)	(1.22)	(0.83)	(0.59)	(0.81)	(0.34)	(2.49)

stem from lower correlations between the common risk factors. Consider for a moment the whole economy categorized into industries. Each of those industries consists of firms from various credit ratings. As Panel C in Table IV shows this credit rating bares a significant source of heterogeneity within a given industry. Interestingly, only the finance industry remains homogenous. Their homogeneity stems from a facilitated liquidity access and very

Table IV

Homogeneity of credit rating/industry/firm size common risk factors

Jennrich (1970) test for equality of correlation matrices. It tests the difference between an estimate of a partition of (common risk factors) correlation matrix Ω and a matrix of ones. The partitioning is done according to the dimension tested for homogeneity. Thus if the homogeneity within credit rating is analyzed, the Ω is broken in such way that only the correlations within a given credit rating remain.

	Credit rating									
	1	2	3	4	5	6	7	8	9	10
Panel A: Industry related homogeneity in credit rating										
χ^2	138.37	$21{\cdot}10^{28}$	110.28	29.03	104.40	1,953.50	228.35	2,785.00	35.98	168.43
df	45	45	45	45	45	45	45	45	45	45
p-value	0.00	0.00	0.00	0.97	0.00	0.00	0.00	0.00	0.83	0.00
Panel B: Firm size related homogeneity in credit rating										
χ^2	24.29	15.62	771.52	68.69	44.80	1,061.80	214.94	113.01	189.32	14.88
df	21	21	21	21	21	21	21	21	21	21
p-value	0.28	0.79	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.83

Panel C: Credit rating related homogeneity in industry										
	Industry									
	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)	(I)	(J)
χ^2	448.66	621.77	770.89	678.61	6,098.10	$253{\cdot}10^{3}$	$7{,}805{\cdot}10^{4}$	11.35	2,429.40	2.42
df	45	45	45	45	45	45	45	45	45	45
p-value	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	1.00

Panel D: Credit rating related homogeneity in firm size							
	Firm size						
	$\leq$5	6-10	11-20	21-30	31-50	51-100	>100
χ^2	149.81	480.08	$13{\cdot}10^{5}$	4.62	179.24	47.09	4.48
df	45	45	45	45	45	45	45
p-value	0.00	0.00	0.00	1.00	0.00	0.39	1.00

distinctive risk factors which influence all firms in finance industry regardless of the credit rating, i.e. money provision, regulation or credit cycle.

In addition to industry, heterogeneity within credit rating arises typically from firm size. Intuitively micro firms which are great in number should operate in an almost prefect competitive environment while the larger ones should benefit from market power. Table V shows the sensitivity to common risk factors for obligor classes separated with respect to credit rating and firm size. With respect to those sensitivity parameters we find strong evidence of firm size related homogeneity in credit ratings. On the other hand, if the segmentation was made according to firm size, the assumption of class homogeneity would have been violated. Thus the credit rating contains significant information about the sensitivity to common risk

Table V

Sensitivity to the credit rating & firm size common risk factors

The credit rating is constructed to represent deciles of the firms' risk distribution where 1 represents the lowest and 10 the highest credit risk. The firm size stands for number of employees in a firm. Sensitivity w_k without the firm size/credit rating related heterogeneity is reported at the bottom of the table. Significant difference to w_k without the credit rating related heterogeneity (Panel C) is denoted by * at the 90% level, ** at the 95% level and *** at 99% level. Significant difference to w_k without the firm size related heterogeneity (Panel B) is denoted by † at the 90% level, †† at the 95% level and ††† at 99% level. Bootstrapped S.E. in parenthesis ($\times 10^{-2}$).

	Sensitivity w_k (%)									
Firm size	Credit rating 1	2	3	4	5	6	7	8	9	10
Panel A: w_k to the credit rating & firm size common risk factors										
≤5	4.69***	5.43***	5.69***	5.69***	6.07***	6.61***$_{\dagger\dagger}$	6.85***	6.99***	7.55***$_{\dagger}$	7.53***
	(1.62)	(1.07)	(0.92)	(0.81)	(0.88)	(0.78)	(0.80)	(0.75)	(0.77)	(0.84)
6-10	3.20***	4.83	5.15***	6.20***	6.16***	5.37***	7.34***	7.39***	6.49***	7.32***
	(2.15)	(1.64)	(1.54)	(1.42)	(1.44)	(1.58)	(1.41)	(1.40)	(1.43)	(1.69)
11-20	2.49***	2.55***	2.54***	3.66***	5.05***	4.22***	6.50***	5.82**	4.91***	6.22***
	(2.02)	(2.06)	(2.02)	(2.11)	(2.01)	(2.08)	(1.83)	(2.00)	(2.20)	(2.23)
21-30	4.58*	6.27***	2.27***	5.66	6.72***	3.16***	5.07***	4.55***	5.89***	9.68***
	(2.68)	(3.04)	(3.24)	(3.58)	(3.70)	(3.73)	(3.82)	(3.87)	(3.54)	(3.21)
31-50	3.82***	5.77***	0.00***	6.26***	6.92***	6.98***	6.72***	5.33***	5.72***	5.82***
	(2.70)	(3.57)	(3.27)	(3.79)	(3.89)	(4.04)	(3.81)	(3.87)	(3.75)	(3.83)
51-100	6.59***	6.66***	7.74***	6.43***	8.52***	7.62***	8.14***	6.01	10.83***	7.91*
	(4.31)	(4.53)	(4.04)	(3.93)	(3.79)	(4.04)	(4.18)	(4.23)	(4.14)	(4.86)
>100	12.59***	10.54***	5.79***	3.53***	5.12**	10.20***	7.68***	9.52***	11.95***	10.00***
	(4.76)$_{\dagger}$	(4.56)	(4.38)	(4.39)	(4.63)	(4.75)	(5.16)	(5.50)	(6.07)	(7.18)
Panel B: w_k assuming no firm size related heterogeneity										
Firm sizes	4.31	4.73	4.51	5.41	5.20	5.02	5.93	5.82	6.14	7.07
	(0.64)	(0.66)	(0.60)	(0.56)	(0.55)	(0.55)	(0.53)	(0.53)	(0.55)	(0.62)

Panel C: w_k assuming no credit rating related heterogeneity

	Firm size ≤5	6-10	11-20	21-30	31-50	51-100	>100
Credit ratings	6.10	5.24	4.21	5.57	5.69	7.72	8.53
	(0.30)	(0.50)	(0.60)	(1.03)	(1.12)	(1.30)	(1.53)

factors but not the firm size. Panel B and panel D in Table IV however find only moderate support for the homogeneity across the credit rating and firm size common risk factors. From a risk management perspective it allows for some diversification benefits.

Also, in general we find that the smaller the firm size and the higher the riskiness of a firm there is more evidence of common risk factors' influence on asset value. But the high and significant sensitivity parameter for the largest and most creditworthy firms in our dataset reveals a non-monotonic relationship between common risk factors, creditworthiness

Table VI
Single vs. multi-factor model

Jennrich (1970) test for equality of correlation matrices. It tests the difference between an estimate of common risk factors correlation matrix Ω and a matrix of ones. The obligor classes are divided with respect to credit rating and industry, or credit rating and firm size.

	Credit rating & industry	Credit rating & firm size
χ^2	63,086.00	982.53
df	4950	2415
p-value	0.00	1.00

and firm size. This non-monotonic relationship is inconsistent with the Basel II formula for minimum capital requirement which assumes a strictly decreasing asset correlation function in the domain of creditworthiness and firm size (Basel Committee on Banking Supervision (2005)).

A large bulk of the existing literature (i.e. Gordy (2000, 2003), Lopez (2004)) and regulatory frameworks such as Basel II (Basel Committee on Banking Supervision (2005)) assume a single factor model. This assumption translates into a situation in which only a single economy-wide common risk factor exists and all obligors are subject to its changes. It is counterintuitive to claim that for example all industries were depending on risk factors which strike at the same time with the same strength. It is hard to believe that weather risk associated with agriculture industry, demographic risk with construction industry, oil price risk with transportation industry or liquidity risk associated with finance industry are perfectly correlated.

Plausibility of the single factor assumption was already challenged by Dietsch and Petey (2009) with their multi-factor model of concentration risk. In terms of our model in which the correlation matrix Ω is estimated in an unconstrained manner we statistically test for a single risk factor if all common risk factors were perfectly correlated. In order to test the validity of this simplifying assumption for US small businesses we use Jennrich (1970) test for equality of correlation matrices. The outcomes of the test are shown in Table VI. Our results call into question the assumption of a single common risk factor in US retail portfolios. This assumption is violated for the obligors segmented according to their credit rating and industry. As expected those two dimensions capture some of information differentiating obligors' risk types. However, there is no empirical evidence in favor of the second type of segmentation done according to credit rating and firm size. We find those dimensions redundant where risk factors are perfectly correlated. Also, in view of the above results we create the homogenous obligor classes with respect to two criteria: credit rating and industry.

Furthermore, we answer a question related to which risk dominates in small businesses: systematic or idiosyncratic. Given that small businesses correspond to a significant part of the US economy one could expect that their aggregate behavior matches the economy swings. On the other hand each small business has individual qualities as its location, business network, faithful clients that are stable over the business cycle and often decide about firm's success or failure. Bakery at the corner or a dentist in downtown can do fine even during recession. Table III shows that across the whole sample period small businesses have a tendency to dependent merely on the idiosyncratic risk. It is the direct neighborhood and obligor's characteristics that often decide about the success or failure of the small business (also in Phelan (2011)). Even though the sample period covers whole business cycle: with expansion in 2005-2006, throughout the 2007-2009 recession and recovery in 2010, we observe that the estimated sensitivities to the common risk factors remain low and vary between 0.00-18.41% explaining only 0.00-3.39% of the asset variability. The remaining 96.61-100.00% of small business risk is due to changes in the firm specific characteristics.[2] Those results are striking especially in the light of the crisis which affected the whole economy without exception. Reason for that resides in the relatively high default probabilities in small business. Although the probabilities of default were on average on a high level during the crisis, the uncertainty about default decreased and became more of a certainty. Second reason for finding low sensitivity of US small business to systemic risk factor stems from the fact that the dataset is quasi-exhaustive and captures the limit of diversification in the US economy. As Dietsch and Petey (2004) we believe financial institutions observe higher asset correlations

[2]The low values of sensitivity parameters w_k remain robust to changes in the default definition to a less conservative one which considers only events of losses acquired by a debt holder. Also for US geographic regions the values of sensitivity parameters w_k remain low. Intuitively, it is expected that geographic proximity in the activity of small businesses would cause them to be more susceptive to common risk factors. However, the results for US states show that the idiosyncratic risk in small business loans prevails.

in their portfolios due to a possible further diversification on their books.

Moreover, we find that in general retail trade and transportation are the most sensitive to the economic environment. To get a better feeling where this high level of sensitivities comes from, let us look closer on the unique characteristics of those industries. Retail trade relies heavily on consumer spending which varies with the mood swings of the economy. Transportation on the other hand can be considered as a denominator of the economy with prices of its services adding up to the economic environment. Overall the sensitivity varies with the credit rating and industry from 0.00-12.00% with the exception of an outlier: mining which exhibits the highest but insignificant sensitivity of 18.41%. As expected agriculture and mining, two industries with a rather inelastic demand, were not subject to changes in common risk factors.

On the asset correlation side, presented in Table VII, we find that during the whole analyzed period the implied asset correlation averaged around 0.41% with the lowest values of 0.00% for mining and agriculture and a statistically significant maximum of 0.78% for the highest risk retail trade businesses. Most importantly regardless of the small business' riskiness and industry our estimates are significantly lower than any available estimates for corporate firms. For example McNeil and Wendin (2007) report asset correlation between corporate firms ranging from 6.30-10.90% that is about fourteen times larger than our estimate for small businesses. The considerably lower asset correlation between small businesses has important regulatory consequences which are discussed further in this section.

We turn our attention to the development of small business riskiness over the course of the recent financial crisis. Intuitively, the time of deteriorated macroeconomic conditions should trigger clustering of defaults and increased uncertainty about firms' creditworthiness. For this purpose we apply a moving window technique in which the overall sample period is subdivided into three phases according to NBER business cycle reference dates. First phase: pre-crisis covers June 2005 till September 2007. The crisis phase is from December 2007 until June 2009 and post-crisis phase covers September 2009 until December 2010. This treatment allows to estimate the model separately in those three phases and to outline the evolution of

Table VII

Asset correlation and default rates per credit rating & industry

The values reported cover period from June 2005 to December 2010. The credit rating is constructed to represent deciles of the firms' risk distribution where 1 represents the lowest and 10 the highest credit risk. Industry is defined as in Table I. Bootstrapped S.E. in parenthesis ($\times 10^{-2}$).

		Asset correlation ρ_{ii} (%) *within obligor class* and default rates $\bar{p}$ (%)									
Industry		Credit rating 1	2	3	4	5	6	7	8	9	10
(A)	ρ_{ii}	0.14	0.00	0.31	0.00	0.60	0.70	0.58	0.47	0.40	0.38
		(0.43)	(0.36)	(0.50)	(0.40)	(0.68)	(0.67)	(0.72)	(0.65)	(0.63)	(0.68)
	$\bar{p}$	5.94	5.68	5.50	6.17	6.52	8.09	9.99	11.15	16.49	23.49
(B)	ρ_{ii}	0.00	3.39	0.08	0.00	0.43	0.00	0.00	0.00	0.00	2.58
		(1.62)	(5.12)	(3.36)	(3.78)	(4.54)	(4.32)	(4.77)	(6.65)	(6.83)	(9.73)
	$\bar{p}$	9.16	9.05	8.71	11.96	13.34	15.03	18.33	17.21	23.32	31.57
(C)	ρ_{ii}	0.30	0.31	0.23	0.25	0.41	0.35	0.38	0.41	0.61	0.76
		(0.30)	(0.24)	(0.20)	(0.22)	(0.23)	(0.21)	(0.21)	(0.21)	(0.23)	(0.27)
	$\bar{p}$	8.65	7.30	8.06	8.84	9.53	10.82	12.19	14.56	20.46	30.82
(D)	ρ_{ii}	0.34	0.20	0.29	0.38	0.21	0.22	0.45	0.37	0.50	0.40
		(0.30)	(0.30)	(0.30)	(0.32)	(0.27)	(0.25)	(0.35)	(0.28)	(0.31)	(0.31)
	$\bar{p}$	13.38	12.62	12.99	13.79	14.37	15.56	17.39	19.55	24.11	32.36
(E)	ρ_{ii}	0.00	0.05	0.43	0.66	0.00	0.32	0.27	0.55	0.60	0.80
		(0.48)	(0.59)	(0.65)	(0.67)	(0.42)	(0.51)	(0.47)	(0.50)	(0.57)	(0.57)
	$\bar{p}$	10.32	10.65	10.71	10.81	11.06	11.67	13.34	15.01	18.06	26.01
(F)	ρ_{ii}	0.19	0.42	0.05	0.33	0.41	0.07	0.42	0.21	0.51	0.40
		(0.24)	(0.34)	(0.20)	(0.34)	(0.35)	(0.23)	(0.34)	(0.27)	(0.35)	(0.34)
	$\bar{p}$	13.11	11.89	12.77	13.86	14.10	14.76	16.49	18.49	22.90	30.26
(G)	ρ_{ii}	0.14	0.45	0.40	0.50	0.59	0.46	0.51	0.41	0.43	0.78
		(0.21)	(0.30)	(0.24)	(0.24)	(0.26)	(0.22)	(0.22)	(0.18)	(0.18)	(0.26)
	$\bar{p}$	10.67	9.84	10.50	11.08	11.14	12.21	13.10	14.63	18.21	25.68
(H)	ρ_{ii}	0.14	0.23	0.07	0.45	0.21	0.24	0.75	0.48	0.58	0.58
		(0.21)	(0.26)	(0.17)	(0.29)	(0.26)	(0.28)	(0.41)	(0.34)	(0.42)	(0.43)
	$\bar{p}$	9.25	8.09	8.94	9.21	9.46	10.63	11.63	14.42	18.40	25.63
(I)	ρ_{ii}	0.14	0.14	0.18	0.25	0.25	0.25	0.29	0.29	0.32	0.52
		(0.09)	(0.09)	(0.09)	(0.10)	(0.10)	(0.09)	(0.10)	(0.10)	(0.11)	(0.15)
	$\bar{p}$	7.93	7.62	8.17	8.99	9.46	10.29	11.56	13.30	17.40	24.73
(J)	ρ_{ii}	0.54	0.00	0.00	0.33	0.31	0.35	1.40	1.44	0.38	0.94
		(0.72)	(0.83)	(1.14)	(1.64)	(2.02)	(1.95)	(2.18)	(2.29)	(1.97)	(1.61)
	$\bar{p}$	21.29	21.90	21.47	23.51	21.83	19.56	15.72	16.47	17.42	22.28

joint default risk. Figure 3 addresses two important elements of default risk in a portfolio of loans: location and spread of defaults. Both of them are expected to vary over the different phases of the crisis. First one is associated with the probabilities of default illustrated in Panel a, c and e and resembles expected loss. The second one is associated with the asset correlation illustrated in Panel b, d and f and resembles unexpected loss. Clustering of

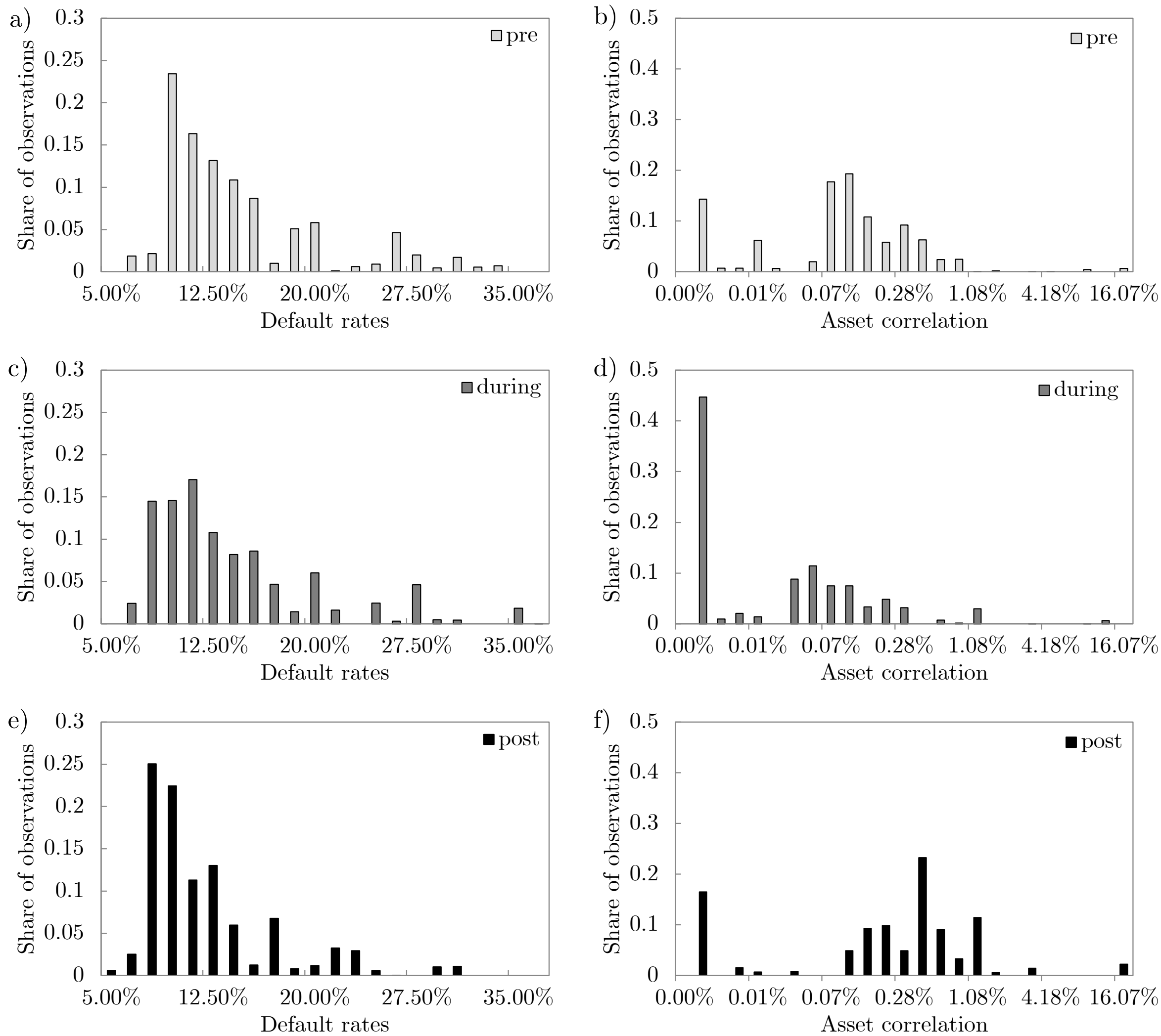

Figure 3: Default rates $\bar{p}$ and asset correlation ρ_{ii} per credit rating & industry in the pre-, during and post-crisis phases. The pre-crisis phase covers June 2005 till September 2007; crisis is from December 2007 until June 2009 and post-crisis phase covers September 2009 until December 2010.

defaults can be revealed by both of them either by increased default frequencies or by high asset correlation and thus higher uncertainty of defaults. In the crisis eve the probabilities of default are high with average of 13.47% (Panel a) and fell slightly to 13.14% during the crisis (Panel c). This decline in defaults does not support clustering of defaults in crisis phase. However it should not be interpreted as a sign of economic recovery either but rather an indicator of the creditworthiness of those firms which resisted the crisis [3]. It is after the trough that the probability of default averaged at its low value of 11.23% (Panel e).

[3]Probability of default is a forward looking measure assigned to those firms whose contracts are not in default. Those firms which are in default have probability of default equal unity.

Moreover, the predicted increased uncertainty about firms' creditworthiness finds no support in the data either. We observe that before the economic turmoil the asset correlation stayed on average on a low level (0.25%). It declined even further to 0.13% during the crisis when the mass of the asset correlation moved left with many obligors exhibiting virtually no correlation between each other. It stems from low exposure to common risk factors and resembles the fact that in this phase the analyzed population was composed of firms which lasted the strike of the crisis without a default. Obviously those firms which withstood the crisis showed less sensitivity to economic conditions but instead a substantial reliance on the firm characteristics. Those results remain in context of the financial crisis which set traps in form of weak sales and change in consumer tastes. Those firms which managed to right-size and organize their financial houses (Phelan (2011)) dampened the negative impact of common risk factors in those phases. On the other hand, after the economic turmoil the average asset correlation soared up to 0.68% exposing the dependencies between small businesses. At that time the mass of the asset correlation shifted right and the obligors began to experience higher asset correlation. This delay in response of implied asset correlations parameters to the economic downturn can be interpreted as evidence that in the recent crisis small businesses were suffering its consequences rather than inducing it.

The discussion about location and spread of defaults as elements of portfolio default risk continues in Figure 4. It is a comparison of Monte Carlo generated loss distributions plotted for the three phases defined above and over the whole analyzed period. We simulate panels of default indicators for a portfolio of 10,000 firms distributed across credit ratings and industries proportionally to the historical data. To that end we use the estimates of θ and of default thresholds $\Phi^{-1}(\bar{p}_k)$ which are phase-specific. The density estimates are given by Gaussian kernel smoothing (with interval length of 10). From Basel perspective the pricing of loan exposures and provisions should cover losses up to the location of our loss distribution. On the other hand, if there are any losses associated to the spread of the loss distribution they should be covered by the capital requirement.

The results show that those values evolved over the different phases of crisis. We ob-

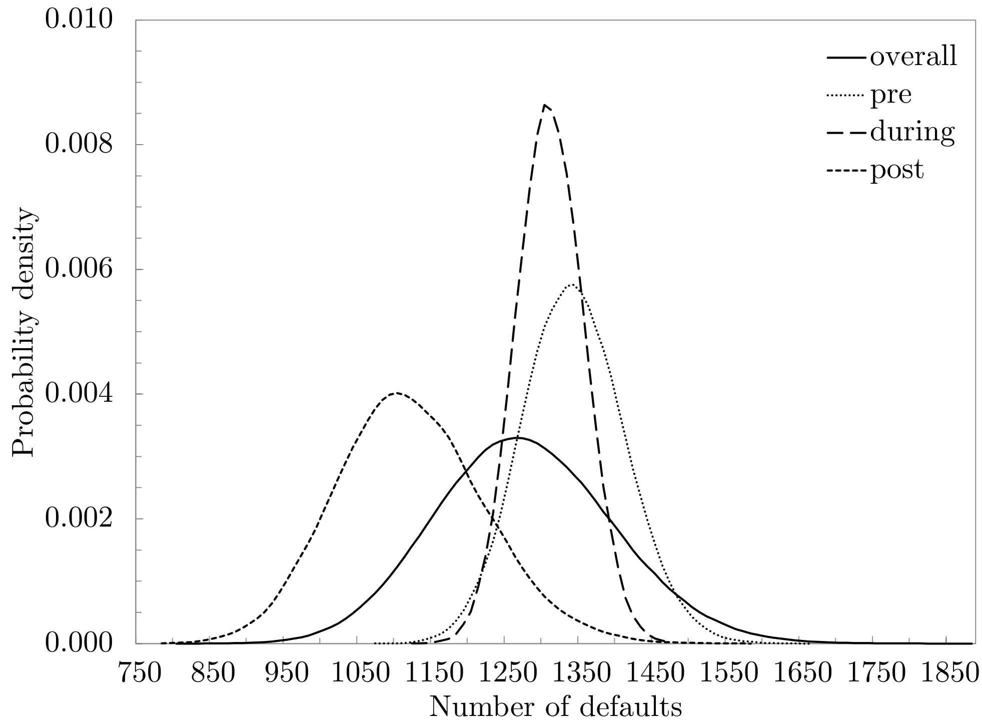

Figure 4: Portfolio loss distribution comparison. Density of number of defaults for the pre-crisis (dotted line), during the crisis (dashed line), post-crisis (square-dotted line) phases and over the whole analyzed period (solid line). The density estimates are given by Gaussian kernel smoothing (with interval length of 10) of the Monte Carlo generated loss distribution. The pre-crisis phase covers Q2 2005 till Q3 2007; crisis is from Q4 2007 until Q2 2009 and post-crisis phase covers Q3 2009 until Q4 2010.

serve shifts in both location and spread of the loss distribution as we move through different phases of the crisis. The pre-crisis phase was characterized by a relatively high location and moderate spread of loss distribution. In this phase the average realized number of defaults was 1,345 with 99.9$^{\text{th}}$ percentile of losses at 1,560 defaults. Interestingly, the least uncertain level of defaults occurred during the crisis. At that time the distance between realized number of defaults (1,314) and 99.9$^{\text{th}}$ percentile (1,450) reached its minimum indicating low capital requirement but nevertheless high provisions. It turns out the crisis was informative for portfolio management in a sense that firms which withstood the deterioration of macroeconomic conditions did not go systematically into default. It was due to their higher resistance to changes in the common risk factors that correlation in retail portfolios was on a low level. Next the post-crisis phase was characterized by low location of loss distribution (1,121) signaling economic recovery which however was accompanied by high uncertainty with 99.9$^{\text{th}}$ percentile at 1,453 defaults. Typically, the location of loss distribution (thus

Table VIII
Capital requirement for corporate debt in the US

The default rates $\bar{p}$ are an average over time of observed default frequencies. Estimation of asset correlation ρ_{ii} *within obligor class* is based on sample of annual default rates provided by S&P. The time span is 2005-2010. Monte Carlo S.E. in parenthesis ($\times 10^{-2}$). K_m stands for capital requirement computed with the regulatory formula but with our estimates of asset correlation, K_r is the regulatory one. In computation of capital requirement we assume $LGD = 0.50$ and effective maturity $M = 3$. Significant difference to K_r is denoted by * at the 90% level, ** at the 95% level and *** at 99% level. Panel B displays results of the paired difference test for a difference between the K_r and K_m. The test results are robust to changes in LGD and remain robust for $M \leq 14$.

Panel A: Capital requirement for corporate sub-portfolios

	AA	AA-	A+	A	A-	BBB+	BBB	BBB-
$\bar{p}$ (%)	0.19	0.17	0.13	0.16	0.16	0.16	0.24	0.30
ρ_{ii} (%)	20.72	20.28	19.34	11.57	20.11	11.54	9.57	13.20
	(6.33)	(6.31)	(6.40)	(4.60)	(6.46)	(4.40)	(3.78)	(4.60)
K_r (%)	4.14	3.92	3.45	3.83	3.83	3.79	4.72	5.31
K_m (%)	3.64	3.33	2.73	1.69	3.22	1.66	1.77	2.89
	(1.25)	(1.17)	(1.00)	(0.72)	(1.16)	(0.68)	(0.73)	(1.09)
Difference (%)	0.50	0.58	0.72	2.15	0.61	2.13	2.95	2.42
t-statistic	0.40	0.50	0.71	2.99***	0.53	3.13***	4.06***	2.23**

	BB+	BB	BB-	B+	B	B-	CCC/C
$\bar{p}$ (%)	0.68	0.44	0.47	1.61	2.92	6.24	23.97
ρ_{ii} (%)	13.80	11.31	3.83	15.02	16.81	22.18	15.92
	(4.50	(4.05)	(1.86)	(4.52)	(5.14)	(5.45)	(6.00)
K_r (%)	7.59	6.30	6.55	10.16	11.88	14.92	22.24
K_m (%)	4.93	3.08	1.20	8.82	13.33	23.38	25.81
	(1.66)	(1.14)	(0.50)	(2.56)	(3.71)	(4.72)	(5.09)
Difference (%)	2.66	3.23	5.35	1.34	-1.45	-8.47	-3.57
t-statistic	1.60	2.82***	10.67***	0.52	0.39	1.79*	0.70

Panel B: Paired difference test ($H_0 : \bar{K}_r - \bar{K}_m = 0$)

	LGD	M	Mean (%)	SD (%)	t	df	p-value
Pair $K_r - K_m$	0.5	3	0.74	3.28	0.88	14	0.39
	0.9	3	1.34	5.92	0.87	14	0.40
	0.5	14	3.17	7.04	1.74	14	0.10

provisions level) is considered not to bare uncertainty which instead is associated with the spread of loss distribution. However, what we observe are considerable shifts in the location of loss distribution related to provisions level. In practice from risk management perspective it should mean that the capital a financial institution holds accounts for shifts in location parameter as well.

Next, we show robustness of our estimator which can be applied to portfolio of corporate debt as well. We illustrate here that although its simplicity it shows to be effective in retail as well as in corporate environment. But mostly we show that the estimator proposed produces similar results for corporate exposures as does the Basel II regulatory framework which demonstrates its reliability. To that end we use the public information on US corporate default rates per credit rating provided by S&P. The S&P reports payment history of about 3 thousand US firms during a period of six years from 2005 to 2010 and cover a broad range of industries. Both S&P and our study weight the default events by the number of obligors rather than the nominal value of default. We exclude AAA and AA+ rating from the analysis due to no defaults in those rating categories during the analyzed period. For consistency with Basel methodology (Gordy (2000, 2003)) the estimation of our model follows per sub-portfolios composed of obligors from one homogenous obligor class. In its essence this procedure is equivalent to estimation of a single factor model.

Panel A in Table VIII shows the resulting asset correlation estimates together with the default rates, capital requirements for corporate exposures and a difference between our model and Basel approach. In general the results show that although corporate firms exhibit low probabilities of default relative to retail debt, they are heavily exposed to changes in economic conditions. The asset correlation varies between 3.83-22.18% and averages at 15.01% which confirms a substantial interdependence in corporate exposures. But most importantly the capital requirements for corporate exposures implied by our estimates of asset correlation are in line with the regulatory ones which is shown both in Panel A and Panel B in Table VIII. The paired difference test reported confirms that our model and Basel II formula produce on average similar outcomes. We find no significant difference between the capital requirement computed according to regulatory formula and the one computed using our estimates of asset correlation.

Given the consistency of the Basel II and the proposed model in corporate portfolios one could expect to find matching estimates in case of retail portfolios as well. To illustrate the implication of the model on capital requirements in financial institutions holding retail portfolios, we use the results from Table VII and contrast them with outcomes of the Basel II regulatory formula. Table IX suggests that small businesses are subject to inefficient capital

Table IX
Capital requirement for retail portfolios

Credit rating is constructed to represent deciles of the firms' risk distribution where 1 represents the lowest and 10 the highest credit risk. The time span is 2005-2010. K_m (%) stands for capital requirement computed with the regulatory formula but with our estimates of asset correlation, K_r (%) is the regulatory one. We take the asset correlation as in Table VII and assume $LGD = 0.50$. The later parameter does not affect the ratio K_r/K_m. Bootstrapped S.E. in parenthesis ($\times 10^{-2}$). Paired difference test ($\text{H}_0 : \bar{K}_r - \bar{K}_m = 0$) is highly significant with mean of 5.50% and t-statistics of 41.65.

	Credit rating:	1	2	3	4	5	6	7	8	9	10
(A)	K_r	6.01	5.98	5.96	6.04	6.09	6.33	6.71	6.97	8.21	9.48
	K_m	0.73	0.00	1.07	0.00	1.78	2.27	2.35	2.25	2.64	3.09
		(0.91)	(0.80)	(0.88)	(0.90)	(1.13)	(1.25)	(1.57)	(1.65)	(2.05)	(2.56)
	K_r/K_m	8.23	NA	5.55	NA	3.43	2.79	2.86	3.10	3.11	3.07
(B)	K_r	6.54	6.52	6.45	7.16	7.49	7.88	8.59	8.36	9.46	10.34
	K_m	0.00	6.33	0.71	0.00	2.41	0.00	0.00	0.00	0.00	9.56
		(2.76)	(5.77)	(4.39)	(5.50)	(6.51)	(6.57)	(7.48)	(8.68)	(9.10)	(10.80)
	K_r/K_m	NA	1.03	9.06	NA	3.10	NA	NA	NA	NA	1.08
(C)	K_r	6.44	6.20	6.33	6.48	6.61	6.90	7.22	7.78	9.00	10.28
	K_m	1.48	1.34	1.23	1.36	1.89	1.88	2.11	2.46	3.73	5.00
		(0.85)	(0.58)	(0.59)	(0.65)	(0.58)	(0.63)	(0.64)	(0.67)	(0.75)	(0.92)
	K_r/K_m	4.35	4.64	5.16	4.77	3.51	3.67	3.42	3.16	2.41	2.06
(D)	K_r	7.50	7.32	7.40	7.59	7.73	8.00	8.40	8.83	9.57	10.39
	K_m	2.13	1.53	1.92	2.29	1.71	1.84	2.88	2.79	3.62	3.63
		(1.07)	(1.13)	(1.05)	(1.05)	(1.13)	(1.12)	(1.24)	(1.14)	(1.21)	(1.55)
	K_r/K_m	3.52	4.79	3.85	3.32	4.51	4.35	2.92	3.17	2.64	2.86
(E)	K_r	6.78	6.86	6.87	6.89	6.95	7.09	7.49	7.88	8.54	9.82
	K_m	0.00	0.69	2.07	2.66	0.00	1.86	1.85	2.96	3.46	4.83
		(1.40)	(1.61)	(1.62)	(1.52)	(1.35)	(1.47)	(1.52)	(1.51)	(1.82)	(1.90)
	K_r/K_m	NA	9.98	3.32	2.60	NA	3.81	4.05	2.66	2.46	2.03
(F)	K_r	7.43	7.15	7.35	7.61	7.67	7.82	8.21	8.63	9.40	10.24
	K_m	1.52	2.19	0.73	2.13	2.44	0.95	2.71	2.01	3.59	3.53
		(1.00)	(1.00)	(0.96)	(1.19)	(1.13)	(1.10)	(1.22)	(1.31)	(1.34)	(1.73)
	K_r/K_m	4.88	3.26	10.06	3.57	3.15	8.24	3.03	4.30	2.62	2.90
(G)	K_r	6.86	6.68	6.83	6.96	6.97	7.22	7.43	7.79	8.57	9.78
	K_m	1.15	2.02	1.97	2.33	2.56	2.35	2.61	2.47	2.91	4.73
		(0.84)	(0.77)	(0.67)	(0.61)	(0.62)	(0.64)	(0.63)	(0.59)	(0.63)	(0.82)
	K_r/K_m	5.95	3.31	3.47	2.99	2.73	3.07	2.85	3.16	2.95	2.07
(H)	K_r	6.56	6.33	6.49	6.55	6.60	6.85	7.08	7.74	8.61	9.77
	K_m	1.03	1.20	0.69	1.93	1.31	1.52	3.01	2.67	3.43	4.02
		(0.78)	(0.75)	(0.70)	(0.70)	(0.83)	(0.90)	(0.94)	(1.06)	(1.39)	(1.67)
	K_r/K_m	6.35	5.26	9.45	3.39	5.04	4.51	2.36	2.90	2.51	2.43
(I)	K_r	6.31	6.25	6.35	6.50	6.60	6.78	7.07	7.48	8.40	9.66
	K_m	0.90	0.87	1.06	1.38	1.42	1.49	1.78	1.94	2.42	3.75
		(0.32)	(0.30)	(0.28)	(0.30)	(0.31)	(0.31)	(0.32)	(0.36)	(0.43)	(0.55)
	K_r/K_m	7.00	7.15	6.00	4.71	4.65	4.53	3.98	3.85	3.47	2.57
(J)	K_r	9.14	9.24	9.17	9.49	9.23	8.83	8.04	8.20	8.41	9.30
	K_m	3.56	0.00	0.00	2.89	2.68	2.71	5.11	5.33	2.65	4.90
		(2.45)	(2.93)	(3.46)	(4.32)	(4.72)	(4.51)	(4.15)	(4.39)	(4.25)	(4.08)
	K_r/K_m	2.56	NA	NA	3.28	3.44	3.26	1.57	1.54	3.17	1.90

allocation imposed by the regulator. The results show significant discrepancies in capital requirements implied by the Basel II and the proposed model. Regardless of the creditworthiness of the obligor the Basel II formula significantly overstates the asset correlation and thus capital requirement for sub-portfolios of small businesses which is shown by the highly significant paired difference test. Indeed, we observe that the capital requirement is on average almost four times higher than the data suggests. And it is the more creditworthy small obligors that suffer the highest capital charges relatively to their riskiness. For them the regulatory formula overestimates the capital requirement even by factor of 10.06. As a result those more creditworthy obligors pay for the credit risk of their less creditworthy peers. It also creates inverse incentives for financial institutions that flee to other obligor classes in which loans originated are less costly to hold. Similarly we compute the 'aggregated' capital requirement on a portfolio level which is composed of all the obligor classes in the historical proportions. Here the regulatory capital requirement amounts to 7.31% which is four times more than our multi-factor model implies (2.01%).

As the Basel Committee on Banking Supervision (2005) suggest, the overly high capital requirements for US retail loan portfolios may resemble a need for constructing a uniform framework applicable to a wider range of countries. But most importantly the regulatory formula for retail asset correlation was not fitted to historical loan data. Instead the Basel Committee on Banking Supervision reversed engineered the asset correlation from the information on historical capital that banks held. Our results suggest that the obtained retail asset correlation function imposed by regulator is far from being accurate. Moreover, the resulting inefficiency in capital allocation encourages more financing in the corporate sector rather than in small business economy, an outcome undesired by the policy makers.

From risk management perspective an important feature of the approach proposed is given by a possibility to assess the parameter uncertainty of the capital requirement. For example if we take the prudential value of the capital requirement equal to its estimate plus its uncertainty (here the standard error) the capital requirement increases even by 10.80% for least creditworthy obligors from mining industry. But on average prudential financial

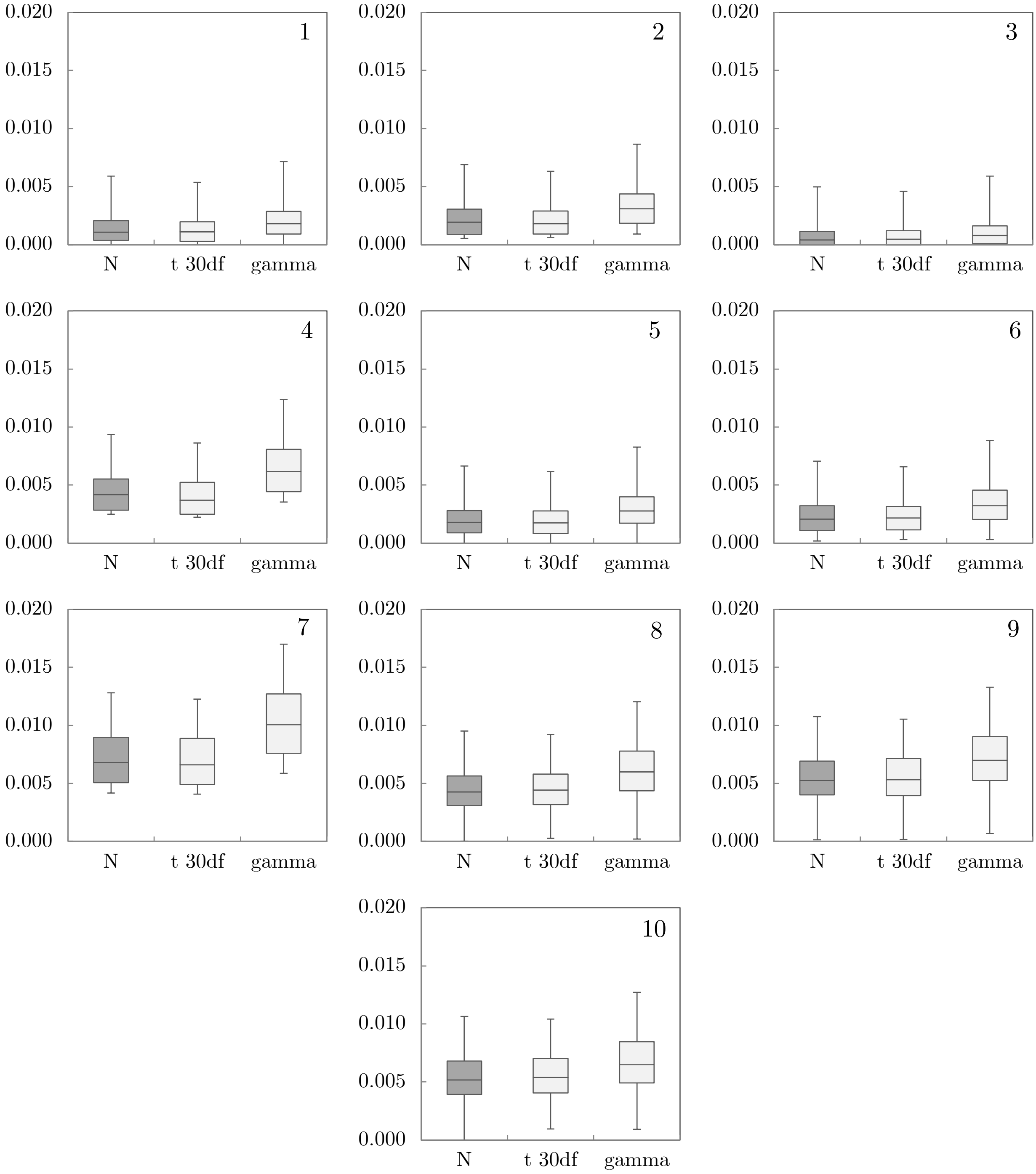

Figure 5: Parameter uncertainty. The estimated uncertainty of ρ_{ii} given the distribution of common and idiosyncratic risk factors. Results based on 1,000 simulated portfolios with $T = 24$ in which true value of asset correlation is fixed at the estimates from Table VII and the risk factors are normally, t-student or gamma distributed.

would hold 1.87% above the model's requirement.

This leads us to another aspect of parameter uncertainty namely uncertainty which stems from normality assumption of common risk factors. Although the normality of risk factors is not a necesity to construct the multifactor model, we derive our estimates of asset correlation for the case in which the common and idiosyncratic risk factors are normally distributed. Figure 5 depicts the performance of the proposed estimator in a world with normally distributed risks next to its robustness to alternative fat-tailed distributions of risk factors. The model proposed is applied here to 1,000 simulated portfolios each consisting of $K = 10$ homogenous obligor classes for 24 simulated time points. We select finance industry as an example. The observations are generated according to the following relationship:

$$D_{it} = \begin{cases} 1 & \text{if } A_{it} < F^{-1}(\bar{p}_k), \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

where A_{it} follows the relationship in (1) and $F(\cdot)$ is the cumulative distribution function of the asset value that for normally distributed risk factors takes the form of $\Phi(\cdot)$. To each of the obligor classes we assign a sensitivity to common risk factors as in Table III: $w = (3.77\%, 4.75\%, 2.62\%, 6.72\%, 4.63\%, 4.94\%, 8.68\%, 6.92\%, 7.63\%, 7.58\%)$ and a corresponding probability of default as in Table VII: $\bar{p} = (9.25\%, 8.09\%, 8.94\%, 9.21\%, 9.46\%, 10.63\%, 11.63\%, 14.42\%, 18.40\%, 25.63\%)$.

The results for normally, t-student and gamma distributed risk factors are ilustrated in Figure 5. In a world in which the common and idiosyncratic risk factors follow normal distribution, the estimated asset correlation uncover the true value of ρ_{ii} with a small dispersion and are centered around it. If the risk factors were to follow a fat-tailed distribution the sensitivity parameters estimated also ocilate around the true value. However, for risk factors following a gamma distribution ρ_{ii}'s are overestimated while for t-student distribution with 30 degrees of freedom ρ_{ii}'s are marginally underestimated. Inspite of the estimates remaining close to the true value, any change in risk factors distribution may have an impact on the portfolio risk Schönbucher (2000)). Thus if a financial institution belives the common risk factors of its portfolio follow a fat-tailed distribution, the proper response is to have a prudential approach to asset correlation estimates and the following capital requirements estimates.

V Concluding remarks

In this paper, we compare the minimum capital requirements implied by the Basel II Accord and our estimates and analyze its development over the course of recent crisis. We find that for every small business the Basel II formula overestimates economic capital. Moreover, it is the most creditworthy small obligors that suffer the highest capital charges relatively to their riskiness. Those most creditworthy obligors turn out to pay for their riskier peers. It can result in distorted lending or risk management practices in financial institutions which hold retail loan portfolios. Our empirical results show that from a credit risk perspective retail exposures are safer investment than the regulator would suggests. In our view it stems from an overly-simplistic way in which Basel II models and estimates the asset correlations in retail loan portfolios.

Secondly, we track the evolution of two important elements of default risk in a portfolio of loans: location and spread of defaults. Interestingly, the crisis eliminated many uncertainties about defaults in a retail loan portfolio. Thus the firms which withstood the deterioration of macroeconomic conditions did not go systematically into default. It was due to their higher resistance to changes in the common risk factors during the crisis that correlation in retail portfolios was on a low level.

Lastly, equipped in a simple yet effective estimation technique, we show an empirical analysis of a representative panel of exposures to US small businesses between 2005 and 2011. We find that in general sensitivity to the common risk factors remains low and small business risk is predominantly subject to idiosyncratic risk even when controlling for different definitions of default event, geographical proximity, industry and firm size heterogeneity. Our results show that only 0.00-3.39% of the asset variability is explained by the economy related risk factors. The remaining 96.61-100.00% of small business risk is due to changes in the firm specific characteristics. But most importantly, regardless of the small business' riskiness, industry or firm size our estimates of asset correlation are significantly lower than any available estimates for corporate firms.

Appendix A Parameter estimation

Given the vector of sensitivity parameters w, the distribution of a single default event in a obligor class k is given by:

$$p_i = P[D_{it} = 1] = P\left[A_{it} < \Phi^{-1}\left(\bar{p}_k\right)\right] = \int_{-\infty}^{\Phi^{-1}(\bar{p}_k)} f(A_{it})dA_{it} \tag{A1}$$

where $f(\cdot)$ is a density function and in our application of the model takes the form of normal probability distribution function and $\Phi(\cdot)$ denotes the cumulative standard normal distribution function. By design for any i and j where $i \neq j$ the probability distribution of a default event in which two obligors fail to meet their payments is modeled as a bivariate normal distribution:

$$f_{ij}(A_{it}; A_{jt}) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}A^T\Sigma^{-1}A\right\} \tag{A2}$$

$$\text{where} \quad A = \begin{bmatrix} A_{it} \\ A_{jt} \end{bmatrix} \tag{A3}$$

$$\text{and} \quad \Sigma = \begin{bmatrix} 1 & w_k w_l \Omega_{kl} \\ w_k w_l \Omega_{kl} & 1 \end{bmatrix} \tag{A4}$$

The above joint density of A_{it} and A_{it} can be transformed by standardizing the vector A and integrating out the effects of the risk factors. Consequently one will obtain the probability of an event in which both obligors default at once:

$$\begin{aligned} p_{kl} &\equiv P[D_{it} = 1, D_{jt} = 1] \\ &= \int_{-\infty}^{\Phi^{-1}(\bar{p}_l)} \Phi\left(\frac{\Phi^{-1}(\bar{p}_k) - \Omega_{kl} w_k w_l y}{\sqrt{1 - \Omega_{kl}^2 w_k^2 w_l^2}}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy \end{aligned} \tag{A5}$$

The expression in (A5) gives the population moment for joint probability of default. The sample moment is derived in the following way. We take the joint probability of default for two firms i and j from two different obligor classes k and l to be an average of all occasions in which both firms are simultaneously in default:

$$\hat{p}_{ij} = \frac{1}{T}\sum_{t=1}^{T}(D_{it} \cdot D_{jt}) \tag{A6}$$

Next, to arrive at sample moment of joint probability of default for two obligor classes, we need to take an average over all possible pairs of firms in both obligor classes:

$$\hat{p}_{kl} = \frac{1}{N_{kt}N_{lt}}\sum_{i\in k}^{N_{kt}}\sum_{j\in l}^{N_{lt}}\frac{1}{T}\sum_{t=1}^{T}(D_{it} \cdot D_{jt}) \tag{A7}$$

where N_{kt} and N_{lt} is the number of firms in obligor class and respectively. Now we change the order of summation which gives us that the sample moment for joint probability of default is an average over time of the product of observed default frequencies:

$$\hat{p}_{kl} = \frac{1}{T}\sum_{t=1}^{T}\frac{\sum_{i\in k}^{N_{kt}} D_{it}}{N_{kt}}\frac{\sum_{j\in l}^{N_{lt}} D_{jt}}{N_{lt}} \tag{A8}$$

$$\Rightarrow \hat{p}_{kl} = \frac{1}{T}\sum_{t=1}^{T}(ODF_{kt} \cdot ODF_{lt})$$

The GMM estimator proposed minimizes the distance between the population and sample moments with respect to the parameter vector θ.

References

Azizpour, Shahriar, Kay Giesecke, and Gustavo Schwenkler, 2012, Exploring the sources of default clustering, Working Paper.

Basel Committee on Banking Supervision, 2005, An explanatory note on the Basel II IRB risk weight functions, Bank for International Settlements.

Black, Fischer, and John C. Cox, 1976, Valuing corporate securities: Some effects of bond indenture provisions, *The Journal of Finance* 31, 351–367.

Botha, Marius, and Gary van Vuuren, 2010, Implied asset correlation in retail loan portfolios, *Journal of Risk Management in Financial Institutions* 3, 156–173.

Carling, Kenneth, Lars Rönnegård, and Kasper Roszbach, 2004, Is firm interdependence within industries important for portfolio credit risk?, Sveriges Riksbank Working Paper Series 168.

Crosbie, Peter, and Jeff Bohn, 2003, Modeling default risk, Moody's KMV Working Paper.

Dietsch, Michel, and Joël Petey, 2002, The credit risk in SME loans portfolios: Modeling issues, pricing, and capital requirements, *Journal of Banking & Finance* 26, 303–322.

———, 2004, Should SME exposures be treated as retail or corporate exposures? A comparative analysis of default probabilities and asset correlations in French and German SMEs, *Journal of Banking & Finance* 28, 773 – 788.

———, 2009, Sector concentration risk in SME credit portfolios: A multifactor approach, Working Paper.

Duffie, Darrell, Andreas Eckner, Guillaume Horel, and Leandro Saita, 2009, Frailty correlated default, *The Journal of Finance* 64, 2089–2123.

Duffie, Darrell, Leandro Saita, and Ke Wang, 2007, Multi-period corporate default prediction with stochastic covariates, *Journal of Financial Economics* 83, 635–665.

Düllmann, Klaus, and Harald Scheule, 2003, Determinants of the asset correlations of german corporations and implications for regulatory capital, Deutsches Bundesbank Working Paper.

Giesecke, Kay, 2006, Default and information, *Journal of Economic Dynamics and Control* 30, 2281–2303.

Glennon, Dennis, and Peter Nigro, 2005, Measuring the default risk of small business loans: A survival analysis approach, *Journal of Money, Credit and Banking* 37, 923–947.

Gordy, Michael B., 2000, A comparative anatomy of credit risk models, *Journal of Banking & Finance* 24, 119 – 149.

——— , 2003, A risk-factor model foundation for ratings-based bank capital rules, *Journal of Financial Intermediation* 12, 199–232.

Gupton, Greg M., Christopher C. Finger, and Mickey Bhatia, 1997, CreditMetrics – technical document, J. P. Morgan.

Jarrow, Robert A., and Fan Yu, 2001, Counterparty risk and the pricing of defaultable securities, *The Journal of Finance* 56, 1765–1799.

Jennrich, Robert I., 1970, An asymptotic chi^2 test for the equality of two correlation matrices, *Journal of the American Statistical Association* 65, 904–912.

Jorion, Philippe, and Gaiyan Zhang, 2009, Credit contagion from counterparty risk, *The Journal of Finance* 64, 2053–2087.

Lee, Joseph, Joy Wang, and Jing Zhang, 2009, The relationship between average asset correlation and default probability, Moody's KMV Working Paper.

Lopez, Jose A., 2004, The empirical relationship between average asset correlation, firm probability of default, and asset size, *Journal of Financial Intermediation* 13, 265–283.

McNeil, Alexander J., and Jonathan P. Wendin, 2007, Bayesian inference for generalized linear mixed models of portfolio credit risk, *Journal of Empirical Finance* 14, 131–149.

Merton, Robert C., 1974, On the pricing of corporate debt: The risk structure of interest rates, *The Journal of Finance* 29, 449–470.

Phelan, William, 2011, Fuelling economic growth. Where's the impediment to returning to the good old days of small-business expansion and more start-ups?, *The RMA Journa* pp. 25–31.

Schönbucher, Philipp J., 2000, Factor models for portofolio credit risk, Bonn Econ Discussion Papers 16/2001.

Steijvers, Tensie, and Mervi Niskanen, 2009, Cash holdings of small and medium sized private family firms: Evidence on the effect of generational evolution, Working Paper.

Vasicek, Oldrich, 1987, Probability of loss on loan portfolio, KMV Working Paper.