Rescaling the mean-reverting 4/2 stochastic volatility model for applications to derivative pricing

Abstract

The 4/2 model, unifying the Heston and 3/2 models, exhibits important features of volatility and is somewhat tractable enough to provide a certain level of pricing procedure. However, a closed-form formula for derivative price is still lacking. We use a rescaling technique to obtain a closed-form formula for approximate derivative price. Our formula has no integral term at all and it can be explicitly calculated by taking derivatives of the Black-Scholes price. Based on the analytic formula, we show that our model is more tractable than the original 4/2 model and yet flexible enough to capture important features of volatility.

JEL classification: C22, C65, G12, G13

Keywords: 4/2 stochastic volatility; Mean-reversion; Closed form formula; Option; Implied volatility

1. Introduction

The original 4/2 stochastic volatility model was proposed by Grasselli (2017) by combining the Heston model (Heston, 1993) and 3/2 model of Heston and Platen (1997) to solve the problems faced by the two models in derivative pricing. The Heston model can provide a semi-analytic solution for pricing many derivatives while reflecting main characteristics of volatility but it can not admit extreme behaviors of stock prices requiring high volatility-of-volatility. So, the Heston model has a practical difficulty of capturing the implied volatility surface for deep in and out-of the-money option strikes. Refer to, for instance, Clark (2011) and Tian (2013) for details. However, the 3/2 model is able to admit abnormal movements in the stock prices so that it can capture the steep volatility skews in equity markets. The implied volatilities of VIX options are shown to be upward-sloping and consistent with market data according to a study by Baldeaux and Badran (2013) on joint modeling of equity and VIX derivatives under the 3/2 model with jumps. But it is found to give rise to computational difficulties since it is a non-affine model as discussed by Drimus (2012) although it still can obtain a certain level of tractability in terms of characteristic functions when pricing derivatives. The Heston model predicts flattening implied volatility skew while the 3/2 model predicts steepening skews when the instantaneous volatility increases. So, the 4/2 stochastic volatility model is designed to unify the complementing two different structures through the superposition of the Heston and 3/2 volatility terms.

Despite its success on capturing most characteristics of volatility, the 4/2 model does not allow a closed-form expression for option price and thus its practical application is very limited. Instead of using a mixture of two different volatility models like the 4/2 model, another way of overcoming the traditional single factor volatility models is to use multi-factor stochastic volatility models. The double-mean-reverting model of Gatheral (2008) and the double Heston model proposed by Christoffersen et al. (2009) are this type of multi-factor models. The double-mean-reverting model is a model well established particularly for capturing the empirical dynamics of variance and the price of options on both SPX and VIX indices consistently with the market. However, this model also doesn't produce a closed-form derivative price formula so that calibrating the model to actual market data might be too slow to be implemented in practice. The double Heston model has a merit of exploring the correlation between the volatility and the smile. However, it also costs too much time for calibration because of a large number of the model parameters.

An incorporation of mean-reversion to the 4/2 model proposed by Escobar-Anel and Gong (2020) provides an improved result that makes computations more efficient. However, a closed-form formula for derivative price is still lacking. One possible way of circumventing this problem is rescaling the model in the way that has been done in recent works by Kil and Kim (2022) and Yoon and Kim (2023). In this paper, we propose to use a rescaled form of the mean-reverting 4/2 model. From this framework, we are able to derive a closed-form analytic formula for European derivative price by using the asymptotic analysis of Fouque et al. (2011). The formula has no integral term and it can be explicitly calculated by taking derivatives of the well-known Black-Scholes price (Black and Scholes 1997, Merton 1997). Based on this result, we show that faster calibration of the 4/2 model is available and yet practical implied volatility structure can be approximated.

This paper is organized as follows. In Section 2, we introduce a mean-reverting 4/2 volatility model given in terms of a small positive parameter and obtain a closed-form pricing formula for European derivatives. In Section 3, we do some numerical experiments about accuracy of our analytic formula, calibration to market data and sensitivity to volatility parameters. We give some concluding remarks in Section 4. Appendix provides proofs of Propositions and Corollary.

2. Mean-reverting 4/2 volatility

For the underlying asset price process S_t, we first consider the following 4/2 stochastic volatility model:

$$\begin{aligned} dS_t &= rS_t dt + \left(a\sqrt{V_t} + \frac{b}{\sqrt{V_t}}\right) S_t\, dW_t, \\ dV_t &= \frac{1}{\epsilon}(\theta - V_t)dt + \frac{\xi}{\sqrt{\epsilon}}\sqrt{V_t}\, d\left(\rho_{sv} W_t + \sqrt{1-\rho_{sv}^2}\, W_t^{\perp}\right), \end{aligned} \tag{2.1}$$

where r (the risk-free interest rate), ϵ, θ and ξ are positive real numbers, $\rho_{sv} \in [-1, 1]$ and a and b are real numbers. W_t and $W_t^{\perp}$ represent independent Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$. The Feller condition $2\theta \geq \xi^2$ is assumed to ensure that V_t is bounded below by zero.

If we take $a \neq 0$ and $b = 0$, then the model (2.1) becomes the Heston model. If take $a = 0$ and $b \neq 0$, then it becomes the 3/2 model. In the model (2.1), ϵ governs the rate of mean-reversion. It is assumed to be a small parameter satisfying $0 < \epsilon \ll 1$ and so the process V_t is fast mean-reverting with an invariant distribution given by $\Phi_{\text{CIR}}(v) = \frac{(2/\xi^2)^{\frac{2\theta}{\xi^2}}}{\Gamma(2\theta/\xi^2)} v^{\frac{2\theta}{\xi^2}-1} e^{-\frac{2v}{\xi^2}}$, where Γ is the gamma function. We note that if ϵ goes to zero, then the variance becomes the constant mean level θ and we recover the Black-Scholes model with constant volatility $a\sqrt{\theta} + b/\sqrt{\theta}$.

Given the underlying price $S_t = s$ and the volatility level $V_t = v$ at time t, the derivative price $P^{\epsilon}(t, s, v)$ is given by

$$P^{\epsilon}(t, s, v) = E^{\mathbb{Q}}\left[e^{-r(T-t)} h(S_T) | S_t = s, V_t = v\right], \tag{2.2}$$

where h is a payoff function. From the Feynman-Kac theorem (see, for example, the book of Oksendal (2013)), this integral form can be transformed into a parabolic PDE problem

$$\mathcal{L}^{\epsilon} P^{\epsilon}(t, s, v) = 0, \quad t < T, \quad \mathcal{L}^{\epsilon} := \partial_t + \mathcal{L}^{\epsilon}_{S,V} - r, \quad P^{\epsilon}(T, s, v) = h(s), \tag{2.3}$$

where $\mathcal{L}^{\epsilon}_{S,V}$ is an infinitesimal generator of the joint diffusion process (S_t, V_t). We decompose the operator $\mathcal{L}^{\epsilon}$ as

$$\mathcal{L}^{\epsilon} = \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2,$$
$$\mathcal{L}_0 := \frac{1}{2}\xi^2 v\partial_{vv} + (\theta - v)\partial_v, \quad \mathcal{L}_1 := \rho_{sv}\xi(av + b)\mathcal{D}_1\partial_v, \quad \mathcal{L}_2 := \partial_t + \frac{1}{2}\left(a\sqrt{v} + \frac{b}{\sqrt{v}}\right)^2 \mathcal{D}_2 + r\left(\mathcal{D}_1 - I\right). \tag{2.4}$$

Here, I is the identity operator and $\mathcal{D}_1$ and $\mathcal{D}_2$ are differential operators defined by $\mathcal{D}_1 = s\partial_s$ and $\mathcal{D}_2 = s^2\partial_{ss}$, respectively. Since the PDE problem (2.3) is a singular perturbation problem, we are interested in an asymptotic solution P^{ϵ} of the form $P^{\epsilon}(t, s, v) = \sum_{i=0}^{\infty} \sqrt{\epsilon}^i P_i(t, s, v)$.

Let $\bar{\sigma}_{4/2}$ denote the average of the volatility term $a\sqrt{V_t} + \frac{b}{\sqrt{V_t}}$ (the superposition of the 1/2 and the 3/2 terms) with respect to the invariant distribution Φ_{CIR}, i.e., $\bar{\sigma}_{4/2} = \langle\left(a\sqrt{\cdot} + \frac{b}{\sqrt{\cdot}}\right)^2\rangle^{\frac{1}{2}}$, and $\mathcal{L}_{\text{BS}}(\bar{\sigma}_{4/2})$ denote the Black-Scholes pricing operator with constant volatility $\bar{\sigma}_{4/2}$, i.e.,

$$\mathcal{L}_{\text{BS}}(\bar{\sigma}_{4/2}) = \partial_t + \frac{1}{2}\bar{\sigma}^2_{4/2}\mathcal{D}_2 + r\left(\mathcal{D}_1 - I\right). \tag{2.5}$$

Then, following the asymptotic analysis of Fouque et al.(2011), one can obtain the following PDEs for P_0 and P_1.

$$\mathcal{L}_{\text{BS}}(\bar{\sigma}_{4/2})P_0(t, s) = 0, \quad \mathcal{L}_{\text{BS}}(\bar{\sigma}_{4/2})P_1(t, s) = F\mathcal{D}_1\mathcal{D}_2P_0, \quad F := \frac{1}{2}\rho_{sv}\xi\left(a\langle\cdot\,\phi^{'}_{ab}(\cdot)\rangle + b\langle\phi^{'}_{ab}(\cdot)\rangle\right) \tag{2.6}$$

where $\phi_{ab}(v)$ is a function defined by

$$\mathcal{L}_0\phi_{ab} = \left(a\sqrt{v} + \frac{b}{\sqrt{v}}\right)^2 - \langle\left(a\sqrt{\cdot} + \frac{b}{\sqrt{\cdot}}\right)^2\rangle. \tag{2.7}$$

Solving the PDEs (2.6), we obtain an approximated derivative price formula as follows.

Proposition 2.1. *Under the risk-neutral dynamics (2.1) of the underlying asset price, the derivative price defined by (2.2) is approximated by* $\tilde{P}^{\epsilon}(t, s) := P_0(t, s) + \sqrt{\epsilon}P_1(t, s)$ *which is*

$$\tilde{P}^{\epsilon}(t, s) = P_{\text{BS}}(\bar{\sigma}_{4/2}) - (T - t)F^{\epsilon}\mathcal{D}_1\mathcal{D}_2P_{\text{BS}}(\bar{\sigma}_{4/2}), \quad F^{\epsilon} := \frac{1}{2}\sqrt{\epsilon}\rho_{sv}\xi\left(a\langle\cdot\,\phi^{'}_{ab}(\cdot)\rangle + b\langle\phi^{'}_{ab}(\cdot)\rangle\right), \tag{2.8}$$

where $P_{\text{BS}}(\bar{\sigma}_{4/2})(t, s)$ *is the solution of the first equation for* $P_0(t, s)$ *in (2.6) with the final condition* $P_0(T, s) = h(s)$.

Proof. See Appendix. □

All the original model parameters are not required to price European derivatives. In fact, from Proposition 2.1, one can notice that the number of necessary parameters for the derivative price is reduced considerably from 6 to 2 as follows: $a, b, \epsilon, \theta, \xi, \rho \implies \bar{\sigma}_{4/2}, F^{\epsilon}$. The parameters $\bar{\sigma}_{4/2}$ and F^{ϵ} are required to be estimated for pricing the derivatives. In order to estimate these parameters, one can utilize calibration from near-the-money European call option implied volatilities. So, we need an implied volatility formula to follow a procedure to fit the skew from the European option prices.

We define the implied volatility $I^{\epsilon}(t, s, v)$ of a call option as a solution of the equation $P_{\text{BS}}(t, s; I^{\epsilon}) = \tilde{P}^{\epsilon}(t, s)$, where $P_{\text{BS}}(t, s; \sigma)$ stands for the Black-Scholes formula for call options with constant volatility σ, and we seek I^{ϵ} in the form

$I^\epsilon(t,s) = \sum_{i=0}^{\infty}(\sqrt{\epsilon})^i I_i(t,s)$ with $I_0 := \bar{\sigma}_{4/2}$. Then we obtain

$$I_1(t,s) = P_1(t,s)\,(\partial_\sigma P_{\rm BS})^{-1} = \frac{F}{\bar{\sigma}_{4/2}}\left(\frac{d_1(t,s)}{\bar{\sigma}_{4/2}\sqrt{T-t}} - 1\right), \tag{2.9}$$

where d_1 is

$$d_1(t,s) = \frac{\log(s/K) + (r + \frac{1}{2}\bar{\sigma}_{4/2}^2)(T-t)}{\bar{\sigma}_{4/2}\sqrt{T-t}}. \tag{2.10}$$

For the second equality in (2.9), we have used the Vega-Gamma relationship, Vega $= (T-t)\bar{\sigma}_{4/2}s^2$Gamma, and the identity $\mathcal{D}_1(\text{Vega}) = (1 - \frac{d_1(t,s)}{\bar{\sigma}_{4/2}\sqrt{T-t}})$Vega for call options. This leads to the following approximate implied volatility surface formula.

$$\tilde{I}^\epsilon(t,s) := I_0 + \sqrt{\epsilon}I_1(t,s) = \bar{\sigma}_{4/2} - \frac{F^\epsilon}{\bar{\sigma}_{4/2}^3}\left(\frac{\log(K/s)}{T-t} - r + \frac{1}{2}\bar{\sigma}_{4/2}^2\right). \tag{2.11}$$

For European call options, therefore, we present the following corollary of Proposition 2.1 about how to calculate the option price and how to estimate explicitly the relevant pricing parameters $\bar{\sigma}_{4/2}$ and F^ϵ from the term structure of implied volatility.

Corollary 2.1. *For a European call option with payoff $h(s) = (s-K)^+$, we have an expression*

$$\tilde{P}^\epsilon(t,s) = N(d_1)s - N(d_2)Ke^{-r(T-t)} - F^\epsilon\frac{se^{-d_1^2/2}}{\bar{\sigma}_{4/2}\sqrt{2\pi}}\left(\sqrt{T-t} - \frac{d_1}{\bar{\sigma}_{4/2}}\right), \tag{2.12}$$

where N is the standard normal cumulative distribution function and $d_2 = d_1 - \bar{\sigma}_{4/2}\sqrt{T-t}$. If the implied volatility $\tilde{I}^\epsilon(t,s)$ for a European call option is expressed by the form

$$\tilde{I}^\epsilon(t,s) = \alpha\left(\frac{\log(K/s)}{T-t}\right) + \beta \tag{2.13}$$

for some constants $\alpha \neq 0$ and β satisfying $\alpha r + \beta > 0$ or $\alpha < 0$, then $\bar{\sigma}_{4/2}$ and F^ϵ are explicitly given by

$$\begin{aligned} \alpha r + \beta > 0: \quad & \bar{\sigma}_{4/2} = \frac{-1 + \sqrt{1 + 2\alpha(\alpha r + \beta)}}{\alpha}, \quad F^\epsilon = \frac{\left(1 - \sqrt{1 + 2\alpha(\alpha r + \beta)}\right)^3}{\alpha^2}, \\ \alpha < 0: \quad & \bar{\sigma}_{4/2} = \frac{-1 - \sqrt{1 + 2\alpha(\alpha r + \beta)}}{\alpha}, \quad F^\epsilon = \frac{\left(1 + \sqrt{1 + 2\alpha(\alpha r + \beta)}\right)^3}{\alpha^2}, \end{aligned} \tag{2.14}$$

in terms of α and β, respectively, under the condition that $\alpha(\alpha r + \beta) \geq -\frac{1}{2}$ holds.

Proof. See Appendix. □

In practice, one can estimate the parameters $\bar{\sigma}_{4/2}$ and F^ϵ required to price the derivatives by using the algebraic relation (2.14) once the slope and intercept coefficients α and β of an affine function of the log-moneyness-to-maturity ratio are estimated through fitting the implied volatilities from observed S&P 500 European call option prices. In particular, it is noteworthy that we don't need to calculate directly the long-run historical volatility $\bar{\sigma}_{4/2}$ for the pricing purpose. It can be estimated by the algebraic formula (2.14) from the calibrated values of the slope and intercept coefficients of the implied volatilities. This way of estimation is valid as long as the derivative contract is not very close to expiration or very far away from the money. Note that the inequality $\beta > r\alpha$ must be satisfied since the long-run mean volatility $\bar{\sigma}_{4/2}$ has to be positive.

Next, we proceed to obtain the second order correction term $P_2(t,s,v)$ of the asymptotic expansion $P^{\epsilon}(t,s,v) = \sum_{i=0}^{\infty} \sqrt{\epsilon}^i P_i(t,s,v)$. The second order term can play an important role in capturing the convexity of the implied volatility observed in option market data since the implied volatility depends quadratically on log-moneyness as discussed below.

Proposition 2.2. *If $\phi_{ab}(v)$ is the solution of (2.7) and satisfies $\langle \phi_{ab}(\cdot)\rangle = 0$ and $\psi_{ab}(v)$ is the solution of*

$$\mathcal{L}_0\psi_{ab} = av\phi_{ab}^{'}(v) + b\phi_{ab}^{'}(v) - \left(a\langle \cdot \phi_{ab}^{'}(\cdot)\rangle + b\langle \phi_{ab}^{'}(\cdot)\rangle\right), \tag{2.15}$$

then $P_2(t,s,v)$ is given by $P_2(t,s,v) = -\frac{1}{2}\phi_{ab}(v)\mathcal{D}_2 P_0 + c(t,s)$, where $c(t,s)$ is the solution of

$$\mathcal{L}_{\mathrm{BS}}(\bar{\sigma}_{4/2})c(t,s) = \left(B_1 (\mathcal{D}_1)^2 \mathcal{D}_2 + B_2 (\mathcal{D}_2)^2 - F^2(T-t)(\mathcal{D}_1)^2 (\mathcal{D}_2)^2\right)P_0, \quad t < T, \quad c(T,s) = 0, \tag{2.16}$$

where B_1 and B_2 are, respectively, given by

$$B_1 = -\frac{1}{2}(\rho\xi)^2\left(a\langle \cdot\, \psi_{ab}^{'}(\cdot)\rangle + b\langle \psi_{ab}^{'}(\cdot)\rangle\right), \quad B_2 = \frac{1}{4}\langle \phi_{ab}(\cdot)\left(\left(a\sqrt{\cdot} + \frac{b}{\sqrt{\cdot}}\right)^2 - \langle\left(a\sqrt{\cdot} + \frac{b}{\sqrt{\cdot}}\right)^2\rangle\right)\rangle. \tag{2.17}$$

Proof. See Appendix. □

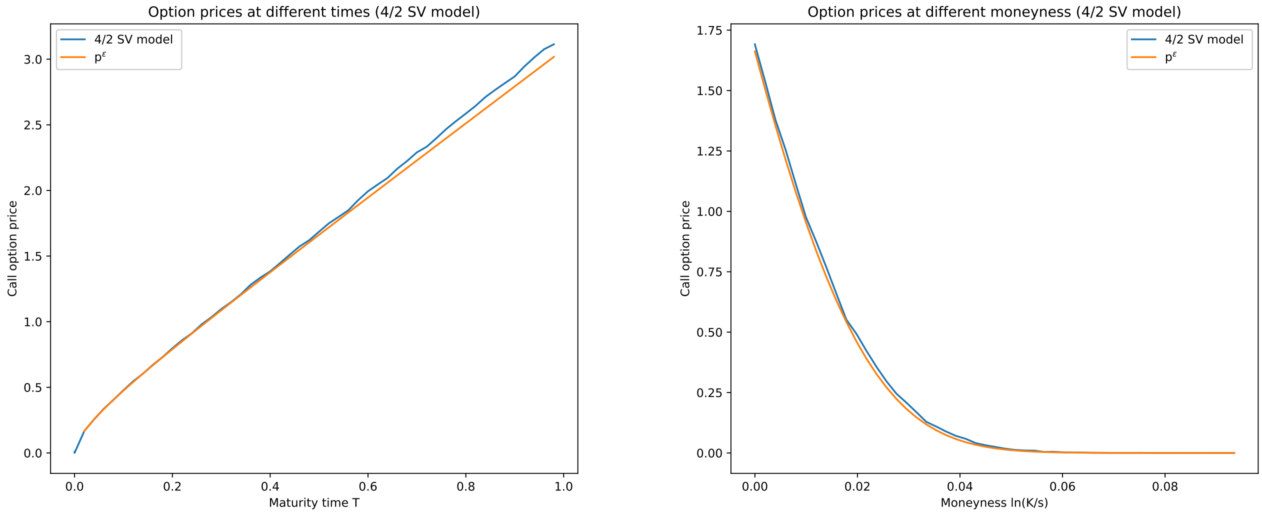
Once we have a solution for P_2, the second order term I_2 of the implied volatility is given by

$$I_2(t,s,v) = \left(P_2(t,s,v) - \frac{1}{2}\frac{\partial^2 P_{\mathrm{BS}}}{\partial\sigma^2} I_1^2(t,s)\right)\left(\frac{\partial P_{\mathrm{BS}}}{\partial\sigma}\right)^{-1}. \tag{2.18}$$

Since $I_1(t,s)$ is linear in the log-moneyness $\log(K/s)$, this second order term is quadratic in the log-moneyness and thus it is able to account for the convex smile effect usually observed in shorter time-to-maturity options.

3. Numerical experiments: accuracy, calibration and sensitivity analysis

In this section, we present some results on four types of numerical experiments. First of all, we conduct Monte-Carlo simulations of the fast mean-reverting 4/2 stochastic volatility model (2.1) to generate values of European-style call options. Then, we compare these option values with those generated by our approximate analytic formula given in Proposition 2.1. For this purpose, the values of the model parameters are chosen according to $r = 0.03$, $a = 0.09099$, $b = 0.00111$, $\epsilon = 0.001$, $\theta = 0.04$, $\xi = 0.123$, $\rho_{sv} = -0.7$ and results are plotted in Figure 1(a). As we can see from Figure 1(a), for short time-to-maturity options, call option values from Monte-Carlo simulations match well with those generated by our approximate analytic formula. For longer time-to-maturity options, the option values from Monte-Carlo simulation are slighter higher than those from the approximate formula. However, in terms of moneyness $\ln(K/s)$, as we can see from Figure 1(b), the two types of options values are very close in all cases.



(a) **(b)**

Fig. 1. Monte-Carlo versus approximate analytic values, with initial asset price $S_0 = 100$ and strike price $K = 100$

Secondly, we use the market data to test our formula in Proposition 2.1. We collect the S&P 500 options data on May 3, 2008. The maturity dates of these options are up to May 3, 2010. According to their maturity times, we split the data into two samples: in-sample (maturity times are from 30 to 270 days) and out-sample (maturity times are 360 and 720 days). The in-sample data are used to estimate the values of α and β in Corollary 2.1 via linear regression of implied volatility. Then, the values of $\bar{\sigma}_{4/2}$ and F^ϵ are calculated by using equation (2.14). The out-sample data are used for comparison. More precisely, we use market implied volatility to calculate the market option values, which are compared with the option values calculated by the formula in Proposition 2.1. The results are shown in Figure 2.

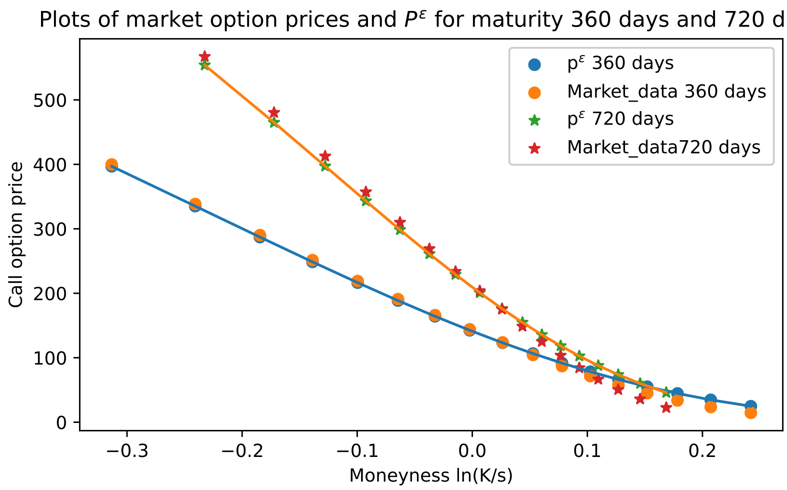
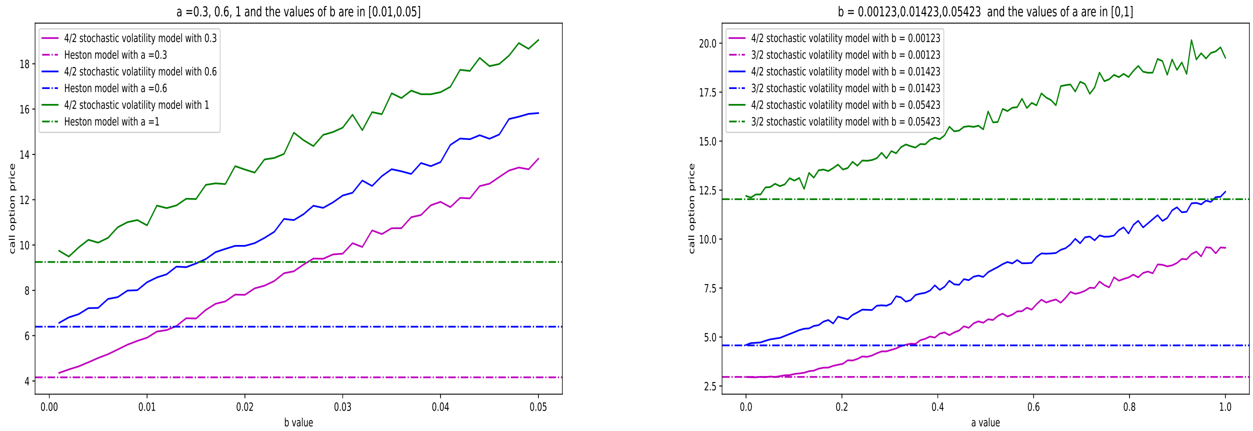


Fig. 2. S&P 500 market values versus approximate analytic values

In the third type of numerical experiments, we use Monte-Carlo simulations to compare the 4/2 stochastic volatility model with the Heston model and the 3/2 stochastic volatility model, with different combinations of values of parameters a and b. The results are presented in Figures 3. In Figure 3(a), the value of parameter a is fixed at 0.3, 0.6 and 1 respectively, and we let the value of parameter b varies between 0 and 0.05. In Figure 3(b), the value of parameter b is fixed at 0.00123, 0.01423 and 0.05423 respectively, and we let the value of parameter a varies between 0 and 1.



(a) **(b)**

Fig. 3. (a) 4/2 model versus Heston model, (b) 4/2 model versus 3/2 model.

Lastly, we compare the market implied volatility data with the approximate implied volatility formula given by (2.11) together with (2.18). As shown in Figure 4, The first order implied volatility approximation approaches the market implied volatility in terms of slope. The first order implied volatility approximation is shown to be almost linear but the second order implied volatility approximation starts capturing the convexity of the market implied volatility and it is closer to the market data than the first order approximation in all time-to-maturities. Inclusion of higher order terms would make the implied volatility move much closer to the market implied volatility.

4. Conclusion

In this paper, we have obtained a closed-form formula for European derivative prices under a rescaled versions of the 4/2 stochastic volatility model. The pricing formulas can be calculated easily starting from the corresponding Black-Scholes derivative prices. The formula is based on fast mean-reverting 4/2 volatility. The corresponding implied volatility formula is a useful tool to approximate the market implied volatility. The market data can be approached by the theoretical formula step by step by adding asymptotic terms. We have shown that the first and second order approximations can capture the slope and the convexity of the market implied volatility, respectively. A possible future work is an extension of the fast mean-reverting 4/2 volatility to two-scale (fast and slow) double-mean-reverting 4/2 volatility.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

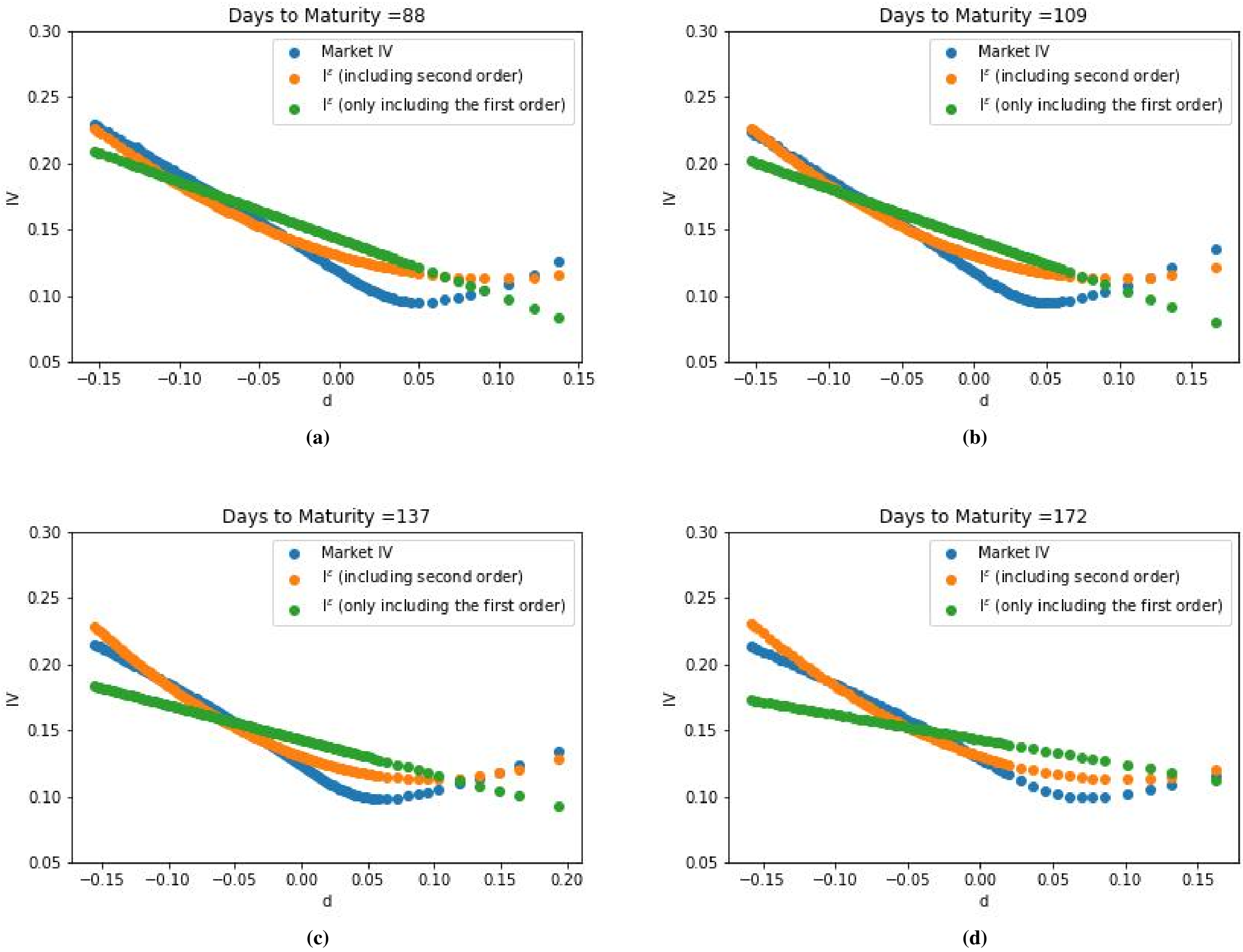


Fig. 4. S&P500 implied volatilities on 1st April, 2019 for days to maturity 88 through 172.

References

[1] Baldeaux, J. and Badran, A., 2014. Consistent modelling of VIX and equity derivatives using a 3/2 plus jumps model. Applied Mathematical Finance, 21(4), 299-312.

[2] Black, F. and Scholes, M., 1976. The pricing of options and corporate liabilities. Journal of Political Economy, 81(3), 637-654.

[3] Clark, I.J., 2011. Foreign exchange option pricing: A Practitioner's guide. John Wiley & Sons.

[4] Christoffersen, P., Heston, S. L. and Jacobs, K., 2009. The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well. Management Science, 55(12), 1914-1932.

[5] Drimus, G., 2012. Options on realized variance by transform methods: a non-affine stochastic volatility model. Quantitative Finance, 12(11), 1679-1694.

[6] Escobar-Anel, M. and Gong, Z., 2020. The mean-reverting 4/2 stochastic volatility model: Properties and financial applications. Applied Stochastic Models in Business and Industry, 36(5), 836-856.

[7] Fouque, J.-P., Lorig, M. and Sircar, R., 2016. Second order multiscale stochastic volatility asymptotics: stochastic terminal layer analysis & calibration. Finance & Stochastics, 20(3), 543-588.

[8] Fouque, J.-P. and Papanicolaou, G. and Sircar, R. and Sølna, K., 2011. Multiscale Stochastic Volatility for Equity, Interest Rate, and Credit Derivatives, Cambridge University Press, Cambridge.

[9] Gatheral, J., 2008. Consistent modeling of SPX and VIX options. In the Fifth World Congress of the Bachelier Finance Society, London, July 18.

[10] Grasselli, M. 2017. The 4/2 stochastic volatility model: A unified approach for the Heston and the 3/2 model. Mathematical Finance, 27, 1013-1034.

[11] Heston, S., 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. Review of Financial Studies, 6, 327-343.

[12] Heston, S., 1997. A simple new formula for options with stochastic volatility. Technical Report, Washington University of St. Louis.

[13] Kil, H-M. and Kim, J.-H., 2022. A closed-form approximation formula for pricing European options under a three-factor model, Probability in the Engineering and Informational Sciences, 36(4), 1214-1240.

[14] Merton, R., 1973. Theory of rational option pricing, Bell Journal of Economics and Management Science, 4(1), 141-183.

[15] Oksendal, B., 2013. Stochastic Differential Equations: An Introduction with Applications. Springer, Berlin.

[16] Platen, E., 1997. A non-linear stochastic volatility model. Financial Mathematics Research Report No. FMRR 005-97, Center for Financial Mathematics, Australian National University, Canberra.

[17] Tian, Y., 2013. The hybrid stochastic-local volatility model with applications in pricing FX options. Available at SSRN 2399935.

[18] Yoon, Y., Kim, J.-H., 2023. A closed form solution for pricing variance swaps under the rescaled double Heston model, Comput. Econ. 61, 429-450.

5. Appendix

In this appendix, we formally prove Propositions and Corollary by using simple operator algebra.

Proof of Proposition 2.1: The solution of the first PDE in (2.6) with the final condition $P_0(T, s) = h(x)$ is exactly the Black-Scholes derivative price with volatility $\bar{\sigma}_{4/2}$. For the correction term $P_1(t, s)$, using the fact that the operators ∂_t, I, $\mathcal{D}_1$ and $\mathcal{D}_2$ commute, we find

$$
\begin{aligned}
\langle \mathcal{L}_2 \rangle \left(-(T-t)F\mathcal{D}_1\mathcal{D}_2P_0\right) &= \left(\partial_t + \frac{1}{2}\,\bar{\sigma}^2_{4/2}\mathcal{D}_2 + r\,(\mathcal{D}_1 - I)\right)\left((t-T)F\mathcal{D}_1\mathcal{D}_2P_0\right) \\
&= F\mathcal{D}_1\mathcal{D}_2P_0 + (t-T)F\mathcal{D}_1\mathcal{D}_2\partial_tP_0 + (t-T)F\mathcal{D}_1\mathcal{D}_2\left(\frac{1}{2}\,\bar{\sigma}^2_{4/2}\mathcal{D}_2 + r\,(\mathcal{D}_1 - I)\right)P_0 \\
&= F\mathcal{D}_1\mathcal{D}_2P_0 + (t-T)F\mathcal{D}_1\mathcal{D}_2\cancelto{0}{\mathcal{L}_{\mathrm{BS}}(\bar{\sigma}_{4/2})P_0} = F\mathcal{D}_1\mathcal{D}_2P_0.
\end{aligned}
$$

So, $-(T-t)F\mathcal{D}_1\mathcal{D}_2P_0$ solves the second equation for $P_1(t, s)$ in (2.6) and satisfies the final condition $P_1(T, s, z) = 0$.

Proof of Corollary 2.1: If we calculate (2.8) directly using the Black-Scholes price $P_{\mathrm{BS}}(\bar{\sigma}_{4/2})$ for the payoff $h(s) = (s-K)^+$, then the formula (2.12) follows immediately. By identifying the two equations (2.11) and (2.13), we obtain two algebraic equations as follows:

$$
\alpha = -\frac{F^\epsilon}{\bar{\sigma}^3_{4/2}}, \quad \beta = \bar{\sigma}_{4/2} + \frac{F^\epsilon}{\bar{\sigma}^3_{4/2}}\left(r - \frac{1}{2}\bar{\sigma}^2_{4/2}\right).
$$

Solving these equations for $\bar{\sigma}_{4/2}$ and F^ϵ yields (2.14). Note that the condition $\alpha r + \beta > 0$ or $\alpha < 0$ is required. Otherwise, the volatility $\bar{\sigma}_{4/2}$ would become negative.

Proof of Proposition 2.2: Plugging $P^{\epsilon}(t,s,v) = \sum_{i=0}^{\infty} \sqrt{\epsilon}^{i} P_i(t,s,v)$ into $\mathcal{L}^{\epsilon} P^{\epsilon}(t,s,v) = 0$, we obtain the PDE systems $\mathcal{L}_0 P_k + \mathcal{L}_1 P_{k-1} + \mathcal{L}_2 P_{k-2} = 0, \quad k = 0,1,2,...$, where $P_{-1} = 0$ and $P_{-2} = 0$ are defined. For the derivation of the second order term P_2, from $\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0$ we observe that $\mathcal{L}_0 P_2 = -\mathcal{L}_2 P_0 = -(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P_0 = -\frac{1}{2}\mathcal{L}_0(\phi_{ab}\mathcal{D}_2 P_0)$ holds. Then P_2 is given in the form $P_2(t,s,v) = -\frac{1}{2}\phi_{ab}(v)\mathcal{D}_2 P_0 + c(t,s)$ for some function $c(t,s)$ independent of v. It remains to find a PDE problem that $c(t,s)$ satisfies. We expect that the desired result comes from the PDE of $k = 4$, i.e., $\mathcal{L}_0 P_4 + \mathcal{L}_1 P_3 + \mathcal{L}_2 P_2 = 0$. From the Fredholm alternative, we have $\langle \mathcal{L}_1 P_3 + \mathcal{L}_2 P_2 \rangle = 0$. From now, we consider the two terms $\mathcal{L}_2 P_2$ and $\mathcal{L}_1 P_3$, respectively. For the first term, we have

$$\begin{aligned}\langle \mathcal{L}_2 P_2 \rangle &= \langle \mathcal{L}_2(-\frac{1}{2}\phi_{ab}\mathcal{D}_2 P_0 + c)\rangle = -\frac{1}{2}\langle \phi_{ab}\mathcal{D}_2\mathcal{L}_2 P_0\rangle + \langle \mathcal{L}_2\rangle c \\ &= -\frac{1}{2}\mathcal{D}_2\langle \phi_{ab}(\mathcal{L}_2 - \langle \mathcal{L}_2\rangle)P_0\rangle + \langle \mathcal{L}_2\rangle c = -B_2\mathcal{D}_2^2 P_0 + \langle \mathcal{L}_2 \rangle c\end{aligned} \tag{5.1}$$

For the second term, using $\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0$ and the Fredholm alternative, we first have

$$\begin{aligned}\mathcal{L}_0 P_3 &= -(\mathcal{L}_1 P_2 - \langle \mathcal{L}_1 P_2 \rangle) - (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P_1 \\ &= \frac{1}{2}\rho_{sv}\xi(av+b)\phi_{ab}^{'}\mathcal{D}_1\mathcal{D}_2 P_0 - \frac{1}{2}\rho_{sv}\xi\langle (a\cdot + b)\phi_{ab}^{'}(\cdot)\rangle \mathcal{D}_1\mathcal{D}_2 P_0 - \frac{1}{2}(\mathcal{L}_0\phi_{ab})\mathcal{D}_2 P_1 \\ &= \frac{1}{2}\rho_{sv}\xi\mathcal{L}_0(\psi_{ab}\mathcal{D}_1\mathcal{D}_2 P_0) - \frac{1}{2}\mathcal{L}_0(\phi_{ab}\mathcal{D}_2 P_1)\end{aligned}$$

which leads to $P_3(t,s,v) = \frac{1}{2}\rho_{sv}\xi\psi_{ab}\mathcal{D}_1\mathcal{D}_2 P_0 - \frac{1}{2}\phi_{ab}\mathcal{D}_2 P_1 + c_1(t,s)$ for some function $c_1(t,s)$ independent of v. Hence, $\langle \mathcal{L}_1 P_3 \rangle$ becomes

$$\langle \mathcal{L}_1 P_3 \rangle = -B_1\mathcal{D}_1^2\mathcal{D}_2 P_0 - F\mathcal{D}_1\mathcal{D}_2 P_1, \tag{5.2}$$

Finally, by inserting (5.1) and (5.2) into $\langle \mathcal{L}_1 P_3 + \mathcal{L}_2 P_2 \rangle = 0$, we obtain $\langle \mathcal{L}_2 \rangle c = \left(B_1\mathcal{D}_1^2\mathcal{D}_2 + B_2\mathcal{D}_2^2\right) P_0 + F\mathcal{D}_1\mathcal{D}_2 P_1$. So, from $P_1 = -(T-t)F\mathcal{D}_1\mathcal{D}_2 P_0$, the function c satisfies the PDE in (2.16). The singular perturbation problem creates a terminal layer near expiration. The appropriate terminal condition for the second order term P_2 is $\langle P_1(T,s,\cdot)\rangle = 0$ as discussed by Fouque et al.(2016). Using the assumption $\langle \phi(\cdot)\rangle = 0$, we obtain the terminal condition $c(T,s) = 0$.