Jump Activity Analysis for Affine Jump-diffusion Models: Evidences from the Commodity Market*

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November 14, 2016

Abstract

The objective of this paper is to perform a joint analysis of jump activity for commodities and their respective volatility indices. Exploiting the property that for affine jump-diffusion models a volatility index, which is quoted on the market, is an affine function of the instantaneous volatility state variable (thus turning this quantity observable), we perform a test of common jumps for multidimensional processes to assess whether an asset and its volatility jump together. Applying this test to the crude oil pair USO/OVX and the gold pair GLD/GVZ we find strong evidence that for these two markets the asset and its volatility have disjoint jumps. This result contrasts with existing results for the equity market and underpins a very specific nature of the commodity market. The results are further confirmed by analysing jump size distributions using a copula methodology.

JEL Classification: G12, G13, C14
Keywords: Affine jump-diffusion models, Volatility indices, Jump activity, Model specification

*Parts of this research were conducted while the first author was visiting fellow at the University of Technology, Sydney and while the second author was visiting Auckland University of Technology under the project grant VRF2015-7. Financial supports from these institutions are gratefully acknowledged.

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1 Introduction

The affine stochastic volatility model proposed by Heston (1993) and extended by Bates (2000) and Duffie et al. (2000) provides an affine framework that is the most frequently used modeling framework for equity derivative products available to date. From the initial continuous time model to the very general specification of Duffie et al. (2000) several improvement were proposed along the way to address some specific empirical properties of equity dynamics. Jump activity is certainly an important aspect as shown by a vast amount of research focusing on this feature. In Bates (2000), stock jumps are shown to be essential to capture crash risk while Eraker et al. (2003) or Eraker (2004) underpin the importance of volatility jumps. Many studies emphasize the importance of jumps, either in the stock or the volatility dynamics, or both, jumps in the stock and volatility dynamics for the model to capture equity empirical properties. Indeed, the affine framework offers a wide range of possibilities to incorporate jumps in the asset dynamics. However, the majority of the models incorporate jumps that can affect either the stock, or the volatility alone, as well as jumps that affect simultaneously the stock and the volatility. Although there exist several possible choices to incorporate jumps within the affine framework, no specific model singles out in the literature due to the different type of tests that can be used. Quite surprisingly, models where the stock and the volatility jumps are not contemporaneous, thus, disjoint, are considered less often.[1]

A large variety of model specifications allowed by the affine framework and the implementations obtained so far can be explained by the evolution of the equity derivative markets. Starting from the equity (index) derivative options traded in the early eighties, followed by the emergence of volatility indices in the early nineties, and now by an increase of actively traded volatility derivative products such as VIX futures and VIX options, the financial derivative markets have grown considerably. This, in turn, lead to a rich data environment that allows, and maybe also requires, more sophisticated modeling frameworks. Many research works have been directed towards modeling volatility and jumps, and pricing in the VIX markets (see e.g. Zhang and Zhu (2006), Zhu and Zhang (2007), Sepp (2008), Lian and Zhu (2013)). Along with the increase of the number of derivative products, the greater availability of high frequency data for financial assets, thus enhancing granularity of the data, has also triggered a large body of works devoted to model identification. In particular, jump activity analysis has attracted a lot of attention among academics, enabling a very

[1]The noticeable exception are models based on the Hawkes process that naturally introduce disjoint jumps, see Aït-Sahalia et al. (2015a) or Kokholm (2016), but it is well known that the computation of the characteristic function for these models requires elaborated simulations of a set of ordinary differential equations.

accurate understanding of the impact of jumps on assets, let us quote without pretending to be exhaustive, Barndorff-Nielsen and Shephard (2004), Huang and Tauchen (2005), Barndorff-Nielsen and Shephard (2006), Jacod and Todorov (2009) and Corsi et al. (2010).

There is no doubt that the SP500 option market along with the VIX, its volatility index, were the precursors of the evolutions observed in the equity derivative markets. It is worth noting that commodity markets followed the same trend, though, with a certain delay. Among all the commodity products that are actively traded, crude oil and gold are certainly the most important ones.[2] As a result, it comes as no surprise that these two assets also possess volatility indices, thus mimicking the development of the pair SP500/VIX. For the crude oil an actively traded asset is the USO (United Stated Oil Fund), which is an exchange-traded fund (ETF), while its volatility index, computed using the VIX methodology, is the OVX (Oil Volatility Index). For the gold an actively traded asset is the GLD (Gold), which is also an ETF, and its volatility index is the GVZ (Gold Volatility Index). While the precedence of the SP500/VIX pair over the other assets may explain why the number of studies focusing on it is larger compared to those dealing with the crude oil and its volatility index or the gold and its volatility index, there exists a fair amount of research available for the commodity markets, see e.g., Geman (2005) and Chevallier and Ielpo (2013) for a general overview. However, if we restrict our consideration to the more specific problem of the use of affine models for the commodity market and its corresponding volatility market then the literature is relatively sparse. For example, Trolle and Schwartz (2009) consider a stochastic volatility model without jumps to analyze commodity options (i.e. the authors do not consider the corresponding volatility index) or Chiarella et al. (2013) and Chiarella et al. (2016) dealing with a continuous stochastic volatility model for the commodity futures market.[3] As a result, regarding the problem of finding a suitable affine model for both, the commodity market and its associated volatility market, the question remains largely unexplored, and taking into account numerical difficulties that arise when analysing jointly these two markets, it is better to first approach the problem with a simple but robust test strategy.

Our paper contributes to the literature by performing a model specification analysis of jump activity for

[2] Crude oil is in fact the world's most actively traded commodity, while gold is famous as a currency, which is also know as the ultimate "safe haven" asset.

[3] Regarding commodity derivative models, some effort has been devoted so far to modeling the convenience yield, see for example Gibson and Schwartz (1990), Cortazar and Schwartz (2003) and Liu and Tang (2010), as it is a very specific and important feature for that market. This consideration partially explains why stochastic volatility models are less often considered in this literature as it adds another modeling complexity. The availability of volatility indices for the commodity market along with the associated derivative products will change that fact.

the commodity markets using the affine framework and the methodology proposed by Jacod and Todorov (2009). We apply the joint jump test to the crude oil market, given by the USO, and its associated volatility market, given by the OVX as well as to the gold market defined by the pair GLD/GVZ. We find that stock jumps and volatility jumps are disjoint; this finding contrasts with the usual common joint jump assumption made in the literature for the affine models. Our results are the opposite of those of Todorov and Tauchen (2011) obtained for the equity pair SP500/VIX, thus, underpinning the very specific behavior of the commodity markets. We further analyze the jump size distributions using a copula approach and confirm the hypothesis of disjoint stock jumps and volatility jumps. Overall, our results point towards a model specification that is rather different from those frequently used in the equity derivative literature.

The paper is organized as follows. We present the key results for affine models with jumps, econometric results for jump detection as well as joint and disjoint jump test specifications and some well-know properties of copulas in Section 2. A description of the empirical data used in our analysis along with some basic descriptive statistics are provided in Section 3. Empirical analysis is performed in Section 4. Section 5 concludes the paper.

2 Modelling Framework

2.1 Model specification

We denote by $(F_t)_{t\geq 0}$ the forward price of the underlying[4] and $(y_t)_{t\geq 0} = (\log(F_t))_{t\geq 0}$ its log return. We assume that under the risk neutral probability Q it follows the dynamics

$$dy_t = -\frac{v_t}{2}dt + \sqrt{v_t}(\rho dw_t^{1,Q} + \sqrt{1-\rho^2}dw_t^{1,Q}) + J_t^{s,Q}dN_t^{s,Q} - \nu_t^{s,Q}dt + \bar{J}_t^{s,Q}dN_t^{c,Q} - \bar{\nu}_t^{s,Q}dt, \tag{1}$$

$$dv_t = \kappa^Q(\theta^Q - v_t)dt + \sigma\sqrt{v_t}dw_t^{1,Q} + J_t^{v,Q}dN_t^{v,Q} + \bar{J}_t^{v,Q}dN_t^{c,Q} \tag{2}$$

with $(w_t^{1,Q}, w_t^{2,Q})_{t\geq 0}$ being standard Brownian motions, $(\kappa^Q, \theta^Q, \sigma) \in \mathbb{R}_+^3$, $(N_t^{s,Q}, N_t^{v,Q}, N_t^{c,Q})_{t\geq 0}$ being a three dimensional counting process with intensities $(\lambda_t^{s,Q}, \lambda_t^{v,Q}, \lambda_t^{c,Q})_{t\geq 0}$ while the random jump sizes affecting the assets are given by $J_t^{s,Q}, J_t^{v,Q}$, $\bar{J}_t^{s,Q}$ and $\bar{J}_t^{v,Q}$. Thus, the notation $(N_t^{s,Q}, J_t^{s,Q})$ is used to represent the

[4]The underlying could be the stock, or in our context, the exchange-traded fund (ETF) USO or GLD. We notice that working with the ETFs and not futures allows us to avoid modelling of the dividend yield, which we are forced to do if we work with futures. There is no dividend yield for an ETF but one has to pay a management fee for holding the ETF, which can be regarded as being similar to a stock that pays a dividend.

situation when jumps are affecting the stock (s); ($N_t^{v,Q}$, $J_t^{s,Q}$) is describing the scenario when jumps are affecting the volatility (v); while ($N_t^{c,Q}$,$\bar{J}_t^{s,Q}$,$\bar{J}_t^{v,Q}$) is used to represent the case of the common jumps, that is, when jumps in the stock and the volatility are contemporaneous (c).

As a result, the compensators for the jumps affecting the stock are given by $\nu_t^{s,Q} = E_t^Q\left[\exp(J_t^{s,Q}) - 1\right]\lambda_t^{s,Q}$ and $\bar{\nu}_t^{s,Q} = \mathbb{E}_t^Q\left[\exp(\bar{J}^{s,Q}) - 1\right]\lambda_t^{c,Q}$ with $\mathbb{E}_t^Q[.]$ denoting the time-t conditional expectation under Q (the conditioning is performed with respect to the filtration generated by the Brownian motion and the jump process). The jump sizes $J_t^{s,Q}$ and $\bar{J}_t^{s,Q}$ are Gaussian under Q, so that we have $\mathbb{E}_t^Q\left[\exp(J_t^{s,Q}) - 1\right] = g^{s,Q} = e^{\mu^{s,Q}+(\sigma^s)^2/2} - 1$, $\mathbb{E}_t^Q\left[\exp(\bar{J}_t^{s,Q}) - 1\right] = \bar{g}^{s,Q} = e^{\bar{\mu}^{s,Q}+(\bar{\sigma}^s)^2/2} - 1$ while $J_t^{v,Q}$ and $\bar{J}_t^{v,Q}$ are exponentially distributed under Q and we have $\mathbb{E}_t^Q\left[J_t^{v,Q}\right] = \mu^{v,Q}$ and $\mathbb{E}_t^Q\left[\bar{J}_t^{v,Q}\right] = \bar{\mu}^{v,Q}$ as the jump distributions are independent of the filtration generated by the Brownian motion and the jump processes. The square root nature of the volatility process imposes some strong constraints on the volatility jump distributions; they must be positive, which also justifies the choice of the exponential law.

The proposed formulation covers many research works on option pricing within the affine framework. The majority of the literature assumes only common jumps, that is, $N_t^{s,Q} = N_t^{v,Q} = 0$, which implies that both, the stock and the volatility jump together through the common jump factor $N_t^{c,Q}$, but with different jump size distributions. The most frequently used distributions are the normal distribution for the (log-)stock while for the volatility it is the exponential distribution in order to be compatible with the required positivity of this process. This specification was first proposed in Duffie et al. (2000). However, Duffie et al. (2000) allows for a much more general specification regarding jump activity and, in particular, nothing prevents from having jumps affecting the stock without impacting the volatility (i.e. $N_t^{s,Q} \neq 0$) and vice-versa (i.e. $N_t^{v,Q} \neq 0$).

Without pretending to be exhaustive and in addition to the works mentioned in the introduction, let us cite the works of Bates (1996), Bakshi et al. (1997), Pan (2002), Broadie et al. (2007), Yun (2011) and Luo and Zhang (2012) where jumps can occur either in the stock dynamics, or in the stock and the variance dynamics (simultaneously), and the jump sizes can either be constant or have a distribution compatible with the state variable that they affect (stock or volatility). However, the literature always assumes that either the stock jumps (but not the volatility), or both, the stock and the volatility jump together. These rather strong assumptions might be justified by the data available in the markets. Indeed, if only equity options are available with only few strikes and maturities, then it might be wise to only consider very parsimonious models and jumps in the volatility should be discarded. On the other hand, the growth of the volatility market can justify the usage of more elaborated models. The VIX in the equity markets is by far the most well known volatility index and VIX futures that are essential tools to trade volatility, entered the market in March 2004 and experienced a rapid increase in trading volumes by 2010. In addition to this richer financial market environment, the availability of high frequency data certainly opens the way to test more sophisticated models and in particular, allows us to develop a better understanding of the correct model specification.

In line with the evolution of the equity index option market and its volatility market counterpart (i.e. the SP500 option market and the VIX derivative market) the commodity market has followed a similar development, though with some lag. Among the different commodity products actively traded on the market, crude oil and gold are certainly the most prominent ones. As a result, it is not surprising for these two assets to have volatility indices computed using the methodology employed for the VIX. The volatility indices (OVX and GVZ), along with the underlying commodities (USO and GLD) are available and quoted at high frequencies thanks to the high liquidity observed on the crude oil and gold option markets. This rich data environment allows us to improve our knowledge of these two key markets and extend previous existing studies on affine models applied to commodity markets, such as e.g. Larsson and Nossman (2011), Arismendi et al. (2016) or Baum and Zerilli (2016).

As mentioned above, the rapid growth of the option market led to the creation of several volatility indices; we will use the generic symbol VOL to denote a volatility index, which is computed using a continuum of call and put options on a given asset. Thanks to the results presented, among others, in Carr and Wu (2006), Aït-Sahalia et al. (2015b) and Bardgett et al. (2016), we can write the following relationship:

$$\text{VOL}_t^2 = \frac{2}{\tau}\mathbb{E}_t^Q\left[\int_t^{t+\tau} v_u du + \left(e^{J_u^{s,Q}} - 1 - J_u^{s,Q}\right) dN_u^{s,Q} + \left(e^{\bar{J}_u^{s,Q}} - 1 - \bar{J}_u^{s,Q}\right) dN_u^{c,Q}\right], \tag{3}$$

where τ is in practice 30 days (expressed in years).

Under standard assumptions made for affine jump-diffusion models, Eq.(3) possesses a very simple form and it is possible to prove that

$$\text{VOL}_t^2 = \beta_0 v_t + \beta_1 \tag{4}$$

with β_0 and β_1 being constants that depend on the model parameters (see Aït-Sahalia et al. (2015b)). For the purpose of our analysis we do not need to compute these constants explicitly, thus, we refer to the literature for their expressions. Regarding Eq.(4) the key remark is that the volatility index, given by the left hand side of this equation, is observable and thanks to this relationship, it makes the right hand side (and more precisely, v_t, the volatility process) observable, too. Thus, the volatility process is no longer a latent factor and it opens a very interesting possibility to perform model specification tests for affine models (since the relationships in Eq.(3) and Eq.(4) heavily exploit the affine property of the model). We notice that Eq.(3) and Eq.(4) hold under the risk neutral measure but since the econometric tests involve the dynamics under the historical probability measure, it remains to investigate the market price of risks.

In order to specify the market price of risk, we follow Aït-Sahalia et al. (2015b) and references therein that provide the standard choices for these quantities, often motivated by parsimonious consideration to ease the estimation procedure. For the Brownian motion we will assume that the market price of risk is given by $\Lambda_t = (\gamma_1\sqrt{(1-\rho^2)v_t}, \gamma_2\sqrt{v_t})^\top$ (with $\top$ denoting the transpose). Regarding the counting processes, Aït-Sahalia et al. (2015b) propose to set the jump intensities under the risk neutral and the historical measures identical for each of the processes, and further specify an affine form of the volatility process v_t. As result, we assume that $\lambda_t^{s,Q} = \alpha_0^s + \alpha_1^s v_t$, $\lambda_t^{c,Q} = \alpha_0^c + \alpha_1^c v_t$ and $\lambda_t^{v,Q} = \alpha_0^v + \alpha_1^v v_t$ with $(\alpha_0^s, \alpha_1^s, \alpha_0^c, \alpha_1^c, \alpha_0^v, \alpha_1^v)$ being all positive. Also, regarding the jump sizes we also follow Aït-Sahalia et al. (2015b) and assume that the jump distributions under P possess the same law as those under Q but with different parameters. More precisely, when we set $\mathbb{E}_t^P[e^{J_t^{s,P}} - 1] = g^{s,P} = e^{\mu^{s,P}+(\sigma^s)^2/2} - 1$, $\mathbb{E}_t^P[e^{\bar{J}_t^{s,P}} - 1] = \bar{g}^{s,P} = e^{\bar{\mu}^{s,P}+(\sigma^s)^2/2} - 1$, the jumps have the same variance but different mean values, while $\mathbb{E}_t^P[J_t^{v,P}] = \mu^{v,P}$ and $\mathbb{E}_t^P[\bar{J}_t^{v,P}] = \bar{\mu}^{v,P}$. All these hypothesis lead to an asset jump risk premium given by $(g^{s,P} - g^{s,Q})(\alpha_0^s + \alpha_1^s v_t) + (\bar{g}^{s,P} - \bar{g}^{s,Q})(\alpha_0^c + \alpha_1^c v_t)$ while for the volatility jump risk premium we have $(\mu^{v,P} - \mu^{v,Q})(\alpha_0^v + \alpha_1^v v_t) + (\bar{\mu}^{v,P} - \bar{\mu}^{v,Q})(\alpha_0^c + \alpha_1^c v_t)$. Therefore, under the historical probability measure the dynamics for (y_t, v_t) take the same form as in Eq.(1)-(2), and by analyzing the joint jump activity of the asset and its corresponding volatility index we can draw conclusions on model specification regarding the important aspect of jumps.

2.2 Daily jump detection

To identify days for which an asset jumps we will follow methodology proposed in Tauchen and Zhou (2011) and report the main equations of the methodology when applied to the asset $y_t = \log(F_t)$. Define

$$r_{t,i} = y_{t,i\Delta} - y_{t,(i-1)\Delta}, \tag{5}$$

where $r_{t,i}$ refers to the i^{th} intra-day return on day t, with Δ being the sampling frequency within each day such that $m = 1/\Delta$ observations occur every day and as $\Delta \to 0$ we have that $m \to \infty$.

Barndorff-Nielsen and Shephard (2004) propose two measures for quadratic variation process namely, the realized variance (RV) and the realized bipower variation (BV) that converge uniformly as $\Delta \to 0$ to different quantities of the jump diffusion process such as

$$RV_t = \sum_{i=1}^{m} r_{t,i}^2 \to \int_{t-1}^{t} v_u du + \int_{t-1}^{t} ({J_u}^{s,P})^2 dN_u^{s,P} + \int_{t-1}^{t} ({\bar{J}_u}^{s,P})^2 dN_u^{c,P}, \tag{6}$$

$$BV_t = \frac{\pi}{2} \frac{m}{m-1} \sum_{i=2}^{m} |r_{t,i}||r_{t,i-1}| \to \int_{t-1}^{t} v_u du. \tag{7}$$

As it is evident from Eq.(6) and Eq.(7), the difference between the realized variance and the realized bipower variation is zero when there is no jump and strictly positive when there is a jump. For detecting jumps, we adopt the ratio test, proposed in Huang and Tauchen (2005) and Andersen et al. (2007), where the test statistic

$$RJ_t \equiv \frac{RV_t - BV_t}{RV_t} \tag{8}$$

is an indicator for the contribution of jumps to the total within-day variance of the process. This test statistic converges in distribution to a standard normal distribution when using an appropriate scaling

$$ZJ_t = \frac{RJ_t}{\sqrt{\left\{ \left(\frac{\pi}{2}\right)^2 + \pi - 5 \right\} \Delta \max\left(1, \frac{TP_t}{BV_t^2}\right)}} \to N(0,1). \tag{9}$$

In Eq.(9) TP_t is the tripower quarticity that is robust to jumps; it is defined in Barndorff-Nielsen and Shephard (2004) as

$$TP_t \equiv m\mu_{4/3}^{-3} \frac{m}{m-2} \sum_{i=3}^{m} |r_{t,i-2}|^{4/3} |r_{t,i-1}|^{4/3} |r_{t,i}|^{4/3} \to \int_{t-1}^{t} v_u^2 du, \tag{10}$$

where

$$\mu_k \equiv 2^{k/2} \frac{\Gamma\left((k+1)/2\right)}{\Gamma\left(1/2\right)}, \quad k > 0.$$

Assuming that there is at most one jump per day and that jump size dominates the return when a jump occurs (Andersen et al. (2007)), daily realized jump sizes can be obtained as

$$\hat{J}_t = \text{sign}(r_t) \times \sqrt{(RV_t - BV_t) \times I_{(ZJ_t \geq \Phi_{\tilde{\alpha}}^{-1})}}, \tag{11}$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution function with $\tilde{\alpha}$ being the level of significance and $I_{(ZJ_t \geq \Phi_{\tilde{\alpha}}^{-1})}$ is an indicator function which takes the value of one if there is a jump on a given day, and zero otherwise.

Once the realized jumps have been established, we can compute the jump mean $\hat{\mu}_J$, the variance $\hat{\sigma}_J$ and intensity $\hat{\lambda}_J$ as follows

$$\hat{\mu}_J = \text{Mean of } \hat{J}_t, \tag{12}$$

$$\hat{\sigma}_J = \text{Standard deviation of } \hat{J}_t, \tag{13}$$

$$\hat{\lambda}_J = \frac{\text{Number of jump days}}{\text{Number of trading days}}. \tag{14}$$

It has been shown in Tauchen and Zhou (2011) that such an approach for estimation of realized jump parameters is robust with respect to the drift and the diffusion function specifications. It makes it easy to specify the jump arrival rate, avoiding sophisticated estimation methods, and yields reliable results under various settings, for instance, when the sample size is either finite, increasing or shrinking.[5]

It is essential to notice that if we consider both, Eq.(2) and Eq.(4), the methodology can also be applied to detect jumps of the process $(v_t)_{t\geq 0}$ using the observations VOL_t. As result, despite the fact that v_t is a latent factor if we only observe the asset $(F_t)_{t\geq 0}$, the availability of a pure volatility product such as VOL_t allows us to turn the volatility $(v_t)_{t\geq 0}$ into an observable process. This, in turn, allows us to greatly simplify the test of whether the asset and its volatility jump together as we can use the methodology proposed by Jacod and Todorov (2009) that we now present.

[5]As mentioned above, jump diffusion models have a long history in finance. More specific to the commodity markets, recent works underpin the importance of jumps (see Larsson and Nossman (2011), Chevallier and Ielpo (2012) and Brooks and Prokopczuk (2013)), find that intensities for commodity price jumps are time varying (refer to Diewald et al. (2015)) and emphasise the importance of jumps in risk management (Chen et al. (2013)) when considering jumps as a modeling strategy for extreme events.

2.3 Joint and disjoint jump test specifications

Given the relationship in Eq.(4) and the dynamics in Eq.(1)-(2) that is also valid under the historical probability measure P, it seems natural to apply the test for common arrival of jumps proposed by Jacod and Todorov (2009) to the following variables:[6]

$$x_t^1 = \log F_t,$$
$$x_t^2 = \text{VOL}_t^2,$$

We denote a vector $x_t = (x_t^1, x_t^2)$. We fix the time horizon T and assume that the processes x_t^1 and x_t^2 are observed over a given time interval $[0, T]$ at discrete times $i = 0, 1, ..., [T/\Delta]$ where Δ is some small time lag, with $\Delta \to 0$. Since in our empirical analysis we will compute test statistics for each day, we set $T = 1$. We can then consider the increments of those processes:

$$\Delta_i x^1 = \log F_{i\Delta} - \log F_{(i-1)\Delta}, \tag{15}$$
$$\Delta_i x^2 = \text{VOL}_{i\Delta}^2 - \text{VOL}_{(i-1)\Delta}^2. \tag{16}$$

We follow the notation of Jacod and Todorov (2009) and define

$$V(f, \Delta)_T = \sum_{i=1}^{[T/\Delta]} f(\Delta_i x) \tag{17}$$
$$V(f, k\Delta)_T = \sum_{i=1}^{[T/k\Delta]} f(x_{ik\Delta} - x_{(i-1)k\Delta}), \tag{18}$$

with $f(x) = (x^1 x^2)^2$ and the integer k in Eq.(18) is set $k = 2$. The rejection regions C^j for the null of the joint jump (H_0 : Jumps in the time series occur at the same time (common jumps)) is specified as a function of the following test statistic:

$$\Phi_n^{(j)} = \frac{V(f, k\Delta)_T}{V(f, \Delta)_T}. \tag{19}$$

Here, the superscript (j) in the test statistic $\Phi_n^{(j)}$ stands for "joint". To specify the rejection region C^d for the null of the disjoint jump (H_0 : Jumps in the time series do not occur at the same time (disjoint jumps)), we have the following test statistic:

$$\Phi_n^{(d)} = \frac{V(f, \Delta)_T}{\sqrt{V(g_1, \Delta)_T V(g_2, \Delta)_T}}, \tag{20}$$

[6]We notice that one could alternatively consider $x_t^1 = F_t$, i.e. not to apply the log-transformation to the underlying series.

where $g_1(x) = (x^1)^2$, $g_2(x) = (x^2)^2$ and the superscript (d) in the test statistic $\Phi_n^{(d)}$ stands for "disjoint".

In order to determine the critical values, we introduce the following notation:

$$\hat{A}'_T = \frac{1}{\Delta} \sum_{i=1}^{[T/\Delta]} f(\Delta_i x) \mathbf{1}_{\{|\Delta_i x^1| \leq \alpha_1 \Delta^{\bar{\omega}}, |\Delta_i x^2| \leq \alpha_2 \Delta^{\bar{\omega}}\}}, \tag{21}$$

where we choose $\alpha_1 > 0$, $\alpha_2 > 0$, and $\bar{\omega} \in (0, 1/2)$ arbitrary (here, we set $\bar{\omega} = 0.49$). For the parameters α_1, α_2, the choice is made following the Remark 5.4 on p. 1807 in Jacod and Todorov (2009). Thus, we set $\alpha_i = 3\sqrt{\overline{BV_i}}$, where $\overline{BV_i}$ denotes an average BV for series i with $i = 1$ representing the commodity index, and $i = 2$ representing the corresponding volatility index. We further define

$$\hat{F}_T = \frac{1}{2\tilde{k}\Delta} \sum_{i=1+\tilde{k}}^{[T/\Delta]-\tilde{k}-1} \sum_{j \in I(i)} \left(\left(\Delta_i x^1 \Delta_j x^2\right)^2 + \left(\Delta_i x^2 \Delta_j x^1\right)^2 \right) \mathbf{1}_{\{M_1 \cap M_2\}} \tag{22}$$

with $\tilde{k}$ chosen in such a way that $\tilde{k}\Delta \to 0$ (here, we choose $\tilde{k} = [1/\sqrt{\Delta}]$); $M_1 = \{|\Delta_i x^1| \geq \alpha_1 \Delta^{\bar{\omega}}\} \cup \{|\Delta_i x^2| \geq \alpha_2 \Delta^{\bar{\omega}}\}$, $M_2 = \{|\Delta_j x^1| \leq \alpha_1 \Delta^{\bar{\omega}}\} \cap \{|\Delta_j x^2| \leq \alpha_2 \Delta^{\bar{\omega}}\}$ and $I(i) = \{i - \tilde{k}, i - \tilde{k} + 1, \ldots, i - 1\} \cup \{i + 2, i + 3, \ldots, i + \tilde{k} + 1\}$.[7] In addition, we denote

$$\hat{F}'_T = \frac{2}{\tilde{k}\Delta} \sum_{i=1+\tilde{k}}^{[T/\Delta]-\tilde{k}-1} \sum_{j \in I(i)} \left(\left(\Delta_i x^1 \Delta_j x^2\right)^2 \times \left(\Delta_i x^1 \Delta_j x^2 + \Delta_i x^2 \Delta_j x^1\right)^2 \right) \mathbf{1}_{\{M_1 \cap M_2\}}. \tag{23}$$

Finally, we set

$$\hat{V}^{(j)} = \frac{\sqrt{\Delta(k-1)\hat{F}'_T}}{V(f, \Delta)_T} \tag{24}$$

for the common jumps test, and

$$\hat{V}'^{(d)} = \frac{\Delta(\hat{F}_T + \hat{A}'_T)}{\sqrt{V(g_1, \Delta)_T V(g_2, \Delta)_T}} \tag{25}$$

for the disjoint jumps test. The corresponding critical values are given by

$$c^{(j)} = \hat{V}^{(j)} / \sqrt{\alpha} \tag{26}$$

for the common jump test and

$$c^{(d)} = \hat{V}'^{(d)} / \alpha \tag{27}$$

[7] Note, to guarantee a positive index, $I(i)$ is only defined for $i > \tilde{k}$.

for the disjoint jump test. In Eq.(26)-(27) α represents the significance level, which will be specified in Section 4 (Empirical Results).[8] Finally, the critical region where the null of common jumps is rejected is determined as

$$C^{(j)} = \{|\Phi^{(j)} - 1| \geq c^{(j)}\}. \tag{28}$$

Similarly, the critical region where the null of disjoint jumps is rejected is determined by

$$C^{(d)} = \{\Phi^{(d)} \geq c^{(d)}\}. \tag{29}$$

2.4 Copula dependency

This section focuses on the copula methodology, the main concepts of dependence modelling and techniques for copula estimation. It will be used to quantify the dependence between jump sizes affecting the underlying and jump sizes affecting the respective volatility index.

Copulas are multivariate distribution functions connecting d one-dimensional uniform-(0,1) marginals to a joint cumulative distribution. According to *Sklar's theorem*, if F is a d-dimensional distribution function with marginals $F_1 \ldots, F_d$, then under some general conditions there exists a copula C with

$$F(x_1, \ldots, x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\} \tag{30}$$

for every $x_1, \ldots, x_d \in \overline{\mathbb{R}}$.

Throughout the paper we will concentrate on two popular copula families: the *elliptical copulas family* and the *Archimedean copulas family*. Elliptical copulas have a dependence structure generated by elliptical distributions such as normal or Student-t. The Gaussian copula generates the dependence structure given by the multivariate normal distribution. In the case of the normal marginals, that is, if $X_j \sim N(0,1)$ and $X = (X_1, \ldots, X_d)^\top \sim N_d(0, \Psi)$, where Ψ denotes a correlation matrix, the explicit expression for the Gaussian copula is given by

$$C_\Psi^{Ga}(u_1, \ldots, u_d) = F_X\{\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\}. \tag{31}$$

Thus, combining normal marginals by using the Gaussian copula leads to the multivariate normal distribution. Note that the Gaussian copula can be used with any other marginal distribution (in which case the resulting multivariate distribution will not be normal).

[8]Note, the significance level α is not related to the choice of the parameters α_1, α_2 specified above.

The Student-t copula generates the dependence structure from the multivariate Student-t distribution. If $X = (X_1, \ldots, X_d)^\top \sim t_d(\nu, \mu, \Sigma)$, i.e. X has a multivariate Student-t distribution with ν degrees of freedom, mean vector μ and positive-definite covariance matrix Σ, the Student-t copula is given by

$$C^t_{\nu,\Psi}(u_1, \ldots, u_d) = t_{\nu,\Psi}\{t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)\}, \tag{32}$$

where t_ν^{-1} is the quantile function from the univariate t-distribution and Ψ is the correlation matrix associated with Σ.[9] Modelling dependence by using elliptical distributions can be found e.g. in McNeil et al. (2005).

In our empirical analysis we will also use the Gumbel and Clayton copulas that belong to the family of Archimedean copulas. The Clayton copula with the dependence parameter $\theta \in (0, \infty)$ is defined by

$$C_\theta(u_1, \ldots, u_d) = \left\{ \left(\sum_{j=1}^{d} u_j^{-\theta} \right) - d + 1 \right\}^{-1/\theta}. \tag{33}$$

As the copula parameter θ goes to infinity, the dependence becomes maximal and as θ goes to zero, we have independence.

The *Gumbel copula* with the dependence parameter $\theta \in [1, \infty)$ is given by

$$C_\theta(u_1, \ldots, u_d) = \exp\left[- \left\{ \sum_{j=1}^{d} (-\log u_j)^\theta \right\}^{1/\theta} \right]. \tag{34}$$

For $\theta = 1$ it reduces to the product copula (i.e. independence): $C_\theta(u_1, \ldots, u_d) = \prod_{j=1}^d u_j$. Maximal dependence is achieved when θ goes to infinity.

Note that the literature suggests different approaches to the copula estimation. In our empirical analysis, we estimate copulas using the *inference for marginals* (IFM) method. The IFM suggests to estimate the marginal parameters in the first step, and then substitute them into a copula to obtain the pseudo log-likelihood function, which is then maximized with respect to the copula dependence parameter θ. The two-step procedure employed by this method makes it computationally efficient. For details on the IFM and the alternative estimation techniques refer to Joe (1997).

[9] Since copula functions remain invariant under strictly increasing transformations of X (e.g. standardisation of the marginal distributions), see Nelsen (1998), the copula of a $t_d(\nu, \mu, \Sigma)$ distribution is identical to that of a $t_d(\nu, 0, \Psi)$.

3 Data Description

In this study we consider commodity exchanged-traded funds (ETFs) and the respective volatility indices for the crude oil (Tickers: USO/OVX) for the period from 15/07/2008 to 18/11/2015 and gold (Tickers: GLD/GVZ) for the period from 04/08/2008 to 18/11/2015. The chosen time frames provide the longest series available from SIRCA.[10] Both underlying commodities (USO and GLD) are the ETFs quoted on New York Stock Exchange (NYSE) Arca; the options used to build the corresponding implied volatility indices (OVX and GVZ) are traded on Chicago Board Options Exchange (CBOE). In our analysis we are restricted to use these commodities as the respective volatility indices are the longest time series that are readily constructed and available from CBOE.[11] For a work underpinning the importance of the OVX for predicting WTI light sweet crude oil futures see Chevallier and Sévi (2013) and Haugom et al. (2014) while for the gold volatility index GVZ see, for example, Badshah et al. (2013). We restrict the computations to 5-minute interval quotes from 10:00 am to 4:00 pm as it is well known that this sampling frequency avoids microstructure noise effects that can cause biases in the estimation of the realized volatility.

[Insert Table 1 here]

Table 1 shows summary statistics for the log-returns of the underlying (defined in Eq.(15)) and the corresponding differences of squares for the volatility (defined in Eq.(16)) for both pairs of commodities and volatility indices. These include means, standard deviations, skewness, kurtosis, minimum and maximum, computed first for each day using 5-minute log-returns (for the indices) and difference in squares (for the volatility indices), and then averaging the resulted quantities across all days in the sample. We observe a negative relationship between the average return and the average standard deviation for the pairs of USO/OVX and GLD/GVZ, attributed to the leverage effect, i.e. with a drop in the underlying, the fear of market crashing kicks in, and implied volatility rises. Volatility index for the crude oil exhibits higher variability compared to gold, which justifies gold's reputation as the ultimate "safe haven" asset, which is especially important during the crisis periods, see Baur and McDermott (2010). We further notice that gold ETF (i.e. GLD) is positively skewed, which is different from financial assets. It indicates that large declines

[10]see `http://www.sirca.org.au/`

[11]A long time frame is required to ensure that the sample contains sufficient number of days on which both series, the ETF and its volatility index jump, in order to guarantee the validity of the statistical test results. If one was interested in analysing other commodities, one could construct volatility indices from options written on those commodities using the methodology developed for the VIX. In that case, most likely the options would be quoted on NYMEX.

in demand do not decrease prices as sharply as large increases in demand can increase them at the right tail of the price distribution. USO, on the contrary, has negative skewness, which points out more extreme negative returns observed in the crude oil market. Large skewness in volatility indices, OVX and GVZ, indicates high volatility movements occurring in the crude oil and gold markets. Notice, however, that the mean level of the volatility, if restricted to the same period (16/03/2011 to 20/11/2015), corresponds to 31.90% and 19.03% per annum for OVX and GVZ, respectively.[12] Large positive kurtosis for both, ETFs and volatility indices suggests non-normality of the data. Finally, high variability in the data is evident from the large range between the minimum and the maximum values. In particular, we confirm our previous observation that the volatility movements in the crude oil market are more pronounced compared to the volatility movements in the gold market.

[Insert Figure 1 here]

Figure 1 shows evolution of USO and GLD ETFs over the considered time period. We notice a sharp drop in the ETF value for the crude oil at the beginning of the sample period (year 2008), which is consistent with the start of the Global Financial Crisis (GFC). At the same, the value of the ETF for gold exhibits a minor increase towards the end of 2008, indicating that when the fear of financial or energy markets crashing kicks in, the demand for gold surges, which in turns generates an increase in the gold's share price. GLD continues to increase until end of 2011 (end of the GFC), which reinforces again the identification of gold as a "safe haven" asset. During the post-GFC stage (from 2012 until end of the sample) gold share price experiences a slight decrease. Crude oil prices have been fairly stable up until mid-2014, but then started to decreases reaching its absolute minimum for the first time since 2008. The reason for this drop is manifold: weak demand for oil due to insipid economic growth, in combination with surging US production, and OPEC's[13] decision not to cut production as a way to prop up prices.

[Insert Figure 2 here]

Figure 2 shows annualised volatility indices for the crude oil (OVX) and gold (GVZ). We observe that the volatilities range between 10% to 100% per annum. We confirm our previous results regarding higher variability in the crude oil market, compared to the gold market.

[12] Note, these quantities are computed by annualizing daily volatilities.

[13] Organization of the Petroleum Exporting Countries, see `http://www.opec.org`.

4 Empirical Results

4.1 Daily jump activity

To perform a joint analysis of ETF and its volatility index jump activity we first need to identify the days for which both series jump. To this end we use Tauchen and Zhou (2011)'s approach briefly presented in the modelling framework section. We compute daily realized variance (RV_t) and realized bipower variation (BV_t) using Eq.(6) and (7), respectively. From these two quantities we extract daily jumps using Eq.(11) (where we set $\tilde{\alpha} = 0.95$). Whenever J_t is non-zero, there is a jump in the underlying time series. While jumps in the index returns can be either positive or negative, we restrict jumps in volatility to be positive. This is consistent with the model specification in Eq.(1) - (2) that imposes a constraint on the sign of the volatility jumps.

[Insert Figure 3 here]

Figure 3 shows in the left panel the jumps sizes (J_t's extracted from Eq.(11)) in USO (top panel) and OVX (bottom panel) across the entire series (1852 observations). The right panel shows jump sizes for days where both series, USO and OVX, jump (49 observations).[14]

[Insert Table 2 here]

In Table 2 we report in lines (i) the total number of days in the samples under consideration; (ii) the number of days when the ETF jumps regardless of whether the corresponding volatility index jumps or not; (iii) the number of days when volatility index jumps (positively) regardless of whether the corresponding ETF jumps or not. For the USO/OVX pair, there are 393 days over 1852 for which the ETF jumps regardless of the evolution of the volatility index while there are 183 days for which the volatility jumps with no constraints on the evolution for the ETF. For the GLD/GVZ pair the corresponding numbers are 427 and 220 out of 1838. Already from these numbers we can conclude that many jumps in the ETF and its volatility do not occur simultaneously. It is a first evidence that the usual common jump hypothesis is not supported by the commodity market data.

[14] The figures for the pair (GLD,GVZ) are not reported here but are available from the authors upon request.

Further in that direction, if we consider the USO/OVX pair, the days for which the ETF jumps but not its volatility index corresponds to 276, given by line (iv) in Table 2, while there are 134 days with jumps in the volatility but not in the stock (line (v) in Table 2). For the pair GLD/GVZ, these numbers correspond to 300 and 158. As a result, there are quite a few days for which jumps occur only in one of the markets (i.e. either ETF or volatility index).

[Insert Table 3 here]

We report in Table 3 the descriptive statistics for the jumps in ETF returns and the squared volatility index returns. These include mean jump sizes, volatility of the jump size and jump intensity. Panel A reports the results computed separately for the days where jumps occur in the commodity ETF (the number of those days is reported in line (ii) of Table 2) and days when jumps occur in the volatility index (the number of those days is reported in line (iii) of Table 2). On average, we observe positive jumps for USO and GLD returns with jumps in USO returns being nearly five times larger in magnitude (5.52×10^{-5}) compared to GLD returns (1.31×10^{-5}). The jump direction for commodity indices is consistent with the sign of the mean returns reported in Table 1 (positive for USO and GLD). The intensity of jumps in the ETF returns are 0.2122 for USO and 0.2323 for GLD. For the volatility indices we observe that the mean jump sizes and respective standard deviations correspond to (66.3433/212.9379) for OVX and (15.3733/39.5035) for GVZ. These results are consistent with high volatilities (standard deviations) documented for OVX, compared to GVZ, reported in Table 1 and Figure 2. Jump intensity for volatility indices ranges from 0.0988 for OVX to 0.1197 for GVZ. Panel B of Table 3 reports the results computed for the days on which both series, the commodity ETF and the corresponding volatility index, jump (the number of the joint jump days is reported in line (vi) of Table 2). We observe that the average jumps for USO (-0.0014) and GLD (-7.62×10^{-4}) are negative and of low intensity (0.0265 for USO; 0.0337 for GLD).[15] The average jump sizes in the volatility index correspond to 40.787 for OVX to 11.9312 for GVZ. The lowest mean jump sizes and the lowest standard deviation are observed for GVZ reinforcing again the identification of gold as a ”safe haven” asset.

[15]This result has been already reported based on Table 2. We also notice that jump intensities for commodity ETFs and respective volatility indices are identical, as we assume that jumps in the ETF and the volatility index occur on the same day.

4.2 Intraday joint and disjoint jump activity

Although for many days the ETF and its volatility do not jump together, there are a few days for which jumps occur on both market. Table 2 line (vi) reports the number of those days, which corresponds to 49 for the USO/OVX pair and 62 for the GLD/GVZ pair. The next step is to determine whether jumps occur at same time. To this end, we follow the methodology described in Section 2.3 for the joint and disjoint jump test specification, and investigate whether each ETF and its corresponding volatility index exhibit jumps occurring simultaneously. For all jump tests we set the significant level $\alpha = 5\%$ in Eq.(26) and Eq.(27).

[Insert Figure 4 here]

[Insert Figure 5 here]

Figure 4 shows using a black solid line the test statistic $C_n^j = |\Phi^{(j)} - 1|$ computed from Eq.(19) (and used to determine the critical region in Eq.(28)) to test the null of joint jumps in the times series of USO and OVX; the red dotted line represents the corresponding critical value $c^{(j)}$ computed from Eq.(26).[16] We notice that the value of the test statistic exceeds the critical value for the majority of cases. More precisely, the number of rejections of the null corresponds to 81.63%, which is reported in Table 4.

[Insert Table 4 here]

This rejection rate for the pair GLD/GVZ reaches 85.48%, indicating that the joint jumps in the ETF and respective volatility index occur only in 18.37% and 14.52% of all cases for USO/OVX and GLD/GVZ, respectively. Figure 5 shows the distribution of the test statistic (top panel) and the critical value (bottom panel) estimated in a non-parametric way.[17] Despite having a long right tail, the distribution of the test statistic indicates that the majority of values concentrate in range 0 to 5, which is much above the critical values ranging from 0 to 2.5, leading to a rejection rate of 85.48%.

[16]The figures for the pair GLD/GVZ look similar, and are available from authors upon request.

[17]Given n sample points $(x_{t_1}, ..., x_{t_n})$, a kernel estimator for the probability density $f(x)$ is defined by $\hat{f}(x, h_n) = \frac{1}{nh_n} \sum_{i=1}^{n} K\left(\frac{x - x_{t_i}}{h_n}\right)$ where $K(\cdot)$ is a kernel function and h_n is a bandwidth parameter controlling the degree of smoothness of the estimator. The density evaluated at point x is estimated as the average of densities centered at the actual data points. The further away the data point is located from the estimated point, the lower is the contribution to the estimated density. We use a Gaussian kernel given by $K(z) = (2\pi)^{-1/2} \exp\{-\frac{1}{2}z^2\}$ and the 'rule-of-thumb' bandwidth, which suggests to set $h_c = 1$ in bandwidth $h_n = h_c \hat{\sigma} n^{-1/5}$, which also depends on the dispersion of the observations $\hat{\sigma}$ and the total number of data points n, see e.g. Epanechnikov (1969).

[Insert Figure 6 here]

[Insert Figure 7 here]

Figure 6 shows the test statistic $\Phi_n^{(d)}$ from Eq.(20) (black solid line) and the critical value $c^{(d)}$ computed from Eq.(27) (red dotted line) to test the null of jumps arriving disjointly in USO and OVX. The bottom panel is a zoomed-in version of the top panel. We observe that the value of the test statistic is below the critical value for the majority of cases, which results in a relatively low rejection rate corresponding to 6.12% (reported in Table 4). For the pair GLD/GVZ this value corresponds to 3.12%. In other words, we conclude that joint jumps in the ETF and respective volatility index occur only in 6.12% and 3.12% of all cases for USO/OVX and GLD/GVZ, respectively. Figure 7 shows distribution of the test statistic (top panel) and the critical value (bottom panel) estimated in a non-parametric way. Again we confirm that the critical values typically fall in range between 0 to 19, thus, substantially exceeding the values of the test statistics taking values in the range between 0 to 1, leading to a low rejection rate of only 6.12%.

Overall, the test results prove our conjecture that for the days when both series jump, the jumps in the ETF and the respective volatility index tend to occur disjointly.

4.3 Joint jump sizes analysis

The final stage of the empirical analysis investigates whether jump sizes in the ETF and the respective commodity volatility index follow some dependence structure. Again, we consider only those days where both series experience a jump, i.e. 49 days for the pair USO/OVX and 62 days for the pair GLD/GVZ.

[Insert Table 5 here]

We report in Table 5 a linear correlation coefficient computed for the jump sizes in ETF and its volatility, as well as copula dependence parameter for the four copula models discussed in Section 2.4. Thereby, we use the IFM procedure for the estimation of copula parameters, assuming an empirical distribution for the marginals (i.e. the ETF and the volatility index). We observe nearly zero correlation (0.0993) between the jump sizes for the commodity pair USO/OVX. Copula parameter, which also measures non-linear dependencies in the time series, also points towards independence (recall that for the Clayton copula independence is achieved when the copula parameter reaches zero, while for the Gumbel case independence is achieved when the dependence parameter reaches one). This result is consistent with the fact reported above that most of the jumps occurring in the time series are disjoint. For the pair GLD/GVZ we observe a weak negative dependence measured by the linear correlation coefficient (it corresponds to -0.14); as well as Gaussian and Student-t copulas, while Clayton and Gumbel copulas point towards independence.

[Insert Figure 8 here]

Finally, Figure 8 shows a scatter plot and a histogram for the jump sizes in USO and OVX for the days where both series jump.[18] Again, we confirm that there is no clear dependence structure between the jump sizes in USO and OVX.

Overall, our results from the analysis of dependence between jump sizes, in conjunction with analysing intraday joint and disjoint jump activity, indicate that most of the jumps occurring (even though on the same days) in the time series of ETFs and respective volatility indices, are disjoint in their nature.

5 Conclusion

This paper presents a model specification test analysis on jump activity for the commodity markets, developed within the affine jump diffusion framework. Thanks to the affine property of the model, the volatility index associated with a set of options written on a commodity is an affine function of the instantaneous volatility, thus, turning this variable observable. Combining high frequency data for both, the commodity index and its volatility index and applying joint and disjoint jump tests proposed by Jacod and Todorov (2009), we carry out a test on the simultaneity of jumps for the asset and its volatility. Applying this methodology to the actively traded crude oil pair USO/OVX and gold pair GLD/GVZ, we find that models with disjoint jumps most likely provide the correct specification for these two markets. Our findings point towards a very specific property of the commodity markets and their dissimilarity to the equity market (proxied by SP500/VIX) where the latter is typically modelled using frameworks allowing for common jumps (thus, excluding disjoint jumps).

Our results provide useful guidance on the affine model specifications to ensure consistency with the dynamics observed in the commodity and volatility markets. It suggests several extensions. First, the fact that we

[18]The figure for the pair GLD/GVZ looks similar, and is available from authors upon request.

found strong evidence for disjoint jumps does not imply that they are not related. Indeed, “correlating” two jump processes is difficult, with a good example given by the problem of correlating default times for credit derivatives products, but it is still possible to perform such a task. Indeed, a good example is a two-dimensional Hawkes process, which is a vector jump process that enables a dependency between the two jump process components through the self-exciting property, and that nevertheless excludes simultaneous jumps. The recent works of Aït-Sahalia et al. (2015a) and Kokholm (2016) suggest that it might be a convenient tool for the commodity market. Second, a natural extension of our work is a joint calibration of derivatives written on the commodity and derivatives written on the volatility index associated with that commodity. It is far from being an exotic objective as, indeed, for both, the crude oil and gold, such derivative markets exist and are the places of important trading activity.

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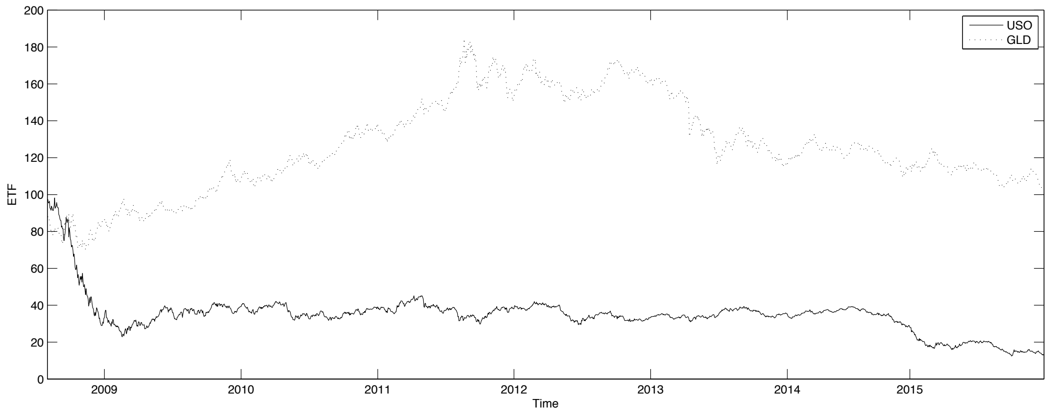
A Appendix

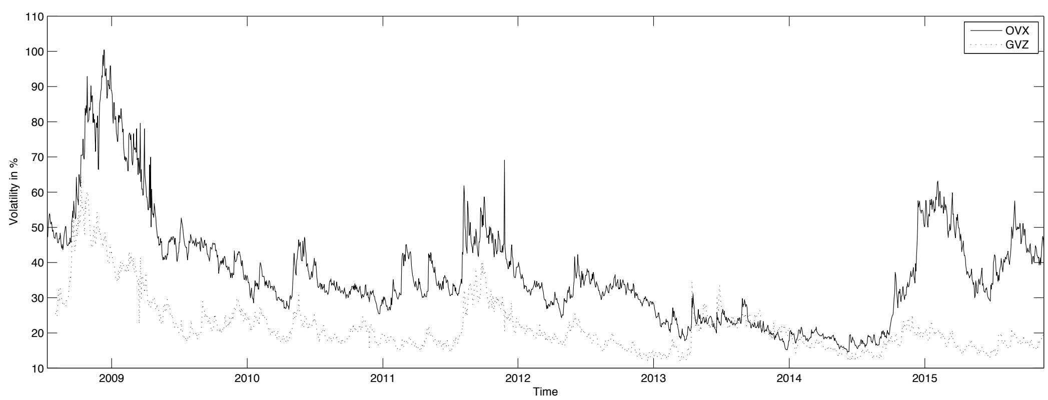
A.1 Figures

Figure 1: ETF plots



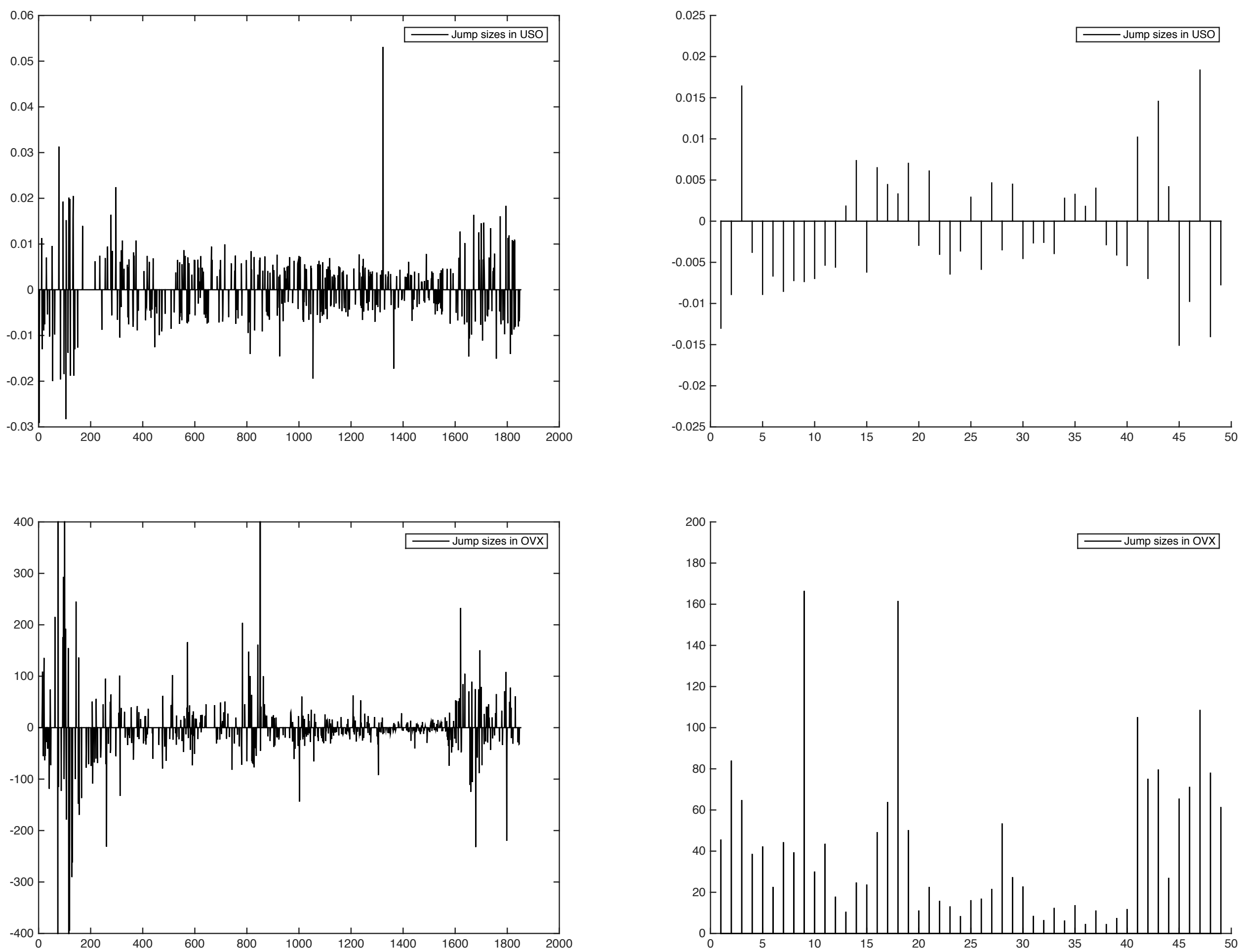
Note. ETF level plots for crude oil (USO) and gold (GLD).

Figure 2: Volatility plots



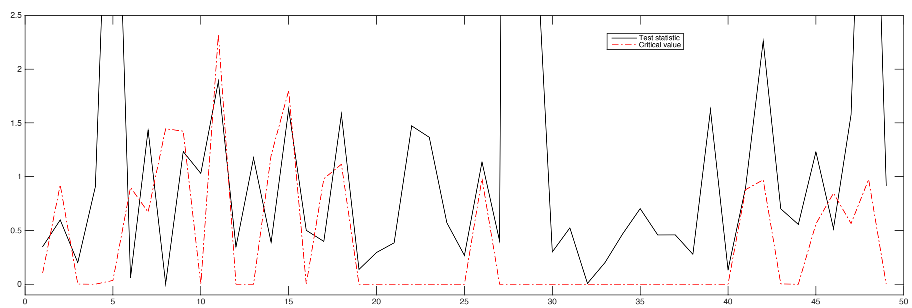
Note. Annualised volatility indices for crude oil (OVX) and gold (GVZ).

Figure 3: Jump plots



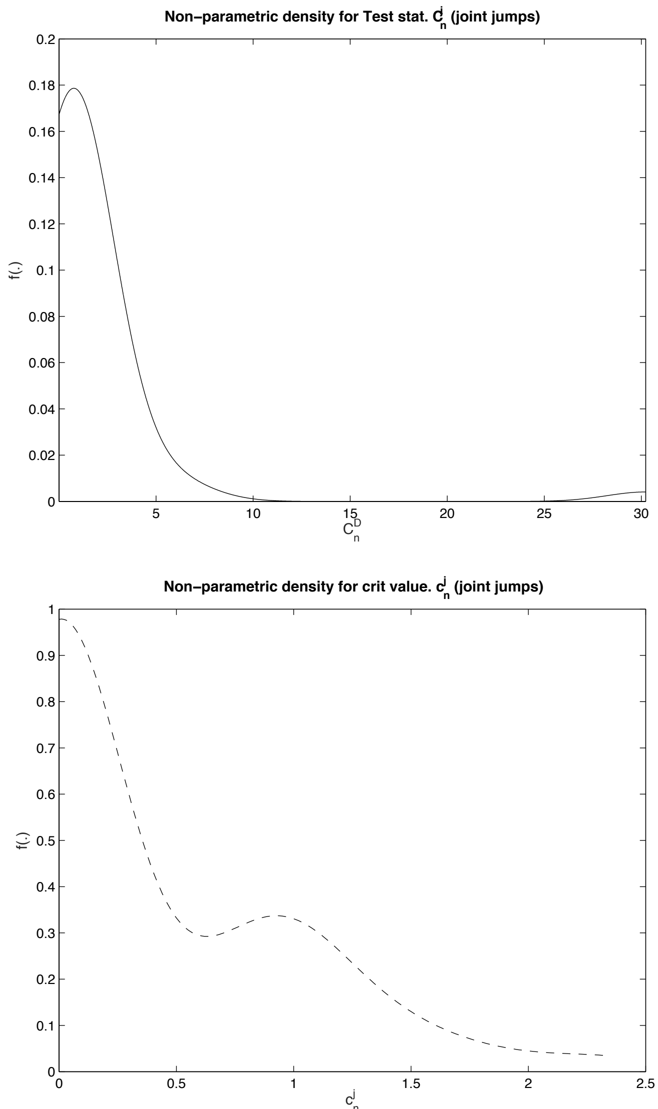
Note. Left panels: jumps sizes in USO (top panel) and OVX (bottom panel) across the entire series (1852 observations), extracted using Eq.(11). The jump is observed when J_t is non-zero. Right panel: jump sizes for days where both series, USO and OVX, jump (49 observations).

Figure 4: Test statistic and critical value for H_0 : Joint jumps



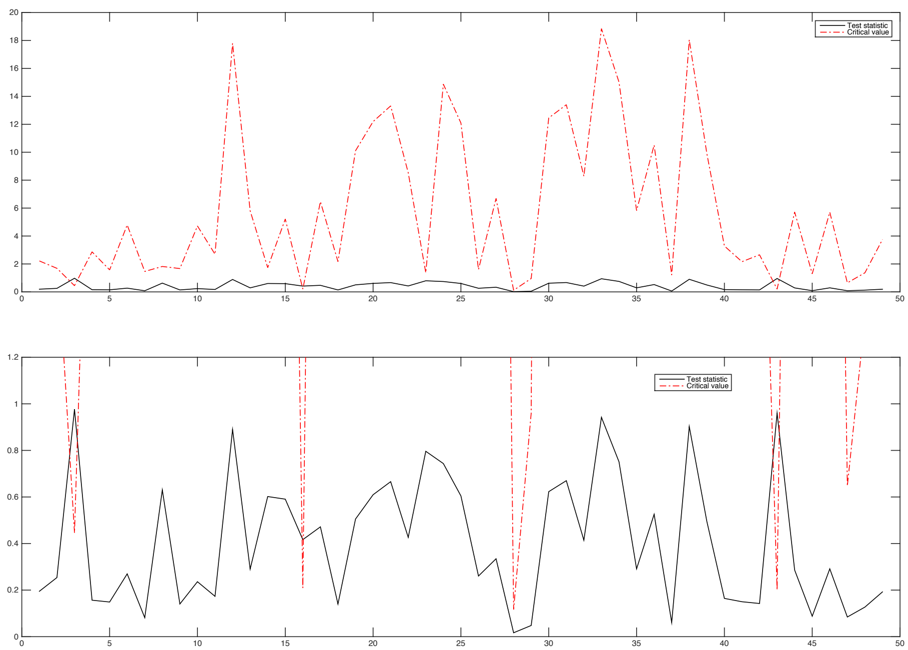
Note. For USO/OVX we plot the test statistics $C_n^j = |\Phi^{(j)} - 1|$ computed from Eq.(19) (black solid line) and the critical value $c^{(j)}$ computed from Eq.(26) (red dotted line) to test H_0 : Jumps arrive jointly in USO and OVX. We document the rejection of the null in 81.63% of all cases (reported in Table 4).

Figure 5: Distribution of the test statistic and the critical value for H_0 : Joint jumps



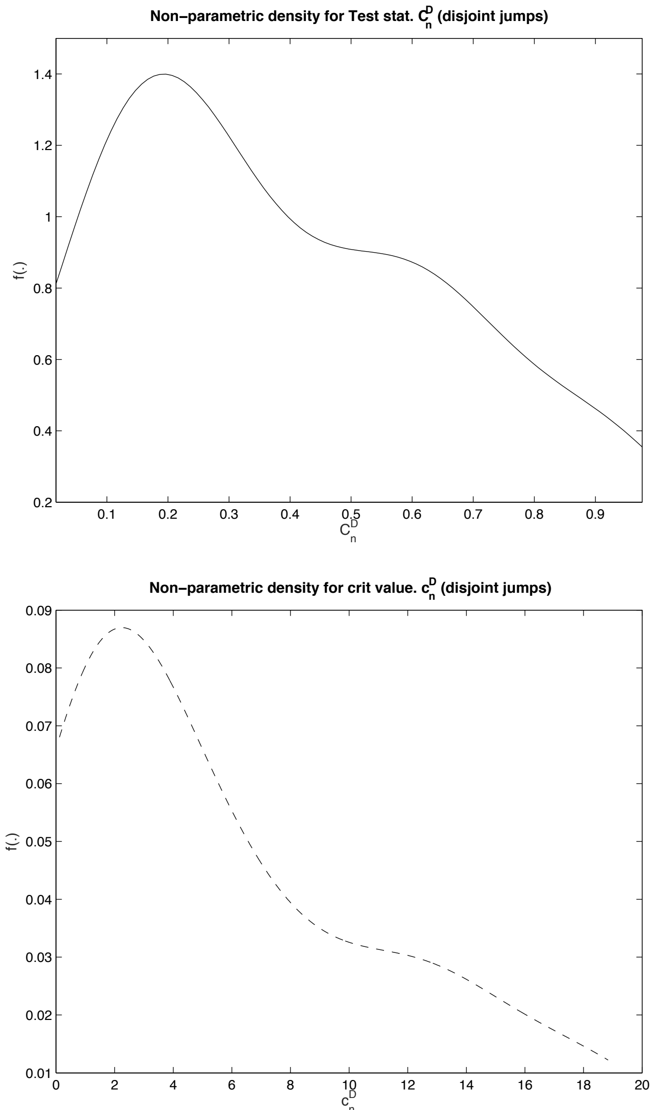
Note. For USO/OVX we plot in the top panel: the non-parametric density estimate for the test statistic $C_n^j = |\Phi^{(j)} - 1|$ computed from Eq.(19) for the H_0 : Joint jumps. Bottom panel: the corresponding non-parametric density estimate for the critical value $c^{(j)}$ computed from Eq.(26). We can observe that the magnitude of the values for the test statistic (upper panel) is typically much higher than the ones for the critical value (bottom panel), which leads to rejection of the H_0 : Joint jumps in 81.63% of all cases (reported in Table 4).

Figure 6: Test statistic and critical value for H_0 : Disjoint jumps



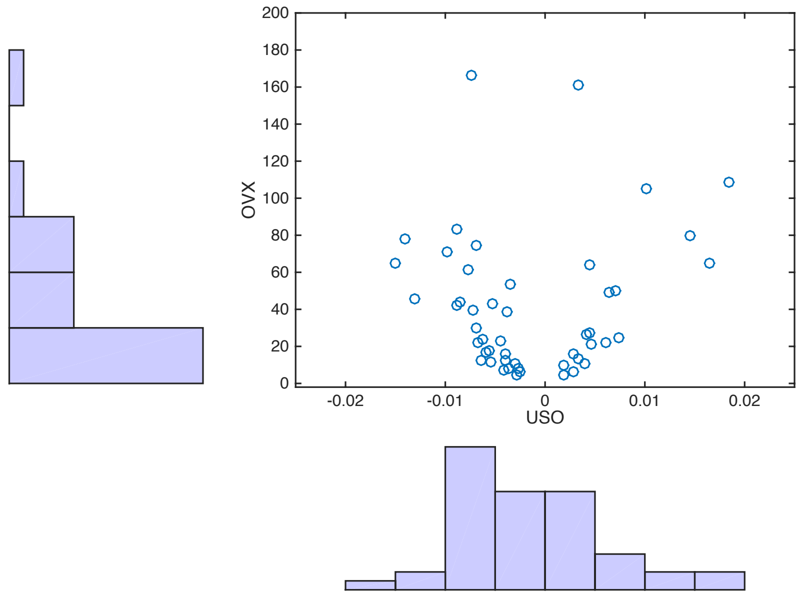
Note. For USO/OVX we plot the test statistics $\Phi_n^{(d)}$ computed from Eq.(20) (black solid line) and the critical value $c^{(d)}$ computed from Eq.(27) (red dotted line) to test H_0 : Jumps arrive disjointly in USO and OVX. The bottom panel is a zoomed-in version of the top panel. We document the rejection of the null in 6.12% of all cases (reported in Table 4).

Figure 7: Distribution of the test statistic and the critical value for H_0 : Disjoint jumps



Note. For USO/OVX we plot in the top panel: the non-parametric density estimate for the test statistic $\Phi_n^{(d)}$ computed from Eq.(20) computed from Eq.(19) for the H_0 : Disjoint jumps. Bottom panel: the corresponding non-parametric density estimate for the critical value $c^{(d)}$ computed from Eq.(27). We can observe that the magnitude of the values for the critical value (bottom panel) is typically much higher than the ones for the test statistic (top panel), which leads to rejection of the H_0 : Joint jumps in only 6.12% of all cases (reported in Table 4).

Figure 8: Scatter plot and histogram for jump sizes in USO and OVX



Note. Scatter plot and histogram for the jump sizes in USO and OVX for days where *both* series jump (but jumps in OVX are positive).

A.2 Tables

Table 1: Summary statistics

	Mean	Std. dev.	Skewness	Kurt.	Min	Max
USO	2.09×10^{-6}	0.0016	-0.0197	5.0314	-0.0048	0.0047
OVX	-0.1838	9.8192	0.1210	6.9646	-30.7985	32.9643
GLD	3.42×10^{-6}	8.01×10^{-4}	0.0020	5.8468	-0.0024	0.0024
GVZ	-0.0083	3.1599	0.1654	7.7313	-10.6269	11.3808

Note. Summary statistics are computed for the log-returns of the underlying ($\Delta_i x^1$ defined in Eq.(15)) and the corresponding squared differences of the volatility ($\Delta_i x^2$ defined in Eq.(16)) for each day and then averaged across days.

Table 2: Jump frequency statistics

Pairs	USO/OVX	GLD/GVZ
(i) Total days	1852	1838
(ii) Jump days (ETF jumps)	393	427
(iii) Jump days (vol. index jumps)	183	220
(iv) ETF jumps, vol. index does not jump	276	300
(v) Vol. index jumps; ETF does not jump	134	158
(vi) Jump days (both series jump)	49	62

Note. Jump frequency statistics include (i) the total number of days in the samples under consideration; (ii) the number of days when ETF jumps regardless of whether the corresponding volatility index jumps or not; (iii) the number of days when volatility index jumps (positively) regardless of whether the corresponding ETF jumps or not; (iv) the number of days when commodity ETF jumps and the corresponding volatility index does not jump; (v) the number of days when volatility index jumps (positively) and the corresponding ETF does not jump; (vi) the number of days when the commodity ETF jumps and the corresponding volatility index jumps (positively).

Table 3: Jump statistics

	USO	GLD	OVX	GVZ
Panel A:	Jumps in the commodity ETF		Jumps in volatility index	
Mean	5.52×10^{-5}	1.31×10^{-5}	66.3433	15.3733
Std. dev.	0.0083	0.0044	212.9379	39.5035
Intensity	0.2122	0.2323	0.0988	0.1197
Panel B:	Jumps in the commodity ETF and volatility index occurring on the same day			
Mean	-0.0014	-7.62×10^{-4}	40.787	11.9312
Std. dev.	0.0075	0.0035	37.427	15.0769
Intensity	0.0265	0.0337	0.0265	0.0337

Note. Descriptive statistics for the jumps in ETF returns and the squared volatility index returns. Panel A reports the results computed separately for the days where jumps occur in the commodity ETF (number of the days is reported in line (ii) of Table 2) and days when jumps occur in the volatility index (number of the days is reported in line (iii) of Table 2). Panel B reports the results computed for those days where both series, the commodity ETF and the corresponding volatility index jump (number of the days is reported in line (vi) of Table 2).

Table 4: Joint and disjoint test results

Pairs	USO/OVX	GLD/GVZ
H_0 : Joint	0.8163	0.8548
H_0 : Disjoint	0.0612	0.0312

Note. Test results: the number of rejections (in %) of the H_0 : Joint jumps (first line) and H_0 : Disjoint jumps (second line).

Table 5: Copula analysis

Model/Pair	USO/OVX	GLD/GVZ
Lin. correl.	0.0993	-0.1472
Gaussian	0.0096	-0.1417
Student-t	-0.0744	-0.1899
Clayton	0.0000	0.0000
Gumbel	1.0958	1.0398

Note. Copula dependence parameter measuring the dependence between jump sizes for ETF and respective volatility index on days when both series jump (but jumps in OVX are positive).