Ad-hoc Analytic Option Pricing under Nonlinear GARCH with NIG Lévy Innovations

By

Sharif Mozumder[a], Michael Dempsey[b], M. Humayun Kabir[c,*] Taufiq Choudhry[d]

[a] Department of Mathematics, University of Dhaka, Bangladesh, Email: sharif_math2000@yahoo.com
[b] School of Economics, Finance and Marketing, RMIT University, Melbourne, Australia, Email: michael.dempsey@rmit.edu.au
[c] School of Economics and Finance, Massey University, Palmerston North, New Zealand, Email: M.H.Kabir@massey.ac.nz
[d] School of Business, University of Southampton, UK, Email: t.choudhry@soton.ac.uk

[*] Corresponding author

Abstract

This paper proposes an approximate closed-form option-pricing model based on a GARCH process with NIG (Normal Inverse Gaussian) Lévy innovations. We develop the mathematical framework and demonstrate how to obtain a closed-form solution to the option price when the return dynamics are characterized by NIG innovations that feed volatility from a GARCH process with NIG innovations. Using a sample of S&P 500 options data, we calibrate the proposed model alongside popular existing models at different points in time and for different sample sizes. We perform both in-sample and out-of-sample fitting. Overall, we find that our model performs significantly better than existing models both in-sample and out-of-sample.

Keywords: Lévy innovations, stochastic volatility, GARCH, calibration, Normal Inverse Gaussian

Ad-hoc Analytic Option Pricing under Nonlinear GARCH with NIG Lévy Innovations

Abstract

This paper proposes an approximate closed-form option-pricing model based on a GARCH process with NIG (Normal Inverse Gaussian) Lévy innovations. We develop the mathematical framework and demonstrate how to obtain a closed-form solution to the option price when the return dynamics are characterized by NIG innovations that feed volatility from a GARCH process with NIG innovations. Using a sample of S&P 500 options data, we calibrate the proposed model alongside popular existing models at different points in time and for different sample sizes. We perform both in-sample and out-of-sample fitting. Overall, we find that our model performs significantly better than existing models both in-sample and out-of-sample.

Keywords: Lévy innovations, stochastic volatility, GARCH, calibration, NIG

1. INTRODUCTION

The well-known drawbacks of the seminal Black-Scholes option-pricing model (1973) are that it fails to account for skewness and heavy tails in the underlying asset and volatility clustering over time. A number of approaches have been suggested that deal with number of drawbacks. Notable amongst these are the jump-diffusion model (Merton, 1976; Psychoyios *et al.*, 2010), the stochastic volatility model (Hull and White, 1987; Heston, 1993; Bates, 2003), the GARCH model of Engle (1982) and Bollerslev (1986) applied to option pricing (e.g., Duan, 1995; Heston and Nandi, 2000; Barone-Adesi *et al.*, 2008), and the Lévy approach (e.g., Geman and Yor, 2001; Geman, 2002; Carr *et al.*, 2002; Dingec and Hormann, 2012).While the jump-diffusion approach considers discrete jumps as well as diffusion in stock prices, the volatility itself is modeled as a second stochastic process under the stochastic volatility approach. The Lévy approach considers the stock prices following a Lévy process. This approach has been further extended by Carr and Wu (2004) and Huang and Wu (2004) who use time-changed Lévy processes to tackle the empirical features of observed option prices. Ornthanalai (2014) finds that models that do not price jump risk under

Lévy processes cannot explain the divergence between the physical and risk-neutral probability measures, and that the model with infinite-activity jumps, rather than Brownian increments, is more relevant in asset pricing.

This paper addresses the interface of the GARCH and Lévy bodies of literature and proposes an alternative – yet complementary – approach to those of Carr and Wu (2004), and Huang and Wu (2004). Our aim is to provide the fast, yet convincing treatment of option pricing under a special-case Lévy process – the Normal Inverse Gaussian (NIG). We intend to achieve this replacing the NIG innovation with the approximate normal innovation. Moolman (2008), Kim *et al.* (2008, 2010), and Ornthanalai (2014) incorporate GARCH within a Lévy process. More recently Badescu *et al.* (2015) propose a non-Gaussian GARCH option-pricing model. However, these studies rely on the Monte Carlo simulation to price options and, as such, do not benefit from an analytic approach. Another notable example in the same spirit is Mercuri (2008), who combines GARCH with tempered stable (TS) innovations and provides an analytical solution comparable to ours. However, his analysis is based on a more limited dataset. More importantly, the GARCH-TS volatility dynamics used by Mercuri (2008) are applicable only for innovations coming from a positive Lévy process[1]. We, on the other hand, introduce a mathematical framework with a nonlinear volatility dynamic approximated by replacing standard NIG innovations with standard normal innovations and demonstrate how ad-hoc analytic solutions can be derived in the presence of both positive and negative Lévy innovations.

---

[1] Kim *et al.* (2008) propose "KR distribution" as one such subclass of the tempered stable distribution, and empirically test and compare with CGMY and the modified tempered stable (MTS) distribution. Kim *et al.* (2010) introduce the rapidly decreasing tempered stable (RDTS) GARCH model with an infinitely divisible distributed innovation, and compare the findings based on classical tempered stable (CTS) GARCH model.

We focus on the pricing of a European option, for which we first need to know the risk-neutral density at maturity. However, a problem with the standard GARCH set-up is that we only know the risk-neutral distribution one-step ahead. To overcome this difficulty, Heston and Nandi (2000) propose a GARCH-like model with normal innovations, for which they compute the characteristic function of the underlying using a recursive procedure, and then employ the Fourier inversion approach of Heston (1993) to price options. Unfortunately, their model is not sufficiently flexible to explain some well-known option biases, particularly those related to short-term maturity options; Christoffersen (2006) conjectures that this limitation might be due to the fact that their single-period innovations are normal. Various attempts have subsequently, been made to relax the normality assumption in the context of the longer-horizon GARCH option pricing. For example, Badescu and Kulperger (2008) replace normality with a semi-parametric approach, and then use Monte Carlo simulation to price the options. This, however, is time-consuming to implement, and also inefficient as it uses only stock price information, not the information contained in option prices.

The present paper seeks to fill this gap by offering an approximate closed-form solution for the pricing of European options under a GARCH framework, where innovations follow a NIG Lévy process. This is practically useful as it makes option pricing much faster in real time. Such a framework may also be useful for GARCH option pricing under other Lévy processes.

The remaining structure of this paper is as follows: Section 2 derives the closed-form GARCH option-pricing formula with NIG-Lévy innovations. Section 3 discusses data, implementation issues, calibration and empirical results. Section 4 concludes.

2. CLOSED-FORM GARCH OPTION PRICING WITH NIG-LÉVY INNOVATIONS

Two important recent contributions in the GARCH-Lévy area are Christoffersen *et al.* (2010) and Christoffersen *et al.* (2012). The former work sets out the broad characterizations of Lévy dynamics, but does not apply their model to options data or offer explicit derivations for the Lévy innovations that have been extensively used in the derivatives pricing literature. The latter study tackles affine GARCH dynamics with Lévy innovations and demonstrates that closed-form pricing is not possible for innovations based on Lévy processes exhibiting both positive and negative jumps.

We develop a mathematical framework to derive approximate closed-form formulae similar to those of Heston and Nandi (2000). More specifically we replace their conditional normal innovations with innovations derived from Normal Inverse Gaussian (NIG) process, which is a form of Lévy process.

Let us assume that the stock price follows the process

$$S_t = S_{t-1} e^{X_t} \quad , \tag{1}$$

where X_t follows a time-varying NIG Lévy process. The characteristic function of (X_t) is that of NIG (α, β, δt), in particular, the random variable (X_1) it is given by[2]

$$E\left(e^{-isX}\right) = \exp\left(-\delta\left\{\sqrt{\alpha^2 - (\beta + is)^2} - \sqrt{\alpha^2 - \beta^2}\right\}\right) \quad . \tag{2}$$

[2] The moments of $NIG(\alpha,\beta,\delta)$ random variable, X_t ,are shown in Appendix 1.

To price an option, however, we also need the conditional distribution of the underlying asset at a multi-period-ahead maturity, *T*. We derive such a distribution following the recursive method developed by Heston and Nandi (2000).Their recursive procedure is based on the idea that the conditional MGF can be expressed as

$$E\left[e^{u\log(S_T)} \mid \Im_t\right] = S_t^u \exp\left[A(t,T,u) + B(t,T,u)\sigma_{t+1}\right]. \tag{3}$$

The goal is to solve for $A(t,T,u)$ and $B(t,T,u)$ for the NIG-Lévy innovations characterizing σ_t [3].

We project equation (1) by one period to obtain

$$E\left[e^{u\log(S_T)} \mid \Im_{t+1}\right] = S_{t+1}^u \exp\left[A(t+1,T,u) + B(t+1,T,u)\sigma_{t+2}\right]. \tag{4}$$

We assume that the general form of the conditional MGF holds for time *t* + 1and use the iterative property of conditional expectations (which is the central feature of the Heston-Nandi recursive approach) to obtain the corresponding expression for the conditional MGF at time *t*:

$$\begin{aligned} E\left[e^{u\log(S_T)} \mid \Im_t\right] &= E\left[E\left[e^{u\log(S_T)} \mid \Im_{t+1}\right] \mid \Im_t\right] \\ &\overset{(2)}{=} E\left[S_{t+1}^u \exp\left[A(t+1,T,u) + B(t+1,T,u)\sigma_{t+2}\right] \mid \Im_t\right] \end{aligned}. \tag{5}$$

Using equation (1), we have $S_{t+1}^u = S_t^u e^{uX_{t+1}}$. Plugging this into equation (5) we obtain

$$\begin{aligned} E\left[e^{u\log(S_T)} \mid \Im_t\right] &= E\left[S_t^u e^{uX_{t+1}} \exp\left[A(t+1,T,u) + B(t+1,T,u)\sigma_{t+2}\right] \mid \Im_t\right] \\ &= S_t^u E\left[e^{uX_{t+1}} \exp\left[A(t+1,T,u) + B(t+1,T,u)\sigma_{t+2}\right] \mid \Im_t\right] \end{aligned}. \tag{6}$$

Hence, no matter how many steps are there between *t* and *T*, we can use equations (4) and (5) recursively to derive the conditional MGF at any maturity *T* given the information available up to

[3] *En passant*, we note that our solution method can be extended to other Lévy innovations such as Variance Gamma (VG) and CGMY processes.

t. A comparison of (3) and (6) then allows us to derive the recursive relations for the coefficients $A(t,T,u)$ and $B(t,T,u)$.

2.1. NIG TIME-CHANGED LÉVY INNOVATIONS

We derive the recursive coefficient relations for the NIG-Lévy innovations. Given that the stock price follows the dynamics (1), the log return process follows in GARCH settings:

$$X_t = r + \lambda\sigma_t - \frac{z_t}{\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}} \quad .$$

(7)

Here, $z_t \mid \Im_{t-1} \sim NIG\left(\alpha,\beta,\delta\sigma_t\right)$ and the volatility process, σ_t follow the GARCH(1,1) specification

$$\sigma_t = \beta_0 + \beta_1 \frac{z_t}{\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}} + \alpha_1\sigma_{t-1}, \tag{8}$$

where the above scaling ensures unit variance for innovations.

Proposition 2.1.1: *For the GARCH dynamics in (7) with* $z_t \mid \Im_{t-1} \sim NIG\left(\alpha,\beta,\delta\sigma_t\right)$, where σ_t is as in (8), the conditional skewness and conditional kurtosis can be obtained as in (A7) and (A8) in Appendix A. [See Appendix A for the proof.]

Proposition 2.1.2: *For the GARCH dynamics in (7) with* $z_t \mid \Im_{t-1} \sim NIG(\alpha, \beta, \delta\sigma_t)$, where σ_t is as in (8), the equivalent martingale relationships among the parameters are as given by (B13)-(B15) and (B18)-(B19). [See Appendix B for the proof.]

Now we use the above two equations (7) and (8) to replace X_{t+1} and σ_{t+2} in equation (6) to obtain the following GARCH NIG-Lévy dynamics:

$$E\left[e^{u\log(S_T)} \mid \Im_t\right]$$

$$= S_t^u E\left[e^{u\left(r+\lambda\sigma_{t+1}-\frac{z_{t+1}}{\sqrt{\alpha^2\delta(\alpha^2-\beta^2)^{-\frac{3}{2}}}}\right)} \exp\left[\begin{array}{l} A(t+1,T,u)+ \\ B(t+1,T,u)\left(\beta_0+\beta_1\dfrac{z_{t+1}}{\sqrt{\alpha^2\delta(\alpha^2-\beta^2)^{-\frac{3}{2}}}}+\alpha_1\sigma_{t+1}\right)\end{array}\right] \mid \Im_t\right] \qquad (9)$$

$$= S_t^u E\left[\exp\left\{\begin{array}{l} ur+\lambda u\sigma_{t+1}+A(t+1,T,u)+\beta_0 B(t+1,T,u) \\ +\alpha_1\sigma_{t+1}B(t+1,T,u)+\dfrac{[\beta_1 B(t+1,T,u)-u]z_{t+1}}{\sqrt{\alpha^2\delta(\alpha^2-\beta^2)^{-\frac{3}{2}}}} \mid \Im_t\end{array}\right\}\right]$$

$$\stackrel{(20)}{=} S_t^u \exp\left\{\begin{array}{l} ur+\lambda u\sigma_{t+1}+A(t+1,T,u)+\beta_0 B(t+1,T,u) \\ +\alpha_1\sigma_{t+1}B(t+1,T,u)-\delta\sigma_{t+1}\left[\sqrt{\alpha^2-\left(\beta+\dfrac{[\beta_1 B(t+1,T,u)-u]}{\sqrt{\alpha^2\delta(\alpha^2-\beta^2)^{-\frac{3}{2}}}}\right)^2}-\sqrt{\alpha^2-\beta^2}\right]\end{array}\right\}$$

since $z_{t+1} \mid \Im_t \sim NIG(\alpha, \beta, \delta\sigma_{t+1})$

$$= S_t^u \exp\left\{\begin{array}{l} ur+A(t+1,T,u)+\beta_0 B(t+1,T,u) \\ +\left\{\lambda u+\alpha_1 B(t+1,T,u)-\delta\left[\sqrt{\alpha^2-\left(\beta+\dfrac{[\beta_1 B(t+1,T,u)-u]}{\sqrt{\alpha^2\delta(\alpha^2-\beta^2)^{-\frac{3}{2}}}}\right)^2}-\sqrt{\alpha^2-\beta^2}\right]\right\}\sigma_{t+1}\end{array}\right\} \qquad (10)$$

Comparing equations (3) and (9), we then obtain the following recursive relations:

$$A(t,T,u) = ur + A(t+1,T,u) + \beta_0 B(t,T,u)$$

$$B(t+1,T,u) = \lambda u + \alpha_1 B(t,T,u) - \delta\left[\sqrt{\alpha^2 - \left(\beta + \frac{[\beta_1 B(t,T,u) - u]}{\sqrt{\alpha^2 \delta(\alpha^2 - \beta^2)^{-\frac{3}{2}}}}\right)^2} - \sqrt{\alpha^2 - \beta^2}\right] \quad (11)$$

Since $z_t \mid \Im_{t-1} \sim NIG(\alpha, \beta, \delta\sigma_t)$ often assumes positive as well as negative values, we restrict ourselves to nonlinear dynamics of the Heston-Nandi type.

Let us assume the following nonlinear risk-neutral dynamics[4]:

$$\begin{aligned}
\sigma_t &= \beta_0 + \beta_1\left[\frac{NIG(\alpha, \beta, \delta\sigma_{t-1})}{\sqrt{\alpha^2 \sigma_{t-1} \delta(\alpha^2 - \beta^2)^{-\frac{3}{2}}}} - \gamma\sqrt{\sigma_{t-1}}\right]^2 + \alpha_1 \sigma_{t-1} \\
&= \beta_0 + \beta_1\left[\frac{NIG(\alpha, \beta, \delta\sigma_{t-1}) - \mu + \mu}{\sqrt{\alpha^2 \sigma_{t-1} \delta(\alpha^2 - \beta^2)^{-\frac{3}{2}}}} - \gamma\sqrt{\sigma_{t-1}}\right]^2 + \alpha_1 \sigma_{t-1} \\
&= \beta_0 + \beta_1\left[stdNIG + \frac{\mu}{\sqrt{\alpha^2 \sigma_{t-1} \delta(\alpha^2 - \beta^2)^{-\frac{3}{2}}}} - \gamma\sqrt{\sigma_{t-1}}\right]^2 + \alpha_1 \sigma_{t-1}
\end{aligned} \quad (12)$$

where μ is the expected value of a $NIG(\alpha, \beta, \delta\sigma_{t-1})$ random variable and is given by equation (A1) in Appendix A. Moreover this modification of the volatility dynamics re-establishes the historical and risk-neutral relation of β_1 and introduces a new historical and risk-neutral relation for the new parameter γ, as described in Appendix B. We can further simplify the above equation to obtain

[4]This introduction of non-linearity does affect the equivalent martingale relationships among the parameters and is demonstrated in Appendix C.

$$\sigma_t = \beta_0 + \beta_1 \left[stdNIG + \left\{ \frac{\beta\sqrt{\delta}}{\alpha\left(\alpha^2 - \beta^2\right)^{-\frac{1}{4}}} - \gamma \right\} \sqrt{\sigma_{t-1}} \right]^2 + \alpha_1 \sigma_{t-1} \quad . \tag{13}$$

A problem with this characterization of volatility, however, is that when σ_t is plugged into equation (6), it does not yield explicit recursive relations for $A(t,T,u)$ and $B(t,T,u)$. Consequently, no closed-form European option valuation is possible without further approximation.[5]

Our proposed solution is to apply an approximation to the dynamics (13) that upholds closed-form valuation techniques similar to those of Heston and Nandi (2000). In particular, when the dynamics (13) are characterized for NIG innovations, we propose an ad-hoc approximation that preserves the characterization of dynamics but replaces the standard NIG innovations by standard Normal innovations:

$$\sigma_t \approx \beta_0 + \beta_1 \left[stdNormal + \left\{ \frac{\beta\sqrt{\delta}}{\alpha\left(\alpha^2 - \beta^2\right)^{-\frac{1}{4}}} - \gamma \right\} \sqrt{\sigma_{t-1}} \right]^2 + \alpha_1 \sigma_{t-1} \quad . \tag{14}$$

This ad-hoc approximation is motivated by Heston-Nandi closed-form pricing based on the following relation involving a standard Normal random variable,

$$E\left[\exp\left\{a(z+b)^2\right\}\right] = \exp\left\{-\frac{1}{2}\ln(1-2a) + \frac{ab^2}{1-2a}\right\}, \tag{15}$$

where $z \sim N(0,1)$.

[5] A similar problem was encountered by Ornthanalai (2010) in his study of GARCH-Lévy dynamics for asset pricing. He concluded that, for Lévy innovations capable of exhibiting both positive and negative jumps, there is no alternative to the Monte-Carlo valuation of derivatives. However, the problem with Monte Carlo is that it requires a long time to price even a single option. To attempt such pricing we need to consider a large number of simulations – at least 5,000 trials are needed – at the expense of huge computational time that renders quick calibration practically infeasible.

We apply the volatility dynamics (14) (which are characterized for NIG innovation but approximated through the replacement of standard NIG variates by standard Normal variates) in the general recursive relation (6):

$$E\left[e^{u\log(S_T)} \mid \Im_t^1\right]$$
$$= S_t^u E\left[e^{u\left(r+\lambda\sigma_{t+1}-\frac{z_{t+1}}{\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}}\right)} \exp\left[\begin{array}{l} A(t+1,T,u)+ \\ B(t+1,T,u)\left(\beta_0+\beta_1\left[z+\left\{\frac{\beta\sqrt{\delta}}{\alpha\left(\alpha^2-\beta^2\right)^{-\frac{1}{4}}}-\gamma\right\}\sqrt{\sigma_{t+1}}\right]^2+\alpha_1\sigma_{t+1}\right)\end{array}\right] \mid \Im_t^1\right]$$

(16)

We then apply the relation (15) to (16) to simplify further:

$$E\left[e^{u\log(S_T)} \mid \Im_t^1\right]$$
$$\overset{(20)}{=} S_t^u \exp\left\{u\left(r+\lambda\sigma_{t+1}\right)-\delta\sigma_{t+1}\left[\sqrt{\alpha^2-\left(\beta+\frac{(-u)}{\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}}\right)^2}-\sqrt{\alpha^2-\beta^2}\right]\right\}$$
$$\exp\left\{\begin{array}{l} A(t+1,T,u)+B(t+1,T,u)\beta_0-\frac{1}{2}\log\left[1-2B(t+1,T,u)\beta_1\right] \\ +B(t+1,T,u)\beta_1\left(\frac{\beta\sqrt{\delta}}{\alpha\left(\alpha^2-\beta^2\right)^{-\frac{1}{4}}}-\gamma\right)^2\sigma_{t+1}\frac{1}{1-2B(t+1,T,u)\beta_1}+\alpha_1 B(t+1,T,u)\sigma_{t+1}\end{array}\right\}$$

since $z_{t+1} \mid \Im_{t-1}^1 \sim NIG\left(\alpha,\beta,\delta\sigma_{t+1}\right)$

$$= S_t^u \exp\left\{\begin{array}{l} ur+A(t+1,T,u)+\beta_0 B(t+1,T,u)-\frac{1}{2}\log\left[1-2B(t+1,T,u)\beta_1\right] \\ +\left[\begin{array}{l} \lambda u+\alpha_1 B(t+1,T,u)-\delta\left[\sqrt{\alpha^2-\left(\beta+\frac{(-u)}{\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}}\right)^2}-\sqrt{\alpha^2-\beta^2}\right] \\ +B(t+1,T,u)\beta_1\left(\frac{\beta\sqrt{\delta}}{\alpha\left(\alpha^2-\beta^2\right)^{-\frac{1}{4}}}-\gamma\right)^2\frac{1}{1-2B(t+1,T,u)\beta_1}\end{array}\right]\sigma_{t+1}\end{array}\right\} \quad (17)$$

A comparison of equations (17) and (3) gives the following recursive relations:

$$A(t,T,u)=ur+A(t+1,T,u)+\beta_0 B(t+1,T,u)-\frac{1}{2}\log[1-2B(t+1,T,u)\beta_1]$$

$$B(t,T,u)=\lambda u+\alpha_1 B(t+1,T,u)-\delta\left[\sqrt{\alpha^2-\left(\beta+\frac{(-u)}{\sqrt{\alpha^2\delta(\alpha^2-\beta^2)^{-\frac{3}{2}}}}\right)^2}-\sqrt{\alpha^2-\beta^2}\right] \quad . \qquad (18)$$

$$+B(t+1,T,u)\beta_1\left(\frac{\beta\sqrt{\delta}}{\alpha(\alpha^2-\beta^2)^{-\frac{1}{4}}}-\gamma\right)^2\frac{1}{1-2B(t+1,T,u)\beta_1}$$

2.2. OPTION PRICING AND CALIBRATION METHODOLOGY

We obtain the option prices through Fourier Inversion as in Heston (1993) and Heston and Nandi (2000). For the closed-form (up to numerical integration) GARCH model with NIG innovations, we denote the model price by c_{cfgnig} (where the subscript alludes to closed-form GARCH-NIG). This model has seven parameters to be estimated[6] $[\beta_0,\beta_1,\alpha_1,\gamma,\alpha,\beta,\delta]$.

Given the parameter constraints, we consider the calibration as a constrained optimization problem rather than a simple nonlinear least-square one. The constraints arise from the GARCH structure as well as the usual NIG parameterization, $\beta_0 \geq 0, \beta_1 \geq 0, \alpha_1 \geq 0$, $\alpha_1+\beta_1<0$, $\alpha>0$, and

[6]The one-day-ahead GARCH variance σ^2_{t+1}, can also be treated as a parameter. However treating one-day-ahead volatility as a parameter is only manageable for a few days' market data, because each new day creates a new parameter to be estimated. So, for calibrations using option records over a long period, we need to directly feed the one-day-ahead volatilities in a dynamic fashion. Heston and Nandi (2000) input these through a GARCH process that forces the calibration to rely heavily on a long time series of asset returns, in addition to the market price of options. By contrast, we implement the same approach using GARCH-NIG closed-form volatility dynamics. Our calibration often provides a better fit than Heston and Nandi's model does. Another advantage of our approach is that it renders past asset returns redundant.

$|\beta| \le \alpha$, i.e. $-\alpha - \beta < 0, \delta > 0$. Thus, the calibration of the model comes from the solution to the following optimization problem[7],

$$Minimize\left[\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(C_{market}^{i} - c_{cfgnig}^{i}\left[\beta_0, \beta_1, \alpha_1, \gamma, \alpha, \beta, \delta\right]\right)^2}\right], \quad (19)$$
$$s.t.\ \ \mathrm{A}\left[\beta_0, \beta_1, \alpha_1, \gamma, \alpha, \beta, \delta\right]' \le b$$

where

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$b = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

We implement 12 different models: (1) BS: Black-Scholes (1973) model, (2) VG: Variance Gamma of Madan *et al.* (1998), (3) NIG: Normal Inverse Gaussian model of Schouten (2003), (4) JD-DE: Exponential double jump model of Kou (2002), (5) CGMY: Carr *et al.'s* (2002) model, (6) HS: Heston's (1993) stochastic volatility model, (7) HN(R): restricted version of Heston and Nandi's (2000) GARCH model, (8) HN(U): unrestricted version of Heston and Nandi's (2000) GARCH model, (9) SC: Scott's (1997) model, (10) BT: Bates' (1996) model, (11) CH:

---

[7] We implement the constrained optimization using the MATLAB function "fmincon".

Christoffersen *et al.'s* (2006) model, and (12) CFGNIG: our proposed closed-form GARCH model with NIG innovations. Table 1 presents the characteristics functions of all models.

3. DATA AND PILOT SURVEY

We use daily records of options written on the S&P500 index traded on the Chicago Board Options Exchange (CBOE). We retrieve the S&P500 index put-and-call option quotes from Thompson Reuter Tick history. The sample period runs from January 2012 through to December 2014. For calibration and out-of-sample assessment we employ data up until June 2015[8]. We use the data on every Wednesday as it is the day of the week least likely to be a holiday, and it is less likely to be impacted by day-of-the-week effects (Ornthanalai, 2014)[9].

We then clean our data using the same rules applied by Heston and Nandi (2000):

- We do not consider any option of a particular moneyness or maturity more than once in our sample. This eliminates a large number of observations.
- We exclude deep out-of-the-money and deep in-the-money options because these are either infrequently traded and/or have low enough prices for the bid-ask spread to constitute a major portion of the price. To be precise, only records having an index-to-strike ratio between 0.9 and 1.1 are included in our sample.
- We only include options that have between six and 100 days to expiration. We eliminate very long-term options because they are not actively traded and are prone to mispricing.

[8]We discuss different aggregation schemes and corresponding in-sample and out-of-sample data requirements in section 5.

[9]Dumas, Fleming and Whaley (1998) discuss the advantages of Wednesday data in detail.

Conversely, we eliminate very short-term options because they have substantial time decay which makes it difficult to reliably determine the volatility parameter.

After applying the filtering rules described above, we have 8,404 options over the time window. We first carry out a pilot survey using the options recorded on Wednesday, July 1, 2015. Since the discrete time models take a considerable time to calibrate, Heston and Nandi (2000) did not consider more than six months' records at a time. In particular, they take a year's set of records broken down into two six-month periods, considering the first six months as an in-sample period and the second six months as out-of-sample. We follow the same procedure, and break down annual records in the same way, using the first six months' records for in-sample calibration and the second six months' records to assess out-of-sample performance. We also consider longer periods of two years' and three years' aggregation of option records. Table 2 presents descriptive statistics for the option quotes based on moneyness and maturity. We define moneyness as *S/K* where *S* is the option price at maturity *t*, and *K* is the strike price.

4. GOODNESS-OF-FIT

To assess in-sample goodness-of-fit, we use the Root Mean Square Measure, *RMSE,* which considers quadratic deviations between model and market prices:

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(C_i^{market} - C_i^{model}\right)^2} \quad . \tag{20}$$

To assess out-of-sample goodness-of-fit, we use a naive measure known as average absolute error (*AAE*), which considers linear deviations between model and market prices:

$$AAE = \sum_{i=1}^{N} \frac{\left| model\ price_i - market\ price_i \right|}{N} . \tag{21}$$

We also assess out-of-sample goodness-of-fit using Mean-Outside-Error (*MOE*), which is used to assess a model's average tendency to over-price or underprice:

$$MOE = \sum_{i=1}^{N} \frac{\left\lfloor \left(model\ price_i - ask_i\right) \mathrm{I}_{\{model\ price_i - ask_i\}} + \left(model\ price_i - bid_i\right) \mathrm{I}_{\{model\ price_i - bid_i\}} \right\rfloor}{N}. \quad (22)$$

When assessing out-of-sample performance, we restrict our reported results to models that explicitly incorporate stochastic volatility. We find that models that incorporate jumps provide excellent fits over very short periods (e.g., one day or a couple of days' observations); but such models perform much worse for long periods relative to models which consider explicit stochastic volatility dynamics. Consequently, we restrict our out-of-sample performance to *seven* different models: Heston's (1993) continuous-time stochastic volatility model, restricted and unrestricted versions of Heston and Nandi's discrete-time GARCH volatility model with normal innovations (Heston and Nandi, 2000), Scott (1997), Bates (1996), Christoffersen *et al.* (2006), and our CFGNIG model.

5. CALIBRATION

We report calibrations using different cross-sections of options recorded on a wider time frame: from January 2012 to December 2014. We calibrate models under three temporal aggregation schemes. The *first* scheme corresponds to calibration using options traded on the first six months of each of the years 2012, 2013 and 2014, and then we assess out-of-sample performance using options traded on the remaining six months of the relevant year. Tables 4A, 5A and 6A report the in-sample performance and tables 4B, 5B and 6B report the out-of-sample performance for the 2012, 2013 and 2014 calibrations, respectively.

Our *second* temporal aggregation scheme calibrates models using information contained in option contracts traded over two-year periods. More precisely, we calibrate the models using options traded during 2012-2013 and 2013-2014. In the first case, we use the first six months' contracts of 2014 to assess models' out-of-sample performance, and in the second case we use contracts for the first six months of 2015 for out-of-sample assessment. Tables 7A and 8A report the in-sample results and 7B and 8B report the out-of-sample results for the 2012-2013 and 2013-2014 calibrations, respectively.

Our *third* aggregation scheme considers the full three-year period of 2012-2014. Table 9A reports the calibration results corresponding to this scheme, and Table 9B presents the corresponding out-of-sample assessment for the period from January 2015 to June 2015.

6. RESULT ANALYSIS

6.1. In-sample calibration results

Table 3 shows the RMSE along with parameter estimates of piloted 1-day calibration for all 12 models for the options traded on Wednesday, July1, 2015. After cleaning the data, we have 78 records to consider on that particular day. Models with time-varying volatility under GARCH with normal innovations outperform other sophisticated characterizations, and models with time-varying volatility under GARCH with NIG innovations outperform models with GARCH volatility with normal innovations.

Comparing the parameter estimates[10] in Tables 3, 4A, 5A, 6A, 7A, 8A and 9A, we find that – with the exception of the one-day calibration in Table 3 – BS-implied volatility is stable during different time periods and over different sample sizes. In the VG model, the σ and υ parameters are moderately stable, but the θ parameter fluctuates from sample to sample. For example, the estimates of θ based on the first half of every year's data in Tables 4A, 5A and 6A are 0.2319, 0.9197 and -1,0296, respectively. The estimate of the same parameter for the period January 2012 -December 2013 calibration is 0.7082 (Table 7A), which turn out to be -0.0603 for the period January 2012-December 2013 calibration (Table 8A). In the NIG model, both α and β parameters fluctuate across different samples, but the δ parameter is more stable. In the JD-DE model, with the exception of the one-day calibration, the σ parameter is stable across the different samples; the λ and p parameters become more stable as sample size increases, but the η_1 and η_2 fluctuate irrespective of sample size. The estimate of p based on the first half of every year's data ranges from 0.7093 to 0.9998, while for calibrations based on two-year and three-year periods the estimates remain close to 0.4500. In the CGMY model, all four parameters fluctuate significantly across the different samples. In the HS, HN(R) and HN(U) models, all the parameters vary significantly from one sample to another. We find similar fluctuations in parameters across samples for both the SC and BT models. In the CH model, with the exception of the η parameter, all the other parameters fluctuate across different samples. Finally, for our proposed GARCH-NIG model, with the exception of the α parameter, all parameters fluctuate across the different samples.

---

[10] In implementing sophisticated alternatives to benchmark BS model we follow the exact procedure as described and followed by the corresponding authors. While calibrating Scotts (1997), Bates (1996) and other models as initial guess in optimization we used the estimates proposed by the corresponding authors. For filtering out the volatility in Heston and Nandi (2000), and Christoffersen's GARCH-IG (2006) models we follow the similar methodologies as proposed by the authors.

In short, with the exception of the BS-implied volatility, all the other models yield parameters that can vary over time and across samples.

Turning now to the RMSE we find that, overall, the BS model provides the worst fit. The RMSE ranges from 1.92 to 1.31, from 2.97 to 2.66, and from 3.72 to 3.65 for calibrations over the sample period of the first six months in 2012 (Table 4A), in 2013 (Table 5A), and in 2014 (Table 6A), respectively, for all models. With the exception of the first half of the 2012 calibration, our proposed GARCH-NIG (CFGNIF) provides the best in-sample fit – RMSE is 1.74 for the first half of 2012, and 2.28 for the first half of 2013. Furthermore, as the sample size increases, the performance improves relative to the performance of other models The RMSEs for the GARCH-NIG model are 1.70 and 2.38 for two-year in-sample calibrations of 2012-2013 and 2013-2014 compared to the RMSEs in the ranges of 2.37-2.20 and 5.68-3.20 for the same sample calibration. For the three-year calibration in Table 9A, the RMSE for our model is 2.26. In sum, our proposed model generally outperforms existing models based on in-sample RMSE fit.

6.2. Out-of-sample Results

Tables 4B, 5B, 6B, 7B, 8B and 9B provide out-of-sample fits for options with three ranges of time-to-maturity: options with maturity < 40 days, options with maturity greater than or equal to 40 days but less than 70 days, and options with maturity greater than or equal to 70 days but less than 100 days. We also report results across different levels of moneyness: out-of-the-money ($0.95 \leq S/K < 0.99$); at-the-money ($0.99 \leq S/K < 1.01$) and in-the-money ($1.01 \leq S/K < 1.05$).

The findings show that our proposed ad-hoc analytic GARCH-NIG model outperforms the other models based on the out-of-sample RMSE criterion in five out of six samples. Only in the sample based on the calibration performed during the first half of 2013 is the GARCH-NIG model beaten and even then only in a single instance. Second, by the other two measures, AAE and MOE, the GARCH-NIG model always outperforms the other models. Third, the goodness-of-fit results achieved using BS, HS, HN(R), and HN(U), SC, BT, CH are of similar order of magnitude amongst the different sample sizes. This would imply, for example, that when it comes to out-of-sample fit, a stochastic volatility model with multiple parameters is not guaranteed to have significant advantage over the single-parameter BS model. Fourth, as the calibration period lengthens, the performance of our proposed model improves relative to the performance of other models: this finding is noticeable in Tables 8B and 9B.

In sum, our proposed ad-hoc analytic GARCH-NIG model outperforms many existing models by a variety of criteria across different sample sizes and time periods. This is due to the unique feature of our model where we are able to incorporate evolving conditional skewness and kurtosis. We are thus able to incorporate the observed empirical characteristics of the asymmetry of the risk-neutral distribution, volatility clustering and fat tails.

7. CONCLUSION

This paper addresses the interface of the GARCH and Lévy bodies of literature and proposes an alternative approach. Our aim is to provide the fast, yet convincing treatment of option pricing under a special-case Lévy process – the Normal Inverse Gaussian (NIG). The approach provides a mathematical framework with a nonlinear volatility dynamic approximated by replacing standard NIG innovations with standard normal innovations, and demonstrates how ad-hoc analytic solutions can be derived in the presence of both positive and negative Lévy innovations. Given that the GARCH-NIG model has more flexibility to describe the conditional evolution of skewness and the conditional evolution of kurtosis, we might expect it to accommodate cross-strike and cross-maturity features better than what other models do. For example, skewness and kurtosis in the Heston-Nandi (2000) model are determined (and hence constrained) by GARCH structural parameters, whereas the GARCH-NIG models skewness and kurtosis in a more flexible time-varying fashion based on non-normal rather than normal innovations. Similarly, the continuous-time SVJ or SVJJ models are limited by their Markovian structure, whereas the GARCH-NIG model is not.

Two natural extensions lend themselves to further investigation for further work. The first and most obvious is to undertake comparable analyses of other Lévy processes – the Variance-Gamma, CGMY, and so forth – and then to compare the performance of these different processes. A second extension, first identified by Bates (2003), is to further investigate and model the extent to which cross-sectional option-pricing patterns could be made quantitatively consistent with the time-series patterns of the underlying asset price. Ideally, the risk-neutral characteristic functions, which are often used to reveal cross-sectional option-pricing patterns, could be used to reveal the time-series properties of the underlying asset as well, and so ensure consistency between the two. Currently, however, this remains a challenge. Part of the problem would appear to be that instantaneous option-price evolution in standard models is not fully captured by underlying asset price movements. Part of the problem might also relate to the fact that the heteroscedasticity of GARCH conflicts with the stationary Markov time-series properties of standard models. Since the GARCH-NIG is free of both of these limitations, we would speculate that the greater flexibility of the GARCH-NIG may hold the key to achieving this objective.

We use daily records of options written on the S&P500 index traded on the Chicago Board Options Exchange (CBOE). We retrieve the S&P500 index put-and-call option quotes from Thompson Reuter Tick history. The sample period runs from January 2012 through to December 2014. For calibration and out-of-sample assessment we apply data up until June 2015. When assessing out-of-sample performance we restrict our reported results to models that explicitly incorporate stochastic volatility. We find that models that incorporate jumps provide excellent fits over very short periods (e.g., one day or a couple of days' observations); but such models perform much worse for long periods relative to models which consider explicit stochastic volatility dynamics. Consequently we restrict our out-of-sample performance assessment to *seven* competing models: Heston's (1993) continuous-time stochastic volatility model, restricted and unrestricted versions of Heston and Nandi's discrete-time GARCH volatility model with normal innovations (Heston and Nandi, 2000), Scott's (1997) model, Bates' (1996) model, Christoffersen *et al.'s* (2006) model, and our novel CFGNIG model.

Our proposed ad-hoc analytic GARCH-NIG model outperforms many existing models by a variety of criteria across different sample sizes and time periods. This is due to the unique feature of our model where we are able to incorporate evolving conditional skewness and kurtosis. We are thus able to incorporate observed empirical characteristics of the asymmetry of the risk-neutral distribution, volatility clustering, and fat tails.

References

Badescu, A., & Kulperger, R.J. (2008). GARCH option pricing: A semiparametric approach. Insurance: Mathematics and Economics, 43, 69-84.

Badescu, A., Elliott, R.J., & Oretga, J.P. (2015). Non-Gaussian GARCH option pricing models and their diffusion limits. European Journal of Operational Research, 247, 820-830.

Barone-Adesi, G., Engle, R., & Mancini, L. (2008). A GARCH option pricing model with filtered historical simulation. Review of Financial Studies, 21, 1223-1258

Bates, D.S. (2003). Empirical option pricing: A Retrospection. Journal of Econometrics, 116, 387-404.

Bates, D.S. (1996). Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche Mark options. Review of Financial Studies, 9, 69-107.

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. Journal of Political Economy, 81, 637-659.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroeskedasticity. Journal of Econometrics, 31, 307-327.

Carr, P., Geman, H., Madan, D. B., & Yor, M. (2002). The affine structure of asset returns: an empirical investigation. Journal of Business, 75, 305-332.

Carr, P., & Wu L. (2004). Time-changed Lévy processes and option pricing. Journal of Financial Economics, 71, 113-141.

Christoffersen, P., Heston, S.L., & Jacobs, C. (2006). Option valuation with conditional skewness. Journal of Econometrics, 131, 253-284.

Christoffersen, P., Dorion, C., Jacobs, P., & Wang, Y. (2010). Volatility Components, Affine Restrictions, and Nonnormal Innovations. Journal of Business & Economic Statistics, 28(4), 483-502.

Christoffersen, P., Elkamhi, R., Feunou, B., & Jacobs, K. (2010). Option valuation with conditional heteroskedasticity and nonnormality. Review of Financial Studies, 23(5), 2139-2183.

Christoffersen, P., Jacobsc, K., & Ornthanalaia, C. (2012). Dynamic jump intensities and risk premiums: Evidence from S&P500 returns and options. Journal of Financial Economics, 106(3), 447–472.

Dingec, K D., & Hormann, W. (2012). A general control variate method for option pricing under Lévy processes. European Journal of Operational Research, 221, 368-377.

Duan, J.C. (1995). The GARCH option pricing model. Mathematical Finance, 5, 13-32.

Dumas, B., Fleming, J., & Whaley, R. (1998). Implied volatility functions: empirical tests. Journal of Finance, 53, 2059–2106.

Engle, R. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of UK inflations. Econometrica, 50, 987-1008.

Geman, H., Madan, D., & Yor, M. (2001). Time changes for Lévy processes. Mathematical Finance, 11, 79-96.

Geman, H. (2002). Pure Jump Lévy processes for asset price modelling. Journal of Banking & Finance.26, 1297-1316.

Gerber, H.U., & Shiu, E.S.W. (1994). Option pricing by Esscher Transforms. Transactions of the Society of Actuaries, 46, 99-191.

Heston, S. (1993). A closed form solution for options with stochastic volatility with applications to bond and currency options. Review of Financial Studies, 6, 327-343.

Heston, S.L., & Nandi, S. (2000). A closed form GARCH option valuation model. Review of Financial Studies, 13, 585-625.

Huang, J.Z., & Wu, L. (2004). Specification analysis of option pricing models based on time-changed Lévy processes. Journal of Finance, 59, 1405-1439.

Hull, J., & White, A. (1987). The pricing of options on assets with stochastic volatilities. Journal of Finance, 42, 281-300.
Kim, K.S., Rachev, S.T., Bianchi, M.L., & Fabozzi, F.J. (2008). Financial market models with Lévy processes and time-varying volatility. Journal of Banking and Finance, 32, 1363-1378.
Kim, K.S., Rachev, S.T., Bianchi, M.L., & Fabozzi, F.J. (2010). Tempered stable and tempered infinitely divisible GARCH models. Journal of Banking and Finance, 34, 2096-2109.
Kou, S. (2002). A jump diffusion model for option pricing. Management Science, 48, 1086-1101.
Madan, D., Carra, P., & Chang, E. (1998). The Variance Gamma process and option pricing model. European Finance Review, 2, 79-105.
Mercuri, L. (2008). Option pricing in a GARCH model with tempered stable innovations. Finance Research Letters, 5, 172-182.
Merton, R. (1976). Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics, 3, 125-144.
Moolman, G.P. (2008). Option pricing: A GARCH model with Lévy innovations. University of Johannesburg. Dissertation.
Ornthanalai, C. (2010). A new class of asset pricing models with Lévy processes: theory and applications. Working Paper, Georgia Institute of Technology.
Ornthanalai, C. (2014). Lévy jump risk: Evidence from options and returns. Journal of Financial Economics, 112, 69-90.
Psychoyios, D., Dotsis, G., & Markellos, R.N. (2010). A jump diffusion model for VIX volatility options and futures. Review of Quantitative Finance and Accounting, 35, 245-269.
Schouten, W. (2003). Lévy Processes in Finance: Pricing Financial Derivatives. John Wiley & sons. Ltd.
Scott, L.O. (1997). Pricing stock options in a jump-diffusion model with stochastic volatility and interest rates: Application of Fourier inversion methods. Mathematical Finance, 7, 413-426.
Shiu, T.K., Tong, H., & Yang, H. (2004). On pricing derivatives under GARCH models: A dynamic Gerber-Shiu approach. North American Actuarial Journal, 8, 17-31.

Table 1
Jump Models and Risk-neutral Characteristic Functions

	Model	Characteristic Functions
VG	Variance Gamma (Madan *et al.*, 1998)	$\Phi_{X_t}^{VG}(s) \quad = \exp\left\{ i\left[r + \frac{1}{\gamma}\ln\left(1 - \theta\gamma - \frac{1}{2}\sigma^2\gamma\right)\right] st - \frac{t}{\gamma}\ln\left(1 - is\theta\gamma + \frac{1}{2}s^2\sigma^2\gamma\right)\right\}$
NIG	Normal Inverse Gaussian model (Schouten, 2003)	$\Phi_{X_t}^{NIG}(s) = \exp\left\{ i \begin{pmatrix} r + \delta\left(\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2}\right) st \\ -\delta\left(\sqrt{\alpha^2 - (\beta + is)^2} - \sqrt{\alpha^2 - \beta^2}\right) \end{pmatrix} \right\}$
JD-DE	Exponential double jump model (Kou, 2002)	$\Phi_{X_t}^{JD-DE}(s) = \exp\left\{ \begin{matrix} i\left(r - \frac{1}{2}b^2 - \lambda\left[\frac{p\eta_1}{\eta_1 + 1} + \frac{(1-p)\eta_2}{\eta_2 + 1} - 1\right]\right) st \\ -\frac{1}{2}b^2 s^2 t + is\lambda t\left[\frac{p\eta_1}{\eta_1 + is} + \frac{(1-p)\eta_2}{\eta_2 + is} - 1\right] \end{matrix} \right\}$
CGMY	Carr *et al.* (2002)	$\Phi_{X_t}^{CGMY}(s) = \exp\left\{ \begin{matrix} i\left(r - C\Gamma(-Y)\left((M-1)^Y - M^Y + (G+1)^Y + G^Y\right)\right) st \\ +Ct\Gamma(-Y)\left((M - is)^Y - M^Y + (G + is)^Y + G^Y\right) \end{matrix} \right\}$
HS	SV model (Heston, 1993)	$f_j = \exp\{C_j + D_j\upsilon + izx\}$ for j = 1, 2 $x = x_t = \log(S_t); \upsilon = \upsilon_t$ $C_j = irz(T-t) + \frac{k\upsilon}{\xi^2}\left\{(b_j - i\rho\xi z + d_j)(T-t) - 2\ln\left[\frac{1 - g_j e^{d_t(T-t)}}{1 - g_j}\right]\right\}$ $D_j = \frac{b_j - iz\rho\xi + d_j}{\xi^2}\left[\frac{1 - e^{d_t(T-t)}}{1 - g_j e^{d_t(T-t)}}\right]; \quad g_j = \frac{b_j - iz\rho\xi + d_j}{b_j - iz\rho\xi - d_j}; \quad d_j = \sqrt{(iz\rho\xi + b_j)^2 - (2iu_j z - z}$ $u_1 = \frac{1}{2}; \quad u_2 = -\frac{1}{2}; \quad b_1 = \kappa + \lambda - \rho\xi; \quad b_1 = \kappa + \lambda$
HN(R)	Restricted Heston and Nandi (2000) GARCH	$f(z) = S_t^z \exp\{A(t;t+T,z) + B(t;t+T,z)\sigma_{t+1}^2\}$, where $A(t;t+T,z)$ and $B(t;t+T,z)$ are given by the recursive relations: $A(t;t+T,z) = A(t+1;t+T,z) + zr + B(t+1;t+T,z)\varpi - \frac{1}{2}\ln(1 - 2\alpha B(t+1;t+T,z))$ $B(t;t+T,z) = z(\lambda + \theta) - \frac{1}{2}\theta^2 + \beta B(t+1;t+T,z) + \frac{\frac{1}{2}(z-\theta)^2}{1 - \alpha B(t+1;t+T,z)}$ The risk neutral version is obtained by plugging in $\lambda = -\frac{1}{2}$ and replacing θ with $\theta^* = \theta + \lambda + \frac{1}{2}$

HN(U)	Unrestricted Heston and Nandi (2000) GARCH	$f(z) = S_t^z \exp\left\{A(t;t+T,z) + B(t;t+T,z)\sigma_{t+1}^2\right\}$, where $A(t;t+T,z)$ and $B(t;t+T,z)$ are given by the recursive relations: $A(t;t+T,z) = A(t+1;t+T,z) + zr + B(t+1;t+T,z)\varpi - \frac{1}{2}\ln\left(1 - 2\alpha B(t+1;t+T,z)\right)$ $B(t;t+T,z) = z(\lambda+\theta) - \frac{1}{2}\theta^2 + \beta B(t+1;t+T,z) + \frac{\frac{1}{2}(z-\theta)^2}{1-\alpha B(t+1;t+T,z)}$ The risk neutral version is obtained by plugging in $\lambda = -\frac{1}{2}$ and replacing θ with $\theta^* = \theta + \lambda + \frac{1}{2}$
SC	Jump-diffusion with Stochastic volatility and Stochastic interest rate model (Scott, 1997)	$\psi\left(z \mid y_1(0), y_2(0)\right) = E\left(e^{-R_T + z\ln S(T)}\right) = E\left(e^{z\ln J(T)}\right)E\left(e^{-R_T + z\ln S^c(T)}\right)$ $E\left(e^{z\ln J(T)}\right) = \exp\left\{\lambda T\left(e^{\mu_J z + (1/2)\sigma_J^2 z^2} - 1\right) - \lambda Tz\left(e^{\mu_J + (1/2)\sigma_J^2} - 1\right)\right\}$ $E\left(\exp\left\{(z-1)R_T + z\xi_T + z\eta_T\right\}\right)$ $= E\left(\exp\left\{(z-1)R_T + (z-1)z\frac{1}{2}\sigma^2\left(1-\rho^2\right)Y_1(T) + z\eta_T\right\}\right)$ $= E\left(e^{(z-1)Y_2(T)}\right)E\left(\exp\left\{\omega Y_1(T) + z\frac{\rho\sigma}{\sigma_1}\left[y_1(0) - \kappa_1\theta_1 T\right]\right\}\right)$, where $\omega = (z-1) + (z-1)z\frac{1}{2}\sigma^2\left(1-\sigma^2\right) + z\left(\frac{\rho\sigma}{\sigma_1}\kappa_1 - \frac{1}{2}\rho^2\sigma^2\right)$ For any complex s_1 and s_2 with nonnegative real parts $E\left(e^{-s_1 Y_j(T) - s_2 y_j(T)} \mid y_j(0) = y_j\right) = \exp\left\{a_j(T) - b_j(T)y_j\right\}$ For j=1,2, with $a_j(T) = \frac{2\kappa_j\theta_j}{\sigma_j^2}\ln\left[\frac{2\gamma_j e^{(1/2)\left[\kappa_j - \gamma_j\right]T}}{2\gamma_j e^{-\gamma_j T} + \left[\kappa_j + \gamma_j + \sigma_j^2 s_2\right]\left(1 - e^{-\gamma_j T}\right)}\right]$ $b_j(T) = \frac{\left(1-e^{-\gamma_j T}\right)\left[2s_1 - \kappa_j s_2\right] + \gamma_j s_2\left(1 + e^{-\gamma_j T}\right)}{2\gamma_j e^{-\gamma_j T} + \left[\kappa_j + \gamma_j + \sigma_j^2 s_2\right]\left(1 - e^{-\gamma_j T}\right)}$ with $\gamma_j = \sqrt{\kappa_j^2 + 2\sigma_j^2 s_1}$
BT	Jump with SV model (Bates, 1996)	$f_j(\Phi \mid S,V,T) \equiv E\left[e^{\Phi\ln(S_T/S_0)} \mid P_j\right]$ for j=1,2 $= \exp\left\{C_j(T;\Phi) + D_j(T;\Phi)V + \lambda^* T\left(1+\overline{k^*}\right)^{\mu_j + 1/2} \times \left[\left(1+\overline{k^*}\right)^{\Phi} e^{\delta^2\left(u_j\Phi + \phi^2/2\right)} - 1\right]\right\}$ $C_j(T;\Phi) = \left(b - \lambda^*\overline{k^*}\right)\Phi T - \frac{\alpha T}{\sigma_v^2}\left(\rho\sigma_v\Phi - \beta_j - \gamma_j\right) - \frac{2\alpha}{\sigma_v^2}\ln\left[1 + \frac{1}{2}\left(\rho\sigma_v\Phi - \beta_j - \gamma_j\right)\frac{1-e^{\gamma_j T}}{\gamma_j}\right]$ $D_j(T;\Phi) = -2\frac{\mu_j\Phi + \frac{1}{2}\Phi^2}{\rho\sigma_v\Phi - \beta_j - \gamma_j\frac{1+e^{\gamma_j T}}{1-e^{\gamma_j T}}}; \quad \gamma_j = \sqrt{\left(\rho\sigma_v\Phi - \beta_j\right)^2 - 2\sigma_v^2\left(\mu_j\Phi + \frac{1}{2}\Phi^2\right)}$ $\mu_1 = +\frac{1}{2}, \ \mu_2 = -\frac{1}{2}, \ \beta_1 = \beta^* - \rho\sigma_v, \ and \ \beta_2 = \beta^*$

CH	IG-GARCH model (Christoffersen *et al.*, 2006)	$f(z) = S_t^z \exp\{A(t;t+T,z) + B(t;t+T,z)\sigma_{t+1}^2\}$ $A(t) = A(t+\Delta) + \phi r\Delta + wB(t+\Delta) - \frac{1}{2}\ln\left(1-2a\eta^4 B(t+\Delta)\right)$ $B(t) = bB(t+\Delta) + \phi v + \eta^{-2} - \eta^{-2}\sqrt{\left(1-2a\eta^4 B(t+\Delta)\right)\left(1-2cB(t+\Delta)-2\eta\phi\right)}$ The risk-neutral version corresponds to Christoffersen et al. (2006): $v = \left(\frac{1}{\eta^2}\right)\left(\sqrt{(1-2\eta)}-1\right)$

Table 2
S&P 500 Index Options data, 2012-2014

Panel A: Number of option contracts				
		6≤ Maturity <40	40≤Maturity<70	70≤Maturity≤ 100
Out-of-the-money	0.90 ≤ *S/K*< 0.99	698	996	967
At-the-money	0.99 ≤ *S/K*< 1.01	879	1,232	1,023
In-the-money	1.01 ≤ *S/K*< 1.10	654	989	966
All		2,231	3,217	2,956
Panel B: Average option prices				
		6≤ Maturity <40	40≤Maturity<70	70≤Maturity≤ 100
Out-of-the-money	0.90 ≤ *S/K*< 0.99	8.85	15.13	27.82
At-the-money	0.99 ≤ *S/K*< 1.01	16.63	28.89	46.98
In-the-money	1.01 ≤ *S/K*< 1.10	12.67	24.73	44.11
Panel C: Average implied volatility				
		6≤ Maturity <40	40≤Maturity<70	70≤Maturity≤ 100
Out-of-the-money	0.90 ≤ *S/K*< 0.99	0.18	0.17	0.16
At-the-money	0.99 ≤ *S/K*< 1.01	0.23	0.22	0.23
In-the-money	1.01 ≤ *S/K*< 1.10	0.28	0.26	0.25

Note: We use European put-and-call options on the S&P500 index. The prices are taken from nonzero trading volume quotes on each Wednesday during the January, 2012 to December, 2014 period. We clean our data using the same rules applied by Heston and Nandi (2000). The implied volatilities are calculated using the Black-Scholes formula.

Table 3
Pilot calibration with S&P500 index options

Model	RMSE	Parameters							
		(σ)							
BS	13.59	0.5172							
		(0.0048)							
		(σ)	(θ)	(v)					
VG	12.74	0.0899	-3.2635	0.0279					
		(0.0008)	(0.0009)	(0.0006)					
		(α)	(β)	(δ)					
NIG	12.85	418.3021	-409.7441	1.0347					
		(0.0251)	(0.0250)	(0.0180)					
		(σ)	(λ)	(p)	(η_1)	(η_2)			
JD-DE	12.61	3.1230e-04	33.4824	0.4741	14.8251	14.6700			
		(0.0577)	(0.2407)	(0.2540)	(0.2047)	(0.1888)			
		(C)	(G)	(M)	(Y)				
CGMY	12.21	1.7432e8	3.5792e1	9.6529e5	-6.0337				
		(0.0890)	(0.0870)	(0.0880)	(0.0060)				
		(κ)	(θ)	(σ)	(ρ)	(V_0)			
HS	11.07	11.069	1.1198	0.6980	-0.9900	0.2992			
		(0.0200)	(0.0486)	(0.0460)	(0.0471)	(0.0058)			
		(α_1)	(β_1)	(β_0)	(γ)	(λ)			
HN(R)	7.37	0.5876	4.150e-6	9.318e-6	299.993	-300.493			
		(0.151e-23)	(0.082e-23)	(0.152e-23)	(0.181e-23)	(0.292e-23)			
		(α_1)	(β_1)	(β_0)	(γ)	(λ)			
HN(U)	7.26	0.957	2.567e-9	2.567e-9	3.6154	7.3960			
		(0.001)	(1.404e-7)	(1.395e-7)	(0.0040)	(0.005)			
		(κ_1)	(κ_2)	(θ_1)	(θ_2)	(σ_1)	(σ_2)	(σ)	
SC	7.08	24.8511	0.3145	0.0231	1.0517e-4	0.0448	5.0000	1.1052	
		(2.4210)	(2.3783)	(0.0021)	(0.8692)	(0.1000)	(0.3645)	(0.0235)	
		(ρ)	(μ_j)	(σ_j)	(λ_j)				
		-0.0742	-7.1904	0.2390	1.4867				
		(0.0121)	(0.8222)	(0.0463)	(1.8719)				
BT	11.01	(V_0)	(V_{bar})	(α)	(β)	(ρ)	(λ)	(μ_λ)	(σ_λ)
		0.2634	0.4450	3.2173	0.9999	-0.9999	2.4704e-14	-0.3000	2.1930e-6
		(2.4717e-4)	(1.2740e-3)	(0.1720)	(0.0096)	(0.0675)	(0.0196)	(0.0304)	(0.0585)
		(ω)	(b)	(a)	(c)	(η)			
CH	11.18	1.2382-10	0.0095	2.5751e2	4.6612e-5	-0.0066			
		(3.6639e-6)	(0.0086)	(3.2579e2)	(1.4100e-6)	(7.5544e-5)			
		(α_1)	(β_1)	(β_0)	(γ)	(α)	(β)	(δ)	
CFGNIG	7.04	0.9772	2.56e-9	2.56e-9	-4.3189	32.0734	30.3958	21.4260	
		(8.579e-4)	(7.804e-7)	(1.124e-6)	(0.0348)	(0.0249)	(0.0191)	(0.0194)	

Notes: Calibration is carried out by applying the FRFT approach to price options. We consider options traded on July1, 2015. After cleaning the data, we have 78 records to consider on that particular day. The models in column (1) are defined in Table 1 with their risk-neutral Characteristic Functions.

Table 4A: Calibration with options traded over the period January 2012-June 2012

Model	RMSE	Parameters							
		(σ)							
BS	2.00	0.0678							
		(0.0059)							
		(σ)	(θ)	(v)					
VG	1.81	0.0144	0.2319	0.0828					
		(0.0148)	(0.0132)	(0.0119)					
		(α)	(β)	(δ)					
NIG	1.82	1621.6	1584.1	0.0743					
		(0.0178)	(0.0177)	(0.0130)					
		(σ)	(λ)	(p)	(η_1)	(η_2)			
JD-DE	1.90	0.0678	0.0813	0.9998	186.1660	215.3270			
		(0.0063)	(0.0190)	(0.0166)	(0.0178)	(0.0702)			
		(C)	(G)	(M)	(Y)				
CGMY	1.81	43.5396	558.8773	57.5013	-0.2960				
		(0.7759)	(0.7200)	(0.7389)	(0.0493)				
		(κ)	(θ)	(σ)	(ρ)	(V_0)			
HS	1.84	0.1247	0.0427	0.1032	0.9900	0.0041			
		(0.0897)	(0.0482)	(0.0823)	(0.1212)	(0.0010)			
		(α_1)	(β_1)	(β_0)	(γ)	(λ)			
HN(R)	1.95	2.2204e-6	2.5670e-9	1.8374e-5	419.0836	-419.5836			
		(2.5719e-6)	(5.2326e-7)	(1.9018e-6)	(1.0037e-5)	(1.0037e-5)			
		(α_1)	(β_1)	(β_0)	(γ)	(λ)			
HN(U)	1.43	0.4583	2.5670e-9	1.4438e-5	419.1233	-3.5116			
		(0.0058)	(5.0629e-7)	(2.9606e-6)	(0.0111)	(0.0071)			
		(κ_1)	(κ_2)	(θ_1)	(θ_2)	(σ_1)	(σ_2)	(σ)	
SC	1.31	4.070	36.6468	0.0520	0.0506	0.7175	4.1559	1.0870	
		(0.6871)	(3.2024)	(0.1886)	(0.0421)	(0.1240)	(0.3303)	(0.0197)	
		(ρ)	(μ_j)	(σ_j)	(λ_j)				
		-0.0448	0.0116	0.0390	2.0905				
		(0.0108)	(8.1800e-3)	(3.4200e-3)	(0.4393)				
		(V_0)	(V_{bar})	(α)	(β)	(ρ)	(λ)	(μ_λ)	(σ_λ)
BT	1.41	0.0028	0.0045	1.2608	2.6029	-0.4517	0.3748	0.0702	2.7452e-4
		(0.0004)	(0.0004)	(0.0500)	(0.0113)	(0.0603)	(0.0341)	(0.0070)	(0.0049)
		(ω)	(b)	(a)	(c)	(η)			
CH	1.92	6.0511e-9	0.1247	4.4807e4	2.3749e-6	-0.0031			
		(3.1309e-7)	(0.0032)	(1.7118e4)	5.1860e-8	(1.0022e-4)			
		(α_1)	(β_1)	(β_0)	(γ)	(α)	(β)	(δ)	
CFGNIG	1.54	0.9937	2.5670e-15	2.5670e-15	-0.2690	98.6165	-67.0771	39.8413	
		(0.0059)	(1.1929e-7)	1.2268e-7	(5.5020)	(0.0079)	(0.0146)	(0.0677)	

Notes: We consider options traded on every Wednesday. After cleaning the data, we have 1,179 option contracts. We applied the FRFT approach to price options which significantly reduces the calibration time. The models in column (1) are defined in Table 1 with their risk-neutral Characteristic Functions.

Table 4B

Out-of-sample valuation errors for call options traded in the second half of 2012

	RMSE	AAE	MOE	RMSE	AAE	MOE	RMSE	AAE	MOE
	Days to maturity < 40			40 ≤ Days to maturity < 70			70 ≤ Days to maturity ≤ 100		
0.95<=(*S*/*K*)<0.99									
BS	1.041	0.675	-0.305	2.002	1.500	-0.828	2.283	1.878	-1.123
HS	1.237	0.941	-0.097	1.734	1.328	-0.151	1.962	1.626	-0.324
HN(R)	1.272	0.978	-0.117	1.812	1.371	-0.469	2.063	1.688	-0.882
HN(U)	1.389	1.059	0.375	1.759	1.384	0.137	1.997	1.642	-0.624
SC	1.234	0.919	0.044	1.717	1.313	-0.139	1.855	1.528	-0.362
BT	1.229	0.920	0.064	1.722	1.326	-0.098	1.821	1.490	-0.379
CH	1.280	1.010	-0.254	1.819	1.382	-0.504	2.016	1.641	-0.831
CFGNIG	1.033	0.815	0.338	1.026	0.752	0.308	2.470	2.178	-1.504
0.99<=(*S*/*K*)<1.01									
BS	2.196	1.786	-0.575	2.377	1.961	-0.654	2.078	1.721	-0.583
HS	2.393	1.833	0.661	2.356	1.929	-0.639	2.380	2.077	-1.012
HN(R)	2.433	1.837	0.826	2.223	1.814	-0.261	1.947	1.579	-0.440
HN(U)	2.535	1.933	0.970	2.329	1.908	-0.510	2.804	2.467	-1.506
SC	2.439	1.863	0.745	2.265	1.853	-0.427	2.126	1.843	-0.756
BT	2.394	1.841	0.646	2.307	1.891	-0.532	2.049	1.725	-0.623
CH	2.348	1.778	0.701	2.218	1.806	-0.256	1.890	1.495	-0.366
CFGNIG	2.026	1.389	0.758	2.045	1.828	-1.058	3.342	3.193	-2.226
1.01<=(*S*/*K*)<1.05									
BS	1.816	1.439	0.187	2.450	1.938	0.681	2.398	1.704	0.714
HS	1.786	1.469	-0.229	2.174	1.743	-0.212	2.260	1.918	-0.414
HN(R)	1.738	1.394	-0.065	2.273	1.819	0.450	2.310	1.673	0.591
HN(U)	2.106	1.823	0.616	2.707	2.240	1.119	3.867	3.581	-2.547
SC	2.054	1.771	-0.574	2.260	1.866	-0.520	2.845	2.564	-1.407
BT	2.283	1.989	-0.845	2.274	1.864	-0.576	2.217	1.891	-0.549
CH	1.751	1.414	-0.108	2.305	1.841	0.509	2.390	1.693	0.696
CFGNIG	1.568	1.343	-0.308	1.828	1.555	0.520	2.366	2.143	-0.819

Notes: The models are calibrated on first half of the same year. Total number of contracts available for the second half is 1456. RMSE is the root mean square, AAE is the average absolute error and MOE is the mean outside error. The models in column (1) are defined in Table 1 with their risk-neutral Characteristic Functions.

Table 5A: Calibration with options traded over the period January 2013-June 2013

Model	RMSE	Parameters							
		(σ)							
BS	2.97	0.0750							
		(0.0050)							
		(σ)	(θ)	(ν)					
VG	2.91	0.0410	0.9197	0.0046					
		(0.0012)	(0.0084)	(0.0009)					
		(α)	(β)	(δ)					
NIG	2.96	96.1269	28.6945	0.4713					
		(0.0893)	(0.1018)	(0.0629)					
		(σ)	(λ)	(p)	(η_1)	(η_2)			
JD-DE	2.97	0.0750	0.0782 (17.2433)	0.9998 (28.8536)	202.6066	293.0997			
		(0.0050)			(18.3369)	(28.4823)			
		(C)	(G)	(M)	(Y)				
CGMY	2.96	80.6780	1879.1 (0.3425)	182.2822 (0.3419)	0.1689				
		(0.6236)			(0.0272)				
		(κ)	(θ)	(σ)	(ρ)	(V_0)			
HS	2.93	0.3451	0.0399 (0.0181)	0.0255	0.9900 (0.1040)	0.0042			
		(0.0808)		(0.0794)		(0.0009)			
		(α_1)	(β_1)	(β_0)	(γ)	(λ)			
HN(R)	2.98	2.2204e-16	2.5670e-9	2.2606e-5	419.1042	-419.6042			
		(1.6569e-5)	(3.0063e-6)	(5.81165e-6)	(1.4440e-4)	(2.0761e-4)			
		(α_1)	(β_1)	(β_0)	(γ)	(λ)			
HN(U)	2.94	0.6476	2.5670e-9	8.7016e-6	416.6406	-1.3804			
		(0.0082)	(2.4114e-7)	(1.3053e-6)	(0.0073)	(0.0078)			
		(κ_1)	(κ_2)	(θ_1)	(θ_2)	(σ_1)	(σ_2)	(σ)	
SC	2.86	4.9567	0.0440	4.2713e-14	1.8288	0.9412	0.6595	1.0887	
		(2.1148)	(1.1799)	(7.1582e-3)	(0.3009)	(0.1573)	(0.2262)	(0.0543)	
		(ρ)	(μ_j)	(σ_j)	(λ_j)				
		-0.0352	-0.0179	0.0523	2.8238				
		(0.0339)	(0.0418)	(0.0330)	(0.4691)				
		(V_0)	(V_{bar})	(α)	(β)	(ρ)	(λ)	(μ_λ)	(σ_λ)
BT	2.86	0.0039	0.0122	1.5908	0.0363	-1.0000	0.0318	0.1592	6.6391e-6
		(0.0008)	(0.0007)	(0.0554)	(0.0439)	(0.0520)	(0.0375)	(0.0265)	(0.0100)
		(ω)	(b)	(a)	(c)	(η)			
CH	2.66	2.2646e-9	0.1842	2.3667e4	4.7407e-6	-0.0030			
		(3.5311e-7)	(0.0048)	(1.4497e4)	(3.7593e-3)	(0.0002)			
		(α_1)	(β_1)	(β_0)	(γ)	(α)	(β)	(δ)	
CFGNIG	1.74	0.9999	2.567e-15	02.567e-15	-0.5123	42.6940	2.4246	28.4220	
		(0.0039)	(1.0508e-7)	(1.0505e-7)	(1.2015))	(0.5236)	(0.7626)	(0.2833)	

Notes: We consider options traded on every Wednesday. After cleaning we have 1607 option contracts. We applied the FRFT approach to price options which significantly reduces the calibration time. The models in column (1) are defined in Table 1 with their risk-neutral Characteristic Functions.

Table 5B

Out-of-sample valuation errors for call options traded in the second half of 2013

	RMSE	AAE	MOE	RMSE	AAE	MOE	RMSE	AAE	MOE
	Days to maturity < 40			40 ≤ Days to maturity < 70			70 ≤ Days to maturity ≤ 100		
0.95<=(*S*/*K*)<0.99									
BS	1.390	0.994	0.371	1.862	1.576	0.429	2.126	1.837	0.226
HS	1.762	1.395	0.284	2.013	1.738	0.728	2.828	2.579	1.411
HN(R)	1.930	1.518	0.626	2.002	1.714	0.756	2.211	1.972	0.456
HN(U)	2.106	1.656	0.944	2.194	1.877	1.059	2.294	2.048	0.629
SC	1.810	1.407	0.401	1.896	1.631	0.532	2.631	2.413	1.122
BT	1.770	1.393	0.348	1.981	1.711	0.692	2.695	2.473	1.252
CH	1.819	1.454	0.104	1.785	1.496	0.146	2.175	1.879	0.163
CFGNIG	1.419	1.039	0.282	0.629	0.463	-0.067	1.142	0.954	-0.344
0.99<=(*S*/*K*)<1.01									
BS	2.885	2.090	0.258	2.453	1.945	-0.489	2.092	1.513	-0.339
HS	3.445	2.560	1.223	2.442	1.977	-0.351	2.242	1.839	0.243
HN(R)	3.662	2.819	1.647	2.427	2.028	-0.163	2.067	1.533	-0.229
HN(U)	3.826	2.978	1.870	2.435	2.040	-0.149	2.105	1.538	-0.418
SC	3.568	2.681	1.429	2.426	2.000	-0.223	2.386	2.025	0.486
BT	3.443	2.567	1.234	2.436	1.999	-0.247	2.362	2.006	0.468
CH	3.567	2.767	1.579	2.501	2.199	0.263	2.496	2.193	0.696
CFGNIG	2.987	1.827	1.286	1.599	1.448	-0.615	1.940	1.867	-0.928
1.01<=(*S*/*K*)<1.05									
BS	1.932	1.410	-0.024	2.004	1.708	-0.049	2.048	1.735	-0.019
HS	2.216	1.625	-0.508	1.994	1.567	-0.380	1.941	1.645	0.069
HN(R)	2.159	1.572	-0.346	1.951	1.618	-0.218	1.994	1.661	-0.131
HN(U)	2.212	1.618	-0.379	2.029	1.608	-0.597	2.247	1.658	-0.837
SC	2.377	1.789	-0.627	1.997	1.582	-0.327	1.830	1.558	0.106
BT	2.457	1.877	-0.762	2.018	1.549	-0.420	1.878	1.603	0.143
CH	2.148	1.658	-0.074	2.602	2.344	2.602	3.207	2.783	1.753
CFGNIG	1.770	1.303	-0.252	1.442	1.211	-0.419	1.429	1.231	-0.245

Notes: The models are calibrated on first half of the same year. Total number of contracts available for the second half is 1,606. RMSE is the root mean square error, AAE is the average absolute error and MOE is the mean outside error. The models in column (1) are defined in Table 1 with their risk-neutral Characteristic Functions.

Table 6A: Calibration with options traded over the period January 2014 - June 2014

Model	RMSE	Parameters							
		(σ)							
BS	3.74	0.0775							
		(0.0044)							
		(σ)	(θ)	(ν)					
VG	3.72	0.0185	-1.0296 (0.0023)	0.0055					
		(0.0009)		(0.0007)					
		(α)	(β)	(δ)					
NIG	3.72	874.7253	-753.9378	0.6988					
		(0.1114)	(0.1092)	(0.0787)					
		(σ)	(λ)	(p)	(η_1)	(η_2)			
JD-DE	3.72	0.0623	9.7943 (3.0777)	0.7093 (0.9281)	97.5983	82.5284			
		(0.0035)			(17.9164)	(26.8777)			
		(C)	(G)	(M)	(Y)				
CGMY	3.72	97.3378	153.0980	620.8816 (0.0141)	0.0700				
		(0.0216)	(0.0128)		(0.0106)				
		(κ)	(θ)	(σ)	(ρ)	(V_0)			
HS	3.70	0.2971	0.0405 (0.0195)	0.0343	-0.9900	0.0049			
		(0.0716)		(0.0698)	(0.0800)	(0.0008)			
		(α_1)	(β_1)	(β_0)	(γ)	(λ)			
HN(R)	3.70	2.2204e-16	2.5670e-9	2.2606e-5 (1.361e-6)	419.1042	-419.6042			
		(0.0040)	(4.0221e-7)		(0.0020)	(0.0042)			
		(α_1)	(β_1)	(β_0)	(γ)	(λ)			
HN(U)	3.67	0.4992	2.0502e-6	1.8777e-6 (5.4676e-7)	419.2132	-1.4783			
		(0.0055)	(9.4270e-8)		(0.0054)	(0.0059)			
		(κ_1)	(κ_2)	(θ_1)	(θ_2)	(σ_1)	(σ_2)	(σ)	
SC	3.65	3.1931	0.0181	0.0157	3.5021	0.7294	0.6451	1.0918	
		(1.1683)	(1.1741)	(7.6600e-3)	(0.3924)	(0.2692)	(0.2158)	(0.0485)	
		(ρ)	(μ_j)	(σ_j)	(λ_j)				
		-0.0542	-0.0287	0.0663	1.8423				
		(0.0219)	(0.1305)	(0.0623)	(0.3042)				
BT	3.66	(V_0)	(V_{bar})	(α)	(β)	(ρ)	(λ)	(μ_λ)	(σ_λ)
		0.0033	0.0085	1.7041	0.0024	-0.3497	0.5000	-0.06557	5.3097e-4
		(0.0009)	(0.0009)	(0.0664)	(0.0588)	(0.0610)	(0.0388)	(0.0369)	(0.0139)
		(ω)	(b)	(a)	(c)	(η)			
CH	2.95	3.8406e-8	0.2356	2.5195e4	3.8319	-0.0026			
		(5.6594e-7)	(0.0169)	(9.8254e3)	(3.9210e-7)	(0.0004)			
		(α_1)	(β_1)	(β_0)	(γ)	(α)	(β)	(δ)	
CFGNIG	2.28	0.9999	2.567e-15	7.5968e-8 (1.0371e-7)	-0.4985	35.2867	12.1942	26.4248	
		(0.0035)	(1.0341e-7)		(0.2071)	(0.1800)	(0.1616)	(0.1162)	

Notes: We consider options traded on every Wednesday. After cleaning we have 1,578 option contracts. We applied the FRFT approach to price options which significantly reduces the calibration time. The models in column (1) are defined in Table 1 with their risk-neutral Characteristic Functions.

Table 6B

Out-of-sample valuation errors for call options traded in the second half of 2014

	RMSE	AAE	MOE	RMSE	AAE	MOE	RMSE	AAE	MOE
	Days to maturity < 40			40 ≤ Days to maturity < 70			70 ≤ Days to maturity ≤		
0.95<=(*S*/*K*)<0.99									
BS	8.722	6.432	-5.688	15.421	14.066	-12.774	19.676	18.433	-16.942
HS	8.739	6.740	-5.930	15.336	14.028	-12.736	18.819	17.592	-16.100
HN(R)	8.281	6.258	-5.405	14.960	13.647	-12.355	19.229	18.016	-16.525
HN(U)	7.722	5.650	-4.717	14.340	13.051	-11.759	18.745	17.571	-16.079
SC	8.846	6.829	-5.994	15.706	14.440	-13.149	19.023	17.835	-16.343
BT	8.867	6.924	-6.118	15.388	14.101	-12.809	18.825	17.602	-16.110
CH	8.935	6.987	-6.181	15.808	14.595	-13.304	19.569	18.431	19.569
CFGNIG	2.555	1.798	-0.797	2.856	2.470	-1.294	3.054	2.526	-1.267
0.99<=(*S*/*K*)<1.01									
BS	12.878	11.132	-9.469	18.636	17.532	-16.097	20.177	18.962	-17.424
HS	12.188	10.336	-8.206	18.089	16.951	-15.516	18.653	17.366	-15.827
HN(R)	11.571	9.720	-7.421	17.697	16.571	-15.136	19.370	18.123	-16.585
HN(U)	10.702	8.816	-6.240	16.677	15.547	-14.113	18.729	17.483	-15.944
SC	11.973	10.087	-7.812	17.951	16.802	-15.367	18.384	17.081	-15.542
BT	12.066	10.222	-8.082	17.849	16.692	-15.257	18.560	17.255	-15.717
CH	11.864	10.028	-7.891	17.401	16.202	-14.767	18.032	16.663	-15.125
CFGNIG	5.363	4.458	-1.444	4.752	4.527	-3.096	3.698	3.511	-1.989
1.01<=(*S*/*K*)<1.05									
BS	10.051	8.511	-6.512	17.848	17.120	-15.644	18.208	17.125	-15.347
HS	10.381	9.025	-7.103	17.556	16.821	-15.344	16.792	15.628	-13.849
HN(R)	9.850	8.501	-6.545	17.340	16.616	-15.139	17.463	16.373	-14.594
HN(U)	9.045	7.702	-5.695	16.647	15.929	-14.452	17.147	16.135	-14.356
SC	9.882	8.492	-6.517	17.096	16.341	-14.865	16.564	15.469	-13.689
BT	10.245	8.927	-7.003	17.343	16.604	-15.127	16.836	15.711	-13.932
CH	9.752	8.383	-6.439	15.926	15.079	-13.603	15.106	13.748	-11.969
CFGNIG	5.060	4.204	-1.879	5.403	5.153	-3.723	4.374	3.874	-2.309

Notes: The models are calibrated on first half of the same year. Total number of contracts available for the second half is 1,505. RMSE is the root mean square error, AAE is the average absolute error and MOE is the mean outside error. The models in column (1) are defined in Table 1 with their risk-neutral Characteristic Functions.

Table 7A: Calibration with options traded over the period January 2012-December 2013

Model	**RMSE**	**Parameters**							
		(σ)							
BS	2.38	0.0716							
		(0.0051)							
		(σ)	(θ)	(ν)					
VG	2.35	0.0191	0.7082 (0.0044)	0.0094					
		(0.0020)		(0.0014)					
		(α)	(β)	(δ)					
NIG	2.36	928.6679	825.1195	0.4566					
		(0.2445)	(0.2841)	(0.1729)					
		(σ)	(λ)	(p)	(η_1)	(η_2)			
JD-DE	2.38	0.0716	3.7968 (0.2842)	0.4730 (0.1913)	1533.5 (0.2525)	1533.5			
		(0.0052)				(0.3417)			
		(C)	(G)	(M)	(Y)				
CGMY	2.37	0.0272	94.9083 (0.0355)	53.3706 (0.0219)	1.3614				
		(0.0039)			(0.0257)				
		(κ)	(θ)	(σ)	(ρ)	(V_0)			
HS	2.34	0.1870	0.0460 (0.0311)	0.0379 (0.0744)	0.9900	0.0042			
		(0.0775)			(0.1109)	(0.0009)			
		(α_1)	(β_1)	(β_0)	(γ)	(λ)			
HN(R)	2.36	2.2204e-16	2.5670e-9	2.0504e-5 (2.2987e-	419.0868	-419.5868			
		(1.9020e-4)	(5.7663e-7)	6)	(1.0363e-4)	(1.1255e-4)			
		(α_1)	(β_1)	(β_0)	(γ)	(λ)			
HN(U)	2.28	2.2204e-16	2.5670e-9	2.3835e-5	419.9706	-1.7080			
		(0.0137)	(6.9763e-7)	(3.6808e-6)	(0.0070)	(0.0146)			
		(κ_1)	(κ_2)	(θ_1)	(θ_2)	(σ_1)	(σ_2)	(σ)	
SC	2.20	2.9011	23.6459	0.0341	0.0439	0.8455	3.1537	1.1059	
		(0.5828)	(1.4393)	(0.0487)	(0.0652)	(0.0610)	(0.1553)	(0.0143)	
		(ρ)	(μ_j)	(σ_j)	(λ_j)				
		-0.0486	-0.0089	0.0514	1.9186				
		(8.4836e-3)	(0.0177)	(2.5914e-3)	(0.2911)				
		(V_0)	(V_{bar})	(α)	(β)	(ρ)	(λ)	(μ_λ)	(σ_λ)
BT	2.21	0.0040	0.0090	1.5558	0.0163	-0.9998	0.0278	0.1406	0.0624
		(0.0003)	(0.0004)	(0.0268)	(0.0124)	(0.0280)	(0.0184)	(0.0181)	(0.0108)
		(ω)	(b)	(a)	(c)	(η)			
CH	2.26	4.5896e-9	0.1746	2.3772e4	4.4427e-6	-0.0036			
		(5.5982e-7)	(0.0080)	(9.3547e3)	(1.5881e-7)	(0.0002)			
		(α_1)	(β_1)	(β_0)	(γ)	(α)	(β)	(δ)	
CFGNIG	1.70	0.9980	2.567e-15	2.5670e-15	-0.5126 (0.1251)	296.3627	-0.1393	312.6878	
		(0.0045)	(1.0223e-7)	(1.0219e-7)		(0.0105)	(0.0105)	(0.0049)	

Notes: We consider options traded on every Wednesday. After cleaning we have 5,848 option contracts. We applied the FRFT approach to price options which significantly reduces the calibration time. The models in column (1) are defined in Table 1 with their risk-neutral Characteristic Functions.

Table 7B

Out-of-sample valuation errors for call options traded in the first half of 2014

	RMSE	AAE	MOE	RMSE	AAE	MOE	RMSE	AAE	MOE
	Days to maturity < 40			40 ≤ Days to maturity < 70			70 ≤ Days to maturity ≤ 100		
0.95<=(*S*/*K*)<0.99									
BS	2.2074	1.3461	-0.3481	3.5136	2.5294	-0.5713	6.0532	4.4878	-2.1437
HS	2.3409	1.7433	-0.2749	3.4392	2.5483	-0.1206	5.6243	4.2388	-1.0760
HN(R)	2.3316	1.7525	-0.0699	3.4108	2.4899	-0.2603	5.9557	4.4072	-1.8840
HN(U)	2.3430	1.7792	0.2012	3.3987	2.5578	0.1331	5.9631	4.4374	-1.6571
SC	2.3481	1.7391	-0.2560	3.4379	2.5111	-0.2678	5.6706	4.2390	-1.3575
BT	2.3328	1.7364	2.3328	3.4277	2.5207	-0.1776	5.5903	4.1933	-1.1842
CH	2.3703	1.7905	2.3703	3.4509	2.4766	-0.4876	5.8634	4.3433	-1.8508
CFGNIG	1.7614	1.3704	0.5146	0.8670	0.5876	-0.0750	1.8078	1.3718	-0.5850
0.99<=(*S*/*K*)<1.01									
BS	4.9875	3.6377	-1.1268	5.1376	3.8793	-2.0494	7.8154	5.9639	-4.3866
HS	5.1212	3.9012	0.0874	5.0129	3.7564	-1.8327	7.4242	5.6176	-3.9375
HN(R)	5.0971	3.9383	0.3924	4.8939	3.6482	-1.6270	7.6899	5.8229	-4.2110
HN(U)	5.1114	3.9686	0.4809	3.6647	3.6647	-1.6885	8.0528	6.1289	-4.6553
SC	5.1297	3.9301	0.2348	4.8760	3.6431	-1.5567	7.2749	5.4617	7.2749
BT	5.1149	3.8980	0.1224	4.9357	3.6922	-1.6765	7.1858	5.4037	-3.5855
CH	5.0451	3.8668	0.2805	4.8131	3.5861	-1.4313	7.3278	5.4846	7.3278
CFGNIG	3.7954	2.4969	1.2636	2.5060	1.9955	-1.0942	3.1106	2.6789	-1.5454
1.01<=(*S*/*K*)<1.05									
BS	4.1194	2.6545	-0.9448	4.5360	3.1752	-0.8124	2.8609	2.2658	-0.3348
HS	4.4929	2.9341	-1.5443	4.7061	3.2040	-1.3687	2.9403	2.3457	-0.6483
HN(R)	4.3820	2.8330	-1.3732	4.5830	3.1361	-1.0614	2.8860	2.2934	-0.4853
HN(U)	4.5581	3.0042	-1.5867	4.9044	3.3146	-1.8031	3.5061	2.8174	-1.6319
SC	4.5682	3.0134	-1.6054	4.6385	3.1563	-1.2176	2.8202	2.2906	-0.7487
BT	4.7533	3.2237	-1.8719	4.7742	3.2364	4.7742	2.8525	2.3511	-0.6721
CH	4.3026	2.7859	-1.2687	4.4820	3.1552	-0.6103	2.8248	2.2474	0.1080
CFGNIG	2.7516	1.8567	-0.6468	2.8950	1.9977	-1.0025	1.8051	1.5212	-0.3290

Notes: The models are calibrated using 2005 and 2006 contracts. Total number of contracts available for the second half is 1,578. RMSE is the root mean square error, AAE is the average absolute error and MOE is the mean outside error. The models in column (1) are defined in Table 1 with their risk-neutral Characteristic Functions.

Table 8A: Calibration with options traded over the period January 2013-December 2014

Model	RMSE	Parameters							
		(σ)							
BS	5.68	0.0895							
		(0.0044)							
		(σ)	(θ)	(v)					
VG	5.68	0.0900	-0.0603 (0.2097)	0.0283					
		(0.0045)		(0.1193)					
		(α)	(β)	(δ)					
NIG	5.68	70.5699	-8.0693 (0.2034)	0.5663					
		(0.0852)		(0.0553)					
		(σ)	(λ)	(p)	(η_1)	(η_2)			
JD-DE	5.68	0.0892	0.0105 (0.1637)	0.4473 (0.7376)	13.7655 (0.8875)	16.3763			
		(0.0044)				(0.0044)			
		(C)	(G)	(M)	(Y)				
CGMY	5.68	9.9238e-4	12.9536 (0.0176)	90.5655 (0.0662)	1.8597				
		(0.0001)			(0.0089)				
		(κ)	(θ)	(σ)	(ρ)	(V_0)			
HS	5.68	8.8081e-4	0.0530	0.0097	-0.9900 (0.1491)	0.0080			
		(0.1341)	(0.1491)	(0.0650)		(0.0009)			
		(α_1)	(β_1)	(β_0)	(γ)	(λ)			
HN(R)	4.86	0.9861	2.5670e-9	2.5670e-9	418.8949 (1.1373e-4)	419.3949			
		(2.2635e-5)	(1.1691e-7)	(1.1557e-6)		(2.2635e-5)			
		(α_1)	(β_1)	(β_0)	(γ)	(λ)			
HN(U)	4.86	0.9863	2.5670e-9	2.5670e-9	417.9965 (0.0092)	-0.5434			
		(0.0044)	(1.8702e-8)	(1.3887e-7)		(0.0039)			
		(κ_1)	(κ_2)	(θ_1)	(θ_2)	(σ_1)	(σ_2)	(σ)	
SC	5.64	0.0033	3.5918	6.7499e-9	2.0340e-12	0.0340	5.0000	0.4737	
		(0.5547)	(1.0299)	(0.8842)	(0.2704)	(0.0730)	(0.1841)	(0.0156)	
		(ρ)	(μ_j)	(σ_j)	(λ_j)				
		-0.0033	-0.0319	0.0548	2.9313				
		(0.0131)	(0.1512)	(0.0158)	(0.6380)				
		(V_0)	(V_{bar})	(α)	(β)	(ρ)	(λ)	(μ_λ)	(σ_λ)
BT	5.66	0.0066	0.0045	1.7214	0.0713	-0.9999	0.3315	0.0087	0.0970
		(0.0006)	(0.0005)	(0.0293)	(0.0206)	(0.0274)	(0.0180)	(0.0126)	(0.0062)
		(ω)	(b)	(a)	(c)	(η)			
CH	3.20	4.4059e-10	0.1728	1.0643e4	4.7444e-6	-0.0025			
		(2.4836e-7)	(0.0070)	(7.4221e3)	(2.0112e-7)	(1.4089e-4)			
		(α_1)	(β_1)	(β_0)	(γ)	(α)	(β)	(δ)	
CFGNIG	2.38	0.99999	8.7545e-8	2.567e-15	-0.5402 (0.0504)	36.0597	10.2048	26.8947	
		(0.0031)	(1.1055e-7)	(1.1071e-7)		(0.2314)	(0.2314)	(0.0479)	

Notes: We consider options traded on every Wednesday. After cleaning we have 6,296 option contracts with a mean option price of 24.1201 and average implied volatility of 0.0931. The models in column (1) are defined in Table 1 with their risk-neutral Characteristic Functions.

Table 8B

Out-of-sample valuation errors for call options traded in the first half of 2015

	RMSE	AAE	MOE	RMSE	AAE	MOE	RMSE	AAE	MOE
	Days to maturity < 40			40 ≤ Days to maturity < 70			70 ≤ Days to maturity ≤ 100		
0.95<=(*S*/*K*)<0.99									
BS	10.361	9.329	-8.531	18.369	17.974	-16.718	22.584	22.278	-21.049
HS	10.026	9.164	-8.366	18.211	17.829	-16.574	22.533	22.234	-21.006
HN(R)	5.801	4.892	-2.457	10.401	9.466	-6.959	15.099	14.170	-12.555
HN(U)	5.798	4.889	-2.471	10.404	9.464	-6.975	15.095	14.167	-12.558
SC	10.400	9.579	-8.781	18.536	18.159	-16.904	22.867	22.595	-21.367
BT	10.005	9.111	-8.313	18.359	17.978	-16.722	22.809	22.523	-21.295
CH	11.456	10.554	-9.756	20.935	20.673	-19.417	24.937	24.736	-23.508
CFGNIG	2.166	1.969	-1.202	2.838	2.634	-1.521	1.789	1.508	-0.666
0.99<=(*S*/*K*)<1.01									
BS	14.801	14.349	-13.196	21.161	20.899	-19.461	23.627	23.411	-22.261
HS	13.676	13.048	-11.895	20.832	20.566	-19.127	23.364	23.143	-21.993
HN(R)	8.018	7.085	-5.076	11.667	10.632	-7.761	16.173	15.455	-13.629
HN(U)	8.036	7.104	-5.108	11.694	10.664	-7.824	16.208	15.489	-13.686
SC	13.598	12.992	-11.835	20.567	20.297	-18.858	23.201	22.979	-21.829
BT	13.697	13.070	-11.917	20.758	20.492	-19.053	23.019	22.791	-21.641
CH	14.021	13.422	-12.27	20.304	20.021	-18.582	20.093	19.736	-18.586
CFGNIG	3.669	3.575	-2.422	4.251	4.182	-2.744	3.047	2.940	-1.837
1.01<=(*S*/*K*)<1.05									
BS	11.758	10.768	-9.456	19.242	18.839	-17.015	20.609	20.255	-18.919
HS	11.816	10.917	-9.608	19.106	18.705	-16.881	20.354	19.996	-18.661
HN(R)	7.418	6.418	-3.978	11.202	10.305	-7.173	13.972	13.192	-11.323
HN(U)	7.449	6.454	-4.040	11.274	10.379	-7.294	14.061	13.282	-11.445
SC	11.249	10.374	-9.063	18.412	18.003	-16.179	20.398	20.075	-18.739
BT	11.883	11.029	-9.718	18.766	18.362	-16.538	19.653	19.288	-17.952
CH	10.794	9.878	-8.569	15.144	14.428	-12.604	13.125	12.336	-10.999
CFGNIG	3.338	3.019	-1.746	4.022	3.854	-2.077	2.985	2.766	-1.524

Notes: The models are calibrated using 2006 and 2007 contracts. Total number of contracts available for the second half is 943. RMSE is the root mean square error as defined, AAE is the average absolute error, and MOE is the mean outside error. The models in column (1) are defined in Table 1 with their risk-neutral Characteristic Functions.

Table 9A: Calibration with options traded over the period January 2012-December 2014

Model	RMSE	Parameters							
		(σ)							
BS	5.06	0.0846							
		(0.0046□							
		(σ)	(θ)	(ν)					
VG	5.06	0.0801	0.0801	0.0023					
		(0.0028)	(0.1206)	(0.0022)					
		(α)	(β)	(δ)					
NIG	5.06	252.3847	74.8615	1.5681					
		(0.2445)	(0.2841)	(0.1729)					
		(σ)	(λ)	(p)	(η_1)	(η_2)			
JD-DE	5.06	0.0845	0.1382 (0.0128)	0.4519	155.8497 (0.0124)	190.3719			
		(0.0049)		(0.0094)		(0.0141)			
		(C)	(G)	(M)	(Y)				
CGMY	5.06	0.0240	94.9079	53.3726 (0.0201)	1.4418				
		(0.0026)	(0.0264)		(0.0185)				
		(κ)	(θ)	(σ)	(ρ)	(V_0)			
HS	5.06	0.0015	0.0543 (0.1419)	0.0128	0.9900 (0.1419)	0.0071			
		(0.1244)		(0.0683)		(0.0009)			
		(α_1)	(β_1)	(β_0)	(γ)	(λ)			
HN(R)	4.96	0.9348	2.5670e-9	1.3022e-6	355.1512 (5.9926e-5)	-355.6512			
		(5.9897e-5)	(2.2001e-7)	(1.1262e-6)		(1.3546e-4)			
		(α_1)	(β_1)	(β_0)	(γ)	(λ)			
HN(U)	4.95	0.9494	2.5670e-9	1.0208e-6	419.0905 (0.0139)	-1.0947			
		(0.0067)	(3.9161e-8)	(2.6777e-7)		(0.0072)			
		(κ_1)	(κ_2)	(θ_1)	(θ_2)	(σ_1)	(σ_2)	(σ)	
SC	5.01	0.8581	7.5879	4.5504e-6	6.2495e-5	0.3936	0.8972	0.9294	
		(0.5094)	(1.1454)	(0.3362)	(0.1559)	(0.0712)	(0.1199)	(0.0158)	
		(ρ)	(μ_j)	(σ_j)	(λ_j)				
		-0.0117	-0.0278	0.0687	2.6217				
		(7.7636e-3)	(0.1190)	(9.9038e-3)	(0.4188)				
		(V_0)	(V_{bar})	(α)	(β)	(ρ)	(λ)	(μ_λ)	(σ_λ)
BT	5.03	0.0070	0.0050	1.6956	0.0607	-0.9999	0.0100	0.5683	0.2890
		(0.0004)	(0.0004)	(0.02630)	(0.0124)	(0.0226)	(0.0175)	(0.0218)	(0.0136)
		(ω)	(b)	(a)	(c)	(η)			
CH	2.49	4.2055e-14	0.4992	1.0487e4	2.1296e-6	-0.0022			
		(1.1371e-7)	(0.0018)	(4.8175e3)	7.3134e-8	8.8909e-5			
		(α_1)	(β_1)	(β_0)	(γ)	(α)	(β)	(δ)	
CFGNIG	2.26	0.99999	2.1836e-8	2.6181e-8)	-0.5406 (0.0518)	38.5655 (0.0412)	7.5215	26.4156	
		(0.0034)	(1.0925e-7))	(1.0933e-7)			(0.0375)	(0.0380)	

Notes: We consider options traded on every Wednesday. After cleaning we have 8,404 option contracts. We applied the FRFT approach to price options which significantly reduces the calibration time. The models in column (1) are defined in Table 1 with their risk-neutral Characteristic Functions.

Table 9B

Out-of-sample valuation errors for call options traded in the first half of 2015

	RMSE	AAE	MOE	RMSE	AAE	MOE	RMSE	AAE	MOE
	Days to maturity < 40			40 ≤ Days to maturity < 70			70 ≤ Days to maturity ≤ 100		
0.95<=(*S*/*K*)<0.99									
BS	10.887	9.794	-8.996	19.402	19.008	-17.752	24.028	23.726	-22.498
HS	10.474	9.577	-8.779	19.072	18.681	-17.425	23.812	23.499	-22.271
HN(R)	8.995	8.013	-7.216	17.618	17.285	-16.029	23.975	23.683	-22.455
HN(U)	8.784	7.815	-7.017	17.189	16.843	-15.587	23.748	23.438	-22.209
SC	10.641	9.724	-8.926	19.340	18.960	-17.704	24.158	23.853	-22.626
BT	10.379	9.456	-8.658	19.175	18.788	-17.532	24.134	23.838	-22.609
CH	11.379	10.474	-9.676	20.736	20.445	20.736	25.259	25.052	-23.825
CFGNIG	1.918	1.673	-0.934	2.713	2.475	-1.371	1.894	1.588	-0.742
0.99<=(*S*/*K*)<1.01									
BS	15.549	15.095	-13.942	22.312	22.066	-20.627	25.045	24.838	-23.688
HS	14.345	13.697	-12.544	22.075	21.821	-20.386	25.084	24.877	-23.727
HN(R)	12.446	11.680	-10.527	20.660	20.428	-18.989	25.038	24.843	-23.693
HN(U)	12.375	11.622	-10.469	20.537	20.299	-18.859	25.260	25.065	-23.915
SC	14.221	13.588	-12.435	21.732	21.477	-20.038	24.875	24.656	-23.506
BT	14.256	13.608	-12.455	21.915	21.665	-20.226	24.743	24.531	-23.381
CH	14.534	13.926	-12.773	21.424	21.166	21.424	22.504	22.236	-21.086
CFGNIG	3.606	3.505	-2.352	4.354	4.291	-2.852	3.363	3.276	-2.160
1.01<=(*S*/*K*)<1.05									
BS	12.169	11.151	-9.839	20.031	19.622	-17.798	21.604	21.247	-19.911
HS	12.376	11.462	-10.151	20.249	19.855	-18.031	21.906	21.557	-20.221
HN(R)	11.041	10.072	-8.768	19.367	18.969	-17.145	21.775	21.438	-20.103
HN(U)	11.277	10.338	-9.029	19.814	19.439	-17.615	22.603	22.293	-20.957
SC	11.977	11.129	-9.818	19.592	19.200	-17.376	22.137	21.802	-20.466
BT	12.511	11.656	-10.345	20.060	19.682	-17.858	21.428	21.089	-19.753
CH	11.773	10.892	-9.581	17.528	17.001	-15.177	16.956	16.446	-15.111
CFGNIG	3.4930	3.186	-1.914	4.317	4.173	-2.361	3.419	3.242	-1.930

Notes: The models are calibrated using options traded on 2012-2014. Total number of contracts available for the second half is 943. RMSE is the root mean square error as defined, AAE is the average absolute error, and MOE is the mean outside error. The models in column (1) are defined in Table 1 with their risk-neutral Characteristic Functions.

Appendix A

Moments of NIG Lévy Innovation

A $NIG(\alpha,\beta,\delta)$ is infinitely divisible and the associated Lévy process has the distribution of increments over $[s, t + s]$ characterized by a $NIG(\alpha,\beta,t\delta)$. Schouten (2003) shows that the first four moments of the $X \sim NIG(\alpha,\beta,\delta)$ random variable are:

$$E[X] = \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}} \tag{A1}$$

$$V[X] = \alpha^2\delta\left(\alpha^2 - \beta^2\right)^{-\frac{3}{2}} \tag{A2}$$

$$Skew[X] = 3\beta\alpha^{-1}\delta^{-\frac{1}{2}}\left(\alpha^2 - \beta^2\right)^{-\frac{1}{4}} \tag{A3}$$

$$Kurt[X] = 3\left(1 + \frac{\alpha^2 + 4\beta^2}{\delta\alpha^2\sqrt{\alpha^2 - \beta^2}}\right). \tag{A4}$$

With the moments of the NIG random variable as in (A1)- (A4), the first two conditional moments of the log-returns become:

$$\begin{aligned} E[X_t \mid \Im_{t-1}] &= E\left[r + \lambda\sigma_t - \frac{z_t}{\sqrt{\alpha^2\delta\left(\alpha^2 - \beta^2\right)^{-\frac{3}{2}}}} \mid \Im_{t-1}\right] \\ &= r + \lambda\sigma_t - \frac{\frac{\delta\sigma_t\beta}{\sqrt{\alpha^2 - \beta^2}}}{\sqrt{\alpha^2\delta\left(\alpha^2 - \beta^2\right)^{-\frac{3}{2}}}} \\ &= r + \left(\lambda - \frac{\beta\sqrt{\delta}}{\alpha\left(\alpha^2 - \beta^2\right)^{-\frac{1}{4}}}\right)\sigma_t \end{aligned} \tag{A5}$$

$$V[X_t \mid \Im_{t-1}] = V\left[r + \lambda\sigma_t - \frac{z_t}{\sqrt{\alpha^2\delta(\alpha^2-\beta^2)^{-\frac{3}{2}}}} \mid \Im_{t-1} \right]$$

$$= \frac{\alpha^2\delta\sigma_t(\alpha^2-\beta^2)^{-\frac{3}{2}}}{\alpha^2\delta(\alpha^2-\beta^2)^{-\frac{3}{2}}} \qquad . \qquad \text{(A6)}$$

$$= \sigma_t$$

Thus the dynamics of X_t are a scaled and shifted version of $z_t \mid \Im_{t-1} \sim NIG(\alpha, \beta, \delta\sigma_t)$. The conditional skewness and kurtosis are:

$$Skew[X_{t+1} \mid \Im_{t-1}] = 3\beta\alpha^{-1}(\delta\sigma_t)^{-\frac{1}{2}}(\alpha^2-\beta^2)^{-\frac{1}{4}} \qquad \text{(A7)}$$

$$Kurt[X_{t+1} \mid \Im_{t-1}] = 3\left(1 + \frac{\alpha^2+4\beta^2}{\delta\sigma_t\alpha^2\sqrt{(\alpha^2-\beta^2)}}\right). \qquad \text{(A8)}$$

The existence of conditional skewness and conditional kurtosis ensures that smile-skew patterns can be modeled when log-return dynamics follow a GARCH with NIG-Lévy innovations.

Appendix B

GARCH with NIG Lévy Innovation and Equivalent Martingale Measure

We select an Equivalent Martingale Measure (EMM) that depends on finding a solution to the conditional Esscher equation (Gerber *et al.*, 1994; Shiu *et al.*, 2004)

$$\frac{M_{X_t|\Im_{t-1}}\left(\hat{\theta}_t+1\right)}{M_{X_t|\Im_{t-1}}\left(\hat{\theta}_t\right)}=e^{r_t} \tag{B1}$$

where $M_{X_t|\Im_{t-1}}(s)$ is the conditional moment-generating function (MGF) defined as

$$M_{X_t|\Im_{t-1}}(s)=E\left[e^{sX_t}\mid \Im_{t-1}\right]. \tag{B2}$$

In the case of GARCH dynamics with NIG-Lévy innovations (7), the conditional Esscher equation (B1) becomes

$$\frac{E_{t-1}\left[e^{\left(\hat{\theta}_t+1\right)\left(r+\lambda\sigma_t-\frac{z_t}{\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}}\right)}\right]}{E_{t-1}\left[e^{\hat{\theta}_t\left(r+\lambda\sigma_t-\frac{z_t}{\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}}\right)}\right]}=e^{r_t}$$

$$\Rightarrow\frac{E_{t-1}\left[e^{-\left(\hat{\theta}_t+1\right)\left(\frac{z_t}{\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}}\right)}\right]}{E_{t-1}\left[e^{-\hat{\theta}_t\left(\frac{z_t}{\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}}\right)}\right]}=e^{-\lambda\sigma_t}\quad, \tag{B3}$$

which simplifies to (B4) using the constant $c = \frac{-1}{\sqrt{\alpha^2 \delta \left(\alpha^2 - \beta^2\right)^{-\frac{3}{2}}}}$:

$$\frac{E_{t-1}\left[e^{(\hat{\theta}_t + 1) z_t c}\right]}{E_{t-1}\left[e^{\hat{\theta}_t z_t c}\right]} = e^{-\lambda \sigma_t} \tag{B4}$$

Inserting equation (2) into (B4) we obtain:

$$\begin{aligned} e^{-\lambda\sigma_t} &= \frac{\exp\left(-\delta\left\{\sqrt{\alpha^2 - \left(\beta + (\hat{\theta}_t + 1)c\right)^2} - \sqrt{(\alpha^2 - \beta^2)}\right\}\right)}{\exp\left(-\delta\left\{\sqrt{\alpha^2 - \left(\beta + \hat{\theta}_t c\right)^2} - \sqrt{(\alpha^2 - \beta^2)}\right\}\right)} \\ &= \exp\left(-\delta\left\{\sqrt{\alpha^2 - \left(\beta + (\hat{\theta}_t + 1)c\right)^2} - \sqrt{\alpha^2 - \left(\beta + \hat{\theta}_t c\right)^2}\right\}\right) \end{aligned} \tag{B5}$$

Given the parameters of the NIG-Lévy process, α, β, δ and the GARCH volatility process σ_t in equation (8), the solution $\hat{\theta}_t$ of (B5) can be used to describe the distribution of log-returns as

$$\tilde{M}_{X_s|\Im_{s-1}}(l) = \frac{M_{X_s|\Im_{s-1}}\left(l + \hat{\theta}_t\right)}{M_{X_s|\Im_{s-1}}\left(\hat{\theta}_t\right)} \quad . \tag{B6}$$

Given our assumed distribution for innovations and the fact that the GARCH one-period-ahead volatility is known, (B6) becomes

$$\begin{aligned} \tilde{M}_{X_s|\Im_{s-1}}(l) &= \frac{E_{t-1}\left[e^{(\hat{\theta}_t + l)(r + \lambda\sigma_t + z_t c)}\right]}{E_{t-1}\left[e^{\hat{\theta}_t (r + \lambda\sigma_t + z_t c)}\right]}; z_t \mid \Im_{t-1} \sim NIG(\alpha, \beta, \delta\sigma_t) \\ &= \frac{e^{l(r+\lambda\sigma_t)} E_{t-1}\left[e^{(\hat{\theta}_t + l) c z_t}\right]}{E_{t-1}\left[e^{\hat{\theta}_t c z_t}\right]} \\ &\overset{(34)\,\&\,(35)}{=} e^{l(r+\lambda\sigma_t)} \exp\left(-\delta\left\{\sqrt{\alpha^2 - \left(\beta + (\hat{\theta}_t + l)c\right)^2} - \sqrt{\alpha^2 - \left(\beta + \hat{\theta}_t c\right)^2}\right\}\right) \\ &= e^{l(r+\lambda\sigma_t)} \exp\left(-\delta\left\{\sqrt{\alpha^2 - \left(\beta + \hat{\theta}_t c + lc\right)^2} - \sqrt{\alpha^2 - \left(\beta + \hat{\theta}_t c\right)^2}\right\}\right) \end{aligned} \tag{B7}$$

Comparing equations (2) and (B7), we recognize that under-EMM innovations are NIG-distributed with a new characterization, $\beta' = \beta + \hat{\theta}_t c$.

We also need to see what other parameters are influenced by this new characterization. We start with the dynamics of the volatility under the martingale measure:

$$\begin{aligned}
\sigma'_t = \tilde{V}\left[X_t \mid \Im_{t-1}\right] &= \tilde{V}\left[r + \lambda\sigma_t - \frac{z_t}{2\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}} \mid \Im_{t-1}\right] \\
&= \frac{\alpha^2\delta\sigma_t\left(\alpha^2-\beta'^2\right)^{-\frac{3}{2}}}{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}} \\
&= \left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}}\sigma_t
\end{aligned} \tag{B8}$$

Thus, the market and real measures are related through the characterization $\beta' = \beta + \hat{\theta}_t c$, and the market and real volatility processes are related through $\sigma'_t = \left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}}\sigma_t$.

We next identify the remaining parameters that require new characterization to keep the return dynamics equivalent. Under the real measure we have

$$\begin{aligned}
X_t &= r + \lambda\sigma_t - \frac{z_t}{\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}} \\
\Rightarrow X_t &= r + \lambda\left[\frac{\alpha^2-\beta^2}{\alpha^2-\beta'^2}\right]^{-\frac{3}{2}}\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}}\sigma_t - \frac{z_t}{\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}} \\
\Rightarrow X_t &= r + \lambda\left[\frac{\alpha^2-\beta^2}{\alpha^2-\beta'^2}\right]^{-\frac{3}{2}}\sigma'_t - \frac{z_t}{\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}} \\
\Rightarrow X_t &= r + \lambda'\sigma'_t - \frac{z_t}{\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}} \\
&\text{where } \lambda' = \lambda\left[\frac{\alpha^2-\beta^2}{\alpha^2-\beta'^2}\right]^{-\frac{3}{2}}
\end{aligned} \tag{B9}$$

Hence, we introduce α', β', δ' , replacing α, β, δ for the dynamics to be characterized by a martingale measure. We can achieve this from the following:

$$\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}} = \alpha^2\delta\left[\frac{\beta^2}{\beta'^2}\right]^{-\frac{3}{2}}\left(\frac{\beta^2}{\beta'^2}\alpha^2-\beta'^2\right)^{-\frac{3}{2}}$$

$$= \left(\alpha^2\frac{\beta^2}{\beta'^2}\right)\left(\delta\frac{\beta'}{\beta}\right)\left(\frac{\beta^2}{\beta'^2}\alpha^2-\beta'^2\right)^{-\frac{3}{2}}$$

$$= \alpha'^2\delta'\left(\alpha'^2-\beta'^2\right)^{-\frac{3}{2}} \qquad \text{(B10)}$$

where $\alpha' = \alpha\frac{\beta'}{\beta}$ and $\delta' = \delta\frac{\beta'}{\beta}$

We now have the equivalent dynamics for log-returns under the martingale measure

$$X_t = r + \lambda'\sigma'_t - \frac{z_t}{\sqrt{\alpha'^2\delta'\left(\alpha'^2-\beta'^2\right)^{-\frac{3}{2}}}} \text{ with } z_t \mid \Im_{t-1} \sim NIG\left(\alpha',\beta',\delta'\sigma'_t\right) \qquad \text{(B11)}$$

where the parameters of the equivalence-maintaining martingale dynamics are related to those of the market dynamics through

$$\sigma'_t = \left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}}\sigma_t \qquad \text{(B12)}$$

$$\beta' = \beta + \hat{\theta}_t c \qquad \text{(B13)}$$

$$\alpha' = \alpha\frac{\beta'}{\beta} \qquad \text{(B14)}$$

$$\delta' = \delta\frac{\beta'}{\beta}. \qquad \text{(B15)}$$

As a last step, we need to work out the corresponding changes in the GARCH parameters. The GARCH dynamics are

$$\sigma_t = \beta_0 + \beta_1\frac{z_t}{\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}} + \alpha_1\sigma_{t-1}$$

$$\Rightarrow \sigma_t\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}} = \beta_0\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}} + \beta_1\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}}\frac{z_{t+1}}{\sqrt{\alpha^2\delta\left(\alpha^2-\beta^2\right)^{-\frac{3}{2}}}} + \alpha_1\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}}\sigma_{t-1}$$

B(16)

Thus, the equivalent GARCH volatility dynamics under the martingale measure can be written as

$$\sigma_t' = \beta_0' + \beta_1' \frac{z_t}{\sqrt{\alpha'^2 \delta' \left(\alpha'^2 - \beta'^2\right)^{-\frac{3}{2}}}} + \alpha_1 \sigma_{t-1}' \text{ with } z_t \mid \Im_{t-1} \sim NIG\left(\alpha', \beta', \delta' \sigma_t'\right) , \qquad \text{(B17)}$$

where

$$\beta_0' = \beta_0 \left[\frac{\alpha^2 - \beta'^2}{\alpha^2 - \beta^2} \right]^{-\frac{3}{2}} \qquad \text{(B18)}$$

$$\beta_1' = \beta_1 \left[\frac{\alpha^2 - \beta'^2}{\alpha^2 - \beta^2} \right]^{-\frac{3}{2}} \quad . \qquad \text{(B19)}$$

We now need the corresponding risk-neutral characterization to be used in option pricing. First note that with the scaling factor $u = -\frac{1}{\sqrt{\alpha^2 \delta \left(\alpha^2 - \beta^2\right)^{-\frac{3}{2}}}}$ equation (7) becomes:

$$X_t = r + \lambda \sigma_t + u z_t . \qquad \text{(B20)}$$

The MGF under the risk-neutral measure can then be obtained from the following expression for the NIG characteristic function:

$$\begin{aligned} E_t^Q\left(e^{u z_{t+1}}\right) = E_t^Q\left(e^{i(-iu) z_{t+1}}\right) &= \exp\left\{-\delta' \sigma_{t+1}' \left\{\sqrt{\alpha'^2 - \left(\beta' + i(iu)\right)^2} - \sqrt{\alpha'^2 - \beta'^2}\right\}\right\} \\ &= \exp\left\{-\delta' \sigma_{t+1}' \left\{\sqrt{\alpha'^2 - \left(\beta' + u^2\right)^2} - \sqrt{\alpha'^2 - \beta'^2}\right\}\right\} \end{aligned} \qquad \text{(B21)}$$

Hence:

$$\begin{aligned} E_t^Q\left(e^{X_{t+1}}\right) &= e^{\left(r + \lambda' \sigma_{t+1}'\right)} \exp\left\{-\delta' \sigma_{t+1}' \left\{\sqrt{\alpha'^2 - \left(\beta' + u\right)^2} - \sqrt{\alpha'^2 - \beta'^2}\right\}\right\} \\ &= e^{\left(r + \lambda' \sigma_{t+1}'\right)} \exp\left\{-\delta' \sigma_{t+1}' \left\{\sqrt{\alpha'^2 - \left(\beta' - \frac{1}{\sqrt{\alpha'^2 \delta' \left(\alpha'^2 - \beta'^2\right)^{-\frac{3}{2}}}}\right)^2} - \sqrt{\alpha'^2 - \beta'^2}\right\}\right\} \end{aligned} \qquad \text{(B22)}$$

We want to choose λ' under Q in terms of other risk-neutral parameters such that

$$E_t^Q\left(e^{X_{t+1}}\right) = e^{(t+1-t)r}$$

$$\Rightarrow e^{(r+\lambda'\sigma'_{t+1})}\exp\left\{-\delta'\sigma'_{t+1}\left\{\sqrt{\alpha'^2 - \left(\beta' - \frac{1}{\sqrt{\alpha'^2\delta'\left(\alpha'^2-\beta'^2\right)^{-\frac{3}{2}}}}\right)^2} - \sqrt{\alpha'^2-\beta'^2}\right\}\right\} = e^r \quad \text{(B23)}$$

$$= e^r \exp\left\{\sigma'_{t+1}\left(\lambda' - \delta'\left\{\sqrt{\alpha'^2 - \left(\beta' - \frac{1}{\sqrt{\alpha'^2\delta'\left(\alpha'^2-\beta'^2\right)^{-\frac{3}{2}}}}\right)^2} - \sqrt{\alpha'^2-\beta'^2}\right\}\right)\right\} = e^r$$

Since $\sigma'_{t+1} \neq 0$,

$$\lambda' = \delta'\left\{\sqrt{\alpha'^2 - \left(\beta' - \frac{1}{\sqrt{\alpha'^2\delta'\left(\alpha'^2-\beta'^2\right)^{-\frac{3}{2}}}}\right)^2} - \sqrt{\alpha'^2-\beta'^2}\right\} \quad \text{(B24)}$$

The above equation is the final characterization that we use in the MGF expression, which, in turn, is used in our pricing and calibration.

Appendix C

Relationship between parameters under the Risk-neutral and Historical risk measures

$$\sigma_t = \beta_0 + \beta_1\left[\frac{NIG(\alpha,\beta,\delta\sigma_{t-1})}{\sqrt{\alpha^2\sigma_{t-1}\delta(\alpha^2-\beta^2)^{-\frac{3}{2}}}} - \gamma\sqrt{\sigma_{t-1}}\right]^2 + \alpha_1\sigma_{t-1}$$

As in equation (B16) we multiply both sides of the above equation by $\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}}$:

$$\sigma_t\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}} = \beta_0\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}} + \beta_1\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}}\left[\frac{NIG(\alpha,\beta,\delta\sigma_{t-1})}{\sqrt{\alpha^2\sigma_{t-1}\delta(\alpha^2-\beta^2)^{-\frac{3}{2}}}} - \gamma\sqrt{\sigma_{t-1}}\right]^2$$

$$+\alpha_1\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}}\sigma_{t-1}$$

$$= \beta_0\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}}$$

$$+\beta_1\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}}\left[\frac{NIG(\alpha,\beta,\delta\sigma_{t-1})}{\sqrt{\alpha^2\delta(\alpha^2-\beta^2)^{-\frac{3}{2}}}}\frac{\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{4}}}{\sqrt{\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}}\sigma_{t-1}}} - \frac{\gamma}{\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{4}}}\sqrt{\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}}\sigma_{t-1}}\right]^2$$

$$+\alpha_1\left[\frac{\alpha^2-\beta'^2}{\alpha^2-\beta^2}\right]^{-\frac{3}{2}}\sigma_{t-1}$$

Using equation (B12):

$$\sigma_t' = \beta_0 \left[\frac{\alpha^2 - \beta'^2}{\alpha^2 - \beta^2}\right]^{-\frac{3}{2}}$$

$$+ \beta_1 \left[\frac{\alpha^2 - \beta'^2}{\alpha^2 - \beta^2}\right]^{-\frac{3}{2}} \left[\frac{\alpha^2 - \beta'^2}{\alpha^2 - \beta^2}\right]^{-\frac{3}{2}} \left[\frac{NIG(\alpha', \beta', \delta'\sigma_{t-1}')}{\sqrt{\alpha^2 \delta (\alpha^2 - \beta^2)^{-\frac{3}{2}} \sigma_{t-1}'}} - \frac{\gamma}{\left\{\left[\frac{\alpha^2 - \beta'^2}{\alpha^2 - \beta^2}\right]^{-\frac{3}{4}}\right\}^2} \sqrt{\sigma_{t-1}'} \right]^2 + \alpha_1 \sigma_{t-1}'$$

Using equation (B10):

$$\sigma_t' = \beta_0 \left[\frac{\alpha^2 - \beta'^2}{\alpha^2 - \beta^2}\right]^{-\frac{3}{2}} + \beta_1 \left[\frac{\alpha^2 - \beta'^2}{\alpha^2 - \beta^2}\right]^{-3} \left[\frac{NIG(\alpha', \beta', \delta'\sigma_t')}{\sqrt{\alpha'^2 \delta' (\alpha'^2 - \beta'^2)^{-\frac{3}{2}} \sigma_{t-1}'}} - \frac{\gamma}{\left\{\left[\frac{\alpha^2 - \beta'^2}{\alpha^2 - \beta^2}\right]^{-\frac{3}{4}}\right\}^2} \sqrt{\sigma_{t-1}'} \right]^2 + \alpha_1 \sigma_{t-1}'$$

This can be written as

$$\sigma_t' = \beta_0' + \beta_1' \left[\frac{NIG(\alpha', \beta', \delta'\sigma_t')}{\sqrt{\alpha'^2 \delta' \sigma_{t-1}' (\alpha'^2 - \beta'^2)^{-\frac{3}{2}}}} - \gamma' \sqrt{\sigma_{t-1}'} \right]^2 + \alpha_1 \sigma_{t-1}' ,$$

where

$$\beta_0' = \beta_0 \left[\frac{\alpha^2 - \beta'^2}{\alpha^2 - \beta^2}\right]^{-\frac{3}{2}}, \; \beta_1' = \beta_1 \left[\frac{\alpha^2 - \beta'^2}{\alpha^2 - \beta^2}\right]^{-3} \text{ and } \qquad \gamma' = \frac{\gamma}{\left\{\left[\frac{\alpha^2 - \beta'^2}{\alpha^2 - \beta^2}\right]^{-\frac{3}{4}}\right\}^2}$$

Thus, by modifying the volatility dynamics (introducing nonlinearity) the relationships between some parameters under historical and risk-neutral measures also change – in particular, the GARCH parameter β_1 (historical) and β_1' (risk-neutral) are not related as shown in (B19) in the paper but as shown above. Moreover the new parameter γ (historical) in the modified volatility dynamics becomes γ' (risk-neutral) as shown above.