Enhanced optimal portfolios –
A controlled integration of quantitative predictors*

Lars Kaiser, Aron Veress and Marco Menichetti†

This draft: August 20, 2012
First draft: February 15, 2012

Abstract

No unanimous agreement exists on the optimality of market-capitalization weighted portfolios, nor on the potential benefits of active portfolio management. Starting from the classical Black-Litterman approach, we show that historically generated excess return above the market portfolio can be retained whilst constraining additional downside risk. Weighting factors required for the mixed estimation can be directly derived from predictive regressions in form of the goodness-of-fit measure. This enables an unambiguous determination of certainty levels in a dynamic multi-period framework.

Keywords: Bayesian portfolio construction, Black-Litterman, downside risk, goodness-of-fit, random sampling, enhanced indexing
JEL classification: C11; C22; C53; C61; D24; G11; G12

*We would like to acknowledge feedback and helpful comments by Michael Hanke, Roger Stover, Helene Raymond and seminar participants at Wednesday Afternoon Science at University of Liechtenstein, International Ph.D. seminar in Banking 2012 at University of Fribourg and 29^{th} GdRE Annual International Symposium on Money, Banking and Finance

†Address correspondence to Lars Kaiser, University of Liechtenstein, Institute for Financial Services, Chair in Business Administration, Banking and Financial Management, Fürst-Franz-Josef-Strasse, 9490 Vaduz, Principality of Liechtenstein, Tel: +423 265 1186, Email: lars.kaiser@uni.li

1 Introduction

The true optimality of the market portfolio is heavily discussed in numerous studies and, alongside, multiple alternative passive equity index strategies have emerged. Amongst these are equally-weighted, minimum-variance and fundamental indexation approaches. On theoretical grounds the market-capitalization weighted portfolio is optimal under the capital asset pricing model and market efficiency. However, multiple studies, as reported by C. Chen, Chen, and Bassett (2007) have shown that inefficiencies do exist and can be exploited by sensible adjustments to cap-weights. Alternative studies go even further and replace the capitalization-weighting approach altogether. DeMiguel, Garlappi, and Uppal (2009) show that a nave 1/N portfolio outperforms most other strategies on a risk-adjusted basis, including the market portfolio. Arnott, Hsu, and Moore (2005) provide evidence on the risk-adjusted return superiority of equity indices based on fundamental firm metrics (e.g. income, revenue and sales) and Clarke, de Silva, and Steven Thorley (2006) – amongst others – have shown that minimum-variance portfolios can add value over traditional market-capitalization weighted benchmarks. Arguably, these findings have led to a relativization of the general rule of cap-weighted indices being optimal per se.

Given the non-optimality of capitalization-weighted indices and the industry wide application of the respective, managers have to identify ways how to capitalize on inefficiencies not captured by the market. Quantitative predictors have been tested to entail forecasting power and are widely applied, however, majority of factors do not deliver persistent forecasting quality over time. Theoretically, this falls into place as additional information should be absorbed and accounted for by the market portfolio rapidly. Nevertheless, some factors have shown to have gained there edge over years whilst others could not cement their position in literature and practice. Accepting the fact that some factors entail forecasting ability at different levels of significance and given that forecastability various over different factors, aim has to be to identify these periods and incorporate respective predictions into the portfolio construction in order to enhance the optimal market portfolio.

This study offers a quantitative approach to exploiting market inefficiencies not captured by the market and so can be related, but is not limited to, the field of enhanced indexation. Herefore, we build upon the classical Black and Litterman (1992) (BL) approach by setting a market-capitalization weighted portfolio as the passive prior. Subsequently, adjustments are based on quantitative predictions derived from a factor model with an arbitrary predictor. The posterior vector of expected returns is weighted according to certainty levels (Ω) and scalar (τ), where Ω can be directly derived from the predictive regression in form of the goodness-of-fit measure and, thereby, implicitly accounts for estimation errors in the optimization procedure. This methodology can retain generated excess return over the long-run, whilst restraining downside risk. Supporting results based on a robustly constructed simulation framework also reveal that increasing the number of correlation breakdowns and/or the variance of correlation between assets and predictor leads to a monotonously decreasing convergence of average excess returns towards zero. An empirical setting initialized with a global equity index portfolio enhanced via forecast from multiple universal predictors confirm these findings.

2 Enhanced Optimal Portfolios

We consider an unconstrained investor holding the market portfolio, but is keen to enhance the respective by considering quantitative equity return predictions. We propose an approach on which additional quantitative estimates can be incorporated, thereby, offering upside return potential whilst controlling for risk in form of lower partial moments. Fundamentals of the model rely on the procedure of Bayesian portfolio construction. First the implied returns from the underlying portfolio are backed out via a mean-reversion procedure. Next our return estimates and certainty levels are generated based on an arbitrary quantitative predictor. The revised return vector (posterior) is derived via the mixed estimation procedure according to the weighting factors Ω and τ. In the following lower-case letters refer to scalars, bold-face letters denote vectors and upper-case symbols stand for matrices.

2.1 Prior

The starting point of this model, as for the traditional BL model, is the market portfolio. We make use of the reverse optimization procedure to back out implied equilibrium weights from the market portfolio given by $\mathbf{z} = \gamma\Sigma\mathbf{w}$ ((Black & Litterman, 1992)). We denote the vector of implied optimal portfolio returns ($\mathbf{z}$) as a function of investors risk aversion (γ), covariance matrix (Σ) and the vector of portfolio weights ($\mathbf{w}$).

We specify the covariance matrix based on a rolling window of monthly historical equity returns. The risk aversion factor γ is set to one.[1] The resulting distribution of the prior is $N \sim (\mathbf{z}, \tau\Sigma)$. Where the scalar τ is scaling factor towards the covariance matrix, which reflects the uncertainty in our return estimates and serves as a weighting factor for the mixed estimation procedure. The scaling factor τ enables the investor to specify the acceptable degree of deviation of the posterior from the prior. A small value implies a posterior closely tracking the prior and vice versa. Given the quasi optimality of the market portfolio defined as the prior the uncertainty in the prior measured by τ is small. Consequently, a value

[1] We set risk aversion to 1 as – for this case – expected returns derived from the optimal portfolio are not pre-scaled when entering the Bayesian framework and, therefore, the mixed distribution is solely influenced by the BL specific weighting factors Ω and τ.

specification of $\tau \rightarrow 0$ is commonly applied (He and Litterman (1999) and Idzorek (2002)).

2.2 Quantitative Predictions

Performance of optimized portfolios rely upon generating reasonably good estimates of future asset returns and their covariance. Hereby we focus on the former aspect and attempt to estimate monthly returns for each asset i of the investment set defined by the investors existing portfolio. Methodological framework of generating quantitative predictions ($\hat{\mathbf{r}}$) relies on a classical ordinary least squares approach of the form:[2] $r_{t+1} = \alpha + \beta f_t + \varepsilon_{t+1}$. This model allows for any arbitrary factor (f) to be applied as a source of return forecasts.[3] In order to update the vector of prior return expectations according to quantitative predictions, one requires a quantification of the accuracy of predictions. We propose a method to derive confidences towards quantitative predictions directly from the linear regression and thereby provide an intuitive relation between the certainty levels and expected returns.

Hereto, we apply the goodness-of-fit measures to specify accuracy of predictions and plug them into a certainty matrix Ω, entailing a specific level of certainty for each asset. We utilize the adjusted R^2 as a measure unrelated to the number of independent variables in the equation and point out that it comes along with a virtually standardized 0-1 scale.[4] This enables us to unambiguously determine the quality of estimates and as such offers an intuitive solution for specifying elements of the Ω.

$$\Omega_t = \begin{pmatrix} 1-R^2_{1,t} & 0 & \cdots & 0 \\ 0 & 1-R^2_{2,t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-R^2_{i,t} \end{pmatrix}$$

Fabozzi, Focardi, and Kolm (2006) provide a brief theoretical introduction on incorporating factor models in a Bayesian setting but specify elements of Ω according to the variance of residuals. However, their methodology does not provide an intuitive scale as achieved by R^2 and applicability has not been empirically tested. Furthermore, Connor (1997) applies a

[2]Although we make use of the regular OLS approach, instead of classical standard errors we use adjusted measures as proposed by Newey and West (1987) to overcome heteroscedasticity of error terms during time series regressions.

[3]The only constrained regarding the predictor in the linear- or multiple regression is that the Gauss-Markov assumption have to be fulfilled.

[4]The model can be extended to a multi-factor form in order to increase predictive power. Therefore, we make use of the *adjusted* R^2 measure as correction for the degrees of freedom is especially necessary when performing this approach with multifactor forecasting. With regards to the standardized scale one has to differentiate between the standard- and adjusted R^2 measure, later accounting for the number of explanatory terms. Where the standard form always stays between 0-1 guaranteeing the standardized scale, the other can go out of bounds, but offers more accuracy on the explanatory power of the regression. Ultimately, this leads to a slight reconsideration of the range of feasible τ values.

shrinkage approach to the predictive regression in a Bayesian portfolio setting by recalibrating the regression coefficient according to R^2. Thereby, estimated coefficients are shrunk towards zero, which is intuitive given the application to the market portfolio and respective market efficiency. However, both approaches deviate from the one at hand in their implementation and intuition, respectively. We calculate the additive inverse of R^2 and add it to 1 in order to make the values appropriate to the nature of certainty.

Elements of Ω take large values for unreliable predictions, while small values correspond to less noisy forecasts. As a consequence elements indicating noisy predictions exhibit low effect on the EOP and vice versa. We generate a certainty value for every asset of the investment set and calibrate Ω for every out-of-sample period.

2.3 Posterior

Taking the prior return distribution $N \sim (\mathbf{z}, \tau\Sigma)$ we can subjoin the expected returns derived from predictions given by $N \sim (\hat{\mathbf{r}}, \Omega)$. Applying a rearranged version of Black and Litterman (1992) according to Da Silva, Lee, and Pornrojnangkool (2009, p.3) and adjusting the respective to meet the properties of this study, we derive the posterior return vector as follows:[5]

$$\mathbf{z}^* = \mathbf{z} + \Sigma \left[\frac{\Omega}{\tau} + \Sigma \right]^{-1} \cdot (\hat{\mathbf{r}} - \mathbf{z})$$

The revised return vector is constructed as a weighted average of prior and quantitative predictions according to the weighting factors τ and Ω.

In a final step the updated vector of expected returns is fed into the initial portfolio optimizer to generate the portfolio weights of the EOP. This is achieved by rearranging the formula for deriving the prior in order to back out the revised portfolio weights: $\mathbf{w}^* = (\gamma\Sigma)^{-1}\mathbf{z}^*$. We chose to apply the initial optimizer in order to derive a clear picture of the contribution generated by the quantitative predictions and certainty levels. Thereon, we analyze whether EOP's are capable of generating sustainable excess return over the market portfolio. Furthermore, we evaluate whether this excess return comes at the cost of additional downside risk. Due to unequal distribution characteristics we measure risk by second order raw and central lower partial moments (LPM) based on Nantell and Price (1982). We are particularly interested in the differences between the portfolios:

$$\Delta LPM_{\{h,g\}}(r) = \int_{-\infty}^{h} (r-h)^2 f_{EOP}(r)\, dr - \int_{-\infty}^{g} (r-g)^2 f_{MC}(r)\, dr$$

[5]In contrast to the standard form of the BL model, we drop identity matrix P which assigns the views to the respective assets. Given that we only have *absolute* return estimates on all assets at all times, we can drop this matrix. In the periods where the predictor yields weak predictions on certain assets, the according element of Ω will get close to 1 (i.e. no influence on the optimal portfolio), hence eliminating the need for the identification matrix.

We denote h and g to represent the respective target return, while j denotes the order of LPM and MC the market capitalization-weighted portfolio.

3 Simulation Results and Discussion

Before evaluating the model in an empirical setting, we test for its robustness and parameter sensitivity by means of a Monte-Carlo simulation. This allows us to evaluate the models adequateness on theoretical grounds under common statistical assumptions such as normally i.i.d. distributed variables. Furthermore, it allows a model evaluation under extreme conditions, referring to both positive and negative events. Especially, an in-depth sensitivity analysis yields valuable insights on the models characteristics and potential parameters of caution. Consequently, the setup of this simulation study is as arbitrary as possible – in terms of generating asset returns and the predicting factor – in order to identify and evaluate the deterministic factors of the model.

3.1 Setup and Sampling Properties

We calculate EOPs based on two equally-weighted assets, with returns r^A andr^B. Factor f is to deliver 'forecasts' for both series r^A and r^B based on two univariate OLS models. Independent variable f is a random series with arbitrary first and second order moments. Series r^A and r^B, as well as r^A and f are simulated based on bivariate normal distributions. Deterministic parameters of the model are the number of correlation breakdowns and variance of shocks (σ_u) affecting correlation of coefficient between r^A and f. The simulation set-up is laid out formally in Appendix A. . We consider correlation breakdowns, leading to a shift in moments as for the representation of genuine stock market crashes. Notation for variables is given in subscripts – possibly in braces when multiple series involved –, while parentheses in superscripts state breakdowns as total number of unique periods.

First we define correlation of assets A and B to follow a random walk process with mean 0.75 and upper and lower boundaries of 0.5 and 1, respectively. Furthermore, correlation of asset A and predictor f is also characterized by a stochastic process with an arbitrary mean and standard deviation that changes $k-1$ times after each breakdown and due to the nature of correlation is bounded between -1 and 1. Random shocks of both models follow normal distributions: $\eta \sim \mathcal{N}(0, 0.01)$ and $\upsilon \sim \mathcal{N}(0, \sigma_\upsilon)$.

$$
\begin{aligned}
\rho_{\{r_A,r_B\},t} &= \rho_{\{r_A,r_B\},t-1} + \eta_t \quad \text{s.t. } \rho_{\{r_A,r_B\},1} = 0.75 \text{ and } 0.5 \leq \rho_{\{r_A,r_B\}} \leq 1 \\
\rho^{(k)}_{\{r_A,f\},t} &= \rho^{(k)}_{\{r_A,f\},t-1} + \upsilon_t \quad \text{s.t. } \rho^{(k)}_{\{r_A,f\},1} \sim \mathcal{U}(-1,1) \text{ and } |\rho^{(k)}_{\{r_A,f\}}| \leq 1
\end{aligned}
$$

Now we are able to simulate observations r_A and r_B with covariance conditional on $\rho_{\{r_A,r_B\}}$. We set one asset to be riskier than the other with the following parameters: $\mu_{r_A}, \mu_{r_B} = 0$ and $\sigma_A = 0.35, \sigma_B = 0.15$.

$$\langle r_A, r_B \rangle \sim \mathcal{N}(\kappa, \Sigma_t), \text{ with } \kappa = \begin{pmatrix} \mu_{r_A} \\ \mu_{r_B} \end{pmatrix} \text{ and } \Sigma_t = \begin{pmatrix} \sigma^2_{r_A} & \sigma_{\{r_A,r_B\},t} \\ \sigma_{\{r_A,r_B\},t} & \sigma^2_{r_B} \end{pmatrix}$$

Finally, we generate individual moments for subperiods of predictor f in order to simulate correlation breakdowns. Means are based on a normal distribution around 0, while standard deviation – being strictly positive – follow a uniform distribution between 0 and 1. As such we not only have a variance-covariance matrix (Θ) that is time-varying, but also first and second moments are unique for k periods.

$$\mu_f^{(k)} \sim \mathcal{N}(0,1) \text{ and } \sigma_f^{(k)} \sim \mathcal{U}(0,1)$$

$$\left\langle r_A, f^{(k)} \right\rangle \sim \mathcal{N}\left(\xi^{(k)}, \Theta_t^{(k)}\right), \text{ with } \xi^{(k)} = \begin{pmatrix} \mu_{r_A} \\ \mu_f^{(k)} \end{pmatrix} \text{ and } \Theta_t^{(k)} = \begin{pmatrix} \sigma^2_{r_A} & \sigma^{(k)}_{\{r_A,f\},t} \\ \sigma^{(k)}_{\{r_A,f\},t} & \sigma_f^{(k)2} \end{pmatrix}$$

Hereafter, arbitrary asset returns and a random predictor draw on varying forecasting accuracy. Factor f is to deliver 'forecasts' for both series r_A and r_B based on two univariate OLS models. Deterministic parameters of the model are the number of correlation breakdowns and variance of shocks (σ_v) affecting correlation of coefficient between r_A and f. A sensitivity analysis of the two parameters is conducted with regards to the impact on excess return of EOP relative to the initial investors portfolio. Given this set-up, for any possible combination we run 100 simulations of 1'200 months of random asset returns each.

3.2 EOP Characteristics

Taking a random sample from the set of simulations Figure 1 indicates a representative case illustrating the models characteristics. From correlation levels we can assert the quality (goodness-of-fit) of our predictor with respect to the two assets. Given a high level of absolute correlation our predictor will have high predictive power and consequently our certainty levels assigned to the derived predictions will be high leading to a stronger tilt in portfolio weights and vice versa. This relation is clearest for the high correlation level in period two. EOP starts generating outperformance – after a calibration period identical to the estimation window – reflecting the increase in predictive quality and rising certainty levels derived from adjusted R^2.

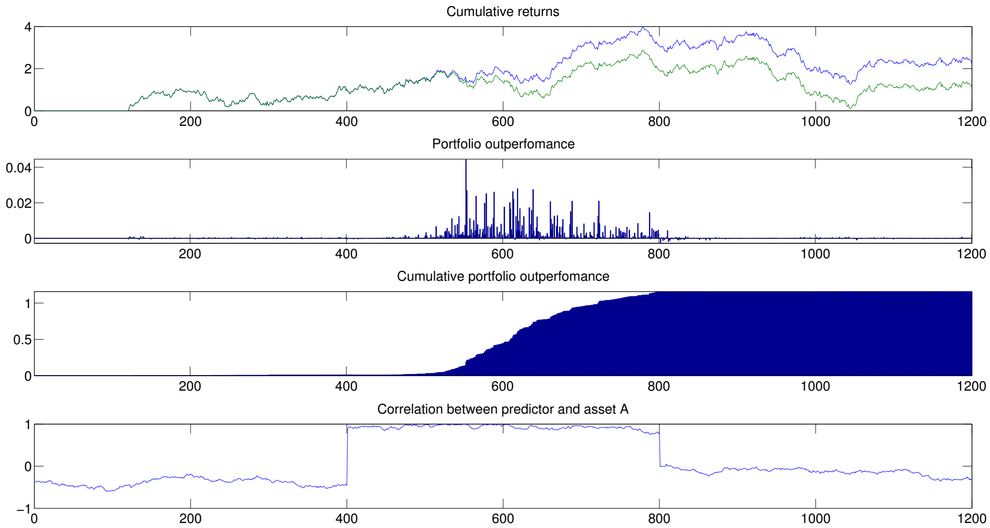

Figure 1: This figure is a composition of: (1) cumulative returns of an equally-weighted portfolio, (2) monthly outperformance of the EOP, (3) cumulative out-performance and (4) correlation and respective breakdowns between the dependently simulated asset A and predictor f. In this setting we enforce $k = 3$ correlation breakdowns, therefore, generating three periods of different correlation means equal to -0.4, +0.9 and 0. We set the variance of correlation to 0.05 for all levels to ensure simulated correlation is close to the set mean for each period.

Cumulative return plot reveals a constant widening of the gap until the second correlation breakdown 800 periods into the sample window. At this point the correlation drops to around 0 and consequently our predictor is jimmied for the remaining months. At this point, the significant contribution of our model comes into place and clarifies what makes this portfolio an EOP. By definition, the EOP cannot be restricted to exhibit temporarily modest negative returns subsequent to the correlation breakdown, during recalibration.

Given the significant correlation breakdown from almost 1 to 0, the model adapts fast by means of a drop in certainty levels assigned to the predictions thereby tilting the EOP towards its prior. Hence, the model is able to preserve the outperformance previously generated even during phases where the quantitative predictor is weak. This is clearly reflected by the constant gap over remaining 400 months where weights of the EOP are approximately identical to the optimal portfolio.

3.3 Sensitivity Analysis

Sensitivity of EOPs regarding the previously mentioned two deterministic parameters – holding everything else constant – is presented in Figure 2. We check for robustness of our assertion that an portfolio optimized by means of this method will generate on average excess return without the burden of additional downside risk – in form of raw lower partial moments – relative to the optimal market portfolio. We prove that these features prevail even when accounting for correlation breakdowns up to a yearly frequency along with rapidly repeating shifts in correlation between predictor and assets. On average positive excess return under both parameter variations can be reported.

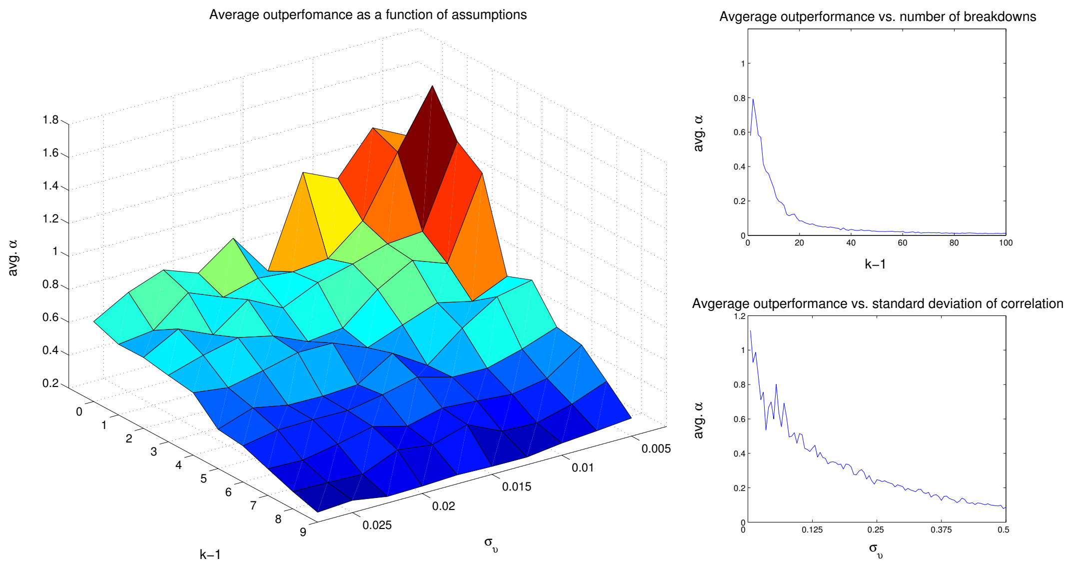

Figure 2: The surface plot shows 10'000 simulation of 100 per parameter combination. Each portfolio simulation is made up of 1'200 months of random asset returns. We depict values for variance of correlation (σ_υ) on the right-horizontal axis, number of correlation breakdown ($k-1$) on the left-horizontal axis and excess return of EOP over the initial (prior) portfolio on the vertical axis. Two graphs to the right are extended sensitivity plots towards the two deterministic parameters.

The surface plot is positive for all variations of either parameter and across all simulations, as indicated by the vertical axis. Right-horizontal axis represents a variation in the variance of correlation between predictor and assets and, therefore, reflects the predictive power of the indicator. Along an increase in variance of correlation, a decline in excess return is observable. This is reasonable as an increase in variation of correlation lowers the predictive quality of the factor. Furthermore, sensitivity towards correlation breakdowns is also according to our expectations. As for the variance of correlation, an increase in breakdowns leads to a reduction in excess return.

Overall, we show that the model is sensitive to the number of correlation breakdowns and variance of correlation between predictor and assets. However, excess returns are strictly positive on average for all tested combinations with a clear convergence towards 0. In terms of return this means an investor cannot be worse off over the long-run allocating according to the EOP. This is confirmed when accounting for lower partial moments, where simulation results prove that both first and second order raw LPMs as well as second order central LPMs are at the maximum equal to the optimal market portfolio. These findings are confirmed in the following empirical setting given the additional difficulty of non-normal equity return distributions.

4 Empirical Evidence

In a second step, we test the described approach on historical equity return observations to provide empirical evidence. Given the models positive characteristics on theoretical grounds, it is of particular interest to observe whether the model is also capable to perform empirically well given the statistically less favorable premises. In order to generate return estimates we make use of common factors of macroeconomic condition. The forecastability of equity returns by means of macroeconomic variables is well documented (N. Chen, Roll, and Ross (1986)). Given the strong interlinkage of economies make such factors a prime choice for the application to an international equity index portfolio. In this context we make use of various leading economic indicators and industrial production indices; these partially coincide with those tested by Sheppard (2008) and N. Chen et al. (1986). Additionally, we make use of a global shipping index which has been tested and shown significant forecasting power by Bakshi, Panayotov, and Skoulakis (2011). For the purpose of this paper we are less interested in the forecasting quality of these predictors, but rather in the models behavior when considering alternative inputs.

4.1 Data

We apply 22 FTSE country indices for an observation period from July 1988 to July 2012 as dependent variables in the predictive regression. Table 1 presents the summary statistics. All data is gathered from DataStream. For the purpose of generating equity return predictions based on indices, we make use of logarithmic growth rates. In this context we derive predictions from 8 different indices, namely: Baltic Dry Index[6] (BDI), Composite Leading Indicator (CLI) for Turkey and Taiwan, Leading Economic Indicators Index - U.S. Conference Board (LEI - USA) and Industrial Production Indices for manufacturing only and whole economy with respect to U.S., France and Bangladesh. We stick to Bakshi et al. (2011) who test for 1 and 3-month log changes as their BDI growth rates and find 3-month rates to yield superior results. For simplicity and without a loss in validity – remember it is not the focus of this paper to test whether these factors do indeed yield good predictions – we make use of 3-month growth rates for all predictors.

$$r_{i,[t\to t+1]} = \alpha_i + \beta_i \, g_{[t-3\to t]} + \varepsilon_{i,t+1}$$

[6]BDI prices are based on weighted averages of twenty global routes and not, as the name might suggest, only on routes around the Baltic states. BDI originated from the Baltic Freight Index (BFI), which was set up in May 1985 to provide a generally accepted base index for freight derivatives. In November 1999 the BFI was replaced by the BDI, which is a constitute representing the average price for the different vessel sizes.

With respect to the estimation window we initialize regressions with a 36-month rolling estimation window. The choice of an appropriate window size and application of a rolling window versus recursive approach is of on-going discussion and cannot be disregarded in the context of this study. For the purpose of this model we suggest a rolling window approach with a short window size in order to generate dynamic adjustments of confidence levels towards our predictions derived from the respective R^2 measure. The benefits of a quick and dynamic adjustment in confidence levels is at the hearth of this model, which is in-line with Rossi and Inoue (2011) reporting higher predictive quality alongside a decrease in window size for economic models.

	obs	mean	std. dev.	skew	kurt	max	min
			Predictors				
BDI	287	-1.97	118.89	-2.37	13.90	102.16	-228.25
CLI - Turkey	287	4.43	7.95	-0.26	4.40	10.19	-9.11
CLI - Taiwan	287	4.30	3.58	-0.05	1.78	3.78	-3.11
IP - USA	287	1.94	2.29	-1.62	7.95	2.12	-4.21
IPM - Bangladesh	287	2.25	2.59	-1.14	4.54	2.63	-3.59
IPM - France	287	0.23	4.44	-0.34	1.28	3.45	-5.19
IPM - USA	287	1.53	5.58	-0.58	3.12	4.50	-8.71
LEI - USA	287	1.59	2.57	-1.23	2.54	1.71	-3.09
			Country Indices				
Australia	287	4.68	28.11	-1.29	8.63	21.39	-46.99
Austria	287	4.73	20.95	-1.74	12.05	14.69	-42.13
Belgium	287	6.35	19.76	-1.05	7.34	19.14	-31.59
Canada	287	9.63	20.24	-0.88	5.93	15.87	-29.91
Denmark	287	5.49	21.00	-0.69	4.32	13.99	-26.19
France	287	5.72	23.90	-0.85	5.06	20.53	-28.39
Germany	287	7.73	27.03	-0.24	4.93	28.08	-34.17
Hong Kong	287	1.71	24.23	-0.92	5.46	17.40	-30.31
Ireland	287	0.59	25.58	-0.30	3.48	19.28	-26.62
Italy	287	-2.08	21.78	0.06	3.66	22.51	-20.69
Japan	287	3.18	24.60	-0.64	4.65	19.04	-29.51
Mexiko	287	8.13	17.65	-0.49	3.74	14.18	-17.25
Netherlands	287	4.15	17.33	-0.36	4.15	13.85	-21.28
New Zealand	287	16.93	32.69	-1.06	6.57	27.85	-43.25
Norway	287	5.00	21.00	-1.20	6.75	14.25	-32.16
Singapore	287	0.09	22.33	-0.37	4.00	24.40	-22.59
South Africa	287	5.26	21.26	-0.77	5.28	15.79	-30.31
Spain	287	7.29	27.47	-1.15	6.97	16.89	-39.65
Sweden	287	5.57	28.48	-0.46	6.09	33.37	-34.85
Switzerland	287	9.51	28.18	-0.70	4.54	21.01	-34.25
UK	287	8.74	26.54	-0.69	4.53	20.66	-32.02
USA	287	6.87	15.14	-0.76	4.56	10.56	-18.86

Table 1: Descriptive statistics presented here are based upon monthly excess returns denoted in US dollars ($) provided by FTSE. The sample contains returns between July 1988 to July 2012. Columns are denoted as follows: Number of observation (obs), annualized percentage mean (mean), annualized percentage standard deviation (std dev), skewness of time series (skew), kurtosis of time series (kurt), highest monthly percentage change (max), lowest monthly percentage change (min). we derive predictions from 8 different indices, namely: Baltic Dry Index (BDI), Composite Leading Indicator (CLI) for Turkey and Taiwan, Leading Economic Indicators Index - U.S. Conference Board (LEI - USA) and Industrial Production Indices for manufacturing only and whole economy with respect to U.S., France and Bangladesh.

4.2 Empirical EOP Realization

Empirical results of EOPs based on a global portfolio composed of 22 equity indices, enhanced by means of 8 alternative predictors yield consistent results alongside findings observable from the simulation. Tables 2 and 3 report figures for EOP portfolios in excess of the underlying capitalization-weighted market portfolio. We report excess figures for the ease of interpretation, as we expect positive signs for all columns except for standard deviation and lower partial moments. Furthermore, we report performance measures for each EOP across various levels of τ.

Note that particular interest is on analysing whether EOPs can generate excess return whilst constraining downside deviation in an empirical setting across all tested predictors and for varying levels of scalar τ. We show that EOPs can indeed generate excess return over the market portfolio whilst not experiencing additional downside risk. This results in a preferable risk-return relation of EOPs relative to the market portfolio, as reflected by positive excess Sharpe ratios.

To assess the impact of τ we show results for value specification between 0.01 and 0.5 for all predictors and can report consistent impact across all tested factors. As τ determines the acceptable deviation of EOP from its underlying and, therefore, influence of this factor is important to understand in order to specify it as optimal as possible. An increase in the scalar reflects an investor's uncertainty in the market portfolio, which results in a stronger tilt of the posterior distribution towards the quantitative return forecasts. Considering additional information in form of quantitative predictions shows to be advantageous, although, at different levels of significance.

An interesting pattern is observable. Whilst return increases throughout an increase in τ, standard deviation shows a U-shaped structure where EOPs can even decrease absolute portfolio risk up to a certain τ specification. The dependency of EOPs on the weighting factor τ is well known, however, our observations gives rise to the possibility that an 'optimal' value for τ exists at which the portfolios risk adjusted return is maximized. With respect to raw and central LPMs we observe a decrease of downside deviation for EOPs relative to the market portfolio. Therefore, EOPs not only generate excess return whilst keeping downside risk constant, but in fact reduce downside risk. This implies, that posterior return estimates underlying our EOPs do entail fewer estimation errors and the respective can be effectively incorporated into the portfolio construction.

The convexity of absolute EOP risk alongside an increase in τ is consistent across all predictors and – taking the BDI based EOP as an example – can be confirmed by Figure 3. This pattern can be explained by two aspects. First of all, the accuracy of predictions can be increased by incorporating additional information into return estimates and, thereby,

tau	0.01	0.05	0.1	0.2	0.3	0.4	0.5
			Baltic Dry Index				
mean	0.06	0.30	0.61	1.20	1.79	2.37	2.94
std. dev.	-0.05	-0.21	-0.38	-0.55	-0.53	-0.34	0.02
raw LPM (1)	-0.06	-0.30	-0.60	-0.87	-1.05	-1.23	-1.40
raw LPM (2)	-0.01	-0.06	-0.11	-0.19	-0.24	-0.27	-0.28
central LPM (2)	-0.01	-0.06	-0.10	-0.15	-0.19	-0.20	-0.19
min	0.37	1.84	3.67	6.27	6.12	5.96	5.81
max	0.02	0.11	0.21	0.42	0.62	5.95	11.32
Sharpe ratio	0.02	0.08	0.16	0.31	0.44	0.55	0.64
alpha	0.75	3.74	7.46	14.80	22.03	29.16	36.18
significance	1.64	1.64	1.65	1.66	1.66	1.67	1.68
tracking error	0.18	0.89	1.77	3.50	5.19	6.84	8.44
			Composite Leading Indicator - Turkey				
mean	0.08	0.39	0.77	1.52	2.25	2.96	3.66
std. dev.	-0.07	-0.30	-0.50	-0.60	-0.32	0.30	1.18
raw LPM (1)	-0.08	-0.42	-0.77	-1.11	-1.41	-1.46	-1.42
raw LPM (2)	-0.02	-0.09	-0.16	-0.25	-0.28	-0.28	-0.27
central LPM (2)	-0.02	-0.08	-0.14	-0.20	-0.22	-0.19	-0.16
min	0.68	3.41	6.62	6.65	6.69	6.72	6.75
max	0.15	0.75	1.48	2.92	6.28	13.55	20.58
Sharpe ratio	0.02	0.10	0.20	0.38	0.52	0.62	0.69
alpha	0.96	4.78	9.51	18.79	27.85	36.72	45.39
significance	1.48	1.48	1.48	1.48	1.49	1.49	1.49
tracking error	0.26	1.27	2.52	4.98	7.37	9.69	11.96
			Composite Leading Indicator - Taiwan				
mean	0.20	1.00	1.98	3.90	5.76	7.57	9.34
std. dev.	-0.09	-0.38	-0.58	-0.46	0.31	1.59	3.22
raw LPM (1)	-0.14	-0.70	-1.39	-2.36	-2.83	-3.21	-3.44
raw LPM (2)	-0.04	-0.17	-0.29	-0.43	-0.48	-0.50	-0.50
central LPM (2)	-0.03	-0.14	-0.24	-0.32	-0.31	-0.28	-0.22
min	1.06	5.26	6.66	6.73	6.80	6.87	6.94
max	0.37	1.82	3.60	8.24	13.06	17.68	22.11
Sharpe ratio	0.05	0.25	0.49	0.91	1.23	1.44	1.56
alpha	2.44	12.11	24.04	47.39	70.10	92.18	113.69
significance	2.80	2.80	2.80	2.80	2.80	2.81	2.81
tracking error	0.34	1.70	3.37	6.64	9.82	12.91	15.92
			Leading Economic Indicators Index - U.S. Conference Board				
mean	0.07	0.37	0.73	1.45	2.16	2.85	3.52
std. dev.	-0.09	-0.40	-0.74	-1.20	-1.38	-1.28	-0.94
raw LPM (1)	-0.09	-0.43	-0.85	-1.50	-1.98	-2.24	-2.35
raw LPM (2)	-0.02	-0.11	-0.20	-0.33	-0.39	-0.41	-0.41
central LPM (2)	-0.02	-0.10	-0.18	-0.29	-0.34	-0.34	-0.32
min	0.66	3.30	6.53	6.47	6.40	6.34	6.28
max	-0.19	-0.73	-0.96	-0.47	0.36	1.17	3.85
Sharpe ratio	0.02	0.11	0.21	0.42	0.60	0.76	0.88
alpha	0.92	4.60	9.15	18.12	26.91	35.53	43.99
significance	1.80	1.80	1.80	1.80	1.81	1.81	1.81
tracking error	0.21	1.06	2.10	4.15	6.16	8.12	10.03

Table 2: Table presents excess values of EOP performance relative to the underlying capitalization-weighted prior. All values are differences of annualized percentages, except minimum and maximum monthly returns, annualized alpha values and alpha significance levels (p-values). Results are based on the out-of-sample portfolio performance between November 1991 and July 2012.

tau	0.01	0.05	0.1	0.2	0.3	0.4	0.5
Industrial Production (Manufacturing) - France							
mean	0.01	0.03	0.06	0.12	0.18	0.25	0.31
std. dev.	-0.02	-0.11	-0.22	-0.39	-0.51	-0.60	-0.65
raw LPM (1)	-0.021	-0.107	-0.210	-0.394	-0.566	-0.682	-0.793
raw LPM (2)	-0.004	-0.019	-0.036	-0.066	-0.088	-0.104	-0.112
central LPM (2)	-0.004	-0.019	-0.036	-0.064	-0.086	-0.100	-0.107
min	0.07	0.33	0.66	1.31	1.95	2.58	3.20
max	-0.11	-0.54	-1.08	-1.66	-1.39	-1.13	-0.87
Sharpe ratio	0.00	0.01	0.03	0.05	0.07	0.09	0.11
alpha	0.09	0.44	0.89	1.79	2.71	3.64	4.59
significance	0.43	0.43	0.44	0.45	0.45	0.46	0.47
tracking error	0.08	0.40	0.80	1.59	2.36	3.12	3.86
US Total Industrial Production Index							
mean	0.08	0.41	0.82	1.63	2.43	3.23	4.01
std. dev.	-0.16	-0.70	-1.17	-1.39	-0.68	0.79	2.80
raw LPM (1)	-0.12	-0.60	-1.20	-1.93	-1.94	-1.70	-1.47
raw LPM (2)	-0.04	-0.18	-0.29	-0.34	-0.33	-0.31	-0.26
central LPM (2)	-0.04	-0.17	-0.27	-0.30	-0.26	-0.21	-0.14
min	1.51	6.44	6.30	6.01	5.73	5.44	5.16
max	-0.38	-0.51	-0.54	-0.58	12.17	26.43	40.47
Sharpe ratio	0.03	0.13	0.26	0.48	0.60	0.64	0.62
alpha	1.06	5.29	10.54	20.96	31.25	41.42	51.46
significance	1.14	1.15	1.15	1.15	1.16	1.16	1.17
tracking error	0.38	1.92	3.81	7.54	11.19	14.77	18.27
Industrial Production (Manufacturing) - U.S.							
mean	0.00	0.00	0.01	0.02	0.04	0.06	0.09
std. dev.	-0.02	-0.10	-0.19	-0.31	-0.36	-0.35	-0.29
raw LPM (1)	-0.02	-0.11	-0.23	-0.45	-0.65	-0.79	-0.92
raw LPM (2)	0.00	-0.01	-0.02	-0.03	-0.03	-0.01	0.01
central LPM (2)	0.00	-0.01	-0.02	-0.03	-0.03	-0.02	0.01
min	-0.13	-0.65	-1.29	-2.56	-3.82	-5.06	-6.29
max	-0.04	-0.22	-0.44	-0.87	-1.12	-0.94	-0.76
Sharpe ratio	0.00	0.01	0.01	0.02	0.03	0.03	0.04
alpha	0.02	0.10	0.22	0.49	0.80	1.15	1.55
significance	0.08	0.08	0.09	0.10	0.11	0.11	0.12
tracking error	0.10	0.50	1.00	1.98	2.94	3.89	4.81
Industrial Production (Manufacturing) - Bangladesh							
mean	0.05	0.22	0.45	0.89	1.33	1.77	2.20
std. dev.	-0.10	-0.48	-0.87	-1.38	-1.52	-1.30	-0.76
raw LPM (1)	-0.08	-0.41	-0.82	-1.49	-1.99	-1.89	-1.69
raw LPM (2)	-0.03	-0.12	-0.21	-0.32	-0.34	-0.32	-0.28
central LPM (2)	-0.02	-0.11	-0.20	-0.30	-0.31	-0.27	-0.22
min	0.82	4.09	6.46	6.34	6.21	6.09	5.97
max	-0.31	-0.47	-0.45	-0.41	-0.37	-0.16	7.53
Sharpe ratio	0.02	0.08	0.15	0.30	0.42	0.50	0.55
alpha	0.59	2.93	5.84	11.61	17.33	22.99	28.59
significance	1.01	1.02	1.02	1.03	1.03	1.04	1.04
tracking error	0.24	1.20	2.39	4.73	7.02	9.25	11.45

Table 3: Table presents excess values of EOP performance relative to the underlying capitalization-weighted prior. All values are differences of annualized percentages, except minimum and maximum monthly returns, annualized alpha values and alpha significance levels (p-values). Results are based on the out-of-sample portfolio performance between November 1991 and July 2012.

reduce estimation errors entailed in solely market implied return estimates. Furthermore, we consider an unconstrained investor which can results in EOPs including short-selling. The degree of short-selling depends of course on the allowable divergence from the market portfolio, represented by scalar τ. Therefore, part of this pattern can be explained by the fact that a certain degree of short-selling can reduce portfolio risk and improve efficiency (Grinold and Kahn (2000), Xu (2007)). In practice various strategies exist – e.g. 130/30, market neutral and long/short – which make use of the preferable portfolio properties achievable by relaxing the short-selling constraint in order to reduce portfolio risk and/or increase return.

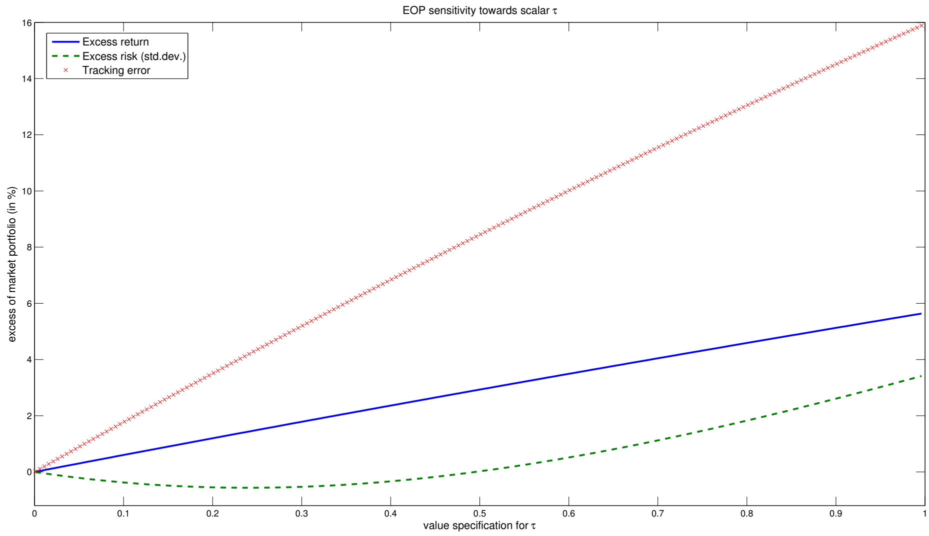

Figure 3: This figure provides a graphical presentation of EOP sensitivity towards a change in weighting factor τ with respect to excess return, excess risk and tracking error (TE). Parameter τ is varied between 0.001 and 1.0 with 200 iterations. This is an illustrative case for EOPs enhanced via BDi derived predictions. Results are representative for the cases of alternatively tested predictors.

Furthermore, we find an almost linear relationship between scalar τ and tracking error with a correlation coefficient of 0.99 and a ratio of approximately 1:16. This implies that investors and/or portfolio managers can make use of τ explicitly to set a tracking error tolerance. Again these finding are consistent across all tested predictors, although, different ratios regarding τ and TE are observable. This feature is especially interesting for the field of enhanced indexing, as an acceptable degree of TE can be specified through scalar τ and, consequently, EOPs yield a certain level of excess return and risk corresponding to the predefined level of TE.

We could show that the market portfolio is not optimal per se and that the consideration of quantitative predictions based on various factors can improve return forecasting accuracy in a portfolio context. Furthermore, we showed that the introduced model in this paper can make use of improved return estimates by enhancing the market portfolio with respect to an increase in return and a constraint towards downside deviation.

The Case for BDI

As a representative sample of the 8 tested predictors, we take a closer look at BDI derived EOPs. Given its extreme jerkiness and unfavorable statistical properties makes the analysis of this predictor particularly insightful. Figure 4 presents portfolio return characteristics on a monthly and cumulated basis. EOP can generate excess return over its market-capitalization weighted prior, driven by the level of certainty in the predictions. Interestingly, the BDI entails low cross-sectional predictive power over large periods of the sample and overall excess return is generated over just a few months. A first conclusion can be drawn on the BDIs added value in a portfolio context. BDI does not generate consistent and reliable cross-sectional equity forecasts over time and given its jerkiness entails detrimental characteristics for application in a portfolio setting. However, since the predictive quality of BDI or any other factor is not at the center of this study, we are still able to show that the models beneficial properties even hold when confronted with unfavorable predictions.

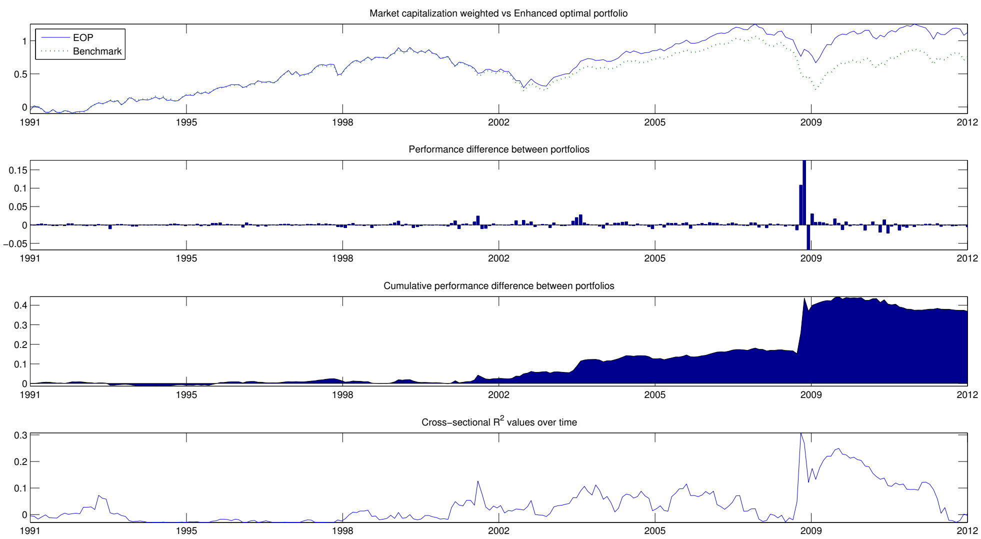

Figure 4: This figure provides a graphical presentation of the results in form of a composite of three separate plots: (1) cumulated portfolio performance, (2) Performance difference between optimal and enhanced portfolio and (3) cumulative performance difference.

Concentrating on plot 3 and 4 of Figure 4 the direct link between cumulated excess return of EOP and cross-sectional adjusted R^2 towards return predictions is observable. BDIs forecasting quality picks up over time, showing an extreme rise to 30% at the end of 2008 and a decline in forecasting power in the aftermath. These jumps and respective impact on the EOP is of particular interest. As expected EOP can generate outperformance where certainty levels are high and even more importantly the model can retain previously generated return – as previously confirmed on theoretical and simulated grounds – even where certainty levels drops. This adjustment speed is highly dependent on the in-sample window and comes at the costs of including fewer observations. Therefore, the model is in-line with the well-known specification problem of estimation window length as reported, as part of an extensive study, by Goyal and Welch (2008).

The pattern observable in the in-depth analysis for the EOP based on BDI derived return forecast is consistent across all tested factors. Of course the level of excess return deviates between predictors, however, all EOPs are capable of generating excess return where predictive power measured by adjusted R^2 increases and is capable of retaining historically generated excess return at a high level even where forecasting power drops sharply. Furthermore, excess return generated by EOPs over the market portfolio could be generated without experiencing additional downside risk. These findings suggest that the market portfolio is not risk-return optimal at all times and portfolios constructed according to the introduced methodology provide a basis upon which these inefficiency can be exploited.

5 Conclusion

We show that the market portfolio is not optimal as proposed by modern portfolio theory and that it can be enhanced in terms of capturing and retaining excess return without the burden of additional downside risk. This is achieved by employing a Bayesian framework and allowing a prior to be enhanced by means of quantitative predictions. We show that the accuracy of predictions can be directly derived from a linear regression in form of adjusted R^2, which established an implicit dependency between quantitative forecasts and their weight on the mixed estimation. Consequently, the model self-adjusts rapidly to changing market conditions by tilting weights towards the underlying portfolio to protect investors from experiencing additional downside deviation relative to their prior.

The model has been tested in a simulated environment under favorable statistical properties – such as normally i.i.d. distributed returns and stationary of predictions – as well as in an empirical setting entailing less favorable statistical properties. Both analysis yield consistent results and provide evidence in favor of two model characteristics: (1) the model is capable of identifying forecasting quality of the quantitative predictor and adjusts itself accordingly, and (2) EOPs based on this new approach are capable of generating excess return and retaining the respective over the long-run whilst not experiencing additional downside risk measured by raw and central LPMs.

References

Arnott, R., Hsu, J., & Moore, P. (2005). Fundamental indexation. *Financial Analysts Journal*, *61(2)*, 83-99.

Bakshi, G., Panayotov, G., & Skoulakis, G. (2011). *The Baltic Dry Index as a Predictor of Global Stock Returns, Commodity Returns, and Global Economic Activity* (Working Paper). McDonough School of Business, Georgetown University.

Black, F., & Litterman, R. (1992). Global Portfolio Optimization. *Financial Analysts Journal*, *48*(5), 28-43.

Chen, C., Chen, R., & Bassett, G. W. (2007). Fundamental indexation via smoothed cap weights. *Journal of Banking & Finance*, *31 (11)*, 3486-3502.

Chen, N., Roll, R., & Ross, S. A. (1986). Economic forces and the stock market. *The Journal of Business*, *3*, 383-403.

Clarke, R., de Silva, H., & Steven Thorley, T. Q. . V. . N. . . p. .-. (2006). Minimum-variance portfolios in the u.s. equity market. *Journal of Investment Management*, *4(3)*, 54-72.

Connor, G. (1997). Sensible return forecasting for portfolio management. *Fianncial Analsyt Journal*, *54(5)*, 44-51.

Da Silva, A., Lee, W., & Pornrojnangkool, B. (2009). The Black-Litterman Model for Active Portfolio Management. *The Journal of Portfolio Management*, *35*(2), 61-70.

DeMiguel, V., Garlappi, L., & Uppal, R. (2009). Optimal versus Naive Diversification: How Inefficient is the 1/N Portfolio Strategy? *The Review of Financial Studies*, *22*(5), 1915-1953.

Fabozzi, F., Focardi, S., & Kolm, P. (2006). Incorporating Trading Strategies in the Black-Litterman Framework. *The Journal of Trading*, *1*(2), 28-37.

Goyal, A., & Welch, I. (2008). A Comprehensive Look at the Empirical Performance of Equity Premium Prediction. *The Review of Financial Studies*, *21*(4), 1455-1508.

Grinold, R. C., & Kahn, R. (2000). The surprising large impact of the long-only constraint. *Barclays Global Investors Investment Insights. May*.

He, G., & Litterman, R. (1999). *The Intuition Behind Black-Litterman Model Portfolios* (Tech. Rep.). Goldman, Sachs & Co. (Investment Management Research).

Idzorek, T. M. (2002). A step-by-step Guide to the Black-Litterman Model: Incorporating user-specified Confidence Levels. In *Forecasting Expected Returns in the Financial Markets* (p. 17-38). Elsevier.

Mankert, C. (2006). *The Black-Litterman Model* (Tech. Rep.). School of Industrial Engineering and Management, Royal Institute of Technology.

Nantell, T., & Price, K. (1982). Mean-lower partial moment asset pricing model: Some empirical evidence. *Journal of Financial and Quantitative Analysis*, *17(5)*, 763-782.

Newey, W., & West, K. (1987). A Simple, Positive Semi-Definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix. *Econometrica*, 703-708.

Rossi, B., & Inoue, A. (2011). Out-of-sample forecast tests robust to the window size choice. *Working Paper*.

Sheppard, K. (2008). Economic factors and the covariance of equity returns. *Working Paper*.

Xu, P. (2007). Does relaxing the long-only constraint increase the downside risk of portfolio alphas? *Journal of Investing*, *16(1)*, 43-50.

A Appendix

A.1 Specification of certainty Matrix Ω

By choosing a classical ordinary least squares (OLS) estimator it is easy to show the relationship between high certainty levels and reliable estimations. For simplicity we use assume regression equations with the same number of independent variables k across indices and also consider time series with the equal lengths.

$$\Omega_t = \begin{pmatrix} 1-R^2_{1,t} & 0 & \cdots & 0 \\ 0 & 1-R^2_{2,t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-R^2_{i,t} \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} \frac{1/(t-k)\sum_{j=1}^{t}\varepsilon^2_{1,j}}{1/(t-1)\sum_{j=1}^{t}(r_{1,j}-\bar{r}_1)^2} & 0 & \cdots & 0 \\ 0 & 1-R^2_{2,t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-R^2_{i,t} \end{pmatrix} \tag{2}$$

$$= \begin{pmatrix} \frac{t-1}{t-k}[\sum_{j=1}^{t}(r_{1,j}-\bar{r}_1)^2]^{-1}\sum_{j=1}^{t}\varepsilon^2_{1,j} & 0 & \cdots & 0 \\ 0 & 1-R^2_{2,t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-R^2_{i,t} \end{pmatrix} \tag{3}$$

Once we substitute the formula for adjusted R^2 into the equation we can back out a matrix that represents diagonal elements of the covariance matrix of the returns, thus its variances.

$$= \frac{1}{t-k} \begin{pmatrix} \sigma^2_1 & 0 & \cdots & 0 \\ 0 & \sigma^2_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2_i \end{pmatrix}^{-1} \begin{pmatrix} \sum_{j=1}^{t}\varepsilon^2_{1,j} & 0 & \cdots & 0 \\ 0 & \sum_{j=1}^{t}\varepsilon^2_{2,j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{j=1}^{t}\varepsilon^2_{i,j} \end{pmatrix} \tag{4}$$

$$= \frac{1}{t-k}\, diag(\Sigma)^{-1} \begin{pmatrix} \sum_{j=1}^{t}\varepsilon^2_{1,j} & 0 & \cdots & 0 \\ 0 & \sum_{j=1}^{t}\varepsilon^2_{2,j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{j=1}^{t}\varepsilon^2_{i,j} \end{pmatrix} \tag{5}$$

Last term of (5) is a matrix with the very elements we minimize during the OLS procedure when predicting equity returns. As the variance and the scalar are predetermined by the sample it is easy to see that our linear predictor singularly maximizes certainty by minimizing Ω.

A.2 Restructuring the Black-Litterman equation

To isolate τ and Ω in the initial Black-Litterman equation we restructure the formula, closely following Mankert (2006):

$$\mathbf{z}^* = \Big[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\Big]^{-1}\Big[(\tau\Sigma)^{-1}\mathbf{z} + P'\Omega^{-1}\hat{\mathbf{r}}\Big] \tag{6}$$

First we multiple (6) with $\tau\Sigma$ and its inverse as an identity matrix:

$$= \Big[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\Big]^{-1}(\tau\Sigma)^{-1}(\tau\Sigma)\Big[(\tau\Sigma)^{-1}\hat{\mathbf{r}} + P'\Omega^{-1}\mathbf{z}\Big] \tag{7}$$

$$= \Big[I + \tau\Sigma P'\Omega^{-1}P\Big]^{-1}\Big[\hat{\mathbf{r}} + \tau\Sigma P'\Omega^{-1}\mathbf{z}\Big] \tag{8}$$

Now we extend the second term by $\tau\Sigma P'\Omega^{-1}P\hat{\mathbf{r}}$ and its additive inverse:

$$= \Big[I + \tau\Sigma P'\Omega^{-1}P\Big]^{-1}\Big[\hat{\mathbf{r}} + \tau\Sigma P'\Omega^{-1}\mathbf{z} + \tau\Sigma P'\Omega^{-1}P\hat{\mathbf{r}} - \tau\Sigma P'\Omega^{-1}P\hat{\mathbf{r}}\Big] \tag{9}$$

$$= \Big[I + \tau\Sigma P'\Omega^{-1}P\Big]^{-1}\Big[\big(I + \tau\Sigma P'\Omega^{-1}P\big)\hat{\mathbf{r}} + \tau\Sigma P'\Omega^{-1}(\mathbf{z} - P\hat{\mathbf{r}})\Big] \tag{10}$$

Once again we multiple with an identity matrix, this time by $\Omega + P'\tau\Sigma P$ and its inverse:

$$= \Big[I + \tau\Sigma P'\Omega^{-1}P\Big]^{-1}\cdots \tag{11}$$

$$\Big[\big(I + \tau\Sigma P'\Omega^{-1}P\big)\hat{\mathbf{r}} + \tau\Sigma P'\Omega^{-1}(\Omega + P'\tau\Sigma P)(\Omega + P'\tau\Sigma P)^{-1}(\mathbf{z} - P\hat{\mathbf{r}})\Big] \tag{12}$$

$$= \Big[I + \tau\Sigma P'\Omega^{-1}P\Big]^{-1}\cdots \tag{13}$$

$$\Big[\big(I + \tau\Sigma P'\Omega^{-1}P\big)\hat{\mathbf{r}} + \big(I + \tau\Sigma P'\Omega^{-1}P\big)\tau\Sigma P(\Omega + P'\tau\Sigma P)^{-1}(\mathbf{z} - P\hat{\mathbf{r}})\Big] \tag{14}$$

After some simple algebra we obtain a modified form of the equation with unified τ and Ω variables:

$$= \mathbf{z} + \tau\Sigma P(\Omega + P'\tau\Sigma P)^{-1}(\mathbf{z} - P\hat{\mathbf{r}}) \tag{15}$$

$$\mathbf{z}^* = \mathbf{z} + \Sigma P\Big[\frac{\Omega}{\tau} + P'\tau\Sigma P\Big]^{-1}(\mathbf{z} - P\hat{\mathbf{r}}) \tag{16}$$

Given that we hold *absolute* views on all assets at all times we can further simplify by dropping out P, which is the identity matrix assigning views to the respective asset where one does not hold a prediction on each asset or states *relative* views:

$$\mathbf{z}^* = \mathbf{z} + \Sigma\Big[\frac{\Omega}{\tau} + \tau\Sigma\Big]^{-1}(\mathbf{z} - \hat{\mathbf{r}}) \tag{17}$$