Cryptocurrency, Mining Pools' Concentration, and Asset Prices

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Abstract

This paper introduces mining pools' concentration into a dynamic asset pricing equilibrium. Our theory builds on the intuitive yet novel insight that modeling competition requires that mining pools clear markets separately from their competitors in equilibrium. Our model predicts that as concentration increases, the cryptocurrency price falls, and its volatility spikes — in line with our empirical analysis of Bitcoin. We further reveal that entry and exit of mining pools affect prices only through their effect on concentration. Lastly, equilibrium reveals that mining pools' total revenues determine the cryptocurrency's value, and if a pricing bubble exists, it amplifies the concentration effects.

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1 Introduction

Cryptocurrencies are a new asset class that differs from classical asset classes, such as commodities, equities, bonds, and currencies, in several critical aspects. Due to these differences, traditional asset pricing models are potentially inadequate for studying cryptocurrencies' asset pricing characteristics, most notably prices and volatilities. This paper proposes a continuous-time asset pricing model that studies mining pools' concentration — one central aspect of cryptocurrencies that has yet to be analyzed.

The vast majority of cryptocurrencies rely on permissionless blockchain protocols. These protocols allow transferring cryptocurrency ownership without relying on the traditional banking system. In permissionless blockchains, transaction validators take the role of banks and intermediaries and add new transactions to the blockchain over time. The blockchain randomly chooses a single transaction validator to record the next block of transactions and rewards it for its effort. Since it is a permissionless blockchain and anyone can become a transaction validator, it is highly competitive. In proof of work blockchains, like Bitcoin, the transaction validators are referred to as miners or, more broadly, mining pools.[1] To increase the probability of recording the next block of transactions, mining pools increase their computing power relative to other mining pools, potentially instigating an arms race. Indeed, in recent years, advances in mining technology have intensified Bitcoin mining pools' competition, resulting in significant variations in mining pools' concentration over time, as Figure 1 illustrates.[2] The main aspect of mining pools central to our analysis is their role as transaction validators, those who increase their cryptocurrency holdings even if they never trade the cryptocurrency.

Our model predicts that as mining pools' concentration increases, the cryptocurrency price falls, and its volatility spikes when the market participants are price takers. We provide novel empirical evidence verifying these predictions on Bitcoin. In line with our prediction

[1]A mining pool is a group of miners that share computing power and split rewards among its members. Mining pools are advantageous since they smooth rewards to their members over time and, as a result, dominate many blockchain ecosystems. Please refer to Cong, He, and Li (2021a) for an extensive discussion. Although we call the transaction validators mining pools throughout the paper, our model's predictions apply to other permissionless blockchain protocols in which the concentration of transaction validators varies over time due to exogenous shocks.

[2]Please see https://bitcoinmagazine.com/business/btc-coms-bitcoin-mining-pool-dominance-threatened-by-poolin for anecdotal evidence.

and evidence: (i) Makarov and Schoar (2021) show that Bitcoin price and mining pools' concentration are negatively related; (ii) Gabaix (2011) and Herskovic, Kelly, Lustig, and Nieuwerburgh (2020) show that volatility and concentration are positively related in publicly listed firms. However, recent evidence from the US product markets runs against our prediction and the evidence in Bitcoin. Grullon, Larkin, and Michaely (2019) among others recently documented that higher concentration increases returns.

Further, our theory predicts, and the empirical analysis verifies, that mining pools' concentration is an essential factor affecting cryptocurrency returns — a factor that the current empirical literature has not yet explored. Liu and Tsyvinski (2021) find that cryptocurrency returns are exposed to cryptocurrency network factors but not cryptocurrency production factors. Their paper constructs production factors of cryptocurrency to proxy for the cost (electricity and computing power) of mining. Consistent with our theory, they show that technological fundamentals affect cryptocurrency valuations.

Lastly, we show that the cryptocurrency pricing implications are similar on the extensive and intensive margin: the entry and exit of mining pools do not affect prices insofar as through their effect on concentration.

We assume that a cryptocurrency's real value — or at least a fraction of it if there is an inflationary bubble — is determined by the sum of all the discounted future services it will provide. These services take several forms, including access to real economies worldwide, and transferring funds across borders and between entities while remaining anonymous.[3]

Mining pools are endowed with these cryptocurrency services over time based on their size relative to the other mining pool, which we refer to as *mining*. When one mining pool size is bigger than the other, it mines more of the cryptocurrency. In addition, our economy features a household sector that cannot mine but instead can trade with mining pools to access the cryptocurrency services. Changes to mining pools' sizes over time determine their concentration and is a critical determinant of the equilibrium.

Our theory builds on the intuitive yet novel insight that introducing competition requires that each mining pool clears its mined cryptocurrency services separately from its competitors in equilibrium. Accordingly, mining pools post fees for trading cryptocurrency services

[3]Our view is similar to Di Tella (2020)'s view in that the real value of money is the present value of expenditures on its liquidity services and is consistent with Cong, Li, and Wang (2021b)s' fundamental-based view in which the demand for transactional benefits determines the value of cryptocurrency tokens.

with other market participants. If market participants demand too many services from a particular mining pool, it increases its fees. Instead, if market participants demand too few services, the mining pool decreases its fees. The process continues until, eventually, supply meets demand in every mining pool individually, and equilibrium reveals the market-clearing fees.

We achieve market clearing at the mining pool level by viewing services that mining pools mine as different goods. Imagine a shop that sells services from two separate service providers (like mining pools). To clear their supply entirely, the two service providers would have to set up relative prices proportional to the inverse of their relative supply; otherwise, an arbitrage opportunity exists. So, if the first supplier has half of the supply of the second supplier, the price for its services would have to be double the price of the second supplier. The cryptocurrency is then a claim to all the future combined services the shop provides. Accordingly, the equilibrium shows that the mining pools' fees are inversely related to their size, which corroborates Cong et al. (2021a)s' findings.

Since each mining pool posts fees to clear services separately from the other mining pools, the sum of all the discounted future services can be represented by the mining pools' revenues: mining pools' fees times their services. The relationship between mining pools' revenues and cryptocurrency prices is instrumental and key to our findings. One novel and potentially testable implication of our model is that the fundamental value of the cryptocurrency is determined by the total revenues of the mining pools. In line with this prediction, Bolt and Van Oordt (2020) documents an inverse relationship between daily volatility of USD/Bitcoin exchange rate and Bitcoin transaction volume, which is a reasonable proxy for mining pools' reveneus.

The prediction arises because exogenous changes to mining pool sizes have opposing effects on mining pools' concentration and total revenues. A shock that increases the sum of mining pools' revenues reduces mining pools' concentration. The endogenous, equilibrium relationship between concentration and total revenues is critical because it establishes the relationship between mining pools' concentration — an empirically observable quantity — and cryptocurrency price and volatility. In our primary analysis, we measure concentration with the established Herfindahl-Hirschman index (HHI).

Our framework allows the cryptocurrency price to differ from its intrinsic, fundamental

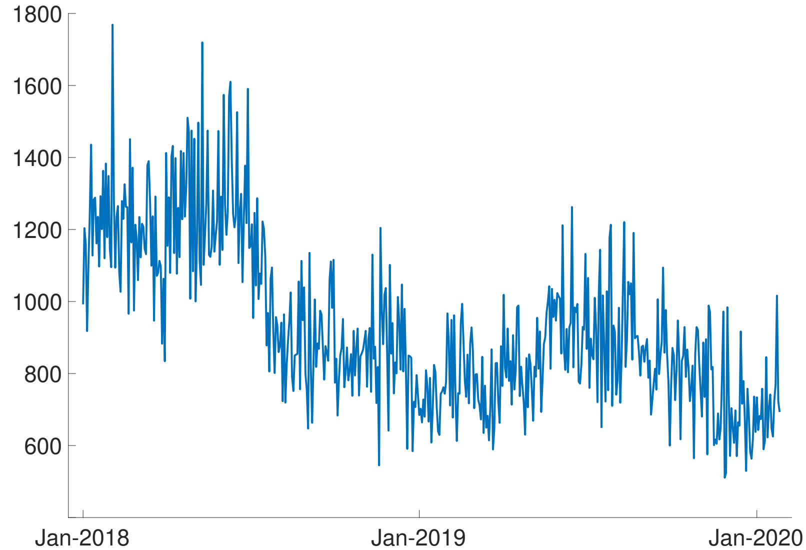

Figure 1. Herfindahl-Hirschman Index over time. A time series plot of daily mining pools' concentration starting from January 2018 to February 2020. To obtain the daily mining pool data, we scrapped https://bitcoinchain.com/pools. When we calculate the Herfindahl-Hirschman index we assume that unknown mining pools are small and each one can either mine one or zero blocks per day.

value and form a bubble. When the bubble is inflationary, only a fraction of the cryptocurrency value is determined by the sum of all the discounted future services it will provide. However, as long as this fraction is strictly positive and at least a tiny fraction of the cryptocurrency price is determined by its fundamentals, our model's predictions apply. Our model predicts that the bubble amplifies the effect of mining pools' concentration on the cryptocurrency price and its return volatility. We model the bubble by allowing the current cryptocurrency price to depend on both future cryptocurrency prices and future services.[4]

The remainder of the paper is organized as follows. Section 2 summarizes the literature; Section 3 sets up the economy with mining pools and competition; Section 4 discusses the equilibrium mechanism; Section 5 introduces the effects of concentration on the cryp-

[4]Tirole (1985) first introduced the interdependency between services and prices to analyze the value of money, and more recently, Biais, Bisière, Bouvard, Casamatta, and Menkveld (2020) adopted it to cryptocurrencies.

tocurrency price and volatility and the effects of the bubble; Section 6 provides empirical support to our main predictions using Bitcoin; Section 7 investigates entry and exit; Section 8 concludes.

2 Related Literature

Our paper fits into the literature studying the asset pricing implications of cryptocurrencies. Despite empirical work on cryptocurrency prices and volatilities, we still know very little about their determinants and, specifically, little about the effects of mining pools' concentration. Our paper is most closely related to Pagnotta (2021) and Biais et al. (2020). Pagnotta (2021) studies the joint determination of bitcoin prices and blockchain security using a game-theoretic setup, and Biais et al. (2020) study the bitcoin equilibrium price using an overlapping generation model with miners, hackers, and investors. These papers emphasize the security aspect of the blockchain as a determinant of cryptocurrency prices, while this paper emphasizes the mining pools' concentration.

Further, Cong et al. (2021b) study the demand side network effects of user adoption on a cryptocurrency token value. They show that the token price determines both the transactional benefit and the intertemporal carry cost of holding tokens, which incentivizes the platform participants to either hold or sell the tokens and, eventually, to pin down the token price through market-clearing with heterogeneous participants. Athey, Parashkevov, Sarukkai, and Xia (2016) model Bitcoin as a medium of exchange of unknown quality that allows users to avoid bank fees when sending remittances. They suggest that Bitcoin exchange rates can be fully determined by two market fundamentals: the steady-state transaction volume of Bitcoin when used for payments and the evolution of beliefs about the likelihood that the technology survives. Schilling and Uhlig (2019) show that a speculative equilibrium where agents hold the cryptocurrency in anticipation of its appreciation exists under some conditions. Sockin and Xiong (2021) model cryptocurrency as membership in a digital platform developed to facilitate transactions between users of certain goods or services. They show that platform users' complementarity makes utility tokens appealing because they prevent the platform from abusing its users. Fanti, Kogan, and Viswanath (2021) study the tradeoffs between staking tokens for transaction validations and utilizing tokens for trades. They show that the tradeoff pins the tokens' value as a function of transaction volume, token velocity, and token supply schedule. Saleh (2021) studies the economics of Proof-of-Stake systems and studies the implications of the consensus mechanism on the token's value. Our model's predictions apply to other permissionless blockchain protocols such as Proof-of-Stake as long as the concentration of the transaction validators varies over time due to exogenous shocks.[5][6]

Our paper also complements earlier work on the production side of cryptocurrencies. Easley, O'Hara, and Basu (2019) focus on the role of transaction fees and their impact on the behavior of miners and users in a game-theoretic setup. Huberman, Leshno, and Moallemi (2021) model how the decentralized design of Bitcoin, particularly the competition among service providers and free entry, helps to protect users from monopoly pricing. Cong and He (2019) analyze the impact of blockchain technology on competition and industrial organization. Cong et al. (2021a) study the rise of mining pools as a risk-sharing mechanism among smaller miners. They find that this risk-sharing mechanism escalates the technological arms race among mining pools. Alsabah and Capponi (2020) study the mining pools' research and development investment decisions. We complement the above papers by focussing on how changes in concentration and entry of mining pools affect cryptocurrency price and volatility.

Our theory builds upon Zapatero (1995) and Pavlova and Rigobon (2007), which study the asset pricing implications in an international finance context. Based on their work, we model cryptocurrency services mined by different mining as different goods to account for the competition between mining pools to attract market participants to clear their mined services separately from the other mining pools. Our paper also relates to Cochrane, Longstaff, and Santa-Clara (2008)s' analysis that studies the asset pricing implications of two Lucas trees. An essential feature our equilibrium inherits from these papers is equilibrium uniqueness, unlike the studies of Biais et al. (2020) and Pagnotta (2021), which lead to multiple equilibria.

[5]The efficiency of Proof-of-Stake protocols relative to Proof-of-Work protocols is not yet clear, as Budish (2018), Gans and Gandal (2019), and Abadi and Brunnermeier (2022) show.

[6]For an extensive survey of the literature on the economics of Bitcoin, please refer to John, O'Hara, and Saleh (2022).

3 An Economy with Mining Pools' Concentration

This section lays out a tractable asset pricing model in which mining pools are competitive and required to clear their mined (endowed) services separately from the other mining pools. The resulting equilibrium studies the cryptocurrency pricing implications of mining pools' concentration. The model builds upon the standard multi-good finite horizon endowment economy, and time t is continuous and goes from zero to T. Two independent Brownian motions drive uncertainty $(Z_t, \bar{Z}_t)$. The first captures shock to services (Z_t), and the second captures shock to the mining pools' relative size $(\bar{Z}_t)$.

3.1 Cryptocurrency Services and the Pricing Bubble

One Lucas tree (Y_t) produces a perishable good that we call services. We assume that a cryptocurrency's real value — or at least a fraction of it if there is an inflationary bubble — is determined by the sum of all the discounted future services it will provide. These services take several forms, including access to real economies worldwide, and transferring funds across borders and between entities while remaining anonymous. Since these services must be utilized at a particular point in time, we view them as perishable goods.

The cryptocurrency price S_t represents a claim on all future services (Y_t) per unit of cryptocurrency; it is endogenous and determined in equilibrium.[7] However, the cryptocurrency price may differ from its intrinsic, fundamental value and form a bubble. In this case, the current cryptocurrency price depends on both future cryptocurrency prices and future services. Formally, we assume that (Y_t) depends on two sources: an exogenous, intrinsic and unrelated to price source (D_t) and the endogenous, non-intrinsic price source (S_t). We aggregate these two sources using a Cobb-Douglas function, such that

$$Y_t \equiv (S_t)^{\beta} (D_t)^{1-\beta}, \tag{1}$$

for a given parameter $\beta \in (0, 1)$. The parameter β represents the bubble size. When $\beta \to 0$, we converge to a standard pure-exchange multi-good economy without a bubble. As

[7] The equilibrium outcome is invariant to coin creation. The dynamics of mining pools' concentration may arise because the blockchain endows newly minted coins to a particular mining pool each competing round.

β increases, the cryptocurrency price diverges away from its intrinsic value and forming a bubble in the cryptocurrency price. With this Cobb-Douglas aggregator, we assume that the feedback effect is less important when prices are high: the marginal feedback effect of the cryptocurrency price (S_t) on its services (Y_t) is smaller when the cryptocurrency price is high ($\partial^2 Y_t / \partial^2 S_t < 0$).

Further, we assume the exogenous source of services follows

$$dD_t = D_t \left(\mu^D dt + \sigma^D dZ_t\right), \tag{2}$$

where μ^D and σ^D are strictly positive constants. Our specification implies that the exogenous source of services (D_t) does not depend on shocks to mining pools relative size, captured by $\bar{Z}_t$. This assumption is for expositional simplicity and can be relaxed in future work. We posit that S_t follows

$$dS_t = S_t \left(\mu_t^S dt + \sigma_t^S dZ_t + \bar{\sigma}_t^S d\bar{Z}_t\right), \tag{3}$$

where μ_t^S, σ_t^S, and $\bar{\sigma}_t^S$ are endogenous processes determined in equilibrium. With (3), we implicitly assume that the cryptocurrency price cannot be zero. In addition to the cryptocurrency, each mining pool has the opportunity to borrow or lend with instantaneous riskless interest rates in their local numeraire goods denoted by r_{1t} and r_{2t}, respectively. These securities are in zero net supply and allow the mining pools to reduce exposure to the Lucas tree. We refer to these securities as bonds and denote their prices by B_{1t} and B_{2t}. Interest rates and bond prices are endogenous and determined in equilibrium.

3.2 Mining Pools and Households

We focus our analysis on two mining pools to keep the model and the equilibrium mechanism transparent. Section 7 extends the model and investigates concentration in the extensive margin, when the number of mining pools increase from two to three. Agents hold the cryptocurrency because it provides access to valuable services. We abstract away from all other potential reasons to hold the cryptocurrency.

Mining pools access the cryptocurrency's services through either mining or trading. We model *mining of the cryptocurrency* in reduced form as an exogenous time-varying endowment distribution process that splits the proportion of cryptocurrency services between the two mining pools at every moment in time. The proportion of services that Pool-1 mines is determined by $\lambda_{1t} \in (0,1)$ (the share of the Lucas (1978) tree endowed to Pool-1 in time t) while Pool-2 mines the residual $\lambda_{2t} \equiv 1 - \lambda_{1t}$ share.

The λ_{1t} process captures the variation in mining pools' relative sizes due to exogenous shocks, like technological shocks. When Pool-1's mining technology improves relative to Pool-2, it mines a more significant share of the services and its size increases at the expense of Pool-2: $\lambda_{1t} > \lambda_{2t}$. We refer to λ_{it} as Pool-i's *size* throughout the analysis and shocks to λ_{it} as *shocks to mining pools' size*. We assume that Pool-1's size process follows

$$d\lambda_{1t} = \lambda_{1t}\lambda_{2t}\left\{\mu_{\lambda 1t}dt + \bar{\sigma}d\bar{Z}_t\right\}, \quad \lambda_{10} \in (0,1). \tag{4}$$

These dynamics ensure that Pool-1 size always moves between zero and one.[8]

We introduce competition between the mining pools by requiring that each mining pool clears its mined cryptocurrency services separately from its competitors in equilibrium. We achieve this goal by viewing services that mining pools mine as different goods.

Like leading equilibrium asset pricing models with intermediaries, such as He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014), where agents maximize consumption over a lifetime, the mining pools, Pool-1, and Pool-2, in this model, derive utility from consuming their services,

$$E\left[\int_0^T e^{-\rho t}\log\left(c_{it}^i\right)dt\right], \quad i = 1,2, \tag{7}$$

where $\rho > 0$ and c_{it}^i is Pool-i's consumption of its own services j at time t.

[8]We obtain λ_{it} dynamics by assuming that each mining pool's absolute size process follows a geometric Brownian motion

$$\frac{dF_i}{F_i} = \mu_i dt + \sigma_i dZ_i, \tag{5}$$

where $i = 1,2$, and Z_i are standard independent Brownian motions uncorrelated with Z. Then, we define the mining process of the first mining pool, which is a relative quantity, as $\lambda \equiv \frac{F_1}{F_1+F_2}$. By applying Itô's Lemma to this definition, we obtain (4). For further simplification, we introduce the standard Brownian motion $\bar{Z} \equiv \frac{\sigma_1 Z_1}{\bar{\sigma}} - \frac{\sigma_2 Z_2}{\bar{\sigma}}$, and

$$\mu_{\lambda 1t} \equiv \left(\mu_1 - \lambda_{1t}\left(\sigma_1\right)^2\right) - \left(\mu_2 - \lambda_{2t}\left(\sigma_2\right)^2\right), \tag{6}$$

where μ_1, μ_2, σ_1, and σ_2 are strictly positive constants, and $\bar{\sigma} \equiv \sqrt{\sigma_1^2 + \sigma_2^2}$.

The households in the economy cannot mine, and their share of services remain fixed and equal to the initial endowment when there are no financial markets. With financial markets, households may trade with the mining pools and increase their share of cryptocurrency services. Accordingly, we assume the households derive utility from consuming services of both mining pools,

$$E\left[\int_0^T e^{-\rho t}\left[\gamma_1 \log\left(c_{at}^1\right)+\log\left(c_{at}^2\right)\right]dt\right]. \tag{8}$$

The parameter γ_1 captures the households' demand-side preference for a particular mining pool's services. We assume it is constant because we focus our analysis on the supply-side effects. Due to this Cobb-Douglas utility function, the expenditure share the households devote to Pool-1 is given by $\frac{\gamma_1}{1+\gamma_1}$, meaning that the households would like to allocate more wealth to Pool-1 when $\gamma_1 > 1$. We refer to γ_1 as *demand bias* and assume that demand bias is always towards Pool-1, $\gamma_1 \geq 1$, without loss of generality. While the equilibrium allows for any $\gamma_1 \geq 1$, we focus the analysis and intuitions on the economy without demand bias, $\gamma_1 = 1$, since the services the two mining pools mine are indistinguishable.

Critically, the households' Cobb-Douglas utility function implies that the households' marginal rate of substitution between the two mined services is convex. The households are willing to substitute one unit of services mined by Pool-1 (c_{at}^1) for more units of serviices mined by Pool-2 (c_{at}^2) when they consume fewer services of Pool-1 and c_{at}^1 is low. This assumption means that if fees were set equal and given that there is no demand bias, households would act like random agents and would allocate resources equally between the mining pools, as one would expect in an economy with two indistinguishable commodities that are being sold for the same price by two identical competing venues. Accordingly, the only reason to allocate more resources to one mining pool is the fees this pool charges.

Furthermore, the mining pools utility function (7) is a special case of the households utility function (8) in which the expenditure share of Pool-1 approaches 1, while the expenditure share of Pool-2 approaches 0. Accordingly, from the mining pools' point of view, services are not substitutable, or substitution is exceedingly costly.

Our theory builds on the intuitive yet novel insight that introducing competition requires that each mining pool clears its mined cryptocurrency services separately from its competitors in equilibrium. Accordingly, mining pools post fees for trading cryptocurrency services with other market participants. If market participants demand too many services from a particular mining pool, it increases its fees. Instead, if market participants demand too few services, the mining pool decreases its fees. The process continues until, eventually, supply meets demand in every mining pool individually, and equilibrium reveals the market-clearing fees, p_{it} for Pool-i, $i = 1, 2$. These fees represent the pay (in units of the numeraire) to obtain one unit of services of Pool-i. In the spirit of commonly used price indices, our numeraire is the simple average of the mining pools' fees p_{1t} and p_{2t}, such that

$$p_{1t} + p_{2t} = \bar{p}, \tag{9}$$

where $\bar{p}$ is an exogenous parameter, which we refer to as the *fees index*. Our choice of numeraire is standard and borrowed from Pavlova and Rigobon (2007). Intuitively, it takes current prices but fixes the quantities to precisely one unit per mining pool.[9]

To see how the cryptocurrency price and the fees are related, imagine a shop that sells services from two separate service providers (mining pools). To clear their supply entirely, the two service providers set up fees competitively. The no arbitrage condition then implies that the cryptocurrency price is a claim to all the future combined services the shop provides:

$$S_t = \mathbb{E}_t \left[\int_t^T \xi_{t,s} \left(p_{1s} \lambda_{1s} Y_s + p_{2s} \lambda_{2s} Y_s \right) ds \right], \tag{10}$$

where $\xi_{t,s} \equiv \xi_t / \xi_s$ is the equilibrium state price density process.

The households and the mining pools are price takers, and without loss of generality, we set the initial supply share to equal the initial wealth share so that

$$\lambda_{10} = \frac{\gamma_1}{1 + \gamma_1}, \quad \lambda_{20} = \frac{1}{1 + \gamma_1}. \tag{11}$$

This assumption is innocuous; it simplifies the exposition without affecting the economic mechanism since it allows the constants in the propositions to cancel out.[10]

Further, we let W_{a0} and W_{i0}, $i = 1, 2$, be the households and the mining pools' value of

[9] We use the simple average for expositional clarity; a general weighted average of fees' price index ($X_1 p_{1t} + X_2 p_{2t} = \bar{p}$), such as the well known Lowe and Laspeyres price indices, would not change our qualitative results when $X_1, X_2 > 0$.

[10] A general initial distribution of the cryptocurrency ($W_{a0} = F_a S_0$, $W_{10} = F_1 S_0$, and $W_{20} = F_2 S_0$, such that $F_a + F_1 + F_2 = 1$ and with $F_a, F_1, F_2 > 0$ would not affect the equilibrium qualitative outcomes.

the initial endowments, respectively. We assume that the households are endowed with $1-\hat{\lambda}$ shares of services at time 0; of those $1-\hat{\lambda}$ shares, a λ_{10} fraction is of Pool-1 good, and a λ_{20} fraction is of Pool-2 good. The two mining pools are initially endowed with the residual $\hat{\lambda}$ shares of services; of those $\hat{\lambda}$ shares, a λ_{10} fraction is of Pool-1 good, and a λ_{20} fraction is of Pool-2 good. Thus, $W_{a0} = (1-\hat{\lambda})S_0$, $W_{10} = \hat{\lambda}\lambda_{10}S_0$, and $W_{20} = \hat{\lambda}\lambda_{20}S_0$.

Starting with these initial endowments, at every instant of time t, each market participant chooses a nonnegative consumption and a portfolio of available securities $(\pi_{it}^S, \pi_{it}^{B_1}, \pi_{it}^{B_2})$ to maximize their utility subject to their dynamic budget constraint, which takes the following standard form

$$dW_{it} = W_{it}\pi_{it}^S \frac{dS_t + p_{1t}\lambda_{1t}Y_t dt + p_{2t}\lambda_{2t}Y_t dt}{S_t} + W_{it}\pi_{it}^{B_1}\frac{dB_{1t}}{B_{1t}} + W_{it}\pi_{it}^{B_2}\frac{dB_{2t}}{B_{2t}} - p_{it}c_{it}^i dt, \tag{12}$$

for Pool-i $= 1, 2$, and

$$dW_{at} = W_{at}\pi_{at}^S \frac{dS_t + p_{1t}\lambda_{1t}Y_t dt + p_{2t}\lambda_{2t}Y_t dt}{S_t} + W_{at}\pi_{at}^{B_1}\frac{dB_{1t}}{B_{1t}} + W_{at}\pi_{at}^{B_2}\frac{dB_{2t}}{B_{2t}} - p_{1t}c_{at}^1 dt - p_{2t}c_{at}^2 dt, \tag{13}$$

for the households. The quantity π_{it}^j is endogenous and represents a fraction of W_{it} invested at time t in security j, and $i = 1, 2, a$.

Remark 1 (**Proof of Stake**). *The main aspect of mining pools central to our analysis is their role as transaction validators, those who increase their cryptocurrency holdings even if they never trade the cryptocurrency. Although we call the transaction validators mining pools, which is a feature of proof of work protocols, our model's predictions apply to other blockchain protocols, such as proof of stake, as long as the concentration of transaction validators varies over time due to exogenous shocks.*

4 Equilibrium with Mining Pools' Concentration

We define equilibrium in a standard way: cryptocurrency price, mining pools' fees, bonds prices, portfolio holdings, and consumption choices are such that (i) the mining pools and households choose their optimal consumption of cryptocurrency services and security holdings for given prices, and (ii) mining pools services clear individually for each mining pool (goods markets clear), the cryptocurrency, and bonds markets clear.

We start the equilibrium analysis by characterizing the mining pool fees. The mining pools compete with each other by posting fees to trade their services with the other market participants. Equilibrium determines the market-clearing fees so that the supply of services meets the demand for services in each mining pool. The following proposition summarizes the key results.

Proposition 1 (**Fees**)**.** *The mining pools post fees given by*

$$p_{1t} = \bar{p}\frac{\frac{\gamma_1}{1+\gamma_1}\frac{1}{\lambda_{1t}}}{\frac{\gamma_1}{1+\gamma_1}\frac{1}{\lambda_{1t}} + \frac{1}{1+\gamma_1}\frac{1}{\lambda_{2t}}}, \quad p_{2t} = \bar{p}\frac{\frac{1}{1+\gamma_1}\frac{1}{\lambda_{2t}}}{\frac{\gamma_1}{1+\gamma_1}\frac{1}{\lambda_{1t}} + \frac{1}{1+\gamma_1}\frac{1}{\lambda_{2t}}}, \tag{14}$$

to trade their services. The fees of Pool-i decrease as that mining pool becomes larger,

$$\frac{\partial p_{1t}}{\partial \lambda_{1t}} < 0, \qquad \frac{\partial p_{2t}}{\partial \lambda_{1t}} > 0, \tag{15}$$

while it increases as the households shift its preference to that pool, $\frac{\partial p_{1t}}{\partial \gamma_1} > 0$*,* $\frac{\partial p_{2t}}{\partial \gamma_1} < 0$*.*

Proposition 1 reveals that an increase in Pool-1's size ($\lambda_{1t} \uparrow$) directly reduces that mining pool's fees and indirectly increases Pool-2's fees. A typical supply shift channel in which an increased supply translates to lower fees and in line with Cong et al. (2021a)s' findings, whereby mining pool size is inversely related to the fees charged by that mining pool.

When the mined services are indistinguishable ($\gamma_1 = 1$), the proposition reveals that the two mining pools set up relative fees (p_{1t}/p_{2t}) proportional to the inverse of their relative size ($\lambda_{2t}/\lambda_{1t}$). So, if Pool-1 has half the size of Pool-2, the price for Pool-1's services is double that of Pool-2's services. When the services are distinguishable ($\gamma_1 > 1$), the fees also depend on the demand bias parameter (γ_1): when the households prefer Pool-1's services, that mining pool fees increase. It is a typical demand channel in which stronger demand to services translates to higher prices.

Due to the negative relationship between mining pool size and its fees, and since the sizes (λ_{it}) and the fraction of fees ($p_{it}/\bar{p}$) are bounded in $(0, 1)$, there is a single crossing between the mining pool size and its fraction of fees, as Figure 2 illustrates. Accordingly, we define the crossing point by λ_{1*},

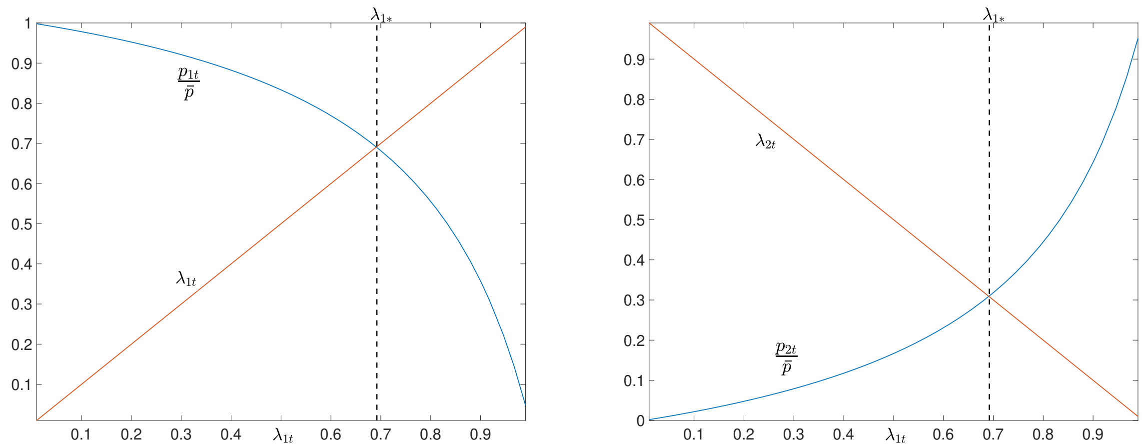

Figure 2. Fees. These figures reveal that as the size of Pool-i increases, the mining pool fees decrease. At $\lambda_{1t} = \lambda_{1*}$, the decrease in Pool-i fees exactly offsets the increase in its size, $\partial(p_{1t}/\bar{p}) = -\partial\lambda_{1t}$. The left panel represents Pool-1, while the right panel Pool-2. Parameter values are: $D_0 = 1$, $D_t = 2$, $\beta = 0.6$, $\rho = 0.98$, $\sigma = \bar{\sigma} = 0.4$, $T = 3$, $t = 1$, $\gamma_1 = 5$, and $\bar{p} = 5$.

$$\frac{p_{1t}\left(\lambda_{1*}\right)}{\bar{p}} \equiv \lambda_{1*}. \tag{16}$$

Intuitively, when Pool-1's size equals its fraction of fees (16), the increase in Pool-1's size exactly offsets the decrease in its fraction of fees. As a result, the fees multiplied by the mining pool's size attains its maximum at $\lambda_{1t} = \lambda_{1*}$. Economically, the mining pool size times its fees represents the revenue this mining pool attains in equilibrium relative to the total revenue available. More explicitly, we define the *revenue shares* of Pool-1 and Pool-2 and the *total revenue share* as

$$V_{1t} \equiv p_{1t}\lambda_{1t}, \quad V_{2t} \equiv p_{2t}\lambda_{2t}, \quad V_t \equiv V_{1t} + V_{2t}, \tag{17}$$

respectively.

Following Proposition 1, a shock to the mining pools' size has two competing effects. It increases one mining pool size, but at the same time, it decreases its fees. The revenue share summarizes these two competing effects and reveals which force dominates and is more important for pricing. When the increase in the mining pool size effect dominates, the mining pool's revenue share increases. Instead, when the decrease in the mining pool fees effect dominates, the mining pool's revenue share decreases. Alternatively, one could translate the two competing effects to (i) an increase in one mining pool's size, and (ii) a decrease in the other mining pool's size. What matters for pricing is the effect on the total reveneue shares. An increase in the total reveneue shares has a positive effect on prices, while a decrease has a negative effect on prices, as Figure 3 illustrates.

A simple manipulation shows that the revenue shares and the total revenue share are given by

$$V_{1t} = \bar{p}\frac{\frac{\gamma_1}{1+\gamma_1}}{\frac{\gamma_1}{1+\gamma_1}\frac{1}{\lambda_{1t}} + \frac{1}{1+\gamma_1}\frac{1}{\lambda_{2t}}}, \quad V_{2t} = \bar{p}\frac{\frac{1}{1+\gamma_1}}{\frac{\gamma_1}{1+\gamma_1}\frac{1}{\lambda_{1t}} + \frac{1}{1+\gamma_1}\frac{1}{\lambda_{2t}}}, \quad V_t = \bar{p}\frac{1}{\frac{\gamma_1}{1+\gamma_1}\frac{1}{\lambda_{1t}} + \frac{1}{1+\gamma_1}\frac{1}{\lambda_{2t}}}. \tag{18}$$

The following Proposition verifies our intuition.

Proposition 2 (**Revenue Share**)**.** *The mining pools' revenue shares and the total revenue share are given in (18), attain their maximum when the relative fees equal the size,* $\lambda_{1t} = \lambda_{1*}$*, where*

$$\lambda_{1*} = \frac{\sqrt{\gamma_1}}{1+\sqrt{\gamma_1}}, \quad \lambda_{2*} = \frac{1}{1+\sqrt{\gamma_1}}, \tag{19}$$

and $\lambda_{1*} \geq \lambda_{2*}$ *due to demand bias towards Pool-1 (*$\gamma_1 \geq 1$*).*

The proposition reveals that when $\lambda_{1t} = \lambda_{1*}$, the distribution of services across the mining pools is optimal. When services are indistinguishable and $\gamma_1 = 1$, the mining pools generate the highest revenue shares when their sizes are equal ($\lambda_{1t} = \lambda_{2t} = 1/2$). Of course, exogenous shocks to the mining pools' sizes knock them out of the optimal balance.

The equilibrium dynamics reveal that when Pool-2 size becomes exceedingly small ($\lambda_{2t} \to 0$), and it mines few cryptocurrencies, equilibrium mandates Pool-2's fees to increase towards the fees index ($p_{2t} \to \bar{p}$) to ensure that Pool-2's services are not attractive and the market clearing condition is satisfied. At the same time, Pool-1 size becomes exceedingly large ($\lambda_{1t} \to 1$), it mines almost all of the cryptocurrencies, and equilibrium mandates the fees to decrease towards zero ($p_{1t} \to 0$) to attract all the demand to Pool-1's services.

Since there are only two mining pools in our economy, when one mining pool's fraction of fees equal its size, it must also be true for the other mining pool,

$$\frac{p_{1t}}{\bar{p}} = \lambda_{1t} \iff 1 - \frac{p_{1t}}{\bar{p}} = 1 - \lambda_{1t} \iff \frac{p_{2t}}{\bar{p}} = \lambda_{2t}. \tag{20}$$

Therefore, both mining pools must attain their maximum revenue share at λ_{1*}, eventually giving rise to the inverted U-shape function of the total revenue share with respect to Pool-1's size, λ_{1t}. Furthermore, it is apparent that the fees index parameter ($\bar{p}$) is not responsible for the inverted U-shape revenue share because the result applies for relative fees ($p_{it}/\bar{p}$).

Demand bias toward Pool-1 ($\gamma_1 > 1$) increases this mining pool's maximal revenue share ($\lambda_{1*} > \frac{1}{2}$). The intuition comes from the fact that the revenue share summarizes the competing effects of the mining pool's size and fees and attains its maximum when the decrease in relative fees offsets the increase in size. So, when the expenditure share in Pool-1 ($\frac{\gamma_1}{1+\gamma_1}$) is larger, the mining pool fees are also higher (Proposition 1), requiring a larger mining pool size to offset it ($\lambda_{1*} \uparrow$).

Next, we connect mining pools' concentration to their total revenue shares, and show that shocks that increase concentration decrease the total revenue share. We utilize the established Herfindahl-Hirschman index (HHI) to measure concentration, when mining pools' services are indistinguishable and there is no demand bias. However, when there is demand bias ($\gamma_1 > 1$), we centralize the HHI to ensure that it reaches its minimum when the distribution of services between the mining pools is optimal. In this way, the minimal concentration is independent of the demand bias parameter.

Definition 1 (**Concentration**). *Let H_t be the measure of concentration that attains its minimum when mining pools' sizes equal their relative fees, $\lambda_{1t} = \lambda_{1*}$,*

$$H_t \equiv \frac{\lambda_{1t}^2}{\lambda_{1*}} + \frac{\lambda_{2t}^2}{\lambda_{2*}}, \qquad 1 \leq H_t \leq \frac{1}{\lambda_{2*}}, \qquad \lambda_{2*} \leq \lambda_{1*}. \tag{21}$$

As λ_{1t} moves towards λ_{1*}, it reduces concentration (H_t), while moving away from λ_{1*} increases it.

To develop our intuition about the concentration measure (21), assume that mining pools' services are indistinguishable and there is no demand bias ($\gamma_1 = 1$). In that case, the concentration measure (H_t) coincides with the well-known HHI. It is minimized when the mining pools have the same size ($\lambda_{1t} = \lambda_{2t} = 1/2$), and it is maximized when there is only one mining pool, ($\lambda_{1t} = 1$ or $\lambda_{2t} = 1$). Notice that the minimum concentration is attained precisely at λ_{1*}, when the distribution of services between the mining pools is optimal. When there is demand bias ($\gamma_1 > 1$), we normalize the HHI to ensure that the minimum concentration is still attained when the distribution of services is optimal.

Whenever a shock to mining pools' sizes arrives, it changes the mining pools' concentration, and, at the same time, the total revenue shares. Critically, whenever a shock to the mining pools' sizes increases concentration it decreases the total revenue shares

$$\frac{\partial H_t}{\partial \lambda_{1t}} > 0 \iff \frac{\partial V_t}{\partial \lambda_{1t}} < 0. \tag{22}$$

Much like the mining pools' total revenue shares, our concentration measure (H_t) summarizes the two competing effects and reveals which force dominates and is more important for pricing: (i) an increase in one mining pool's size and (ii) a decrease in the other mining pool's size. What matters is the effect on concentration. A decrease in concentration has a positive effect on prices because the total revenue shares increase, while an increase in concentration has a negative effect on prices because the total revenue shares decrease. The revenue shares attain maximum precisely when concentration attains its minimum, as Proposition 2 reveals and Figure 3 illustrates.

Equilibrium reveals that the total revenue share determines how a shock to the mining pools' sizes propagates in the economy, as evident by the following no-arbitrage implicit pricing equation.

$$S_t = \mathbb{E}_t \left[\int_t^T \xi_{t,s} \left(p_{1s}\lambda_{1s}Y_s + p_{2s}\lambda_{2s}Y_s \right) ds \right] = \mathbb{E}_t \left[\int_t^T \xi_{t,s} V_s Y_s ds \right], \tag{23}$$

where $\xi_{t,s} \equiv \xi_t / \xi_s$.

The no-arbitrage condition (23) implies that the cryptocurrency price is a claim to all the future combined services, and equilibrium determines that price so that the total demand for the cryptocurrency services meets the total supply of services. Indeed, the second term reveals that the cryptocurrency price (S_t) equals the sum of the discounted future services the cryptocurrency provides, where the deflator is the equilibrium discount factor, ξ. The total available services are Y_t, which is splitted between the mining pools based on their sizes

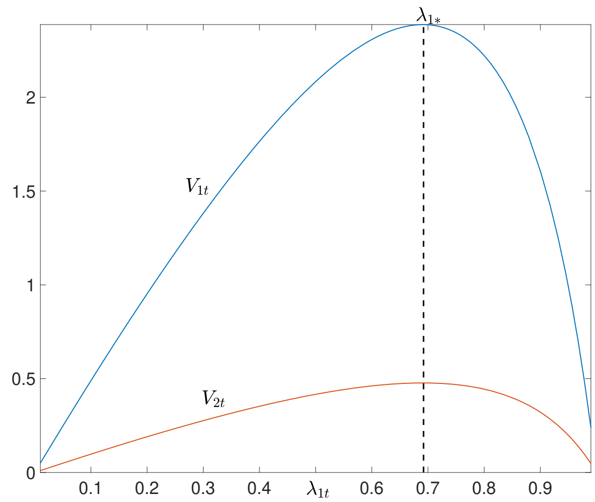

Figure 3. Concentration and Revenue Shares. Whenever a shock to the mining pools' sizes arrives, it increases one mining pool's size and decreases the other mining pool's size. The equilibrium fees of both mining pools readjust to reflect the new distribution of services. Both the total revenue shares and mining pools' concentration determine the direction of the two competing effects. The figure furhter reveals that both the revenue shares and the total revenue shares attain their maximum at $\lambda_{1t} = \lambda_{1*}$. Parameter values are as in Figure 2.

λ_{1t} and λ_{2t}. Therefore, Pool-1's services equal $\lambda_{1t}Y_t$, and Pool-2's services equal $\lambda_{2t}Y_t$, at time t. Since the mining pools compete and require to clear their services individually, they post equilibrium market clearing fees. Accordingly, the pricing equation requires multiplying the mining pools' services by their respective equilibrium fees, eventually leading to the first equality in (23).

By plugging the revenue shares definitions (17) in the second term and noticing that the total revenue share (V_t) times the services (Y_t) represent the mining pools' total revenues, we derive the third term. This term reveals that the mining pools' total revenues (VY) determine the cryptocurrency price. A prediction implying that the cryptocurrency value (or a fraction of it if there is an inflationary bubble) can be determined by the revenues of mining pools, thereby providing a new testable implication for cryptocurrency pricing.

Of course, other potential state variables besides the total revenue share (V_t) may determine how a shock to the mining pools' sizes propagates because we have yet to characterize the state price density. As we reveal soon, the total revenue share is the only mechanism for mining pools' sizes shocks.

Plugging the definition of Y_t (1) into the no-arbitrage pricing equation (23) leads to

$$S_t = \mathbb{E}_t \left[\int_t^T \xi_{t,s} V_s \left((S_s)^\beta (D_s)^{1-\beta} \right) ds \right], \tag{24}$$

and reveals how future cryptocurrency prices feed back and determine the current cryptocurrency price when there is a bubble ($\beta > 0$). At first glance, the feedback in the pricing equation appears intractable. However, the logarithmic utility functions substantially improve the tractability of the pricing equation and eventually lead to a closed-form and precise characterization of the cryptocurrency price.

To further develop our equilibrium mechanism, it is helpful to first discuss the (stochastic) discount factor in an implicit form as a function of the price-to-services ratio.

Lemma 1 (**Implicit Discount Factor**). *The prevailing equilibrium discount factor is given by*

$$\xi_{0,t} = S_0 \left(\frac{\rho e^{-\rho t}}{1 - e^{-\rho T}} \right) \frac{1}{V_t Y_t} = S_0 \left(\frac{\rho e^{-\rho t}}{1 - e^{-\rho T}} \right) \frac{1}{D_t} \frac{1}{V_t} \left(\frac{S_t}{D_t} \right)^{-\beta}, \tag{25}$$

where $\xi_{s,t} \equiv \xi_t / \xi_s$, for $s \leq t$.

In line with our intuitions and as (25) unravels, the discount factor, the process that determines equilibrium prices, depends on prices by itself. Future cryptocurrency prices feed back into the current price through the discount factor, introducing a bubble into the cryptocurrency price controlled by the parameter β. It is clear from (25) that when there is no bubble ($\beta = 0$), there is no feedback, and the discount factor is fully specified without depending on endogenous prices. We derive the discount factor explicitly in the next section after we characterize the cryptocurrency price.

Lemma 1 furhter reveals that the discount factor is inversely related to the exogenous services (D_t) — a feature similar to a traditional asset pricing model — implying that high values of services characterize good states of the world when there is no bubble ($\beta = 0$). Since mining pools compete and equilibrium requires that they clear their services individually, the discount factor is inversely related to mining pools' total revenue shares (V_t), implying that high values of total revenue shares characterize good states when there is no bubble.

However, there is an additional effect through the price–to–services ratio when there is a bubble ($\beta > 0$), revealing that the discount factor is inversely related to the price–to–services raio and implying that high values characterize good states of the world.

Next, we investigate whether the bubble inflates or deflates the cryptocurrency price. To answer that question, we first manipulate the services (1), and reveal that the price–to–services ratio answers that question.

$$Y_t = D_t \left(\frac{S_t}{D_t} \right)^{\beta}. \tag{26}$$

As (26) reveals, when $S_t > D_t$, the bubble increases the available services ($Y_t \uparrow$) and, therefore, inflates the cryptocurrency price. The reverse happens when $S_t < D_t$. The bubble decreases the available services ($Y_t \downarrow$) and deflates the cryptocurrency price.

Since prices are the sum of all discounted future services, one would anticipate that the cryptocurrency price is always greater than current exogenous services ($S_t > D_t$). However, since the mining pools compete and must clear their services individually by posting fees, the cryptocurrency price is not necessarily higher than exogenous services. Imagine an extreme case whereby Pool-1's size approaches one and owns nearly all the services. To clear markets, Pool-1's fees are nearly zero, and as a result, so is Pool-1's revenue share. Since the other mining pool size is nearly zero, Pool-2's revenue share is also nearly zero, resulting in a near-zero total revenue share. Plugging a near-zero total revenue share into our no-arbitrage condition (23), we observe that the cryptocurrency price is nearly zero and certainly below the strictly positive exogenous services, as Figure 3 illustrates. The resulting intuition suggests that when concentration becomes extreme, the cryptocurrency bubble deflates its prices, while the bubble inflates prices when concentration is not extreme. We formalize this intuition in the following section once we characterize the cryptocurrency price and volatility.

5 Implications of Mining Pools' Concentration

So far, we have analyzed the equilibrium mechanism and the implicit pricing formula. This section completes the equilibrium characterization. Despite the feedback between future cryptocurrency prices and the current price, our equilibrium admits precise closed-form expressions. We start by characterizing the equilibrium cryptocurrency price and the discount factor.

Proposition 3 (**Cryptocurrency Price and Discount Factor**). *When the total revenue share attains its maximum, the concentration attains its minimum, and whenever a shock to the mining pools' sizes increases concentration it decreases the total revenue shares (22). The equilibrium cryptocurrency price is given by*

$$S_t = \left(\frac{1-e^{-\rho(T-t)}}{\rho}\right)^{\frac{1}{1-\beta}} D_t \left(V_t\right)^{\frac{1}{1-\beta}} . \tag{27}$$

The discount factor between time 0 *and* t *is given by*

$$\xi_{0,t} = \bar{\xi} e^{-\rho t} \left(\frac{1-e^{-\rho(T-t)}}{\rho}\right)^{-\frac{\beta}{1-\beta}} \frac{1}{D_t} \left(\frac{1}{V_t}\right)^{\frac{1}{1-\beta}} , \tag{28}$$

where V_t *is the total revenue shares (18),* H_t *is the concentration measure (21), and* $\bar{\xi} \equiv S_0 \left(\frac{\rho}{1-e^{-\rho T}}\right)$*. The discount factor increases with concentration, and the cryptocurrency price decreases with concentration, even when there is no bubble* ($\beta = 0$)*.*

The shocks to mining pools' sizes have two opposing forces since when Pool-1's size increases, Pool-2's size decreases in relative terms. It is not immediately clear whether a shock that increases the size of one mining pool propagates positively or negatively to prices. The equilibrium mechanism reveals, and Proposition 3 verifies that when a shock to the mining pools' sizes increases the total revenue share and decreases mining pools' concentration (22), it propagates positively to the cryptocurrency prices. The proposition further reveals that the cryptocurrency price peaks when concentration is minimized and collapses to zero when concentration is maximized, even without a bubble ($\beta = 0$). Figure 4 illustrates this idea.

Similarly, the discount factor reveals that the economy achieves its best economic state precisely when mining pools' concentration is minimized, and the discount factor reaches its minimum. The economic state deteriorates as concentration gets further away from its minimum and reaches the worst economic state when concentration is the furthest away from the minimum. When mining pools' services are indistinguishable and there is no demand bias ($\gamma_1 = 1$), equilbrium achieves its worst economic state when either mining pool completely dominates and becomes a monopoly ($\lambda_{1t} = 0$ or 1) since both extreme states are equidistant from the minimum concentration state, achieved at $\lambda_{1*} = 1/2$. When there is a demand bias ($\gamma_1 > 1$), the worst economic state occurs when Pool-2 becomes a monopoly, and Pool-1 vanishes ($\lambda_{1t} = 0$) since this extreme state is the furthest away from the minimum concentration state, achieved at $\lambda_{1*} > 1/2$.

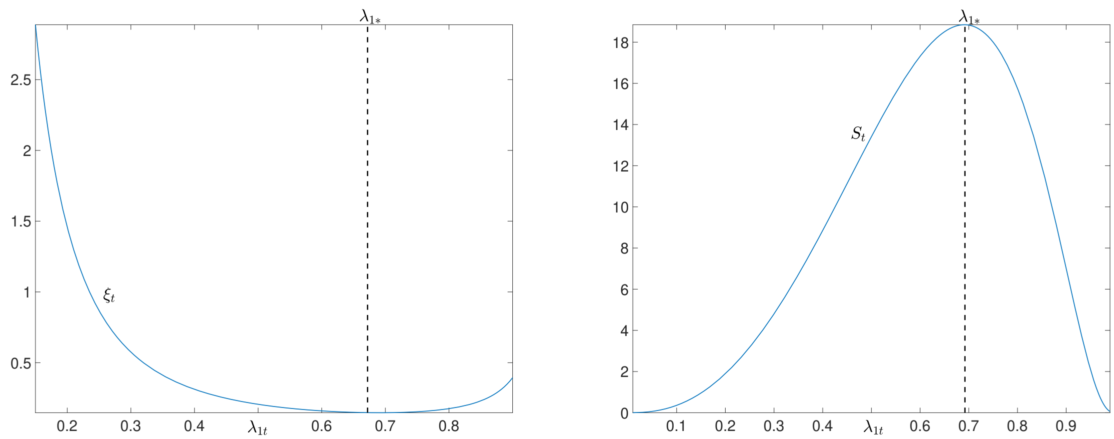

Figure 4. Discount Factor and the Cryptocurrency Price. When concentration is minimized, the cryptocurrency price peaks and the discount factor reaches its minimum. Alternatively, when concentration is maximized, the cryptocurrency price collapses to zero. Parameter values are as in Figure 2.

The cryptocurrency shock sensitivity complements the cryptocurrency price analysis. When concentration leans towards Pool-2 ($\lambda_{1t} < \lambda_{1*}$), the cryptocurrency shock sensitivity is positive, and therefore, a positive shock to Pool-1's size ($\lambda_{1t} \uparrow$) reduces concentration and increases the cryptocurrency price. Due to symmetry, when concentration leans towards Pool-1 ($\lambda_{1t} > \lambda_{1*}$), the shock sensitivity is negative, and therefore, a positive shock to Pool-2's size ($\lambda_{1t} \downarrow$) reduces concentration and increases the cryptocurrency price.

As the concentration drifts away from the optimum, the effect of a shock to the mining pools' sizes heightens, resulting in a more extreme cryptocurrency return volatility. This result further highlights the equilibrium's workings. As mining pools' total revenue share deteriorates and concentration becomes extreme, the cryptocurrency price drops. The drop in the cryptocurrency price makes it an attractive investment to all the market participants. To reduce the cryptocurrency's attractiveness, equilibrium increases its volatility to the point where market participants are indifferent between reducing and increasing their cryptocurrency position, restoring the equilibrium. Overall, a shock to the mining pools' size distribution that reduces concentration increases the cryptocurrency price but decreases its total return volatility even without a bubble ($\beta = 0$). Proposition 4 summarizes our findings, and Figure 5 illustrates them.

Similarly, when concentration drifts towards the optimum, the effect of a shock to the mining pools' sizes shrinks and is eventually turned off entirely when concentration is minimized. At the optimum, a shock to the mining pools' sizes does not affect the cryptocurrency price. To sustain the elevated cryptocurrency price, equilibrium decreases the cryptocurrency volatility to its minimal point to ensure market participants would still support the elevated cryptocurrency price.

Proposition 4 (**Cryptocurrency Volatility**)**.** *The cryptocurrency shock sensitivities are given by*

$$\bar{\sigma}_t^S = \frac{1}{1-\beta}\left(\frac{p_{1t}}{\bar{p}}\lambda_{2t} - \frac{p_{2t}}{\bar{p}}\lambda_{1t}\right)\bar{\sigma}, \qquad \sigma_t^S = \sigma^D. \tag{29}$$

The cryptocurrency is insensitive to shocks to the mining pools' size distribution when mining pools' concentration is minimized. The absolute-term magnitude of the cryptocurrency shock sensitivity to mining pools' size distribution shocks increases with concentration even without interdependency and $\beta = 0$:

$$\frac{\partial \mid \bar{\sigma}_t^S \mid}{\partial \lambda_{1t}} = \begin{cases} \frac{\partial \bar{\sigma}_t^S}{\partial \lambda_{1t}} < 0 & \text{for } \lambda_{1t} < \lambda_{1*}, \\ -\frac{\partial \bar{\sigma}_t^S}{\partial \lambda_{1t}} > 0 & \text{for } \lambda_{1t} > \lambda_{1*}. \end{cases}$$

Further, the absolute-term magnitude of the size distribution shock sensitivity increases with the interdepency:

$$\frac{\partial \mid \bar{\sigma}_t^S \mid}{\partial \beta} = \begin{cases} \frac{\partial \bar{\sigma}_t^S}{\partial \beta} > 0 & \text{for } \lambda_{1t} < \lambda_{1*}, \\ -\frac{\partial \bar{\sigma}_t^S}{\partial \beta} < 0 & \text{for } \lambda_{1t} > \lambda_{1*}. \end{cases}$$

We now turn our attention to analyzing the implications of the bubble on the cryptocur-

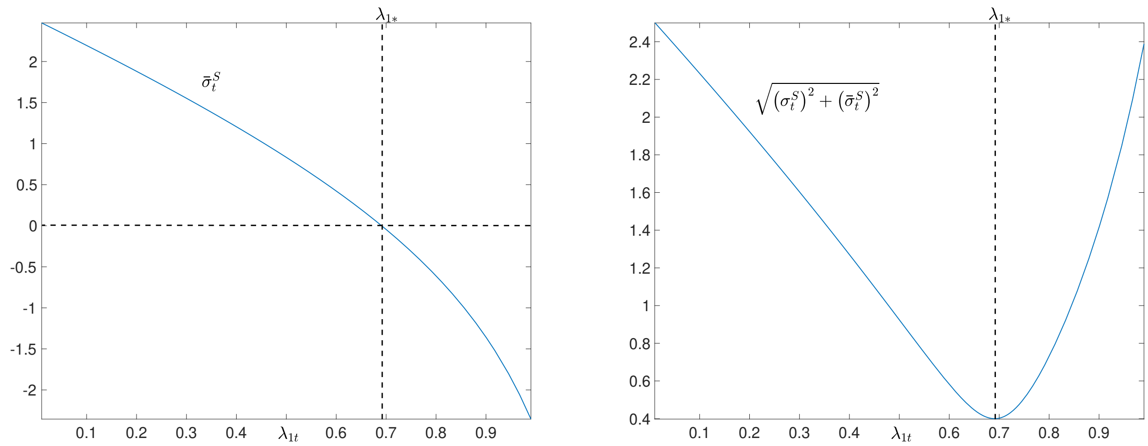

Figure 5. Volatility. On the left figure, we observe the cryptocurrency sensitivity to positive technological shocks. On the right figure, we observe that the total return volatility is minimized when mining pools' concentration is the lowest, and increases with concentration. Parameter values are as in Figure 2.

rency price. The bubble inflates prices when $S_t(\beta > 0) > S_t(\beta = 0)$, where $S_t(\beta = 0)$ represents the cryptocurrency intrinsic value (the value that depends on services), while $S_t(\beta > 0)$ represents the cryptocurrency price with a bubble and all else equal. In that case, only a fraction α_t of the cryptocurrency price S_t depends on services

$$\frac{S_t(\beta = 0)}{S_t(\beta > 0)} = \alpha_t \in (0, 1). \tag{30}$$

However, the bubble ($\beta > 0$) either inflates or deflates the cryptocurrency price relative to an economy without a bubble ($\beta = 0$). In the inflationary case, the cryptocurrency price diverges from the value of the discounted exogenous services. We call this case an *inflationary bubble* because as the bubble parameter increases and β approaches one, the cryptocurrency price approaches infinity.[11] In the deflationary case, instead, the cryptocurrency price is depressed relative to the intrinsic, fundamental value. We call this case a *deflationary bubble* because as the interdependency increases and β approaches one, the cryptocurrency price collapses to zero.

Interestingly, the fees index ($\bar{p}$) determines the bubble type. There are two possible

[11] The definition of an inflationary bubble is in-line with the definition of a bubble in the literature, such as Scheinkman and Xiong (2003).

scenarios. When the fees index is sufficiently high, the cryptocurrency price shifts between an inflationary and deflationary bubble, depending on the concentration of the mining pools. In states where the mining pools' concentration is not extreme, the cryptocurrency is in an inflationary bubble and shifts to a deflationary bubble as mining pools' concentration crosses a threshold and becomes extreme. When the fees index is low, instead, the economy is always in a deflationary bubble, regardless of concentration. The following Proposition summarizes our findings.

Proposition 5 (**Bubble**). *The cryptocurrency price is in an inflationary bubble if and only if the fees index is sufficiently elevated* ($\bar{p} > P_t$) *and mining pools' concentration is not extreme* ($\underline{\lambda}_t < \lambda_{1t} < \bar{\lambda}_t$)*. An increased interdependency implies*

(i) a more substantial effect of concentration on the return volatility,

$$\frac{\partial}{\partial \beta}\left(\frac{\partial \sqrt{\left(\bar{\sigma}_t^S\right)^2 + \left(\sigma_t^S\right)^2}}{\partial \lambda_{1t}}\right) < 0 \iff \lambda_{1t} < \lambda_{1*}. \tag{31}$$

(ii) a more substantial effect of concentration on the cryptocurrency price, if the cryptocurrency price is in an inflationary bubble,

$$\frac{\partial}{\partial \beta}\left(\frac{\partial S_t}{\partial \lambda_{1t}}\right) > 0 \iff \lambda_{1t} < \lambda_{1*}. \tag{32}$$

(iii) a higher return volatility, $\frac{\partial \sqrt{\left(\bar{\sigma}_t^S\right)^2 + \left(\sigma_t^S\right)^2}}{\partial \beta} > 0$.

(iv) a higher cryptocurrency price, $\frac{\partial S_t}{\partial \beta} > 0$*, if and only if the cryptocurrency is in an inflationary bubble.*

The functions P_t*,* $\underline{\lambda}_t$*, and* $\bar{\lambda}_t$ *are deterministic functions of time characterized in (A.39), (A.40), and (A.41).*

Proposition 5 verifies our intuitions derived from the implicit discount factor (1) and the price–to–services ratio (26). When the ratio is bigger than one ($S_t > D_t$), the cryptocurrency price feeds back positively to its services and introduces the inflationary bubble. In contrast, when the ratio is smaller than one ($S_t < D_t$), the cryptocurrency price feeds back negatively to its services and introduces the deflationary bubble.

Accordingly, when the fees index is sufficiently high ($\bar{p} > P$), the total revenue share becomes higher ($V_t \uparrow$), and the cryptocurrency price is guaranteed to surpass the exogenous services when the mining pools' concentration is not extreme ($\underline{\lambda} < \lambda_{1t} < \bar{\lambda}$). The effects are stronger the bigger the bubble: in an extreme case where the bubble approaches its maximum ($\beta \to 1$), the positive feedback becomes so strong that it pushes the cryptocurrency price towards infinity. In contrast, when the fees index is low ($\bar{p} < P$), the total revenue share decreases, and the cryptocurrency price never surpasses the exogenous services regardless of the mining pools' concentration. The effects are again stronger the bigger the bubble: in an extreme case where the interdependency approaches its maximum ($\beta \to 1$), the negative feedback becomes so strong that it pushes the cryptocurrency price towards zero. Figure 6 illustrates these two cases in an example.

The bubble amplifies the concentration effects on the cryptocurrency return volatility. Proposition 5 reveals that a bigger bubble implies a more substantial effect of mining pools' concentration on the total volatility, regardless of the bubble type (31). Though, the effect of a shock to the mining pools' sizes shrinks and is eventually turned off entirely when concentration is minimized. A similar effect occurs with the cryptocurrency price is in an inflationary bubble, revealing that the bigger the bubble, the more substantial the effects of mining pools' concentration on the cryptocurrency price, (32). In a deflationary bubble, the two effects are competing: on the one hand, a lower concentration ($\partial\lambda_{1t}, \lambda_{1t} < \lambda_{1*}$) amplifies the cryptocurrency price, but on the other hand, a bigger bubble ($\partial\beta$) depresses the cryptocurrency price.

6 Empirical Evidence from Bitcoin

This section presents novel empirical evidence to support our model's prediction. To start the analysis, we illustrate the importance of mining pools' concentration to the cryptocurrency price. Figure 7 plots the Bitcoin/Ethereum price ratio against the Herfindahl-Hirschman index (HHI) of Bitcoin mining pools. We look at the ratio of Bitcoin to Ethereum price, the second-biggest cryptocurrency in market capitalization, to take care of the systemic cryptocurrency shocks. The red line is the fitted linear regression line. The figure clearly shows that the Bitcoin price drops relative to the Ethereum price when Bitcoin mining pools'

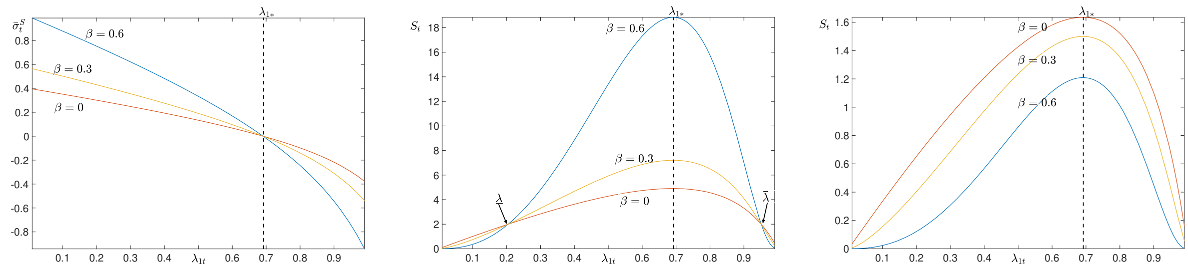

Figure 6. In the left panel, the cryptocurrency total return volatility becomes more sensitive to shocks to the mining pools' size distribution when the interdependency increases. In the middle panel, the fees index is sufficiently high ($\bar{p} = 5$) and an inflationary bubble emerges when the mining pools' size distribution is not extreme ($\underline{\lambda} < \lambda_{1t} < \bar{\lambda}$). As the mining pools' sizes shift away from the optimum and towards the extreme, the cryptocurrency exits the inflationary bubble and enters the deflationary bubble; $\underline{\lambda}$ and $\bar{\lambda}$ identify the switching points. On the right panel, the fees index is not high enough ($\bar{p} = \frac{5}{3}$), and a deflationary bubble emerges. The rest of the parameters are as in Figure 2.

concentration increases, in line with our model's predictions.

To understand the economic magnitude of the effect, Table 1 shows that when the HHI index increases by *227.8* points (one standard deviation), the Bitcoin price falls relative to the Ethereum price by -*0.63* standard deviations. Further, to illustrate the positive relationship between mining pools' concentration and the cryptocurrency volatility, we estimate the monthly Bitcoin return standard deviations using daily returns and the monthly HHI as the average of the daily HHIs. By doing so, we find a monthly correlation coefficient of *0.54*, which suggests that Bitcoin return standard deviation and Bitcoin mining pools'

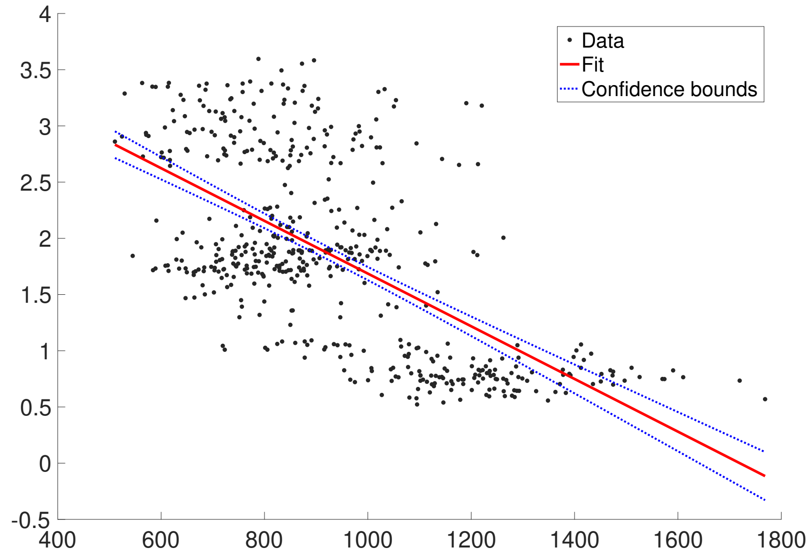

Figure 7. Scatterplot of the BTC/ETH dollar prices (y-axis) on the Herfindahl-Hirschman index measure of Bitcoin mining pools (x-axis) with the fitted regression line. One standard deviation increase in concentration decreases the Bitcoin price by 0.624 standard deviations relative to the Ethereum price.

concentration are highly correlated.

Table 1. Bitcoin/Ethereum Price On The Bitcoin's Mining Pools' HHI

	BTC/ETH	HHI	
Mean	1.84	933	
Standard deviation	0.84	228	
Corr(BTC/ETH, HHI)			-0.63
Daily observations	521	521	521
Corr(σ(BTC), HHI)			0.54
Monthly observations			25
Dependent Variable	BTC/ETH		
constant	4.77(***)		
	(0.33)		
HHI	-0.63(***)		
	(0.06)		
R^2	0.40		

Notes: To derive Corr(σ(BTC), HHI), we compute σ(BTC), the monthly BTC return standard deviation, from the daily Bitcoin returns, and define the monthly HHI as the average of the daily HHI's. The bottom table presents the OLS regression of Bitcoin dollar price relative to the Ethereum dollar price on the Herfindahl-Hirschman index. Variables are Standardized. Newey-West standard errors in parenthesis. (***)

We downloaded the Bitcoin and Ethereum price data from Yahoo Finance and we scrapped the website https://bitcoinchain.com/pools for the daily mining pool data. Our sample period runs from 01/01/2018 until 01/28/2020 and includes US trading days only. Finally, we calculated the Herfindahl-Hirschman index measure using the number of blocks each mining pool has written, where we assume that unknown mining pools are small and each one can mine either one or zero blocks per day.

So far, we have separately shown that the cryptocurrency price decreases with HHI, while the cryptocurrency volatility increases with HHI. We continue with a joint estimation of the conditional Bitcoin expected return and the volatility using the GARCH(1,1) model of Bollerslev (1986). We include the Ethereum standardized return and the HHI index as external regressors to the mean equation and the HHI index as an external regressor to the volatility equation. Figure 8 shows the standardized Bitcoin return volatility estimate.

Table 2 presents the parameter estimates for three specifications that vary the mean equation. One thing unequivocally clear from all those specifications is that the HHI jointly decreases the Bitcoin return and increases the Bitcoin volatility. To understand the economic magnitude of the combined effect, the first column of Table 2 implies that when the HHI index increases by 228 points (one standard deviation), the conditional Bitcoin daily expected return falls by roughly 1.9%, while the conditional daily return standard deviation increases by roughly 7.24

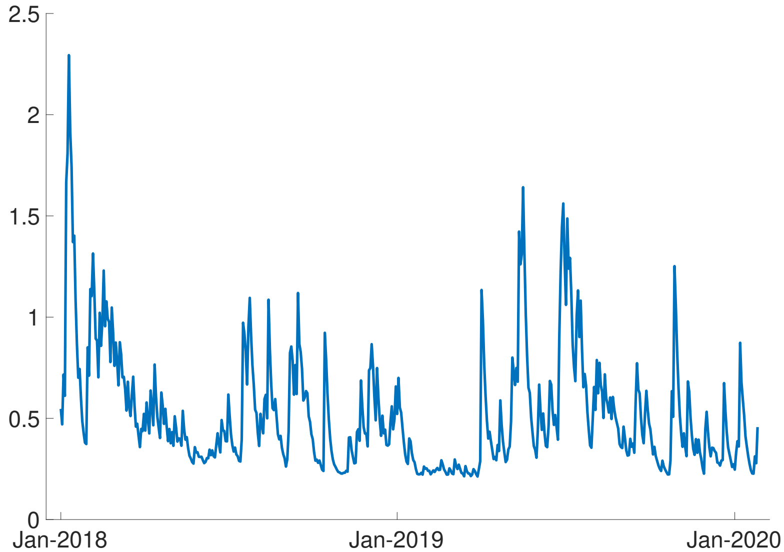

Figure 8. Standardized Bitcoin return volatility estimate using a GARCH(1,1) model.

Table 2. Estimates of a GARCH(1,1) Model

Dependent Variable	$r(\text{BTC})_t$	$r(\text{BTC})_t$	$r(\text{BTC})_t$
Mean equation			
constant	0.0891(***)	0.0164(***)	0.0581(***)
	(0.0202)	(0.0033)	(0.0051)
$r(\text{ETH})_t$	0.7748(***)	0.7933(***)	0.7811(***)
	(0.1700)	(0.0366)	(0.0287)
HHI_t	-8.2×10^{-5}(***)	—	-4.3×10^{-5}(***)
	(1.7×10^{-5})	—	(4×10^{-6})
ar_1	—	—	0.8562(***)
	—	—	(0.0123)
ma_1	—	—	-0.8093(***)
	—	—	(0.0088)
Variance equation			
constant	2.9×10^{-5}	2×10^{-6}	2.64×10^{-4}(***)
	(0.0114)	(11.9×10^{-5})	(0.85×10^{-4})
HHI_t	2.3×10^{-5}(**)	2.8×10^{-5}(***)	2.5×10^{-5}(**)
	(1.2×10^{-5})	(9×10^{-6})	(1.0×10^{-5})
α_1	0.436(***)	0.4349(***)	0.4313(***)
	(0.1415)	(0.0633)	(0.0441)
β_1	0.5822(***)	0.5625(***)	0.5824(***)
	(0.0973)	(0.0307)	(0.0387)

7 Extension

So far, the main analysis focuses on the cryptocurrency pricing implications of concentration in the intensive margin when there are two mining pools and without entry. While the resulting equilibrium mechanism is transparent, the economic setup is unsuitable for investigating the pricing implications in the extensive margin when the number of mining pools changes. This section extends the model to three mining pools and investigates the pricing implications of concentration in the extensive margin when the third mining pool enters the economy.

The section reveals that the effects of mining pools' concentration laid out in the primary analysis carry over to a setup with multiple mining pools and apply to the extensive margin. Similar to the main body, shocks to mining pools' sizes propagate to the cryptocurrency price through the total revenue share, and these shocks have the opposite effect on the concentration. As a result, when a third mining pool emerges and takes market power from both incumbent mining pools, total revenue shares increase, and concentration decreases, implying that cryptocurrency price increases and volatility drops. Concentration effects are amplified when there is a bigger bubble, much as the primary analysis predicts.

We assume that the incumbent mining pools know that a third mining pool exists, but its size is negligible. Thus, we model entry as the size of the third mining pool increases away from zero ($\lambda_{3t} \uparrow$). Accordingly, the two mining pools' economy extends straightforwardly to a three mining pools' economy, and the economics and intuitions carries over from the main analysis.

There is one Lucas tree that produces perishable services. But instead of two mining pools, there are three mining pools that compete to own a share of services at the expense of the other mining pools. With three mining pools, there are two exogenous time-varying dynamic processes that determine the proportion of the service tree that Pool-1 and Pool-2 own at every moment in time, and Pool-3 owns the residual share. Similar to the main analysis, we assume that Pool-1's and Pool-2's size processes follow

$$d\lambda_{1t} = \lambda_{1t}\Big\{\mu_{\lambda 1t}dt + \lambda_{2t}\bar{\sigma}_1 d\bar{Z}_{1t} + \lambda_{3t}\bar{\sigma}_2 d\bar{Z}_{2t}\Big\}, \quad \lambda_{10} \in (0,1), \tag{33}$$

$$d\lambda_{2t} = \lambda_{2t}\Big\{\mu_{\lambda 2t}dt - (\lambda_{1t} + \lambda_{3t})\,\bar{\sigma}_1 d\bar{Z}_{1t} + \lambda_{3t}\bar{\sigma}_2 d\bar{Z}_{2t}\Big\}, \quad \lambda_{20} \in (0,1), \tag{34}$$

and $\lambda_{3t} \equiv 1 - \lambda_{1t} - \lambda_{2t}$, where $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are strictly positive constants and the Brownian motions $\bar{Z}_1$ and $\bar{Z}_2$ are correlated with correlation equaling $\rho > 0$, explicitly given in (36).[12] This characterization ensures that mining pool sizes are strictly between zero and one and their sum equals one all the time:

$$\lambda_{it} \in (0,1), \quad \Sigma_i \lambda_{it} = 1, \quad i = 1,2,3, \quad t \in [0,T], \tag{37}$$

We assume that mining pools' services are indistinguishable and are no demand biases, implying that households derive utility from consuming services of the three mining pools,

$$E\left[\int_0^T e^{-\rho t}\left[\log\left(c_{at}^1\right) + \log\left(c_{at}^2\right) + \log\left(c_{at}^2\right)\right]dt\right]. \tag{38}$$

The mining pools' objectives are similar to the main analysis, given in (7). Our concentration measure (21) coincides with the well-known Herfindahl-Hirschman index, which straightforwardly extends to an economy with three mining pools

$$H_t = \lambda_{1t}^2 + \lambda_{2t}^2 + \lambda_{3t}^2. \tag{39}$$

[12] We obtain λ_{it} dynamics by assuming that each mining pool's size process follows a geometric Brownian motion

$$\frac{dF_i}{F_i} = \mu_i dt + \sigma_i dZ_i, \tag{35}$$

where $i = 1,2,3$, and Z_i are standard independent Brownian motions uncorrelated with Z. Then, we define the size of Pool-1 and Pool-2 as $\lambda_1 \equiv \frac{F_1}{F_1+F_2+F_3}$, and $\lambda_2 \equiv \frac{F_2}{F_1+F_2+F_3}$, respectively. By applying Itô's Lemma to this definition, we obtain (33) and (34). For further simplification, we introduce the standard Brownian motions $\bar{Z}_1 \equiv \frac{\sigma_1 Z_1}{\bar{\sigma}_1} - \frac{\sigma_2 Z_2}{\bar{\sigma}_1}$ and $\bar{Z}_2 \equiv \frac{\sigma_1 Z_1}{\bar{\sigma}_2} - \frac{\sigma_3 Z_3}{\bar{\sigma}_2}$, where $\bar{\sigma}_1 \equiv \sqrt{\sigma_1^2 + \sigma_2^2}$ and $\bar{\sigma}_2 \equiv \sqrt{\sigma_1^2 + \sigma_3^2}$. Notice that $\bar{Z}_1$ and $\bar{Z}_2$ are correlated with

$$\rho = \frac{\sigma_1^2}{\bar{\sigma}_1\bar{\sigma}_2}. \tag{36}$$

The remaining economic ingredients are identical to the main economic setup in Section 3. Proposition 8 in the internet Appendix B reports the equilibrium quantities with three mining pools, verifies that our equilibrium with two mining pools straightforwardly extends to three mining pools, and the equilibrium mechanism and intuitions are generalized.

It is worth emphasizing that households' demand biases would bias toward the incumbents and against the entrant, reducing the entrant's effect on equilibrium quantities. Therefore, it is reasonable to assume that demand biases are time varying and shrink as the third mining pool size increases. We leave this extension to future work.

Lastly, the following proposition summarizes the equilibrium effects of the entrant mining pool on the incumbent mining pools. These effects are predicted by the equilibrium effects of concentration in line with the main analysis.

Proposition 6 (**Extensive Margin**). *As a third mining pool emerges and takes market power from incumbents ($\lambda_{1t}, \lambda_{2t} > \frac{1}{3}, \lambda_{3t} < \frac{1}{3}$), concentration decreases and*

(i) The fees of incumbents increase

$$\frac{\partial p_{1t}}{\partial \lambda_{3t}} > 0, \qquad \frac{\partial p_{2t}}{\partial \lambda_{3t}} > 0. \tag{40}$$

(ii) Total revenue shares increase, resulting in a higher cryptocurrency price and a lower discount factor even without a bubble.

(iii) The absolute-term magnitude of the cryptocurrency shock sensitivities to mining pools' sizes shocks increase even without a bubble:

$$\frac{\partial \mid \bar{\sigma}^S_{1t} \mid}{\partial \lambda_{3t}} < 0, \qquad \frac{\partial \mid \bar{\sigma}^S_{2t} \mid}{\partial \lambda_{3t}} < 0.$$

(iv) A bigger bubble implies that the effect of concentration on the return volatility is more substantial

$$\frac{\partial}{\partial \beta} \left(\frac{\partial \sqrt{\left(\bar{\sigma}^S_{1t}\right)^2 + \left(\bar{\sigma}^S_{2t}\right)^2 + \left(\sigma^S_t\right)^2}}{\partial \lambda_{3t}} \right) < 0. \tag{41}$$

(v) A bigger bubble implies that the effect of concentration on the cryptocurrency price

increases, if the cryptocurrency price is in an inflationary bubble,

$$\frac{\partial}{\partial \beta}\left(\frac{\partial S_t}{\partial \lambda_{3t}}\right) > 0. \tag{42}$$

8 Conclusion

This paper incorporates two central ingredients of cryptocurrencies, mining pools and a cryptocurrency price bubble, into a traditional asset pricing model to study the implications of mining pools' concentration on the price and return volatility of cryptocurrencies.

The main aspect of mining pools central to our analysis is their role as transaction validators. Those who increase their cryptocurrency exposure even without trading. Although we call the transaction validators mining pools, which is a feature of proof of work protocols, our model's predictions apply to other blockchain protocols, such as proof of stake, as long as the concentration of transaction validators exogenously varies over time.

We present all the equilibrium quantities in precise closed-form expressions and find that as mining pools' concentration increases, the cryptocurrency price falls, and its return volatility spikes. We further show that the cryptocurrency pricing implications are similar both on the extensive and intensive margin: the entry and exit of mining pools do not affect prices insofar as through their effect on concentration. Our empirical analysis of Bitcoin verifies the model's predictions.

Our framework allows the cryptocurrency price to differ from its intrinsic, fundamental value and form a bubble. When the bubble is inflationary, only a fraction of the cryptocurrency value is determined by the sum of all the discounted future services it will provide. However, as long as this fraction is strictly positive and at least a tiny fraction of the cryptocurrency price is determined by its fundamentals, our model's predictions apply. Our model predicts that the bubble amplifies the effect of mining pools' concentration on the cryptocurrency price and its return volatility.

Our theory builds on the intuitive yet novel insight that modeling competition in a traditional asset pricing economy requires that each market participant clear markets separately from its competitors in equilibrium. Accordingly, the mining pools post fees for trading cryptocurrency services with other market participants. The equilibrium shows that the mining pools' fees are inversely related to their size, corroborating empirical findings. Importatnly, since each mining pool posts fees to clear services separately from the other mining pools, the sum of all the discounted future services can be represented by the mining pools' revenues: mining pools' fees times their services. One novel and potentially testable implication of our model is that the fundamental value of the cryptocurrency is determined by the total revenues of the mining pools.

The paper highlights the importance of mining pools' concentration to cryptocurrencies' asset pricing underpinnings. Nevertheless, since mining pool data is hard to come by, mining pools' concentration is explored by relatively few papers empirically. It would be interesting to extend our analysis beyond permissionless blockchains, such as Bitcoin, and investigate mining pools' concentration on permissioned blockchains. In those blockchains, mining pools can coordinate to control concentration and, by doing that, regulate the cryptocurrency price and return volatility advantageously. Theoretically, extending the model to study the tradeoffs mining pools face in choosing which cryptocurrency to mine across a menu of cryptocurrencies would be interesting.

A Proofs

In this section we show how to derive the equilibrium quantities. We conjecture and later verify that the security market is dynamically complete. As such, there exist a unique state price density process, ξ, and the no arbitrage condition always holds.

Proof of Lemma 1 (Implicit Discount Factor) . Following the martingale method, we restate the dynamic budget constraints (12) and (13) as

$$\xi_t W_{it} = E_t\left[\int_t^T \xi_v p_{iv} c^i_{iv} dv\right], \quad i = 1, 2, \tag{A.1}$$

$$\xi_t W_{at} = E_t\left[\int_t^T \xi_v \left(p_{1v} c^1_{av} + p_{2v} c^2_{av}\right) dv\right]. \tag{A.2}$$

The mining pool i chooses c^i_{it} to maximize its utility subject to the budget constraint (A.1) evaluated at time $t = 0$, while the households choose c^1_{at}, c^2_{at} to maximize (8) subject to the budget constraint (A.2) evaluated at time $t = 0$. We obtain the following first order conditions for the consumption choices of the mining pools and the households

$$\frac{e^{-\rho t}}{c_{1t}^1} = \frac{1}{y_1}\xi_t p_{1t}, \tag{A.3}$$
$$\frac{e^{-\rho t}}{c_{2t}^2} = \frac{1}{y_2}\xi_t p_{2t}, \tag{A.4}$$
$$\frac{e^{-\rho t}\gamma_1}{c_{at}^1} = \frac{1}{y_a}\xi_t p_{1t}, \tag{A.5}$$
$$\frac{e^{-\rho t}}{c_{at}^2} = \frac{1}{y_a}\xi_t p_{2t}, \tag{A.6}$$

where y_i denotes the Lagrange multiplier for mining pool i, and y_a denotes the Lagrange multiplier for the households. By utilizing the market clearing conditions in consumption goods,

$$c_{1t}^1 + c_{at}^1 = \lambda_{1t} Y_t, \tag{A.7}$$
$$c_{2t}^2 + c_{at}^2 = \lambda_{2t} Y_t, \tag{A.8}$$

we find that the pool-specific state price densities are

$$\xi_t p_{1t} = (\lambda_{1t} Y_t)^{-1} e^{-\rho t} (y_1 + \gamma_1 y_a), \tag{A.9}$$
$$\xi_t p_{2t} = (\lambda_{2t} Y_t)^{-1} e^{-\rho t} (y_2 + y_a). \tag{A.10}$$

By using the numeraire (9), we find the following expression for the state price density

$$\xi_t = e^{-\rho t}\frac{1}{\bar{p}}\frac{1}{Y_t}\left(\frac{y_1 + \gamma_1 y_a}{\lambda_{1t}} + \frac{y_2 + y_a}{\lambda_{2t}}\right). \tag{A.11}$$

We find the following Lagrange multipliers by plugging the optimal consumptions (A.3), (A.4), (A.5), (A.6) into the appropriate budget constraint (A.1), (A.2), and using the initial endowments:

$$y_1 \frac{1 - e^{-\rho T}}{\rho} = \xi_0 W_{10} = \xi_0 \hat{\lambda} \lambda_{10} S_0, \tag{A.12}$$
$$y_2 \frac{1 - e^{-\rho T}}{\rho} = \xi_0 W_{20} = \xi_0 \hat{\lambda} \lambda_{20} S_0, \tag{A.13}$$
$$(\gamma_1 + 1)\, y_a \frac{1 - e^{-\rho T}}{\rho} = \xi_0 W_{a0} = \xi_0 \left(1 - \hat{\lambda}\right) S_0. \tag{A.14}$$

We substitute the Lagrange multipliers into the state price density (A.11), utilize the expression for Y_t (1), and obtain the following desired result:

$$
\begin{aligned}
\xi_t &= e^{-\rho t}\frac{1}{Y_t}\frac{1}{\bar{p}}\left(\frac{y_1+\gamma_1 y_a}{\lambda_{1t}}+\frac{y_2+y_a}{\lambda_{2t}}\right) \\
&= \frac{1}{\bar{p}}\xi_0 S_0\left(\frac{\rho e^{-\rho t}}{1-e^{-\rho T}}\right)\frac{1}{Y_t}\left(\frac{1}{\lambda_{1t}}\left\{\hat{\lambda}\lambda_{10}+\frac{\gamma_1}{\gamma_1+1}\left(1-\hat{\lambda}\right)\right\}+\frac{1}{\lambda_{2t}}\left\{\hat{\lambda}\lambda_{20}+\frac{1}{\gamma_1+1}\left(1-\hat{\lambda}\right)\right\}\right) \qquad \text{(A.15)} \\
&= \frac{\xi_0 S_0}{\bar{p}}\left(\frac{\rho e^{-\rho t}}{1-e^{-\rho T}}\right)\frac{1}{Y_t}\left(\frac{\lambda_{10}}{\lambda_{1t}}+\frac{\lambda_{20}}{\lambda_{2t}}\right) \\
&= \xi_0 S_0\left(\frac{\rho e^{-\rho t}}{1-e^{-\rho T}}\right)\frac{1}{Y_t}\left(\frac{\gamma_1}{1+\gamma_1}\frac{1}{\bar{p}\lambda_{1t}}+\frac{1}{1+\gamma_1}\frac{1}{\bar{p}\lambda_{2t}}\right) \qquad \text{(A.16)} \\
&= \xi_0 S_0\left(\frac{\rho e^{-\rho t}}{1-e^{-\rho T}}\right)\frac{1}{D_t}\left(\frac{S_t}{D_t}\right)^{-\beta}\left(\frac{\gamma_1}{1+\gamma_1}\frac{1}{\bar{p}\lambda_{1t}}+\frac{1}{1+\gamma_1}\frac{1}{\bar{p}\lambda_{2t}}\right). \qquad \text{(A.17)}
\end{aligned}
$$

□

Proof of Proposition 1 (Fees). We now determine the fees charged by the mining pools. Substituting the Lagrange multipliers (A.12), (A.13) and (A.14) into the mining pool specific state price densities (A.9) and (A.10) we obtain

$$
\xi_t p_{1t} = \left(\lambda_{1t}Y_t\right)^{-1}\rho\left(\frac{e^{-\rho t}}{1-e^{-\rho T}}\right)\xi_0 S_0\left\{\hat{\lambda}\lambda_{10}+\frac{\gamma_1}{\gamma_1+1}\left(1-\hat{\lambda}\right)\right\}, \qquad \text{(A.18)}
$$

$$
\xi_t p_{2t} = \left(\lambda_{2t}Y_t\right)^{-1}\rho\left(\frac{e^{-\rho t}}{1-e^{-\rho T}}\right)\xi_0 S_0\left\{\hat{\lambda}\lambda_{20}+\frac{1}{\gamma_1+1}\left(1-\hat{\lambda}\right)\right\}. \qquad \text{(A.19)}
$$

To isolate the fees, we plug the expression for ξ (A.17) into the equations above, and find

$$
\frac{p_{1t}}{\bar{p}} = \frac{\frac{\gamma_1}{\lambda_{1t}}}{\frac{\gamma_1}{\lambda_{1t}}+\frac{1}{\lambda_{2t}}} = \frac{\frac{\gamma_1}{1+\gamma_1}\frac{1}{\lambda_{1t}}}{\frac{\gamma_1}{1+\gamma_1}\frac{1}{\lambda_{1t}}+\frac{1}{1+\gamma_1}\frac{1}{\lambda_{2t}}}, \qquad \text{(A.20)}
$$

$$
\frac{p_{2t}}{\bar{p}} = \frac{\frac{1}{\lambda_{2t}}}{\frac{\gamma_1}{\lambda_{1t}}+\frac{1}{\lambda_{2t}}} = \frac{\frac{1}{1+\gamma_1}\frac{1}{\lambda_{2t}}}{\frac{\gamma_1}{1+\gamma_1}\frac{1}{\lambda_{1t}}+\frac{1}{1+\gamma_1}\frac{1}{\lambda_{2t}}}. \qquad \text{(A.21)}
$$

By differentiating the fees, it is straightforward to show that $\frac{\partial p_{1t}}{\partial \lambda_{1t}} < 0$, $\frac{\partial p_{2t}}{\partial \lambda_{1t}} > 0$, $\frac{\partial p_{1t}}{\partial \gamma_1} > 0$, and $\frac{\partial p_{2t}}{\partial \gamma_1} < 0$.

□

Proof of Proposition 2 (Revenue Share). Plugging the equilibrium fees into the revenue share definitions (17) we obtain our desired result,

$$
V_{1t} = \bar{p}\frac{\frac{\gamma_1}{1+\gamma_1}}{\frac{\gamma_1}{1+\gamma_1}\frac{1}{\lambda_{1t}}+\frac{1}{1+\gamma_1}\frac{1}{\lambda_{2t}}}, \quad V_{2t} = \bar{p}\frac{\frac{1}{1+\gamma_1}}{\frac{\gamma_1}{1+\gamma_1}\frac{1}{\lambda_{1t}}+\frac{1}{1+\gamma_1}\frac{1}{\lambda_{2t}}}, \quad V_t = \bar{p}\frac{1}{\frac{\gamma_1}{1+\gamma_1}\frac{1}{\lambda_{1t}}+\frac{1}{1+\gamma_1}\frac{1}{\lambda_{2t}}}. \qquad \text{(A.22)}
$$

In order to determine the maximal revenue shares and the total revenue share, we set

$$\frac{\partial}{\partial \lambda_{1t}} \left(\frac{1}{\frac{\gamma_1}{1+\gamma_1} \frac{1}{\lambda_{1t}} + \frac{1}{1+\gamma_1} \frac{1}{\lambda_{2t}}} \right) = 0,$$

and obtain our desired result,

$$\lambda_{1*} = \frac{\sqrt{\gamma_1}}{1+\sqrt{\gamma_1}}, \qquad \lambda_{2*} = \frac{1}{1+\sqrt{\gamma_1}}. \tag{A.23}$$

The point λ_{1*} is the global maximum because the revenue shares and the total revenue share are strictly concave and continuous functions of λ_{1t} for $0 < \lambda_{1t} < 1$. $\square$

Proof of Proposition 3 (Cryptocurrency Price and Discount Factor). We find the cryptocurrency price S_t from the no arbitrage relation, which identifies the cryptocurrency price as the discounted services produced by the two mining pools,

$$S_t = \frac{1}{\xi_t} E_t \left[\int_t^T \xi_s \left\{ p_{1s} \lambda_{1s} Y_s + p_{2s} \lambda_{2s} Y_s \right\} ds \right] = \frac{1}{\xi_t} e^{-\rho t} \frac{1 - e^{-\rho(T-t)}}{1 - e^{-\rho T}} \xi_0 S_0, \tag{A.24}$$

where the equality follows from plugging the individual state price densities (A.18) and (A.19), and noting that $\lambda_{10} + \lambda_{20} = 1$. Substituting the state price density ξ_t (A.15) into the above characterization we obtain the following expression for the cryptocurrency price

$$\begin{aligned} S_t &= Y_t \left[\frac{1 - e^{-\rho(T-t)}}{\rho} \right] \left[\frac{1}{\frac{1}{\bar{p}\lambda_{1t}} \left\{ \hat{\lambda}\lambda_{10} + \frac{\gamma_1}{\gamma_1+1} \left(1 - \hat{\lambda}\right) \right\} + \frac{1}{\bar{p}\lambda_{2t}} \left\{ \hat{\lambda}\lambda_{20} + \frac{1}{\gamma_1+1} \left(1 - \hat{\lambda}\right) \right\}} \right] \\ &= Y_t \left[\frac{1 - e^{-\rho(T-t)}}{\rho} \right] \left[\frac{1}{\frac{\lambda_{10}}{\bar{p}\lambda_{1t}} + \frac{\lambda_{20}}{\bar{p}\lambda_{2t}}} \right] = Y_t \left[\frac{1 - e^{-\rho(T-t)}}{\rho} \right] \left[\frac{\gamma_1 + 1}{\frac{\gamma_1}{\bar{p}\lambda_{1t}} + \frac{1}{\bar{p}\lambda_{2t}}} \right]. \end{aligned} \tag{A.25}$$

The above expression can be further simplified by substituting the expressions for the fees charged by mining pools (14). By doing so, we obtain

$$S_t = Y_t \left[\frac{1 - e^{-\rho(T-t)}}{\rho} \right] \left[\lambda_{1t} p_{1t} + \lambda_{2t} p_{2t} \right]. \tag{A.26}$$

The above expression for the cryptocurrency price depends on Y_t, which then depends on the endogenous cryptocurrency price. We substitute the expression for Y_t (26) and the definition of $V_t \equiv \lambda_{1t} p_{1t} + \lambda_{2t} p_{2t}$ to obtain the desired expression for the cryptocurrency price,

$$S_t = D_t \left(\frac{1 - e^{-\rho(T-t)}}{\rho} \right)^{\frac{1}{1-\beta}} (V_t)^{\frac{1}{1-\beta}}. \tag{A.27}$$

By plugging the total revenue share (18), the closed-form expression of the cryptocurrency price becomes

$$S_t = D_t \left[\frac{1 - e^{-\rho(T-t)}}{\rho} \right]^{\frac{1}{1-\beta}} \left[\frac{\gamma_1 + 1}{\frac{\gamma_1}{\bar{p}\lambda_{1t}} + \frac{1}{\bar{p}\lambda_{2t}}} \right]^{\frac{1}{1-\beta}}. \tag{A.28}$$

By plugging the cryptocurrency price into the implicit discount factor (25), we find that the discount factor is given by

$$\xi_{0,t} = \bar{\xi} e^{-\rho t} \left(\frac{1 - e^{-\rho(T-t)}}{\rho} \right)^{\frac{-\beta}{1-\beta}} \frac{1}{D_t} \left(\frac{1}{V_t} \right)^{\frac{1}{1-\beta}}, \tag{A.29}$$

where $\bar{\xi} \equiv S_0 \left(\frac{\rho}{1-e^{-\rho T}} \right)$. The cryptocurrency price S_0 is obtained from the expression for S_t (27) evaluated at $t = 0$. Lastly, observe that $dS_t/dV_t > 0$ (A.27) and $d\xi_t/dV_t < 0$ (A.29), for any $\beta \in [0, 1)$. We further note that

$$\frac{\partial H_t}{\partial \lambda_{1t}} = (1 + \sqrt{\gamma_1}) \left(\frac{2\lambda_{1t}}{\sqrt{\gamma_1}} - 2(1 - \lambda_{1t}) \right) < 0 \Longleftrightarrow \quad \frac{\partial V_t}{\partial \lambda_{1t}} = -\frac{\bar{p}(1 + \gamma_1)}{\left(\frac{\gamma_1}{\lambda_{1t}} + \frac{1}{1-\lambda_{1t}} \right)^2} \left(-\frac{\gamma_1}{\lambda_{1t}^2} + \frac{1}{(1 - \lambda_{1t})^2} \right) > 0,$$

which leads to

$$\frac{d\xi_t}{d\lambda_{1t}} = \frac{d\xi_t}{dV_t} \frac{dV_t}{d\lambda_{1t}} > 0 \iff \frac{dH_t}{d\lambda_{1t}} > 0, \tag{A.30}$$

$$\frac{dS_t}{d\lambda_{1t}} = \frac{dS_t}{dV_t} \frac{dV_t}{d\lambda_{1t}} < 0 \iff \frac{dH_t}{d\lambda_{1t}} > 0, \tag{A.31}$$

for any $\beta \in [0, 1)$. □

Proof of Proposition 4 (Volatility). By applying Itô's Lemma to both sides of the cryptocurrency price (27) and comparing the volatility terms we find that

$$\sigma_t^S = \sigma^D, \tag{A.32}$$

$$\bar{\sigma}_t^S = \frac{1}{1-\beta} \left(\frac{\frac{\gamma_1}{1+\gamma_1} \frac{1}{\lambda_{1t}}}{\frac{\gamma_1}{1+\gamma_1} \frac{1}{\lambda_{1t}} + \frac{1}{1+\gamma_1} \frac{1}{\lambda_{2t}}} \lambda_{2t} + \frac{\frac{1}{1+\gamma_1} \frac{1}{\lambda_{2t}}}{\frac{\gamma_1}{1+\gamma_1} \frac{1}{\lambda_{1t}} + \frac{1}{1+\gamma_1} \frac{1}{\lambda_{2t}}} \lambda_{1t} \right) = \frac{1}{1-\beta} \left(\frac{p_{1t}}{\bar{p}} \lambda_{2t} - \frac{p_{2t}}{\bar{p}} \lambda_{1t} \right) \bar{\sigma}, \tag{A.33}$$

where the last equality follows from plugging the fees' representation (14). We note that $\bar{\sigma}_t^S = 0$ if and only if $\left(\frac{p_{1t}}{\bar{p}} \lambda_{2t} - \frac{p_{2t}}{\bar{p}} \lambda_{1t} \right) = 0$, which translates to

$$\frac{\gamma_1}{\lambda_{1t}^2} - \frac{1}{\lambda_{2t}^2} = 0,$$

after plugging the fees' representation (14). Solving for the optimal λ_t that satisfies the $\bar{\sigma}_t^S = 0$ requirement we obtain

$$\lambda_{1t} = \frac{\sqrt{\gamma_1}}{1+\sqrt{\gamma_1}}, \tag{A.34}$$

which is the point at which the mining pools' concentration is minimized. We further observe that

$$\frac{\partial \bar{\sigma}_t^S}{\partial \lambda_{1t}} = \frac{1}{1-\beta}\frac{\bar{\sigma}}{\bar{p}}\left(\frac{\partial p_{1t}}{\partial \lambda_{1t}}\lambda_{2t} - p_{1t} - p_{2t} - \frac{\partial p_{2t}}{\partial \lambda_{1t}}\lambda_{1t}\right), \tag{A.35}$$

which implies that $\frac{\partial \bar{\sigma}_t^S}{\partial \lambda_{1t}} < 0$ since the signs of all the elements in the brackets are negative, as (15) reveals. The shock sensitivity magnitude can be expressed as

$$\mid \bar{\sigma}_t^S \mid = \begin{cases} \bar{\sigma}_t^S & \text{for } \lambda_{1t} < \lambda_{1*}, \\ 0 & \text{for } \lambda_{1t} = \lambda_{1*}, \\ -\bar{\sigma}_t^S & \text{for } \lambda_{1t} > \lambda_{1*}. \end{cases}$$

By taking the derivative of the shock sensitivity magnitude and given the sign of $\frac{\partial \bar{\sigma}_t^S}{\partial \lambda_{1t}}$, we obtain our desired result,

$$\frac{\partial \mid \bar{\sigma}_t^S \mid}{\partial \lambda_{1t}} = \begin{cases} \frac{\partial \bar{\sigma}_t^S}{\partial \lambda_{1t}} < 0 & \text{for } \lambda_{1t} < \lambda_{1*}, \\ -\frac{\partial \bar{\sigma}_t^S}{\partial \lambda_{1t}} > 0 & \text{for } \lambda_{1t} > \lambda_{1*}. \end{cases}$$

Lastly,

$$\frac{\partial \bar{\sigma}_t^S}{\partial \beta} = \frac{1}{1-\beta}\bar{\sigma}_t^S > 0, \tag{A.36}$$

which implies

$$\frac{\partial \left|\bar{\sigma}_t^S\right|}{\partial \beta} = \begin{cases} \frac{\partial \bar{\sigma}_t^S}{\partial \beta} > 0 & \text{for } \lambda_{1t} < \lambda_{1*}, \\ -\frac{\partial \bar{\sigma}_t^S}{\partial \beta} < 0 & \text{for } \lambda_{1t} > \lambda_{1*}. \end{cases}. \tag{A.37}$$

□

Proof of Proposition 5 (Bubble). We denote by $S_t\,(\beta = 0)$ and $S_t\,(\beta > 0)$ the cryptocurrency prices when $\beta = 0$ and $\beta > 0$, respectively. Following the cryptocurrency price representation (27), a necessary and sufficient condition for the existence of an inflationary bubble is

$$S_t\,(\beta > 0) > S_t\,(\beta = 0) \quad \Longleftrightarrow \quad \frac{1-e^{-\rho(T-t)}}{\rho}V_t > 1. \tag{A.38}$$

We look for the fees index $\bar{p}$ that satisfies (A.38) when the revenue share is maximized ($\lambda_{1t} = \lambda_{1*}$). That is,

$$\bar{p} > \frac{\left(1+\sqrt{\gamma_1}\right)^2}{(1+\gamma_1)} \frac{\rho}{1-e^{-\rho(T-t)}} \equiv P_t. \tag{A.39}$$

This condition guarantees that there exists some value of λ_{1t} for which there is an inflationary bubble (A.38). If the inflationary bubble exists at the maximum ($\lambda_{1t} = \lambda_{1*}$) it must also exist in the neighbourhood of (λ_{1*}) since the total revenue share is strictly concave in λ_{1t}. We define $\underline{\lambda}_t$ and $\bar{\lambda}_t$ as the lower and upper bounds of this neighbourhood; we find $\underline{\lambda}_t$ and $\bar{\lambda}_t$ by looking for λ_t that solves (A.38) with equality:

$$\underline{\lambda}_t = \frac{\left[\frac{\gamma_1-1}{\gamma_1+1} + F_t\bar{p}\right] - \sqrt{\left[\frac{\gamma_1-1}{\gamma_1+1} + F_t\bar{p}\right]^2 - 4F_t\bar{p}\frac{\gamma_1}{\gamma_1+1}}}{2F_t\bar{p}}, \tag{A.40}$$

$$\bar{\lambda}_t = \frac{\left[\frac{\gamma_1-1}{\gamma_1+1} + F_t\bar{p}\right] + \sqrt{\left[\frac{\gamma_1-1}{\gamma_1+1} + F_t\bar{p}\right]^2 - 4F_t\bar{p}\frac{\gamma_1}{\gamma_1+1}}}{2F_t\bar{p}}, \tag{A.41}$$

$$F_t = \frac{1-e^{-\rho(T-t)}}{\rho}. \tag{A.42}$$

We now turn to prove the four bullet points. To prove (i), we take the cross derivative of the total volatility and find that

$$\frac{\partial}{\partial\beta}\left(\frac{\partial\sqrt{\left(\bar{\sigma}_t^S\right)^2 + \left(\sigma_t^S\right)^2}}{\partial\lambda_{1t}}\right) = \frac{1}{1-\beta}\left(\frac{\partial\bar{\sigma}_t^S}{\partial\lambda_{1t}}\right)\frac{\bar{\sigma}_t^S}{\sqrt{\left(\bar{\sigma}_t^S\right)^2 + \left(\sigma_t^S\right)^2}}\left[2 - \frac{\left(\bar{\sigma}_t^S\right)^2}{\left(\bar{\sigma}_t^S\right)^2 + \left(\sigma_t^S\right)^2}\right] < 0 \iff \lambda_{1t} < \lambda_{1*},$$

since $\frac{\partial\bar{\sigma}_t^S}{\partial\lambda_{1t}} < 0$ all the time, and $\bar{\sigma}_t^S > 0$ if and only if $\lambda_{1t} > \lambda_{1*}$. To prove (ii), we take the cross derivative of the cryptocurrency price and find that

$$\frac{\partial}{\partial\beta}\left(\frac{\partial S_t}{\partial\lambda_{1t}}\right) = \frac{1}{(1-\beta)^2}\frac{\partial V_t}{\partial\lambda_{1t}}\left[\log\left(\frac{1-e^{-\rho(T-t)}}{\rho}V_t\right) + 1 - \beta\right].$$

The square brackets are positive if the cryptocurrency is in an inflationary bubble since $\left(\frac{1-e^{-\rho(T-t)}}{\rho}V_t\right) > 1$, and $\frac{\partial V_t}{\partial\lambda_{1t}} > 0$ if and only if $\lambda_{1t} < \lambda_{1*}$, as the proof of Proposition (2) reveals. To prove part (iii), we differentiate the total volatility with respect to the interdependency parameter and find that

$$\frac{\partial\sqrt{\left(\bar{\sigma}_t^S\right)^2 + \left(\sigma_t^S\right)^2}}{\partial\beta} = \frac{\left(\bar{\sigma}_t^S\right)^2}{\sqrt{\left(\bar{\sigma}_t^S\right)^2 + \left(\sigma_t^S\right)^2}}\frac{1}{1-\beta} > 0,$$

where $\bar{\sigma}_t^S$ is given in (29). Lastly, to prove (iv) we take the derivative of S_t with respect to the interdependency parameter and find that

$$\frac{\partial S_t}{\partial \beta} = D_t \left(\frac{1 - e^{-\rho(T-t)}}{\rho} V_t \right)^{\frac{1}{1-\beta}} \left(\frac{1}{1-\beta} \right)^2 \log \left(\frac{1 - e^{-\rho(T-t)}}{\rho} V_t \right),$$

which implies that $\frac{\partial S_t}{\partial \beta} > 0$ if and only if the cryptocurrency is in an inflationary bubble, as (A.38) reveals. $\square$

Proposition 7 (**Portfolios**). *Markets are dynamically complete; the mining pools and households hold their entire wealth in the cryptocurrency.*

Proof of Proposition 7 (Portfolios). We start by characterizing the dynamics of the bonds. Each bond is riskless in terms of its mining pool's numeraire:

$$dB_{it}^i = r_{it}^i B_{it}^i dt, \quad i = 1, 2. \tag{A.43}$$

Converting these bonds to the numeraire we obtain

$$B_{1t} = p_{1t} B_{1t}^1, \qquad B_{2t} = p_{2t} B_{2t}^2. \tag{A.44}$$

By applying Itô's Lemma to both sides of the above equations, we find that

$$\sigma_t^{B_1} = 0, \quad \bar{\sigma}_t^{B_1} = -\bar{\sigma} \frac{\lambda_{1t}}{\lambda_{1t} + \gamma_1 \lambda_{2t}}, \tag{A.45}$$

$$\sigma_t^{B_2} = 0, \quad \bar{\sigma}_t^{B_2} = \bar{\sigma} \frac{\gamma_1 \lambda_{2t}}{\lambda_{1t} + \gamma_1 \lambda_{2t}}. \tag{A.46}$$

Notice that $\bar{\sigma}_t^{B_1} \neq 0$ and $\bar{\sigma}_t^{B_2} \neq 0$ probability almost surely since $\lambda_{1t} \in (0, 1)$. We define the global bond security B^W, which is locally riskless in the numeraire. This additional security is not required but it simplifies the analysis. It is simply defined as the sum of the two bonds.

To dynamically complete the financial markets and to replicate any financial claim, we require three independent investment opportunities: the cryptocurrency, Pool-1's bond, and the global bond. We denote the vector of portfolio weights of agent i in the cryptocurrency and mining pool one bond by π_i, and the volatility matrix of these two securities by Σ, such that

$$\pi_i \equiv \begin{bmatrix} \pi_i^S \\ \pi_i^{B_1} \end{bmatrix}, \quad \Sigma \equiv \begin{bmatrix} \sigma_t^S & \bar{\sigma}_t^S \\ 0 & \bar{\sigma}_t^{B_1} \end{bmatrix}, \tag{A.47}$$

where $i = 1, 2, a$. Solving for the optimal wealth (A.2) (A.1), we find that $W_{it} \propto \frac{1}{\xi_t}$, and by applying Itô's Lemma to both sides of this equation we obtain

$$\Sigma' \pi_i = \theta_t, \tag{A.48}$$

where θ_t is the vector of market prices of risk. It is given by

$$\theta_t = \begin{bmatrix} \sigma_t^S \\ \bar{\sigma}_t^S \end{bmatrix}, \tag{A.49}$$

which is obtained from applying Itô's Lemma to both sides of (A.29). Since $\sigma_t^S \neq 0$, as Proposition 4 reveals, we invert Σ' and obtain that $\pi_i^S = 1$ and $\pi_i^{B_1} = 0$ for $i = 1, 2, a$.

Following Pavlova and Rigobon (2007), we obtain these interest rates, r_{it}^i, by applying Itô's Lemma to the mining pool specific state price density, $\xi_t p_{it}$, given in (A.9) and (A.10), which leads to

$$\begin{aligned} r_{1t}^1 = {} & \rho + \beta\mu_t^S + (1-\beta)\,\mu_t^D + \frac{1}{2}\beta(\beta-1)\frac{1}{S_t^2}\left[\left(\sigma_t^S\right)^2 + \left(\bar{\sigma}_t^S\right)^2\right] - \frac{1}{2}\beta(1-\beta)\frac{1}{D_t^2}\left(\sigma_t^D\right)^2 + \beta(1-\beta)\sigma_t^D\sigma_t^S \\ & + \lambda_{2t}\left\{\left(\mu_1 - \lambda_{1t}\left(\sigma_1\right)^2\right) - \left(\mu_2 - \lambda_{2t}\left(\sigma_2\right)^2\right)\right\} \\ & - \left(\sigma_t^D\right)^2 - \left[\frac{1}{1-\beta}\lambda_{1t}\lambda_{2t}\bar{\sigma}\frac{1}{\frac{\gamma_1}{\lambda_{1t}} + \frac{1}{\lambda_{2t}}}\left(+\frac{\gamma_1}{\lambda_{1t}^2} - \frac{1}{\lambda_{2t}^2}\right)\right]^2 \\ & - \lambda_{2t}^2\bar{\sigma}^2 - \lambda_{2t}\bar{\sigma}\left[\frac{\beta}{1-\beta}\lambda_{1t}\lambda_{2t}\bar{\sigma}\frac{1}{\frac{\gamma_1}{\lambda_{1t}} + \frac{1}{\lambda_{2t}}}\left(+\frac{\gamma_1}{\lambda_{1t}^2} - \frac{1}{\lambda_{2t}^2}\right)\right] \end{aligned}$$

and

$$\begin{aligned} r_{2t}^1 = {} & \rho + \beta\mu_t^S + (1-\beta)\,\mu_t^D + \frac{1}{2}\beta(\beta-1)\frac{1}{S_t^2}\left[\left(\sigma_t^S\right)^2 + \left(\bar{\sigma}_t^S\right)^2\right] - \frac{1}{2}\beta(1-\beta)\frac{1}{D_t^2}\left(\sigma_t^D\right)^2 + \beta(1-\beta)\sigma_t^D\sigma_t^S \\ & - \lambda_{1t}\left\{\left(\mu_1 - \lambda_{1t}\left(\sigma_1\right)^2\right) - \left(\mu_2 - \lambda_{2t}\left(\sigma_2\right)^2\right)\right\} \\ & - \left(\sigma_t^D\right)^2 - \left[\frac{1}{1-\beta}\lambda_{1t}\lambda_{2t}\bar{\sigma}\frac{1}{\frac{\gamma_1}{\lambda_{1t}} + \frac{1}{\lambda_{2t}}}\left(+\frac{\gamma_1}{\lambda_{1t}^2} - \frac{1}{\lambda_{2t}^2}\right)\right]^2 \\ & - \lambda_{1t}^2\bar{\sigma}^2 - \lambda_{1t}\bar{\sigma}\left[\frac{\beta}{1-\beta}\lambda_{1t}\lambda_{2t}\bar{\sigma}\frac{1}{\frac{\gamma_1}{\lambda_{1t}} + \frac{1}{\lambda_{2t}}}\left(+\frac{\gamma_1}{\lambda_{1t}^2} - \frac{1}{\lambda_{2t}^2}\right)\right]. \end{aligned}$$

□

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B Internet Appendix – Three Mining Pools

Proposition 8 (**Asset Prices with Three Mining Pools**). *The mining pools post fees given by*

$$p_{it} = \bar{p} \frac{\frac{1}{\lambda_{it}}}{\frac{1}{\lambda_{1t}} + \frac{1}{\lambda_{2t}} + \frac{1}{\lambda_{3t}}}, \quad i = 1, 2, 3, \tag{B.1}$$

to trade their services. Mining pools' revenue shares and the total revenue share are give by

$$V_{it} = p_{it}\lambda_{it} = \bar{p} \frac{1}{\frac{1}{\lambda_{1t}} + \frac{1}{\lambda_{2t}} + \frac{1}{\lambda_{3t}}}, \quad V_t = \bar{p} \frac{3}{\frac{1}{\lambda_{1t}} + \frac{1}{\lambda_{2t}} + \frac{1}{\lambda_{3t}}}. \tag{B.2}$$

When the total revenue share attains its maximum, the concentration attains its minimum, and their partial derivatives are negatively related for any feasible λ_{1t} and λ_{2t},

$$\frac{\partial H_t}{\partial \lambda_{it}} < 0 \iff \frac{\partial V_t}{\partial \lambda_{it}} > 0, \quad i = 1, 2. \tag{B.3}$$

The equilibrium cryptocurrency price and discount factor are given by

$$S_t = \left(\frac{1 - e^{-\rho(T-t)}}{\rho} \right)^{\frac{1}{1-\beta}} D_t \left(V_t\right)^{\frac{1}{1-\beta}}, \quad \xi_{0,t} = \bar{\xi} e^{-\rho t} \left(\frac{1 - e^{-\rho(T-t)}}{\rho} \right)^{-\frac{\beta}{1-\beta}} \frac{1}{D_t} \left(\frac{1}{V_t} \right)^{\frac{1}{1-\beta}}, \tag{B.4}$$

where V_t is given in (B.2), and $\bar{\xi} \equiv S_0 \left(\frac{\rho}{1-e^{-\rho T}} \right)$. The discount factor increases with concentration, and the cryptocurrency price decreases with concentration, even without interdependency ($\beta = 0$). The cryptocurrency shock sensitivities are given by

$$\bar{\sigma}^S_{1t} = \frac{1}{1-\beta} \left(\frac{p_{1t} + p_{3t}}{\bar{p}} \lambda_{2t} - \frac{p_{2t}}{\bar{p}} \left(\lambda_{1t} + \lambda_{3t} \right) \right) \bar{\sigma}_1, \quad \bar{\sigma}^S_{2t} = \frac{1}{1-\beta} \left(\frac{p_{1t} + p_{2t}}{\bar{p}} \lambda_{3t} - \frac{p_{3t}}{\bar{p}} \left(\lambda_{1t} + \lambda_{2t} \right) \right) \bar{\sigma}_2, \tag{B.5}$$

$$\sigma^S_t = \sigma^D. \tag{B.6}$$

The cryptocurrency price is in an inflationary bubble if and only if the fees index is sufficiently elevated ($\bar{p} > P_{2t}$), and mining pools' concentration is not extreme (B.51), where P_{2t} is a deterministic function of time, defined in (B.50). An increased interdependency implies

(i) A higher return volatility, $\frac{\partial \sqrt{\left(\bar{\sigma}^S_{1t}\right)^2 + \left(\bar{\sigma}^S_{2t}\right)^2 + \left(\sigma^S_t\right)^2}}{\partial \beta} > 0$.

(ii) A higher cryptocurrency price, $\frac{\partial S_t}{\partial \beta} > 0$, *if and only if the cryptocurrency is in an inflationary bubble.*

Proof of Proposition 8 (Asset Prices with Three Mining Pools). In this section we derive the equilibrium quantities for three mining pools. We conjecture and later verify that the security market is dynamically complete, and therefore, there exist a unique state price density process, ξ, and the no arbitrage condition always holds. The dynamic budget constraints are given by

$$\begin{aligned} dW_{it} =& W_{it}\pi_{it}^{S}\frac{dS_t + p_{1t}\lambda_{1t}Y_t dt + p_{2t}\lambda_{2t}Y_t dt + p_{3t}\lambda_{3t}Y_t dt}{S_t} + W_{it}\pi_{it}^{B_1}\frac{dB_{1t}}{B_{1t}} + W_{it}\pi_{it}^{B_2}\frac{dB_{2t}}{B_{2t}} + W_{it}\pi_{it}^{B_3}\frac{dB_{3t}}{B_{3t}} \\ & - p_{it}c_{it}^{i}dt, \end{aligned} \tag{B.7}$$

for Pool-i $= 1, 2, 3$, and

$$\begin{aligned} dW_{at} =& W_{at}\pi_{at}^{S}\frac{dS_t + p_{1t}\lambda_{1t}Y_t dt + p_{2t}\lambda_{2t}Y_t dt + p_{3t}\lambda_{3t}Y_t dt}{S_t} + W_{at}\pi_{at}^{B_1}\frac{dB_{1t}}{B_{1t}} + W_{at}\pi_{at}^{B_2}\frac{dB_{2t}}{B_{2t}} + W_{at}\pi_{at}^{B_3}\frac{dB_{3t}}{B_{3t}} \\ & - p_{1t}c_{at}^{1}dt - p_{2t}c_{at}^{2}dt - p_{3t}c_{at}^{3}dt, \end{aligned} \tag{B.8}$$

for the households. The quantity π_{it}^{j} is endogenous and represents a fraction of W_{it} invested at time t in security j, and $i = 1, 2, 3, a$. Following the martingale method, we restate these dynamic budget constraints as

$$\xi_t W_{it} = E_t\left[\int_t^T \xi_v p_{iv} c_{iv}^{i} dv\right], \quad i = 1, 2, 3, \tag{B.9}$$

$$\xi_t W_{at} = E_t\left[\int_t^T \xi_v \left(p_{1v}c_{av}^{1} + p_{2v}c_{av}^{2} + p_{3v}c_{av}^{3}\right) dv\right]. \tag{B.10}$$

The mining pool i chooses c_{it}^{i} to maximize its utility subject to the budget constraint (B.9) evaluated at time $t = 0$, while the households choose $c_{at}^{1}, c_{at}^{2}, c_{at}^{3}$ to maximize (38) subject to the budget constraint (B.10) evaluated at time $t = 0$. We obtain the following first order conditions for the consumption choices of the mining pools and the households

$$\frac{e^{-\rho t}}{c_{1t}^{1}} = \frac{1}{y_1}\xi_t p_{1t}, \tag{B.11}$$

$$\frac{e^{-\rho t}}{c_{2t}^{2}} = \frac{1}{y_2}\xi_t p_{2t}, \tag{B.12}$$

$$\frac{e^{-\rho t}}{c_{3t}^{3}} = \frac{1}{y_3}\xi_t p_{3t}, \tag{B.13}$$

$$\frac{e^{-\rho t}}{c_{at}^{1}} = \frac{1}{y_a}\xi_t p_{1t}, \tag{B.14}$$

$$\frac{e^{-\rho t}}{c_{at}^{2}} = \frac{1}{y_a}\xi_t p_{2t}, \tag{B.15}$$

$$\frac{e^{-\rho t}}{c_{at}^{3}} = \frac{1}{y_a}\xi_t p_{3t}, \tag{B.16}$$

where y_i denotes the Lagrange multiplier for mining pool i, and y_a denotes the Lagrange multiplier for the households. By utilizing the market clearing conditions in consumption goods,

$$c_{1t}^1 + c_{at}^1 = \lambda_{1t} Y_t, \tag{B.17}$$

$$c_{2t}^2 + c_{at}^2 = \lambda_{2t} Y_t, \tag{B.18}$$

$$c_{3t}^3 + c_{at}^3 = \lambda_{3t} Y_t, \tag{B.19}$$

we find that the pool-specific state price densities are

$$\xi_t p_{1t} = (\lambda_{1t} Y_t)^{-1} e^{-\rho t} (y_1 + y_a), \tag{B.20}$$

$$\xi_t p_{2t} = (\lambda_{2t} Y_t)^{-1} e^{-\rho t} (y_2 + y_a), \tag{B.21}$$

$$\xi_t p_{3t} = (\lambda_{3t} Y_t)^{-1} e^{-\rho t} (y_3 + y_a). \tag{B.22}$$

By using the numeraire

$$p_{1t} + p_{2t} + p_{3t} = \bar{p}, \tag{B.23}$$

we find the following expression for the state price density

$$\xi_t = e^{-\rho t} \frac{1}{\bar{p}} \frac{1}{Y_t} \left(\frac{y_1 + y_a}{\lambda_{1t}} + \frac{y_2 + y_a}{\lambda_{2t}} + \frac{y_3 + y_a}{\lambda_{3t}} \right). \tag{B.24}$$

Similar to the main analysis, the households and the mining pools are price takers, and without loss of generality, we set the initial supply share to equal the initial wealth share so that

$$\lambda_{10} = \frac{1}{3}, \quad \lambda_{20} = \frac{1}{3}, \quad \lambda_{30} = \frac{1}{3}. \tag{B.25}$$

Further, we let W_{a0} and W_{i0}, $i = 1, 2, 3$, be the households and the mining pools' value of the initial endowments, respectively. We assume that the households are endowed with $1 - \hat{\lambda}$ shares of services at time 0; of those $1 - \hat{\lambda}$ shares, a λ_{10} fraction is of Pool-1 good, and a λ_{20} fraction is of Pool-2 good, and a λ_{30} fraction is of Pool-3 good. The three mining pools are initially endowed with the residual $\hat{\lambda}$ shares of services; of those $\hat{\lambda}$ shares, a λ_{10} fraction is of Pool-1 good, a λ_{20} fraction is of Pool-2 good, and a λ_{30} fraction is of Pool-3 good. Thus, $W_{a0} = (1 - \hat{\lambda}) S_0$, $W_{10} = \hat{\lambda} \lambda_{10} S_0$, $W_{20} = \hat{\lambda} \lambda_{20} S_0$, and $W_{30} = \hat{\lambda} \lambda_{30} S_0$. By plugging the optimal consumptions (B.11), (B.12), (B.13), (B.14), (B.15), (B.16) into the appropriate budget constraint

(B.9), (B.10), and using the initial endowments we find the Lagrange multipliers:

$$y_1 \frac{1-e^{-\rho T}}{\rho} = W_{10} = \hat{\lambda}\lambda_{10}\xi_0 S_0, \tag{B.26}$$

$$y_2 \frac{1-e^{-\rho T}}{\rho} = W_{20} = \hat{\lambda}\lambda_{20}\xi_0 S_0, \tag{B.27}$$

$$y_3 \frac{1-e^{-\rho T}}{\rho} = W_{30} = \hat{\lambda}\lambda_{30}\xi_0 S_0, \tag{B.28}$$

$$3y_a \frac{1-e^{-\rho T}}{\rho} = W_{a0} = \left(1-\hat{\lambda}\right)\xi_0 S_0. \tag{B.29}$$

We substitute the Lagrange multipliers into the state price density, (B.24), utilize the expression for Y_t, (1), and obtain the following desired result:

$$\begin{aligned}\xi_t &= e^{-\rho t}\frac{1}{Y_t}\frac{1}{\bar{p}}\left(\frac{y_1+y_a}{\lambda_{1t}} + \frac{y_2+y_a}{\lambda_{2t}} + \frac{y_3+y_a}{\lambda_{3t}}\right) \\ &= \frac{S_0\xi_0}{\bar{p}Y_t}\left(\frac{\rho e^{-\rho t}}{1-e^{-\rho T}}\right)\left(\frac{1}{\lambda_{1t}}\left\{\hat{\lambda}\lambda_{10} + \frac{1}{3}\left(1-\hat{\lambda}\right)\right\} + \frac{1}{\lambda_{2t}}\left\{\hat{\lambda}\lambda_{20} + \frac{1}{3}\left(1-\hat{\lambda}\right)\right\} + \frac{1}{\lambda_{3t}}\left\{\hat{\lambda}\lambda_{20} + \frac{1}{3}\left(1-\hat{\lambda}\right)\right\}\right).\end{aligned} \tag{B.30}$$

Next, by substituting the Lagrange multipliers (B.26), (B.27), (B.28), and (B.29) into the mining pool specific state price densities (B.20), (B.21) and (B.22), we obtain the mining pools fees

$$\xi_t p_{1t} = (\lambda_{1t}Y_t)^{-1}\rho\left(\frac{e^{-\rho t}}{1-e^{-\rho T}}\right)\xi_0 S_0\left\{\hat{\lambda}\lambda_{10} + \frac{1}{3}\left(1-\hat{\lambda}\right)\right\}, \tag{B.31}$$

$$\xi_t p_{2t} = (\lambda_{2t}Y_t)^{-1}\rho\left(\frac{e^{-\rho t}}{1-e^{-\rho T}}\right)\xi_0 S_0\left\{\hat{\lambda}\lambda_{20} + \frac{1}{3}\left(1-\hat{\lambda}\right)\right\}, \tag{B.32}$$

$$\xi_t p_{3t} = (\lambda_{3t}Y_t)^{-1}\rho\left(\frac{e^{-\rho t}}{1-e^{-\rho T}}\right)\xi_0 S_0\left\{\hat{\lambda}\lambda_{30} + \frac{1}{3}\left(1-\hat{\lambda}\right)\right\}. \tag{B.33}$$

To isolate the fees, we plug the expression for ξ (B.30) into the equations above, and find

$$\frac{p_{1t}}{\bar{p}} = \frac{\frac{1}{\lambda_{1t}}}{\frac{1}{\lambda_{1t}} + \frac{1}{\lambda_{2t}} + \frac{1}{\lambda_{3t}}}, \tag{B.34}$$

$$\frac{p_{2t}}{\bar{p}} = \frac{\frac{1}{\lambda_{2t}}}{\frac{1}{\lambda_{1t}} + \frac{1}{\lambda_{2t}} + \frac{1}{\lambda_{3t}}}, \tag{B.35}$$

$$\frac{p_{3t}}{\bar{p}} = \frac{\frac{1}{\lambda_{3t}}}{\frac{1}{\lambda_{1t}} + \frac{1}{\lambda_{2t}} + \frac{1}{\lambda_{3t}}}. \tag{B.36}$$

By plugging the equilibrium fees into the revenue share definitions (17) we obtain our desired result,

$$V_{it} = \bar{p}\frac{1}{\frac{1}{\lambda_{1t}} + \frac{1}{\lambda_{2t}} + \frac{1}{\lambda_{3t}}}, \quad V_t = \bar{p}\frac{3}{\frac{1}{\lambda_{1t}} + \frac{1}{\lambda_{2t}} + \frac{1}{\lambda_{3t}}}, \quad i = 1, 2, 3. \tag{B.37}$$

The cryptocurrency price S_t is obtained from the no arbitrage condition, which identifies the cryptocurrency price as the discounted services produced by the three mining pools,

$$S_t = \frac{1}{\xi_t} E_t\left[\int_t^T \xi_s \left\{p_{1s}\lambda_{1s}Y_s + p_{2s}\lambda_{2s}Y_s + p_{3s}\lambda_{3s}Y_s\right\} ds\right] = \frac{1}{\xi_t} e^{-\rho t}\frac{1 - e^{-\rho(T-t)}}{1 - e^{-\rho T}}\xi_0 S_0, \tag{B.38}$$

where the equality follows from plugging the individual state price densities (B.31), (B.32), and (B.33), and noting that $\lambda_{10} + \lambda_{20} + \lambda_{30} = 1$. Substituting the state price density ξ_t (B.30) into the above characterization we obtain the following expression for the cryptocurrency price

$$S_t = Y_t \left[\frac{1 - e^{-\rho(T-t)}}{\rho}\right] \left[\frac{3}{\frac{1}{\bar{p}\lambda_{1t}} + \frac{1}{\bar{p}\lambda_{2t}} + \frac{1}{\bar{p}\lambda_{3t}}}\right]. \tag{B.39}$$

The above expression can be further simplified by substituting the expressions for the fees charged by mining pools (B.34), (B.35), and (B.36). By doing so, we obtain

$$S_t = Y_t \left[\frac{1 - e^{-\rho(T-t)}}{\rho}\right] \left[\lambda_{1t}p_{1t} + \lambda_{2t}p_{2t} + \lambda_{3t}p_{3t}\right]. \tag{B.40}$$

The above expression for the cryptocurrency price depends on Y_t, which then depends on the endogenous cryptocurrency price. We substitute the expression for Y_t (26) and the definition of $V_t \equiv \lambda_{1t}p_{1t} + \lambda_{2t}p_{2t} + \lambda_{3t}p_{3t}$ to obtain the desired expression for the cryptocurrency price,

$$S_t = D_t \left(\frac{1 - e^{-\rho(T-t)}}{\rho}\right)^{\frac{1}{1-\beta}} (V_t)^{\frac{1}{1-\beta}}. \tag{B.41}$$

By plugging the total revenue share (18), the closed-form expression of the cryptocurrency price becomes

$$S_t = D_t \left[\frac{1 - e^{-\rho(T-t)}}{\rho}\right]^{\frac{1}{1-\beta}} \left[\frac{3}{\frac{1}{\bar{p}\lambda_{1t}} + \frac{1}{\bar{p}\lambda_{2t}} + \frac{1}{\bar{p}\lambda_{3t}}}\right]^{\frac{1}{1-\beta}}. \tag{B.42}$$

Similarly, we find that the discount factor is given by

$$\xi_{0,t} = \bar{\xi}e^{-\rho t}\left(\frac{1 - e^{-\rho(T-t)}}{\rho}\right)^{\frac{-\beta}{1-\beta}} \frac{1}{D_t}\left(\frac{1}{V_t}\right)^{\frac{1}{1-\beta}}, \tag{B.43}$$

where $\bar{\xi} \equiv S_0\left(\frac{\rho}{1-e^{-\rho T}}\right)$. The cryptocurrency price S_0 is obtained from the expression for S_t evaluated at $t = 0$. Next, we show that the partial derivatives of the concentration and total revenue share are negatively related (B.3).

$$\frac{\partial H_t}{\partial \lambda_{1t}} = 2\left(2\lambda_{1t} - (1-\lambda_{2t})\right) < 0 \quad \Longleftrightarrow \quad \frac{\partial V_t}{\partial \lambda_{1t}} = -\frac{3\bar{p}}{\left(\frac{1}{\lambda_{1t}} + \frac{1}{\lambda_{2t}} + \frac{1}{1-\lambda_{1t}-\lambda_{2t}}\right)^2}\left(-\frac{1}{\lambda_{1t}^2} + \frac{1}{(1-\lambda_{1t}-\lambda_{2t})^2}\right) > 0.$$

Next, observe that $\partial S_t/\partial V_t > 0$ (B.41) and $\partial \xi_t/\partial V_t < 0$ (B.43), for any $\beta \in [0,1)$, leading to

$$\frac{\partial \xi_t}{\partial \lambda_{1t}} = \frac{\partial \xi_t}{\partial V_t}\frac{\partial V_t}{\partial \lambda_{1t}} > 0 \Longleftrightarrow \frac{\partial H_t}{\partial \lambda_{1t}} > 0, \tag{B.44}$$

$$\frac{\partial S_t}{\partial \lambda_{1t}} = \frac{\partial S_t}{\partial V_t}\frac{\partial V_t}{\partial \lambda_{1t}} < 0 \Longleftrightarrow \frac{\partial H_t}{\partial \lambda_{1t}} > 0, \tag{B.45}$$

for any $\beta \in [0,1)$. The result applies for λ_{2t} straightforwardly.

By applying Itô's Lemma to both sides of the cryptocurrency price (B.42) and comparing the volatility terms we find that

$$\sigma_t^S = \sigma^D, \tag{B.46}$$

$$\bar{\sigma}_{1t}^S = \frac{1}{1-\beta}\left(\frac{p_{1t}}{\bar{p}}\lambda_{2t} + \frac{p_{3t}}{\bar{p}}\lambda_{2t} - \frac{p_{2t}}{\bar{p}}\lambda_{1t} - \frac{p_{2t}}{\bar{p}}\lambda_{3t}\right)\bar{\sigma}_1, \tag{B.47}$$

$$\bar{\sigma}_{2t}^S = \frac{1}{1-\beta}\left(\frac{p_{1t}}{\bar{p}}\lambda_{3t} - \frac{p_{3t}}{\bar{p}}\lambda_{1t} + \frac{p_{2t}}{\bar{p}}\lambda_{3t} - \frac{p_{3t}}{\bar{p}}\lambda_{2t}\right)\bar{\sigma}_2. \tag{B.48}$$

We denote by $S_t\,(\beta = 0)$ and $S_t\,(\beta > 0)$ the cryptocurrency prices when $\beta = 0$ and $\beta > 0$, respectively. Following the cryptocurrency price representation (B.41), a necessary and sufficient condition for the existence of an inflationary bubble is

$$S_t\,(\beta > 0) > S_t\,(\beta = 0) \quad \Longleftrightarrow \quad \frac{1-e^{-\rho(T-t)}}{\rho}V_t > 1. \tag{B.49}$$

We look for the fees index $\bar{p}$ that satisfies (B.49) when the revenue share is maximized $\left(\lambda_{1*} = \frac{1}{3}, \lambda_{2*} = \frac{1}{3}\right)$, leading to

$$\bar{p} > \frac{3\rho}{1-e^{-\rho(T-t)}} \equiv P_{2t}. \tag{B.50}$$

This condition guarantees that there exists some values of λ_{1t} and λ_{2t} for which there is an inflationary bubble (B.49). If the inflationary bubble exists at the maximum $(\lambda_{1t} = \lambda_{1*}, \lambda_{2t} = \lambda_{2*})$ it must also exist in the neighbourhood of $(\lambda_{1*}, \lambda_{2*})$ since the total revenue share is strictly concave in λ_{1t} and λ_{2t}. Accordingly, the cryptocurrency price is in an inflationary bubble if and only if the following condition holds:

$$3\bar{p}\left(\frac{1-e^{-\rho(T-t)}}{\rho}\right) > \frac{1}{\lambda_{1t}} + \frac{1}{\lambda_{2t}} + \frac{1}{1-\lambda_{1t}-\lambda_{2t}}. \tag{B.51}$$

We now turn to prove the results involving effects of increased interdependency on return volatility and cryptocurrency price. To prove (i), we differentiate the total volatility with respect to the interdependency parameter and find that

$$\frac{\partial\sqrt{\left(\bar{\sigma}_{1t}^{S}\right)^2+\left(\bar{\sigma}_{2t}^{S}\right)^2+\left(\sigma_{t}^{S}\right)^2}}{\partial\beta}=\frac{\left(\bar{\sigma}_{1t}^{S}\right)^2+\left(\bar{\sigma}_{2t}^{S}\right)^2}{\sqrt{\left(\bar{\sigma}_{1t}^{S}\right)^2+\left(\bar{\sigma}_{2t}^{S}\right)^2+\left(\sigma_{t}^{S}\right)^2}}\frac{1}{1-\beta}>0, \tag{B.52}$$

where $\bar{\sigma}_{1t}^{S}$ and $\bar{\sigma}_{2t}^{S}$ are given in (B.47) and (B.48). Lastly, to prove (ii) we take the derivative of S_t with respect to the interdependency parameter and find that

$$\frac{\partial S_t}{\partial\beta}=D_t\left(\frac{1-e^{-\rho(T-t)}}{\rho}V_t\right)^{\frac{1}{1-\beta}}\left(\frac{1}{1-\beta}\right)^2\log\left(\frac{1-e^{-\rho(T-t)}}{\rho}V_t\right), \tag{B.53}$$

which implies that $\frac{\partial S_t}{\partial\beta}>0$ if and only if the cryptocurrency is in an inflationary bubble, as (B.49) reveals. □

Proof of Proposition 6 (Extensive Margin). Let $(\lambda_{1t},\lambda_{2t},\lambda_{3t}\equiv 1-\lambda_{1t}-\lambda_{2t})$ be the initial distribution of share of mining pools, where $\lambda_{1t},\lambda_{2t}>\frac{1}{3},\lambda_{3t}<\frac{1}{3}$. When Pool-3 increases and takes market share from Pool-1 and Pool-2 we have $d\lambda_{1t},d\lambda_{2t}<0$, and we observe that concentration decreases

$$\begin{aligned}dH_t=&2\lambda_{1t}d\lambda_{1t}+2\lambda_{2t}d\lambda_{2t}-2\left(1-\lambda_{1t}-\lambda_{2t}\right)\left(d\lambda_{1t}+d\lambda_{2t}\right)\\=&2\left(\lambda_{1t}-\lambda_{3t}\right)d\lambda_{1t}+2\left(\lambda_{2t}-\lambda_{3t}\right)d\lambda_{2t}<0.\end{aligned} \tag{B.54}$$

To prove (i), we observe that the change in Pool-1 fees is given by

$$dp_{1t}=-\frac{\bar{p}}{\left(1+\frac{\lambda_{1t}}{\lambda_{2t}}+\frac{\lambda_{1t}}{\lambda_{3t}}\right)^2}\left(\frac{1}{\lambda_{2t}}+\frac{1}{\lambda_{3t}}+\frac{\lambda_{1t}}{\lambda_{3t}^2}\right)d\lambda_{1t}-\frac{\bar{p}}{\left(1+\frac{\lambda_{1t}}{\lambda_{2t}}+\frac{\lambda_{1t}}{\lambda_{3t}}\right)^2}\left(-\frac{\lambda_{1t}}{\lambda_{2t}^2}+\frac{\lambda_{1t}}{\lambda_{3t}^2}\right)d\lambda_{2t}>0, \tag{B.55}$$

which is strictly positive because $\lambda_{3t}<\lambda_{2t}$. A similar outcome is obtained from p_{2t} because $\lambda_{3t}<\lambda_{1t}$.
To prove (ii), we observe that the change in total revenue share is given by

$$dV_t=-\frac{3\bar{p}}{\left(\frac{1}{\lambda_{1t}}+\frac{1}{\lambda_{2t}}+\frac{1}{\lambda_{3t}}\right)^2}\left\{\left(-\frac{1}{\lambda_{1t}^2}+\frac{1}{\lambda_{3t}^2}\right)d\lambda_{1t}+\left(-\frac{1}{\lambda_{2t}^2}+\frac{1}{\lambda_{3t}^2}\right)d\lambda_{2t}\right\}>0, \tag{B.56}$$

which follows from $\lambda_{3t},<\lambda_{1t},\lambda_{2t}$. The cryptocurrency price and the discount factor are given in (B.4). Hence, an increase in total revenue share leads to a higher cryptocurrency price and a lower discount factor, regardless of $\beta\in[0,1)$.
To prove (iii), we observe that the shock sensitivities can be represented as

$$\begin{aligned}\bar{\sigma}_{1t}^{S}=&\left(\frac{1}{1-\beta}\right)\frac{1}{\bar{p}}\left(p_{1t}\lambda_{2t}+p_{3t}\lambda_{2t}-p_{2t}\lambda_{1t}-p_{2t}\lambda_{3t}\right)\bar{\sigma}_1=\left(\frac{1}{1-\beta}\right)\frac{1}{\bar{p}}\left(p_{1t}\lambda_{2t}+p_{3t}\lambda_{2t}+p_{2t}\lambda_{2t}-p_{2t}\right)\bar{\sigma}_1\\=&\left(\frac{1}{1-\beta}\right)\left(\lambda_{2t}-\frac{p_{2t}}{\bar{p}}\right)\bar{\sigma}_1\end{aligned} \tag{B.57}$$

and

$$\bar{\sigma}_{2t}^{S} = \left(\frac{1}{1-\beta}\right)\frac{1}{\bar{p}}\left(p_{1t}\lambda_{3t} - p_{3t}\lambda_{1t} + p_{2t}\lambda_{3t} - p_{3t}\lambda_{2t}\right)\bar{\sigma}_2 = \left(\frac{1}{1-\beta}\right)\frac{1}{\bar{p}}\left(p_{1t}\lambda_{3t} + p_{2t}\lambda_{3t} + p_{3t}\lambda_{3t} - p_{3t}\right)\bar{\sigma}_2$$
$$= \left(\frac{1}{1-\beta}\right)\left(\lambda_{3t} - \frac{p_{3t}}{\bar{p}}\right)\bar{\sigma}_2 \tag{B.58}$$

Given our assumption on mining pools' sizes: $\lambda_{1t}, \lambda_{2t} > \frac{1}{3}, \lambda_{3t} < \frac{1}{3}$, we find that $\lambda_{1t} - \frac{p_{1t}}{\bar{p}} > 0$, $\lambda_{2t} - \frac{p_{2t}}{\bar{p}} > 0$, and $\lambda_{3t} - \frac{p_{3t}}{\bar{p}} < 0$. To see why, observe that

$$3\frac{p_{1t}}{\bar{p}} < 1 \iff \frac{1}{\lambda_{1t}}\left(\frac{3}{\frac{1}{\lambda_{1t}} + \frac{1}{\lambda_{2t}} + \frac{1}{\lambda_{3t}}}\right) \leq \frac{1}{\lambda_{1t}}\left(\frac{\lambda_{1t} + \lambda_{2t} + \lambda_{3t}}{3}\right) = \frac{1}{\lambda_{1t}}\left(\frac{1}{3}\right) < 3\left(\frac{1}{3}\right) < 1, \tag{B.59}$$

where the first inequality follows because the harmonic mean is always less than the arithmetic mean, the equality follows because $\lambda_{1t} + \lambda_{2t} + \lambda_{3t} = 1$, and the last inequality follows because $\lambda_{1t} > \frac{1}{3}$. Consequently, we find that

$$\frac{p_{1t}}{\bar{p}} < \frac{1}{3} < \lambda_{1t}, \qquad \frac{p_{2t}}{\bar{p}} < \frac{1}{3} < \lambda_{2t}. \tag{B.60}$$

Since $\frac{p_{1t}}{\bar{p}} + \frac{p_{2t}}{\bar{p}} + \frac{p_{3t}}{\bar{p}} = 1$, we conclude that $\frac{p_{3t}}{\bar{p}} > \frac{1}{3} > \lambda_{3t}$. Thus,

$$|\bar{\sigma}_{1t}^{S}| = \left(\frac{1}{1-\beta}\right)\left(\lambda_{2t} - \frac{p_{2t}}{\bar{p}}\right)\bar{\sigma}_1, \quad |\bar{\sigma}_{2t}^{S}| = -\left(\frac{1}{1-\beta}\right)\left(\lambda_{3t} - \frac{p_{3t}}{\bar{p}}\right)\bar{\sigma}_2, \tag{B.61}$$

and the change in absolute value of $\bar{\sigma}_{1t}^{S}$ is given by

$$d|\bar{\sigma}_{1t}^{S}| = \left(\frac{1}{1-\beta}\right)\left(d\lambda_{2t} - \frac{1}{\bar{p}}dp_{2t}\right)\bar{\sigma}_1 < 0, \tag{B.62}$$

since $d\lambda_{2t} < 0$ and $dp_{2t} > 0$. The change in absolute value of $\bar{\sigma}_{2t}^{S}$ is given by

$$d|\bar{\sigma}_{2t}^{S}| = -\left(d\lambda_{3t} - \frac{1}{\bar{p}}dp_{3t}\right)\bar{\sigma}_2 < 0, \tag{B.63}$$

since $d\lambda_{3t} > 0$ and $dp_{3t} = -dp_{1t} - dp_{2t} < 0$.

To prove (iv), we start from the derivative of the total volatility with respect to the interdependency, given in (B.52), and find that it is characterized as follows

$$\frac{\partial\sqrt{\left(\bar{\sigma}_{1t}^{S}\right)^2 + \left(\bar{\sigma}_{2t}^{S}\right)^2 + \left(\sigma_t^{S}\right)^2}}{\partial\beta} = \frac{\left(\bar{\sigma}_{1t}^{S}\right)^2 + \left(\bar{\sigma}_{2t}^{S}\right)^2}{\sqrt{\left(\bar{\sigma}_{1t}^{S}\right)^2 + \left(\bar{\sigma}_{2t}^{S}\right)^2 + \left(\sigma_t^{S}\right)^2}}\frac{1}{1-\beta} = \frac{1}{\sqrt{\frac{1}{\left(\bar{\sigma}_{1t}^{S}\right)^2 + \left(\bar{\sigma}_{2t}^{S}\right)^2} + \frac{\left(\sigma_t^{S}\right)^2}{\left(\left(\bar{\sigma}_{1t}^{S}\right)^2 + \left(\bar{\sigma}_{2t}^{S}\right)^2\right)^2}}}\frac{1}{1-\beta}.$$

As a result, a change in concentration propagates to the total volatility only through the following expression:

$$d\left(\left(\bar{\sigma}_{1t}^S\right)^2+\left(\bar{\sigma}_{2t}^S\right)^2\right)=2\bar{\sigma}_{1t}^S d\bar{\sigma}_{1t}^S+2\bar{\sigma}_{2t}^S d\bar{\sigma}_{2t}^S, \tag{B.64}$$

where $\bar{\sigma}_{1t}^S$ and $\bar{\sigma}_{2t}^S$ are given in (B.57) and (B.58), respectively. In light of (B.60), we observe that $\bar{\sigma}_{1t}^S>0$, $\bar{\sigma}_{2t}^S<0$, and the change in $\bar{\sigma}_{1t}^S$ is negative, while the change in $\bar{\sigma}_{2t}^S$ is positive

$$d\bar{\sigma}_{1t}^S=\frac{1}{1-\beta}\left(d\lambda_{2t}-\frac{1}{\bar{p}}dp_{2t}\right)\bar{\sigma}_1<0, \tag{B.65}$$

$$d\bar{\sigma}_{2t}^S=\frac{1}{1-\beta}\left(d\lambda_{3t}-\frac{1}{\bar{p}}dp_{3t}\right)\bar{\sigma}_2>0. \tag{B.66}$$

Hence,

$$d\left(\left(\bar{\sigma}_{1t}^S\right)^2+\left(\bar{\sigma}_{2t}^S\right)^2\right)=2\bar{\sigma}_{1t}^S d\bar{\sigma}_{1t}^S+2\bar{\sigma}_{2t}^S d\bar{\sigma}_{2t}^S<0, \tag{B.67}$$

$$d\left(\left(\bar{\sigma}_{1t}^S\right)^2+\left(\bar{\sigma}_{2t}^S\right)^2\right)^2=2\left(\left(\bar{\sigma}_{1t}^S\right)^2+\left(\bar{\sigma}_{2t}^S\right)^2\right)\left(2\bar{\sigma}_{1t}^S d\bar{\sigma}_{1t}^S+2\bar{\sigma}_{2t}^S d\bar{\sigma}_{2t}^S\right)<0. \tag{B.68}$$

Eventually, this leads to a more substantial effect of concentration on the return volatility.
Lastly, we **prove (v)**. We have derived $\frac{\partial S_t}{\partial \beta}$ in (B.53), which is copied below for convenience.

$$\frac{\partial S_t}{\partial \beta}=D_t\left(\frac{1-e^{-\rho(T-t)}}{\rho}V_t\right)^{\frac{1}{1-\beta}}\left(\frac{1}{1-\beta}\right)^2\log\left(\frac{1-e^{-\rho(T-t)}}{\rho}V_t\right).$$

The total change in $\frac{\partial S_t}{\partial \beta}$ when Pool-3 emerges is given by

$$\begin{aligned}d\left(\frac{\partial S_t}{\partial \beta}\right)=&D_t\frac{1}{1-\beta}\left(\frac{1-e^{-\rho(T-t)}}{\rho}V_t\right)^{\frac{\beta}{1-\beta}}\frac{1-e^{-\rho(T-t)}}{\rho}dV_t\left(\frac{1}{1-\beta}\right)^2\log\left(\frac{1-e^{-\rho(T-t)}}{\rho}V_t\right)\\&+D_t\left(\frac{1-e^{-\rho(T-t)}}{\rho}V_t\right)^{\frac{1}{1-\beta}}\left(\frac{1}{1-\beta}\right)^2\frac{1}{\frac{1-e^{-\rho(T-t)}}{\rho}V_t}\frac{1-e^{-\rho(T-t)}}{\rho}dV_t>0.\end{aligned} \tag{B.69}$$

We have already shown that when Pool-3 emerges, $dV_t>0$, (B.56), and when the cryptocurrency is in an inflationary bubble, $\frac{1-e^{-\rho(T-t)}}{\rho}V_t>1$, (B.49), eventually leading to our desired resutl that an increased interdependency implies that the effect of concentration on the cryptocurrency price increases, if the cryptocurrency price is in an inflationary bubble. □

Proposition 9 (**Portfolios**). *Markets are dynamically complete; the mining pools and households hold their entire wealth in the cryptocurrency.*

Proof of Proposition 9 (Portfolios). We start by characterizing the dynamics of the bonds. Each bond is riskless in terms of its mining pool's numeraire:

$$dB_{it}^i=r_{it}^i B_{it}^i dt, \quad i=1,2,3. \tag{B.70}$$

Converting these bonds to the numeraire we obtain

$$B_{1t} = p_{1t} B_{1t}^1, \qquad B_{2t} = p_{2t} B_{2t}^2, \qquad B_{3t} = p_{3t} B_{3t}^3 \tag{B.71}$$

By applying Itô's Lemma to both sides of the above equations, we find that

$$\sigma_t^{B_1} = 0, \qquad \bar{\sigma}_{1t}^{B_1} = -\bar{\sigma}_1 \frac{\lambda_{1t}}{\lambda_{2t}} \frac{p_{1t}}{\bar{p}}, \quad \bar{\sigma}_{2t}^{B_1} = -\bar{\sigma}_2 \lambda_{1t} \frac{p_{1t}}{\bar{p}} \frac{1}{\lambda_{3t}^2} \left\{ \lambda_{3t} \left(1 - \lambda_{1t}\right) + \lambda_{1t} \right\} \tag{B.72}$$

$$\sigma_t^{B_2} = 0, \qquad \bar{\sigma}_{1t}^{B_2} = -\bar{\sigma}_1 \left(1 - \frac{p_{2t}}{\bar{p}}\right), \quad \bar{\sigma}_{2t}^{B_2} = -\bar{\sigma}_2 \frac{\lambda_{2t}}{\lambda_{3t}} \frac{p_{2t}}{\bar{p}} \tag{B.73}$$

$$\sigma_t^{B_3} = 0, \qquad \bar{\sigma}_{1t}^{B_3} = -\bar{\sigma}_1 \frac{\lambda_{3t}}{\lambda_{2t}} \frac{p_{3t}}{\bar{p}}, \quad \bar{\sigma}_{2t}^{B_3} = \bar{\sigma}_2 \left(1 - \frac{p_{3t}}{\bar{p}}\right). \tag{B.74}$$

Notice that $\bar{\sigma}_{1t}^{B_1}, \bar{\sigma}_{2t}^{B_1}, \bar{\sigma}_{1t}^{B_2}, \bar{\sigma}_{2t}^{B_2}, \bar{\sigma}_{1t}^{B_3}, \bar{\sigma}_{2t}^{B_3} \neq 0$ probability almost surely since $\lambda_{1t}, \lambda_{2t} \in (0, 1)$. We define the global bond security B^W, which is locally riskless in the numeraire. This additional security is not required but it simplifies the analysis. It is simply defined as the sum of the three bonds.

To dynamically complete the financial markets and to replicate any financial claim, we require four independent investment opportunities: the cryptocurrency, Pool-1's bond, Pool-2's bond, and the global bond. We denote the vector of portfolio weights of agent i in the cryptocurrency and mining pool bonds by π_i, and the volatility matrix of these two securities by Σ, such that

$$\pi_i \equiv \begin{bmatrix} \pi_i^S \\ \pi_i^{B_1} \\ \pi_i^{B_2} \end{bmatrix}, \quad \Sigma \equiv \begin{bmatrix} \sigma_t^S & \bar{\sigma}_{1t}^S & \bar{\sigma}_{2t}^S \\ 0 & \bar{\sigma}_{1t}^{B_1} & \bar{\sigma}_{2t}^{B_1} \\ 0 & \bar{\sigma}_{1t}^{B_2} & \bar{\sigma}_{2t}^{B_2} \end{bmatrix}, \tag{B.75}$$

where $i = 1, 2, 3, a$.

Solving for the optimal wealth (B.10) (B.9), we find that $W_{it} \propto \frac{1}{\xi_t}$, and by applying Itô's Lemma to both sides of this equation we obtain

$$\Sigma' \pi_i = \theta_t, \tag{B.76}$$

where θ_t is the vector of market prices of risk. It is given by

$$\theta_t = \begin{bmatrix} \sigma_t^S \\ \bar{\sigma}_{1t}^S \\ \bar{\sigma}_{2t}^S \end{bmatrix}, \tag{B.77}$$

which is obtained from applying Itô's Lemma to both sides of (B.43). Since $\sigma_{1t}^S, \sigma_{2t}^S \neq 0$, as Proposition 8 reveals, we invert Σ' and obtain that $\pi_i^S = 1$ and $\pi_i^{B_1} = 0, \pi_i^{B_2} = 0$ for $i = 1, 2, 3, a$.

Following Pavlova and Rigobon (2007), we obtain the interest rates, r_{it}^i, by applying Itô's Lemma to the mining pool specific state price density, $\xi_t p_{it}$, given in (B.20), (B.21), and (B.22). $\square$