Pricing Path-dependent Options under Stochastic Volatility via Mellin Transform⋆

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Abstract

In this paper, we derive closed-form formulas of first-order approximation for down-and-out barrier and floating strike lookback put option prices under a stochastic volatility model, by using an asymptotic approach. To find the explicit closed-form formulas for the zero-order term and the first-order correction term, we use Mellin transform. We also conduct a sensitivity analysis on these formulas, and compare the option prices calculated by them with those generated by Monte-Carlo simulation.

Keywords: Asymptotic approximation, barrier, lookback, Mellin transform, stochastic volatility

2010 MSC: 91G20, 41A60, 44A99, 91G60

1. Introduction

A standard option gives its owner the right to buy (or sell) some underlying asset in the future for a fixed price. Call options confer the right to buy the asset, while put options confer the right to sell the asset. Path-dependent options represent extensions of this concept. For example, a lookback call option confers

⋆The author Jeong-Hoon Kim gratefully acknowledges the financial support by the National Research Foundation of Korea grant NRF2021R1A2C10040.

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the right to buy an asset at its minimum price over some time period. A barrier option resembles a standard option except that the payoff also depends on whether or not the asset price crosses a certain barrier level during the option's life. Lookback and barrier options are two of the most popular types of path-dependent options.

Following the lead set by Black and Scholes [1] and assuming that the underlying asset price follows a geometric Brownian motion with constant volatility, Merton [2] derived a closed-form pricing formula for down-and-out call options. Reiner and Rubinstein [3] extended Merton's result to other types of barrier options. Goldman et al. [4], Goldman et al. [5], and Conze and Vishwanathan [6] provided closed-form pricing formulas for lookback options. For a good summary for research on path-dependent options under the Black-Scholes framework, refer to [7]. As we know, the assumption that an asset price process follows a geometric Brownian motion with constant volatility does not capture the empirical observations, due to the volatility smile effect. So, it is desirable to overcome this drawback. There are different ways of extending the Black-Scholes model to incorporate the "smile" feature: one way is to consider "local volatility" and the other is to consider "stochastic volatility".

One popular local volatility model is the constant elasticity of variance (CEV) model introduced by Cox [8] and [9], where a closed-form pricing formula for European call options was presented. Davydov and Linetsky [10] derived solutions for barrier and lookback option prices under the CEV process in closed form, and demonstrated that barrier and lookback option prices and hedge ratios under the CEV process can deviate dramatically from the lognormal values. In [11], the pricing of certain path-dependent options was re-examined when the underlying asset follows the CEV diffusion process, by approximating the CEV process using a trinomial method.

In general, the pricing problems of path-dependent options do not have analytic solutions under stochastic volatility. Chiarella et al. [12] considered the problem of numerically evaluating barrier option prices when the dynamics of the underlying are driven by the Heston stochastic volatility model and developed a method of lines approach to evaluate the price as well as the delta and gamma of the option. Park and Kim [13] investigated a semi-analytic pricing method for lookback options in a general stochastic volatility framework. The resultant formula is well connected to the Black–Scholes price that is the first term of a series expansion, which makes computing the option prices relatively efficient. Further, a convergence condition for the expansion was provided with an error bound. Leung [14] and Wirtu et al. [15] derived an analytic pricing formula for floating strike lookback options under the Heston model by means of the homotopy analysis method. The price is given by an infinite series whose value can be determined once an initial term is given well.

In addition, Kato et al. [16] derived a new semi closed-form approximation formula for pricing an up-and-out barrier option under a certain type of stochastic volatility model including SABR model. In a more recent paper by Funahashi and Higuchi [17], a unified approximation scheme was proposed for a single barrier option under local volatility models, stochastic volatility models, and their combinations. The basic idea of their approximation is to mimic a target underlying asset process by a polynomial of the Wiener process. They then translated the problem of solving the first hit probability of the asset price into the problem of solving that of a Wiener process whose distribution of the passage time is known. Finally, utilizing Girsanov's theorem and the reflection principle, they showed that single barrier option prices can be approximated in a closed-form.

The main contribution of this paper is to derive new closed-form approximation formulas for pricing down-and-out put barrier options and floating strike lookback put options under a certain type of stochastic volatility model, which is similar to the one in [16]. To achieve our goal, we apply the asymptotic approach discussed in [18] and Mellin transform. Mellin transform techniques were used by Panini and Srivastav [19] to derive integral equation representations for the price of European and American basket put options. Similarly, Yoon [20] applied Mellin transform to derive a closed form solution of the option price with respect to a European call option and a European put option with the Hull-White stochastic interest rate. Moreover, Kim and Yoon [21] derived a closed-form formula of a second-order approximation for a European corrected option price under stochastic elasticity of variance (SEV) model.

The rest of the paper is organized as follows. Section 2 discusses the model framework and the features of down-and-out and floating strike lookback put options. In Section 3, we give detailed discussions on an asymptotic approach which is used to derive approximations to the risk-neutral values of these types of options. In Section 4, we apply Mellin transform to derive a closed-form formula of the first-order approximation for down-and-out barrier put options. In Section 5, we apply Mellin transform to derive a closed-form formula of the first-order approximation for floating strike lookback put options. Section 6 presents sensitivity and comparison analysis, and demonstrate that the results given by these closed-form formulas match well with those generated by Monte-Carlo simulation. Section 7 gives a brief summary. Details on Mellin transform and derivation of the closed-form formulas in Sections 4 and 5 are provided in Appendices A and B, respectively.

2. Basic Model Set-up and Path-dependent Options

2.1. Stochastic volatility model

Let $\{S_t : t \geq 0\}$ denote the price process of a risky asset on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is the physical probability measure. In this paper, we assume that $\{S_t : t \geq 0\}$ evolves according to the following system of stochastic differential equations:

$$\begin{aligned} dS_t &= \mu S_t dt + f(Y_t) S_t dW_t^s, \\ dY_t &= \alpha (m - Y_t) dt + \beta \left(\rho dW_t^s + \sqrt{1-\rho^2} dW_t^y\right), \end{aligned} \tag{1}$$

where μ, $\alpha > 0$, $\beta > 0$ and m are constants, f is a function having non-zero values and specifying the dependence on the hidden process $\{Y_t : t \geq 0\}$. The processes $\{W_t^s : t \geq 0\}$ and $\{W_t^y : t \geq 0\}$ are independent standard Brownian motions. The constant correlation coefficient ρ with $-1 < \rho < 1$ captures the leverage effect. Here, μ is the drift rate. The mean-reversion process $\{Y_t : t \geq 0\}$ given in Eq. (1) is characterized by its typical time to obtain back to the mean level m of its long-run distribution. The parameter α determines the speed of mean-reversion and β controls the volatility of $\{Y_t : t \geq 0\}$. In the sequel, we shall refer to the above system as the stochastic volatility (SV) model. In Sections 2 and 3, we will not specify the concrete form of f, but assume that f is bounded and smooth enough, e.g., $f \in C_0^2(\mathbb{R})$. Furthermore, f has to satisfy a sufficient growth condition in order to avoid bad behavior such as the non-existence of moments of $\{S_t : t \geq 0\}$. For numerical results in Section 6, we choose f to take a special form as used in [22], [18] and [23].

We apply the well-known Girsanov theorem to change the physical measure $\mathbb{P}$ to a risk-neutral martingale measure $\mathbb{Q}$ by letting

$$dW_t^{s*} = \frac{\mu - r}{f\left(Y_t\right)} dt + dW_t^s y \quad \text{and} \quad dW_t^{y*} = \xi\left(Y_t\right) dt + dW_t^y,$$

where $\xi\left(Y_t\right)$ represents the premium of volatility risk. Then the model equations under the measure $\mathbb{Q}$ can be written as

$$\begin{aligned} dS_t &= rS_t dt + f(Y_t) S_t dW_t^{s*}, \\ dY_t &= \left[\alpha\left(m - Y_t\right) - \beta\left(\rho\frac{\mu - r}{f(Y_t)} + \xi(Y_t)\sqrt{1-\rho^2}\right)\right] dt \\ &\quad + \beta\left(\rho dW_t^{s*} + \sqrt{1-\rho^2} dW_t^{y*}\right). \end{aligned} \tag{2}$$

Note that $\{W_t^{s*} : t \geq 0\}$ and $\left\{W_t^{y*} : t \geq 0\right\}$ are independent standard Brownian motions under $\mathbb{Q}$. As an Ornstein-Uhlenbeck (OU) process, $\{Y_t : t \geq 0\}$ in Eq. (1) has an invariant distribution, which is normal with mean m and variance $\beta^2/2\alpha$. Thus, we can expect that if mean reversion is very fast, i.e., α goes to infinity, the process $\{S_t : t \geq 0\}$ should be close to a geometric Brownian motion. This means that if mean reversion is extremely fast, then the model of Black and Scholes would become a good approximation. In reality, however, it may not be the case. For fast but not extremely fast mean-reversion, the Black-Scholes model needs to be corrected to account for the random characteristics of the volatility of a risky asset. For this purpose, we introduce another small parameter ϵ defined by $\epsilon = 1/\alpha$ as done by [22]. For notational convenience, we put $\nu = \beta/\sqrt{2\alpha}$. With the help of these notations, the model equations under $\mathbb{Q}$ is re-written as

$$\begin{aligned} dS_t &= rS_t dt + f\left(Y_t\right) S_t dW_t^{s*}, \\ dY_t &= \left[\frac{1}{\epsilon}\left(m - Y_t\right) - \frac{\sqrt{2}\nu}{\sqrt{\epsilon}}\Lambda\left(Y_t\right)\right] dt + \frac{\sqrt{2}\nu}{\sqrt{\epsilon}} dW_t^{y*}, \end{aligned}$$

where $\Lambda(\cdot)$, defined by

$$\Lambda(y) := \rho\frac{\mu - r}{f(y)} + \xi(y)\sqrt{1-\rho^2},$$

is the combined market price of risk.

2.2. Path-dependent options

Let $V(T)$ denote the payoff of a put option on the risky asset at its expiration T. Then its risk-neutral price at time $t \in [0, T]$ under our SV model is given by

$$P\left(t, s, y\right) = \mathbb{E}^{\mathbb{Q}}\left(e^{-r(T-t)}V(T)\middle|\ S_t = s, Y_t = y\right).$$

Note that $V(T)$ depends on the type of options. In this paper, we consider two types of path-dependent options: down-and-out put options and floating strike lookback put options. For notational convenience, we put $U_t := \min_{0\le u\le t} S_u$ and $Z_t := \max_{0\le u\le t} S_u$. The payoff of a down-and-out put option is given by

$$DOP(T) := \max\{K - S_T, 0\} \times \mathbb{1}_{U_T > B},$$

where K is the strike price, B is the barrier level satisfying $0 < B < K$ and $\mathbb{1}_{U_T>B}$ is the indicator function. For a floating strike lookback put option, its payoff has the form of $LP_{float}(T) := Z_T - S_T$. Applying Itô's lemma, we can obtain a partial differential equation (PDE) for $P(t, s, y)$ as follows:

$$
\begin{aligned}
0 \quad = \quad & \frac{\partial P}{\partial t} + \frac{1}{2} s^2 f^2(y) \frac{\partial^2 P}{\partial s^2} + r\left(s\frac{\partial P}{\partial s} - P\right) + \frac{\sqrt{2}\rho\nu s}{\sqrt{\epsilon}} f(y) \frac{\partial^2 P}{\partial s \partial y} \\
& + \frac{\nu^2}{\epsilon}\frac{\partial^2 P}{\partial y^2} + \left(\frac{1}{\epsilon}(m-y) - \frac{\sqrt{2}\nu}{\sqrt{\epsilon}}\Lambda(y)\right)\frac{\partial P}{\partial y}. \qquad (3)
\end{aligned}
$$

The boundary conditions for Eq. (3) vary depending on the type of options. For example, the boundary conditions for Eq. (3) when $V(T) = DOP(T)$ are

$$
\begin{cases}
P(T, s, y) = \max\{K - s, 0\}, & s > B, \\
P(t, B, y) = 0, & 0 \le t \le T.
\end{cases}
$$

When $V(T) = LP_{float}(T)$, the boundary conditions become the following

$$
\begin{cases}
\dfrac{\partial P}{\partial z}(t, z, y, z) = 0, & 0 \le t \le T, z > 0, \\
P(T, s, y, z) = z - s, & 0 \le s \le z.
\end{cases}
$$

Note that in this case, P is a function of t, s, y and z (here, $Z_t = z$).

3. Asymptotic Expansions

In this section, we apply an asymptotic expansion approach to establish partial differential equations, which will be used to derive an approximate solution to Eq. (3) and thus find an approximated value of a put option. We begin with re-organizing Eq. (3) in terms of the orders of ϵ as follows:

$$
\frac{1}{\epsilon}\mathcal{L}_0 P + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 P + \mathcal{L}_2 P = 0, \qquad (4)
$$

where the operators $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined by

$$
\begin{aligned}
\mathcal{L}_0 \quad &:= \quad (m-y)\frac{\partial}{\partial y} + \nu^2 \frac{\partial^2}{\partial y^2}, \\
\mathcal{L}_1 \quad &:= \quad \sqrt{2}\rho\nu s f(y) \frac{\partial^2}{\partial s \partial y} - \sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y}, \text{ and} \\
\mathcal{L}_2 \quad &:= \quad \frac{\partial}{\partial t} + \frac{1}{2}s^2 f^2(y)\frac{\partial^2}{\partial s^2} + r\left(s\frac{\partial}{\partial s} - \cdot\right).
\end{aligned}
$$

In order to obtain an efficient approximate solution to P, as that in [24] and [18], we apply the following asymptotic expansion of P:

$$P = P_0 + \sqrt{\epsilon}P_1 + \epsilon P_2 + \epsilon\sqrt{\epsilon}P_3 + \cdots, \tag{5}$$

where P_0, P_1, ... are functions corresponding to varying orders of ϵ. Substituting P in Eq. (5) into the Eq.(4) and re-organizing terms, we obtain

$$\begin{aligned} 0 &= \frac{1}{\epsilon}\mathcal{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}}\left(\mathcal{L}_1 P_0 + \mathcal{L}_0 P_1\right) + \left(\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0\right) \\ &\quad + \sqrt{\epsilon}\left(\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1\right) + \cdots. \end{aligned} \tag{6}$$

Our aim is to find P_0 and P_1.

Firstly, from the $O(1/\epsilon)$-order term in Eq.(6), we get $\mathcal{L}_0 P_0 = 0$. If we assume that P_0 does not grow as fast as $e^{y^2/2}$, as that done in [25, Theorem 4.2], we can show that P_0 is independent of y. Secondly, from the $O(1/\sqrt{\epsilon})$-order term in Eq. (6), we get $\mathcal{L}_1 P_0 + \mathcal{L}_0 P_1 = 0$. Since P_0 is independent of y, then $\mathcal{L}_1 P_0 = 0$. It follows that $\mathcal{L}_0 P_1 = 0$. Again, if we assume that P_1 does not grow as fast as $e^{y^2/2}$, then we can deduce that P_1 is also independent of y.

Next, from the $O(1)$-order term in Eq. 6, we get

$$\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0.$$

Since P_1 is independent of y, we have $\mathcal{L}_1 P_1 = 0$ which implies that

$$\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0. \tag{7}$$

Seeing Eq. (7) as a Poisson equation for P_2 in y, in order for it to have a solution, it is required to satisfy the centring condition

$$\langle \mathcal{L}_2 P_0 \rangle = \langle \mathcal{L}_2 \rangle P_0 = 0, \tag{8}$$

which is equivalent to

$$\frac{\partial P_0}{\partial t} + rs\frac{\partial P_0}{\partial s} + \frac{1}{2}s^2\langle f^2\rangle\frac{\partial^2 P_0}{\partial s^2} - rP_0 = 0. \tag{9}$$

This is an equation for us to determine P_0 term. Here, $\langle\cdot\rangle$ denotes the expectation with respect to the invariant distribution of the process $\{Y_t : t \geq 0\}$, i.e.,

$$\langle h\rangle = \int_{-\infty}^{+\infty} h(y)\Phi(y)dy, \quad \text{where} \quad \Phi(y) = \frac{1}{\sqrt{2\pi\nu^2}}e^{-\frac{(y-m)^2}{2\nu^2}}.$$

Note that small ϵ value corresponds to fast-mean reverting. In this case, Y_t approaches to a constant and $\langle f^2\rangle$ can be regarded as constant variance and then Eq. (9) is the Black-Scholes PDE. Thus, for small ϵ, P_0 represents the put option price under the Black-Scholes model.

Following Eq. (8), we have

$$\mathcal{L}_2 P_0 = \mathcal{L}_2 P_0 - \langle\mathcal{L}_2\rangle P_0 = \frac{1}{2}\left(f^2 - \langle f^2\rangle\right) s^2 \frac{\partial^2 P_0}{\partial s^2},$$

which together with Eq. (7) implies

$$\mathcal{L}_0 P_2 = -\frac{1}{2}\left(f^2 - \langle f^2\rangle\right) s^2 \frac{\partial^2 P_0}{\partial s^2}. \tag{10}$$

The solution to Eq. (10) can be expressed as

$$P_2 = -\frac{1}{2}\left(\phi + c\right) s^2 \frac{\partial^2 P_0}{\partial s^2}, \tag{11}$$

where ϕ is a function of y which only satisfies the equation $\mathcal{L}_0\phi = f^2 - \langle f^2\rangle$ and c is a function of other variables except y.

To derive an equation for P_1, we consider the $O(\sqrt{\epsilon})$-term in Eq. (6) and obtain

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0.$$

This equation can be regarded as a Poisson equation for P_3 in y, and in order for it to have a solution, the following centring condition must be satisfied:

$$\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1\rangle = 0. \tag{12}$$

After we substitute P_2 in Eq. (11) into Eq. (12) and make simplification, we obtain

$$\frac{\partial P_1}{\partial t} + \frac{1}{2}\langle f^2\rangle s^2 \frac{\partial^2 P_1}{\partial s^2} + rs\frac{\partial P_1}{\partial s} - rP_1 = c_1 s^3 \frac{\partial^3 P_0}{\partial s^3} + c_2 s^2 \frac{\partial^2 P_0}{\partial s^2}, \tag{13}$$

where

$$c_1 := \frac{\sqrt{2}}{2}\langle f\phi'\rangle\rho\nu \quad \text{and} \quad c_2 := \frac{\sqrt{2}}{2}\left(2\rho\langle f\phi'\rangle - \langle\Lambda\phi'\rangle\right)\nu. \tag{14}$$

This is an equation for us to determine the first correction term P_1.

We summarize the previous formal analysis as the following theorem.

Theorem 1. *Under the SV model governed by Eq.* (1)*, an approximation of the risk-neutral value P of a path-dependent put option is given by*

$$P = P_0 + \sqrt{\epsilon}P_1 + O(\varepsilon), \tag{15}$$

for small ϵ, where P_0 and P_1 are determined by Eqs. (9) *and* (13) *with corresponding boundary conditions, respectively, such that P_0 is the put option price under the Black-Scholes model with constant effective volatility $\sqrt{\langle f^2\rangle}$ and P_1 is the first-order correction term.*

Finally, as mentioned in Section 2, boundary conditions for Eqs. (8) and (13) depend on the types of options that we consider. We describe the corresponding boundary conditions and solve these equations in the next two sections.

4. Determining P_0 and P_1 for Down-and-out Put Options

In this section, we use Mellin transform to derive analytical expressions of the P_0 and P_1 terms for down-and-out put options.

4.1. P_0 term for down-and-out put options

In order to use Mellin transform to calculate the P_0 term for down-and-out put options, noting that P_0 is independent of y under our assumption, we first follow the method in [26] and use the boundary condition,

$$P(T, s, y) = \max\{K - s, 0\}, \quad \text{for} \quad s > B,$$

to set up the boundary condition of P_0 for $s \geq 0$ as follows:

$$P_0\left(T, s\right) := \left(K - s\right)\mathbb{1}_{B<s<K} - \left(\frac{B}{s}\right)^{k_1-1}\left(K - \frac{B^2}{s}\right)\mathbb{1}_{\frac{B^2}{K}<s<B}, \tag{16}$$

where $k_1 = 2r/\langle f^2 \rangle$. Now, we apply Mellin transform to Eq. (9) to convert this PDE into the following ODE:

$$\frac{d\hat{P}_0}{dt} + \left(\frac{1}{2} \langle f^2 \rangle (w^2 + w) - rw - r \right) \hat{P}_0 = 0. \tag{17}$$

The solution to Eq. (17) is given by

$$\hat{P}_0 \left(t, w \right) = \hat{\theta}(w) e^{\frac{1}{2} \langle f^2 \rangle \left(w^2 + (1-k_1) w - k_1 \right) (T-t)}, \tag{18}$$

where $\hat{\theta}$ is a function of w, determined by the boundary condition (16).

Next, we take inverse Mellin transform of Eq. (18) and obtain

$$P_0(t, s) = P_0(T, s) * \mathcal{M}^{-1} e^{\lambda (w+\eta)^2 + \delta},$$

where

$$\lambda = \frac{1}{2} \langle f^2 \rangle \left(T - t \right), \ \eta = \frac{1 - k_1}{2}, \ \delta = -\lambda \eta^2 - r \left(T - t \right)$$

and the operation $*$ means the convolution.. Applying Table A.2 in Appendix A and the boundary condition given in Eq. (16), we have

$$\begin{aligned} P_0 \left(t, s \right) &= P_0 \left(T, s \right) * \left(\frac{e^{\delta} s^{\eta}}{2\sqrt{\lambda \pi}} e^{-\frac{1}{4\lambda} (\ln s)^2} \right) \\ &= \int_B^K \left(K - u \right) e^{\delta} \left(\frac{s}{u} \right)^{\eta} \left(\frac{1}{2\sqrt{\lambda \pi}} e^{-\frac{1}{4\lambda} \left(\ln\left(\frac{s}{u} \right) \right)^2} \right) \frac{du}{u} - \\ &\quad \int_{\frac{B^2}{K}}^{B} \left(\frac{B}{u} \right)^{k_1 - 1} \left(K - \frac{B^2}{u} \right) e^{\delta} \left(\frac{s}{u} \right)^{\eta} \left(\frac{1}{2\sqrt{\lambda \pi}} e^{-\frac{1}{4\lambda} \left(\ln\left(\frac{s}{u} \right) \right)^2} \right) \frac{du}{u}. \end{aligned} \tag{19}$$

After some careful calculation, for down-and-out put options, we derive a closed-form expression of the P_0 term as follows:

$$\begin{aligned} P_0(t, s) &= K e^{-r(T-t)} \left(\Phi \left(-\Delta_- \left(\frac{s}{K} \right) \right) - \Phi \left(-\Delta_- \left(\frac{s}{B} \right) \right) \right) - \\ &\quad s \left(\Phi \left(-\Delta_+ \left(\frac{s}{K} \right) \right) - \Phi \left(-\Delta_+ \left(\frac{s}{B} \right) \right) \right) - \\ &\quad K e^{-r(T-t)} \left(\frac{B}{s} \right)^{k_1 - 1} \left[\Phi \left(\Delta_- \left(\frac{B}{s} \right) \right) - \Phi \left(\Delta_- \left(\frac{B^2}{sK} \right) \right) \right] + \\ &\quad B \left(\frac{B}{s} \right)^{k_1} \left[\Phi \left(\Delta_+ \left(\frac{B}{s} \right) \right) - \Phi \left(\Delta_+ \left(\frac{B^2}{sK} \right) \right) \right], \end{aligned} \tag{20}$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution and

$$\Delta_{\pm}(x) = \frac{1}{\sqrt{\langle f^2\rangle (T-t)}}\left[\ln(x) + \left(r \pm \frac{1}{2}\langle f^2\rangle\right)(T-t)\right].$$

Note that P_0 given in Eq. (20) is precisely the same as the price of a down-and-out put option given in the literature, e.g., [27] (Chapter 26, p.606) or [28] (Chapter 4), if we let $\sigma^2 = \langle f^2\rangle$. For details of the derivation of formula (20), we refer the reader to Appendix B.

4.2. P_1 term for down-and-out put options

For down-and-out put options, the boundary conditions for P_1 are

$$\begin{cases} P_1(T,s) &= 0, \quad for \;\; s \geq B, \\ P_1(t,B) &= 0, \quad for \;\; 0 < t < T. \end{cases}$$

We again follow the method in [26] and extend the boundary conditions $P_1(T,s) = 0$, for $s \geq B$ as $P_1(T,s) = 0$ for all $s \geq 0$.

Next, we apply Mellin transform to Eq. (13) to get

$$\frac{d\hat{P}_1}{dt} + \left(\frac{1}{2}\langle f^2\rangle\left(w^2+w\right) - rw - r\right)\hat{P}_1 = \left(-c_1 w\,(w+1)\,(w+2) + c_2 w\,(w+1)\right)\hat{P}_0.$$

Solving this equation, we obtain

$$\hat{P}_1(t,w) = \left[c_1(T-t)\,w^3 - (c_2 - 3c_1)(T-t)\,w^2 - (c_2 - 2c_1)(T-t)\,w\right]\hat{P}_0(t,w).$$

Finally, applying inverse Mellin transform, we obtain an explicit closed-form expression of P_1 as follows

$$\begin{aligned} P_1(t,s) &= \mathcal{M}^{-1}\left(\hat{P}_1(t,w)\right) \\ &= c_1(T-t)\left(-s\frac{d}{ds}P_0(t,s) - 3s^2\frac{d^2}{ds^2}P_0(t,s) - s^3\frac{d^3}{ds^3}P_0(t,s)\right) \\ &\quad - (c_2 - 3c_1)(T-t)\left(s\frac{d}{ds}P_0(t,s) + s^2\frac{d^2}{ds^2}P_0(t,s)\right) \\ &\quad - (c_2 - 2c_1)(T-t)\left(-s\frac{d}{ds}P_0(t,s)\right), \end{aligned} \tag{21}$$

where P_0 is given in the previous section, c_1 and c_2 are given in Eq. (14).

We summarize the above analysis and calculation on down-and-out put options in the following theorem.

Theorem 2. *Under the SV model governed by Eq.* (1)*, an approximation of the risk-neutral value P of a down-and-out barrier put option is given by*

$$P = P_0 + \sqrt{\epsilon} P_1 + O(\varepsilon), \tag{22}$$

where P_0 and P_1 are given by Eqs. (20) *and* (21)*, respectively.*

5. Determining P_0 and P_1 for Lookback Put Options

In this section, we use Mellin transform to derive analytical expressions of the P_0 and P_1 terms for floating strike lookback put options.

5.1. P_0 term for lookback put options

For lookback floating strike put options, the boundary conditions of P_0 are

$$\begin{cases} \dfrac{\partial P_0}{\partial z}(t, z, z) &= 0, \\ \dfrac{\partial P_0}{\partial z}(T, s, z) &= 1, \quad for \ \ 0 < s < z. \end{cases}$$

Similar to the case of down-and-out put options, we extend the second boundary condition to $0 < s < \infty$ as follows:

$$\frac{\partial P_0}{\partial z}(T, s, z) := \mathbb{1}_{s<z} - \left(\frac{z}{s}\right)^{k_1 - 1} \cdot \mathbb{1}_{z<s}, \quad \text{for} \quad 0 < s < \infty.$$

Then, by integrating each side of the last equation, we can obtain

$$P_0(T, s, z) = \int_s^z -\left(\frac{\xi}{s}\right)^{k_1 - 1} d\xi = -\frac{1}{k_1}\left(\frac{z}{s}\right)^{k_1} s + \frac{1}{k_1} s \tag{23}$$

for $s > z$. For convenience, we let $u = s/z$ and $Q_0 = P_0/z$. With these notations, Eq. (9) becomes

$$\frac{\partial Q_0}{\partial t} + \frac{1}{2} u^2 \langle f^2 \rangle \frac{\partial^2 Q_0}{\partial u^2} + ru \frac{\partial Q_0}{\partial u} - rQ_0 = 0, \tag{24}$$

with boundary conditions

$$Q_0(T, u) = -\frac{1}{k_1} u^{1-k_1} + \frac{1}{k_1} u, \quad \text{for} \quad u > 1, \tag{25}$$

and $Q_0(T, u) = 1$, for $0 < u < 1$.

Note that except the boundary conditions, Eq. (24) is identical to Eq. (9). Applying Mellin transform in the same way as that for the case of down-and-out put options, we can derive the solution to Eq. (24) as follows:

$$Q_0(t,u) = \hat{\theta}(w) * \mathcal{M}^{-1} e^{\lambda(w+\eta)^2+\delta}.$$

Again, applying Table A.2 and P_0 given in Eq. (16), we have

$$\begin{aligned} Q_0(t,u) &= Q_0(T,u) * e^{\delta} z^{\eta} \left(\frac{1}{2\sqrt{\pi}} \lambda^{-\frac{1}{2}} e^{-\frac{1}{4\lambda}(\ln z)^2} \right) \\ &= \int_0^1 (1-\xi)\, e^{\delta} \left(\frac{u}{\xi}\right)^{\eta} \left(\frac{1}{2\sqrt{\pi}} \lambda^{-\frac{1}{2}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2} \right) \frac{d\xi}{\xi} + \qquad (26) \\ & \quad \int_1^{\infty} \left(\frac{-1}{k_1}\xi^{1-k_1} + \frac{\xi}{k_1} \right) e^{\delta} \left(\frac{u}{\xi}\right)^{\eta} \left(\frac{1}{2\sqrt{\pi}} \lambda^{-\frac{1}{2}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2} \right) \frac{d\xi}{\xi}. \end{aligned}$$

After calculating integrals, for floating strike lookback put options, we derive a closed-form expression of the P_0 term as follows:

$$\begin{aligned} P_0(t,s,z) &= z e^{-r(T-t)} \Phi\left(-\Delta_- \left(\frac{s}{z}\right) \right) - s\Phi\left(-\Delta_+ \left(\frac{s}{z}\right) \right) \qquad (27) \\ & \quad - \frac{z}{k_1} \left(\frac{s}{z}\right)^{1-k_1} e^{-r(T-t)} \Phi\left(-\Delta_- \left(\frac{z}{s}\right) \right) + \frac{s}{k_1} \Phi\left(\Delta_+ \left(\frac{s}{z}\right) \right), \end{aligned}$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution. Note that P_0 given in Eq. (27) is precisely the same as the price of a floating strike put option given in the literature, e.g., [27] (Chapter 26, p.608) or [28] (Chapter 4), if we let $\sigma^2 := \langle f^2 \rangle$. Details of the derivation of this formula can be found in Appendix B.

5.2. P_1 term for lookback put options

For floating strike lookback put options, the boundary conditions for P_1 are

$$\begin{cases} P_1(T,s,z) = 0, & for \quad 0 < s < z, \\ \dfrac{\partial P_1}{\partial z}(t,z,z) = 0, & for \quad 0 < t < T \quad and\ z > 0. \end{cases}$$

Just like that for the P_0-term for floating strike lookback put options, we let $u = s/z$ and $Q_1 = P_1/z$.. With these notation changes, Eq. (13) is converted to the following

$$\frac{\partial Q_1}{\partial t} + \frac{1}{2}\langle f^2\rangle u^2 \frac{\partial^2 Q_1}{\partial u^2} + ru\frac{\partial Q_1}{\partial u} - rQ_1 = c_1 u^3 \frac{\partial^3 Q_0}{\partial u^3} + c_2 u^2 \frac{\partial^2 Q_0}{\partial u^2} \tag{28}$$

with $Q_1(T, u) = 0$ for $0 < u < 1$.

Note that Eq. (28) is essentially the same as Eq. (13), except the notational difference. So, we have

$$\begin{aligned} Q_1(t,u) \quad = \quad & c_1(T-t)\left(-u\frac{d}{du}Q_0(t,u) - 3u^2\frac{d^2}{du^2}Q_0(t,u) - u^3\frac{d^3}{du^3}Q_0(t,u)\right) \\ & -(c_2-3c_1)(T-t)\left(u\frac{d}{du}Q_0(t,u) + u^2\frac{d^2}{dz^2}Q_0(t,u)\right) \\ & -(c_2-2c_1)(T-t)\left(-u\frac{d}{du}Q_0(t,u)\right), \end{aligned} \tag{29}$$

where Q_0 is given previously. Consequently, we have

$$\begin{aligned} P_1(t,s,z) \quad = \quad & c_1(T-t)\left(-s\frac{d}{ds}P_0(t,s,z) - 3s^2\frac{d^2}{ds^2}P_0(t,s,z) - s^3\frac{d^3}{ds^3}P_0(t,s,z)\right) \\ & -(c_2-3c_1)(T-t)\left(s\frac{d}{ds}P_0(t,s,z) + s^2\frac{d^2}{ds^2}P_0(t,s,z)\right) \\ & -(c_2-2c_1)(T-t)\left(-s\frac{d}{ds}P_0(t,s,z)\right), \end{aligned} \tag{30}$$

where c_1 and c_2 are the same as those defined previously.

We summarize the above analysis and calculation on floating strike lookback put options in the following theorem.

Theorem 3. *Under the SV model governed by Eq.* (1)*, an approximation of the risk-neutral value P of a floating strike lookback put option is given by*

$$P = P_0 + \sqrt{\epsilon}P_1 + O(\varepsilon), \tag{31}$$

where P_0 and P_1 are given by Eqs. (27) *and* (30)*, respectively.*

6. Numerical Results and Sensitivity Analysis

In this section, we conduct a numerical study to investigate the sensitivity of the first-order correction term P_1 and our approximation results $P_0+\sqrt{\epsilon}P_1$ with respect to the initial value of underlying asset. This means that we set $t = 0$ throughout this section. We also compare the results given by our closed form formulas with those generated by the Monte-Carlo simulation.

Table 1: The role and numerical value of parameters

Parameter	Role	Value
r	risk-free interest rate	0.035
B	barrier level	1500
K	put option strike price	2700
c_1	as defined in Section 3	-0.004
c_2	as defined in Section 3	-0.018

First of all, as done by [22], [18] and [23], we choose f to take the following form

$$f(y) = 0.35\left(\tan^{-1}(y) + \frac{\pi}{2}\right)/\pi + 0.05.$$

Secondly, the values of other parameters used in this section are given in Table 1, whenever they are required to be fixed.

Here, we do not choose precise values of β and ρ, and particular forms of $\xi(y)$ (in Section 2) and $\phi(y)$ (in Section 3) to calculate the above values of c_1 and c_2. Instead, c_1 and c_2 are calibrated from the term structure of the implied volatility surface as described in the book of [22]. Specifically, the implied volatility I^ϵ of a European vallina call option with fast mean-reverting stochastic process can be approximated by the following formula

$$I^\epsilon = a\frac{\ln(\frac{K}{s})}{T-t} + b + o(\sqrt{\epsilon})$$

with

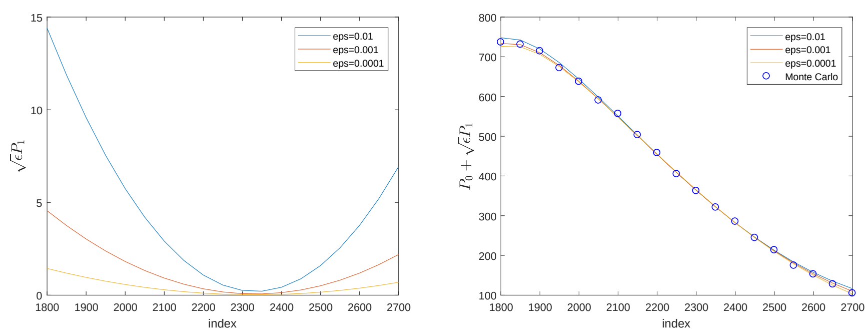
$$a = -\frac{c_1}{\langle f^2\rangle^{3/2}} \quad \text{and} \quad b = \sqrt{\langle f^2\rangle} + \frac{c_1}{\langle f^2\rangle^{3/2}}\left(r + \frac{3}{2}\langle f^2\rangle\right) - \frac{c_2}{\sqrt{\langle f^2\rangle}}.$$

The parameters a and b are estimated as the slope and intercept of the regression fit of the observed implied volatilities as a linear function of logmoneyness-to-maturity-ratio $\ln(K/s)/(T-t)$. From the calibrated values a and b on the observed implied volatility surface, the parameters c_1 and c_2 are obtained as

$$c_1 = -a\sigma\langle f^2\rangle^{3/2} \quad \text{and} \quad c_2 = \sqrt{\langle f^2\rangle}((\sqrt{\langle f^2\rangle} - b) - a(r + \frac{3}{2}\langle f^2\rangle)).$$

Thirdly, note that when $t = 0$, $s = z$. Hence, in this case, the formula for P_0 given by Eq. (27) is simplified.

Figure (1A) shows how the $\sqrt{\epsilon}P_1$-term for a down-and-out barrier put option changes with respect to a variation of ϵ values. As we can see, for fixed ϵ, when s increases, P_1 decreases first, and then increases after it hits its trough. When ϵ gets smaller (equivalently, the mean-reverting speed gets larger), $\sqrt{\epsilon}P_1$ approaches to a zero. Figure (1B) shows how the value of $P_0 + \sqrt{\epsilon}P_1$ for a down-and-out put option varies with respect to the change of ϵ values. As we can see, when the value of ϵ changes from 0.01 to 0.0001, the value of $P_0 + \sqrt{\epsilon}P_1$ does not vary much. In fact, the values of $P_0 + \sqrt{\epsilon}P_1$ match well with the result of Monte-Carlo simulation in all cases. Furthermore, in all cases, the value of $P_0 + \sqrt{\epsilon}P_1$ declines as s increases.

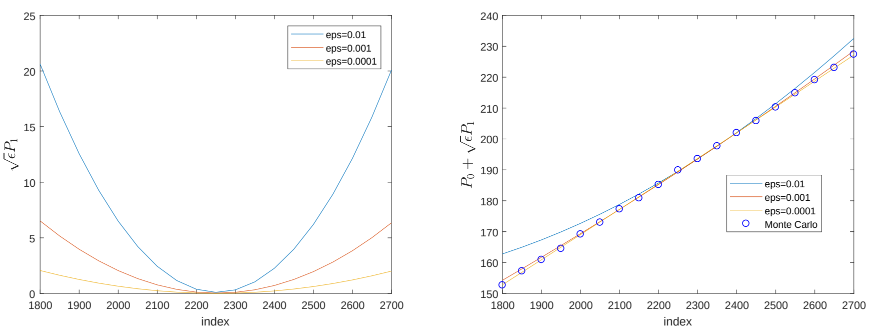


(A) (B)

Figure 1: Plots of $\sqrt{\epsilon}P_1$ and $P_0 + \sqrt{\epsilon}P_1$ with different values of ϵ, against the initial value of the underlying asset, for down-and-out put option

Figure (2A) shows how the $\sqrt{\epsilon}P_1$-term for a floating strike lookback put changes with respect to a variation of ϵ values. In a similar pattern, for a fixed ϵ-value, when s increases, P_1 decreases first and then increases after it hits its trough.

Similar to the case of down-and-out put options, when ϵ gets smaller (equivalently, the mean-reverting speed gets larger), $\sqrt{\epsilon}P_1$ approaches to zero. Figure (2B) shows how the value of $P_0 + \sqrt{\epsilon}P_1$ for a floating strike put varies with respect to the change of ϵ values. When the value of ϵ changes from 0.01 to 0.001, the value of $P_0 + \sqrt{\epsilon}P_1$ varies. But, when the value of ϵ changes from 0.001 to 0.0001, the value of $P_0 + \sqrt{\epsilon}P_1$ does not vary much. The values of $P_0 + \sqrt{\epsilon}P_1$ match well with the result of Monte-Carlo simulation when $\epsilon = 0.001$ or 0.0001. Furthermore, in all cases, the value of $P_0 + \sqrt{\epsilon}P_1$ increases as s increases.



(A) (B)

Figure 2: Plots of $\sqrt{\epsilon}P_1$ and $P_0 + \sqrt{\epsilon}P_1$ with different values of ϵ, against the initial value of the underlying asset, for floating strike put option

7. Conclusion Remarks

This article establishes explicit closed-form solutions for first order approximations of down-and-out barrier and floating strike lookback put option prices under a stochastic volatility model by means of Mellin transform. The zero-order terms in the solutions for the prices of both types of put options coincide with those in [27] or [28] under the classical Back-Scholes model. Our numerical analysis shows that the results given by those explicit closed-form solutions match well with those generated by Monte-Carlo simulation. This confirms the accuracy of the approximation. Furthermore, we also discussed the sensitivity of the first-order error terms and the approximation with respect to the underlying asset price and the mean-reverting speed of the OU-process which governs the volatility.

Appendix A. Mellin Transform

The Mellin transform is an integral transform that may be regarded as the multiplicative version of the two-sided Laplace transform. It is often used in the theory of asymptotic expansions. For a locally Lebesgue integrable function $h : \mathbb{R}^+ \to \mathbb{R}$, the Mellin transform denoted by $\mathcal{M}h$ or $\hat{h}$, is given by

$$\hat{h}(w) = \left(\mathcal{M}h\right)(w) := \int_0^{+\infty} s^{w-1} h(s)\, ds, \quad w \in \mathbb{C},$$

and if $a < \mathrm{Re}(w) < b$ and c such that $a < c < b$ exists, the inverse of the Mellin transform is expressed by

$$h(s) = \left(\mathcal{M}^{-1}\hat{h}\right)(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^{-w} \hat{h}(w)\, dw.$$

In this paper, we use the following properties of Mellin transform.

Table A.2: List of properties of Mellin transform used in this paper

function	Mellin tansform
h	$\hat{h}$
sh'	$-w\hat{h}$
s^2h''	$w(w+1)\hat{h}$
$s^3h^{(3)}$	$-w(w+1)(w+2)\hat{h}$
$\frac{e^\delta s^\eta}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}(\ln s)^2}$	$e^{\lambda(w+\eta)^2+\delta}$
$sh' + s^2h''$	$w^2\hat{h}$
$-sh' - 3s^2h'' - s^3h^{(3)}$	$w^3\hat{h}$

Here, λ, η and δ are not related to w or s, and h', h'' and $h^{(3)}$ are the first-order, second-order and third-order derivatives of h, respectively.

Appendix B. Derivation of Formulas (20) and (27)

In this appendix, we give detailed derivation of the formulas (20) and (27).

Appendix B.1. Derivation of formula (20)

From Eq. (19), we know that

$$\begin{aligned} P_0(t,s) &= \int_B^K (K-u)\, e^{\delta} \left(\frac{s}{u}\right)^{\eta} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{s}{u}\right)\right)^2}\right) \frac{du}{u} - \\ & \int_{\frac{B^2}{K}}^{B} \left(\frac{B}{u}\right)^{k_1-1} \left(K - \frac{B^2}{u}\right) e^{\delta} \left(\frac{s}{u}\right)^{\eta} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{s}{u}\right)\right)^2}\right) \frac{du}{u}. \end{aligned}$$

By letting $v = \ln u$, we convert the first integral to

$$\begin{aligned} & \int_{\ln B}^{\ln K} (K - e^{v}) s^{\eta} e^{\delta} e^{-\eta v} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}(\ln s - v)^2}\right) dv \\ = & \frac{s^{\eta} e^{\delta}}{2\sqrt{\lambda\pi}} \left(\int_{\ln B}^{\ln K} K e^{-\frac{1}{4\lambda}(v^2 - 2v\ln s + (\ln s)^2 + 4\lambda\eta v)} dv \right. \\ & \left. - \int_{\ln B}^{\ln K} e^{-\frac{1}{4\lambda}(v^2 - 2v\ln s + (\ln s)^2 + 4\lambda(\eta-1)v)} dv \right) \\ = & \frac{s^{\eta} e^{\delta}}{2\sqrt{\lambda\pi}} \left(\int_{\ln B}^{\ln K} K e^{-\frac{1}{4\lambda}(v - \ln s + 2\lambda\eta)^2 + \lambda\eta^2 - \eta\ln s} dv \right. \\ & \left. - \int_{\ln B}^{\ln K} e^{-\frac{1}{4\lambda}[v - \ln s + 2\lambda(\eta-1)]^2 + \lambda(\eta-1)^2 - (\eta-1)\ln s} dv \right), \end{aligned}$$

we further apply the following changes of variables

$$x' := \frac{v - \ln s + 2\lambda\eta}{\sqrt{2\lambda}} \quad \text{and} \quad x'' := \frac{v - \ln s + 2\lambda(\eta-1)}{\sqrt{2\lambda}}$$

to get

$$\begin{aligned} & \int_{\ln B}^{\ln K} (K - e^{v}) s^{\eta} e^{\delta} e^{-\eta v} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}(\ln s - v)^2}\right) dv \\ = & \frac{e^{\delta}}{\sqrt{2\pi}} \left(K e^{\lambda\eta^2} \int_{\frac{\ln(\frac{B}{s}) + 2\lambda\eta}{\sqrt{2\lambda}}}^{\frac{\ln(\frac{K}{s}) + 2\lambda\eta}{\sqrt{2\lambda}}} e^{-\frac{x'^2}{2}} dx' - s e^{\lambda(\eta-1)^2} \int_{\frac{\ln(\frac{B}{s}) + 2\lambda(\eta-1)}{\sqrt{2\lambda}}}^{\frac{\ln(\frac{K}{s}) + 2\lambda(\eta-1)}{\sqrt{2\lambda}}} e^{-\frac{x''^2}{2}} dx'' \right) \\ = & K e^{\delta + \lambda\eta^2} \left[\Phi\left(\frac{\ln(\frac{K}{s}) + 2\lambda\eta}{\sqrt{2\lambda}}\right) - \Phi\left(\frac{\ln(\frac{B}{s}) + 2\lambda\eta}{\sqrt{2\lambda}}\right) \right] \\ & - s e^{\delta + \lambda(\eta-1)^2} \left[\Phi\left(\frac{\ln(\frac{K}{s}) + 2\lambda(\eta-1)}{\sqrt{2\lambda}}\right) - \Phi\left(\frac{\ln(\frac{B}{s}) + 2\lambda(\eta-1)}{\sqrt{2\lambda}}\right) \right]. \end{aligned}$$

Now, if we plug into δ, η and λ into the above formula, we derive

$$
\begin{aligned}
&\int_{\ln B}^{\ln K} (K - e^v) s^\eta e^\delta e^{-\eta v} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}(\ln s - v)^2} \right) dv \\
&= K e^{-r(T-t)} \left[\Phi\left(-\Delta_- \left(\frac{s}{K} \right) \right) - \Phi\left(-\Delta_- \left(\frac{s}{B} \right) \right) \right] \\
&-s \left[\Phi\left(-\Delta_+ \left(\frac{s}{K} \right) \right) - \Phi\left(-\Delta_+ \left(\frac{s}{B} \right) \right) \right].
\end{aligned}
$$

Similarly, we can evaluate the second integral

$$
\int_{\frac{B^2}{K}}^{B} \left(\frac{B}{u} \right)^{k_1 - 1} \left(K - \frac{B^2}{u} \right) e^\delta \left(\frac{s}{u} \right)^\eta \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{s}{u}\right)\right)^2} \right) \frac{du}{u}
$$

to obtain

$$
\begin{aligned}
&K e^{-r(T-t)} \left(\frac{B}{s} \right)^{k_1 - 1} \left[\Phi\left(\Delta_- \left(\frac{B}{s} \right) \right) - \Phi\left(\Delta_- \left(\frac{B^2}{sK} \right) \right) \right] \\
&-B \left(\frac{B}{s} \right)^{k_1} \left[\Phi\left(\Delta_+ \left(\frac{B}{s} \right) \right) - \Phi\left(\Delta_+ \left(\frac{B^2}{sK} \right) \right) \right].
\end{aligned}
$$

Putting these two integrals together yields formula (20).

Appendix B.2. Derivation of formulas (27)

From Eq. (26), we have

$$
\begin{aligned}
Q_0(t,u) \quad = \quad & \int_0^1 (1-\xi)\, e^\delta \left(\frac{u}{\xi} \right)^\eta \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2} \right) \frac{d\xi}{\xi} + \\
& \int_1^\infty \left(-\frac{1}{k_1} \xi^{1-k_1} + \frac{\xi}{k_1} \right) e^\delta \left(\frac{u}{\xi} \right)^\eta \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2} \right) \frac{d\xi}{\xi}.
\end{aligned}
$$

We let $v = \ln \xi$. For the first integral, we have

$$
\begin{aligned}
& \int_0^1 (1-\xi) e^\delta \left(\frac{u}{\xi} \right)^\eta \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2} \right) \frac{du}{u} \\
= \quad & \int_{-\infty}^0 u^\eta (1 - e^v)\, e^{\delta - v\eta} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}(\ln u - v)^2} \right) dv \\
= \quad & \frac{u^\eta e^\delta}{2\sqrt{\lambda\pi}} \left(\int_{-\infty}^0 e^{-\frac{1}{4\lambda}\left(v^2 - 2v\ln u + (\ln u)^2 + 4\lambda\eta v\right)} dv \right. \\
& \left. - \int_{-\infty}^0 e^{-\frac{1}{4\lambda}\left(v^2 - 2v\ln u + (\ln u)^2 + 4\lambda(\eta-1)v\right)} dv \right) \\
= \quad & \frac{u^\eta e^\delta}{2\sqrt{\lambda\pi}} \left(\int_{-\infty}^0 e^{-\frac{1}{4\lambda}(v - \ln u + 2\lambda\eta)^2 + \lambda\eta^2 - \eta\ln u} dv \right. \\
& \left. - \int_{-\infty}^0 e^{-\frac{1}{4\lambda}(v - \ln u + 2\lambda(\eta-1))^2 + \lambda(\eta-1)^2 - (\eta-1)\ln u} dv \right).
\end{aligned}
$$

Next, we let

$$v' := \frac{v - \ln u + 2\lambda\eta}{\sqrt{2\lambda}} \quad \text{and} \quad v'' := \frac{v - \ln u + 2\lambda\left(\eta - 1\right)}{\sqrt{2\lambda}}.$$

Then, we have

$$\begin{aligned}
& \int_0^1 (1-\xi) e^{\delta} \left(\frac{u}{\xi}\right)^{\eta} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2} \right) \frac{du}{u} \\
= \; & \frac{e^{\delta}}{\sqrt{2\pi}} \left(\int_{-\infty}^{\frac{-\ln u + 2\lambda\eta}{\sqrt{2\lambda}}} e^{-\frac{v'^2}{2} + \lambda\eta^2} dv' - u \int_{-\infty}^{\frac{-\ln u + 2\lambda(\eta-1)}{\sqrt{2\lambda}}} e^{-\frac{v''^2}{2} + \lambda(\eta-1)^2} dv'' \right) \\
= \; & e^{\delta + \lambda\eta^2} \Phi\left(\frac{-\ln u + 2\lambda\eta}{\sqrt{2\lambda}} \right) - u e^{\delta + \lambda(\eta-1)^2} \Phi\left(\frac{-\ln u + 2\lambda\left(\eta - 1\right)}{\sqrt{2\lambda}} \right) \\
= \; & e^{-r(T-t)} \Phi\left(-\Delta_{-}\left(\frac{s}{z}\right) \right) - \left(\frac{s}{z}\right) \Phi\left(-\Delta_{+}\left(\frac{s}{z}\right) \right).
\end{aligned}$$

For the second integral, we have

$$\begin{aligned}
& \int_1^{\infty} \left(-\frac{1}{k_1}\xi^{1-k_1} + \frac{\xi}{k_1} \right) e^{\delta} \left(\frac{u}{\xi}\right)^{\eta} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2} \right) \frac{d\xi}{\xi} \\
= \; & \int_0^{\infty} \left(-\frac{1}{k_1} e^{(1-k_1)v} + \frac{1}{k_1} e^{v} \right) e^{\delta} u^{\eta} e^{-v\eta} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}(\ln u - v)^2} \right) dv \\
= \; & \frac{e^{\delta} u^{\eta}}{2k_1\sqrt{\lambda\pi}} \int_0^{\infty} \left(-e^{\eta v - \frac{1}{4\lambda}(\ln u - v)^2} + e^{v(1-\eta) - \frac{1}{4\lambda}(\ln u - v)^2} \right) dv \\
= \; & \frac{e^{\delta} u^{\eta}}{2k_1\sqrt{\lambda\pi}} \left(\int_0^{\infty} -e^{-\frac{1}{4\lambda}(v - \ln u - 2\lambda\eta)^2 + \lambda\eta^2 + \eta\ln u} dv \right. \\
& \left. + \int_0^{\infty} e^{-\frac{1}{4\lambda}(v - \ln u - 2\lambda(1-\eta))^2 + \lambda(1-\eta)^2 + (1-\eta)\ln u} dv \right),
\end{aligned}$$

where we use the fact that $k_1 - 1 + \eta = -\eta$. Further, we introduce a new variable

$$v''' := \frac{v - \ln u - 2\lambda\eta}{\sqrt{2\lambda}}.$$

Then, we have

$$
\begin{aligned}
&\int_1^{\infty}\left(-\frac{1}{k_1}\xi^{1-k_1}+\frac{\xi}{k_1}\right)e^{\delta}\left(\frac{u}{\xi}\right)^{\eta}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2}\right)\frac{d\xi}{\xi}\\
= \quad & \frac{e^{\delta}u^{\eta}}{k_1\sqrt{2\pi}}\left(\int_{\frac{-\ln u-2\lambda\eta}{\sqrt{2\lambda}}}^{\infty}-e^{-\frac{v'''^2}{2}}e^{\lambda\eta^2+\eta\ln u}dv'''\right.\\
&\left.+\int_{\frac{-\ln u+2\lambda(\eta-1)}{\sqrt{2\lambda}}}^{\infty}e^{-\frac{v''^2}{2}}e^{\lambda(\eta-1)^2+(1-\eta)\ln u}dv''\right)\\
= \quad & -\frac{1}{k_1}e^{\delta+\lambda\eta^2}u^{1-k_1}\Phi\left(\frac{\ln u+2\lambda\eta}{\sqrt{2\lambda}}\right)+\frac{1}{k_1}ue^{\delta+\lambda(\eta-1)^2}\Phi\left(\frac{\ln u+2\lambda\left(1-\eta\right)}{\sqrt{2\lambda}}\right)\\
= \quad & -\frac{1}{k_1}\left(\frac{s}{z}\right)^{1-k_1}e^{-r(T-t)}\Phi\left(-\Delta_-\left(\frac{z}{s}\right)\right)+\frac{1}{k_1}\left(\frac{s}{z}\right)\Phi\left(\Delta_+\left(\frac{s}{z}\right)\right).
\end{aligned}
$$

Putting these two integrals together and using the fact that $P_0 = zQ_0$, we can obtain our formula (27).

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