NON-TRANSFERABLE NON-HEDGEABLE EXECUTIVE STOCK OPTION PRICING

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Abstract. We introduce a method of valuing non-transferable non-hedgeable (NTNH) contingent claims and use it to price executive stock options (ESOs). We use NTNH constraints to break local co-linearity, caused by derivative assets, that renders common portfolio optimization methods ineffective. We are, thus, able to translate portfolios that include NTNH derivatives into portfolios of primary assets (only). We achieve this by replicating derivatives using primary assets, and then integrating NTNH constraints into a single rectangular constraint. Solving the constrained portfolio optimization facilitates the identification of stochastic discount factors that price any contingent claim in such portfolios. Implementing our method, we find subjective prices of NTNH European and American ESOs, both for block exercise and for a continuum of exercise rates. We identify executives' optimal exercise policies. We use these to find the objective price/cost of ESOs to firms. We then run a simulation study of price sensitivities and changes to optimal exercise policies with respect to model parameters and obtain policy implications regarding ESOs' incentivizing efficiency. We identify comparative statics results and policy implications that are sensitive to (high/low) volatility regimes. Moreover, for the first time, we demonstrate that unlike under the block exercise of European and American ESOs, with the continuous partial exercise of American ESOs, under a low volatility regime, subjective prices may be higher than objective ones.

Keywords: Executive Stock Options, Constrained Portfolio Optimization, Stochastic Discount Factor, Non-Hedgeable, Non-Transferable

JEL Codes: G11, G13, C02, C61

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1 Introduction

Executive stock options (ESOs) are call options granted by firms to employees as a form of compensation in addition to base salary, bonuses, and retirement savings. ESOs typically become "vested" (exercisable) over time (Murphy, 1999). ESOs are generally non-transferrable and non-hedgeable (NTNH) for at least two reasons. First, ESOs incentivize executives by aligning their and the owners' interests as executives benefit only when firms' stock prices rise, and the NTNH features prevent executives from eliminating this incentive. Second, NTNH features prevent negative signalling to investors. Holders of NTNH ESOs cannot transfer the ESOs to third parties at any price, nor are they allowed to hedge their value using positions in underlying assets or other derivatives. ESOs are typically forfeited if the executive leaves the firm before vesting.

There are two approaches to pricing contingent claims: arbitrage and equilibrium. The difficulty in pricing ESOs lies in their NTNH feature. On one hand, NTNH constraints nullify arbitrage pricing, disabling arbitrage strategies. On the other hand, NTNH adds constraints to a portfolio's optimization.

The state of the art in this area of research includes two distinct approaches: the stochastic discount factor (SDF) approach and the utility-based approach. Both approaches originate from constrained portfolio optimization. However, the former employs duality techniques to identify SDFs and derive general closed-form pricing formulas, while the latter makes simplifying assumptions and solves problems numerically, giving up a general pricing framework.

Ingersoll (2006) was one of the first to use the SDF approach. He used the duality method of Cvitanić and Karatzas (1992) and Karatzas and Kou (1996) to solve for the optimal constrained portfolio of the undiversified executive. However, his approach includes two restrictive features. First, the manager is infinitely lived; second, the value of executives' firm's stocks and options holdings as a fraction of their total personal wealth must remain constant. In the real world, however, the value of this fraction fluctuates stochastically[1] and the corresponding constraints become stochastic intervals. Carmona et al. (2011), within Ingersoll (2006)'s framework, added job termination as a Poisson process.

Two comprehensive studies using the utility-based approach are Carpenter et al. (2010) and Leung and Sircar (2009). In these articles, an executive's goal is to maximize the expected utility of terminal wealth by choosing the optimal exercise time and an optimal trading strategy before and after exercise. Traded assets include all the primary assets and some non-

[1] There is no chance, of course, that managers' firm's shares and options holdings value are perfectly correlated with the total value of their personal portfolio.

transferable options. Solving the free-boundary problem, the authors obtained the executives' continuation region and the critical stock price boundary, above which the option holders exercise and below which they wait. Carpenter et al. (2010) allowed for only a single block exercise of the option, while Leung and Sircar (2009) allow for a discrete partial exercise. Grasselli and Henderson (2008) considered exercise under transaction costs and concluded that executives should exercise deceasing blocks of options. The above three utility-based models' simplifications include primary assets of only two dimensions and a specific utility form.

Here, we aim to take the merits of both approaches. We use a partial equilibrium approach—constrained portfolio optimization. We show that NTNH constraints break the local co-linearity caused by derivative assets in solving portfolio optimization problems. Thus, we are able to translate portfolios that include NTNH derivatives into portfolios that consist of primary assets only, by replicating derivatives using primary assets and then integrating the NTNH constraints into a single rectangular stochastic constraint. This allows us to study any granting plan of ESOs, allowing, in turn, a general study of optimal ESO exercise policies which is essential for proper ESO pricing. Ingersoll (2006), in contrast, allows for only a constant singleton constraint, which, in turn, allows only for ESO granting plans which require executives to hold a constant fraction of their wealth in the firm's stock (and stock equivalents of ESOs), not facilitating a careful study of optimal ESO exercise policies.

Solving the constrained portfolio optimization problem identifies a SDF that gives the subjective price of any contingent claim written on the primary assets in this portfolio. Furthermore, by using executives' optimal exercise policies, we derive the objective price of ESOs from the firms' perspectives.

We obtain the subjective and objective prices of NTNH American ESOs with a block exercise policy and under a continuous partial exercise policy. The optimal exercise rate is the one that equates the marginal utility increase, from relaxation of the NTNH constraints induced by reduction in the number of outstanding NTNH ESOs, to the marginal utility decrease from loss of option time value due to early exercise. Without NTNH constraints, maximizing the expected utility of terminal wealth and maximizing the option's values induce identical optimal exercise policies. Here, we find that under NTNH constraints, these two different objectives induce different optimal exercise policies. This implies that NTNH constraints break the independence of executive's wealth components. Furthermore, under some parameter settings, parameter changes may cause the terminal expected utility and ESOs subjective price to change in opposite directions.

We simulate NTNH ESOs and execute a comprehensive comparative statics analysis of prices and optimal exercise policies of NTNH American ESOs with continuous partial exercise. We identify policy implications that enhance the ESOs' incentivizing efficiency.

The relevance and importance of ESO pricing extends over several disciplines: asset pricing, derivative pricing, constrained portfolio optimization, agency theory—incentives design, corporate governance—executive compensation, capital structure, optimal reporting, and optimal accounting.

ESOs have become a major component of corporate compensation (Carpenter, et al., 2010). According to Hall and Murphy (2002), in fiscal year 1999, 94 percent of S&P 500 companies granted ESOs, at a value that accounted for 47 percent of total CEOs pay. Cai and Vijh (2005), using Compustat's ExecuComp database, estimate that in 2002, listed firms granted ESOs worth $98 billion based on their Black–Scholes value, representing 1.2 percent of these firms' year-end market value.

Due to the extensive use of ESOs,[2] optional reporting of their cost became mandatory by the Financial Accounting Standard Board (FASB) in 2004. Clearly, accurate and subjective ESO pricing is essential for understanding the incentives they induce.

Furthermore, the methods that we introduce here should not only facilitate the pricing of ESOs with additional features, but should also help solve other pricing problems with NTNH constraints, including pensions, human capital, and real estate (Detemple and Sundaresan, 1999). Clearly, the non-hedgeable constraints apply to many real options, including energy ones.

Other methods have been used in attempts to price ESOs. The FASB proposes a simple method of ESO pricing, substituting the ESOs' expiration dates with exogenously specified expected time to exercise. Huddart and Lang (1996) conclude that this approach overstates the cost of the ESOs to the firm. The drawback of the FASB approach is that it does not properly reflect executives' optimal ESO exercise times conditional on possible stock price paths. It ignores optimal exercise behaviour, for example. Hitting time models extend the FASB model by, first, replacing the exogenously specified expected exercise time with an endogenous first time the stock price hits an (exogenously given deterministic) optimal exercise boundary and, then, aggregating over possible stock price paths (Hull and White, 2004; Cvitanić et al., 2008). While the exercise boundaries in the above models are independent of model parameters, such as the stocks' drift rates and volatilities, Leung and Sircar (2009) show that optimal exercise boundaries are highly sensitive to these parameters.

Extensions of hitting time models are default models, so named because they share features of credit risk valuation models. These default models define early exercise as the first arrival of an exogenous counting process. Jennergren and Näslund (1993) and Carpenter (1998) use the first jump time of an exogenous Poisson process, which serves as a proxy for executives

[2] Please see Statement of Financial Accounting Standards (SFAS) No. 123 (revised 2004), paragraph B79 and B80. See also International Financial Reporting Standard 3 (IFRS 2), Share-based Payment 3, and Securities Exchange Act of 1934 (as amended through P.L. 111-257, approved October 5, 2010), Sec. 16 (C).

exercising the options early due to liquidity shocks, diversification needs, voluntary or involuntary job termination, or other events relevant to the executives but not to unrestricted option holders. Carr and Linetsky (2000) extend the default model by making the intensity of the Poisson process depend on the firm's stock price and time. Sircar and Xiong (2006), within a Carpenter (1998) framework, include ESOs with resetting and reloading provisions.

The above three models—FASB, hitting time, and default—all focus on finding objective prices. They price the ESOs to firms, assuming exogenously given executives' exercise policies, including estimated times to exercise, the exercise boundary, or the exercise process. The price is the expected payoff at the exercise time discounted by the SDF derived from the unconstrained portfolio optimization problem. In contrast, we develop here the executives' subjective fair price of ESOs and identify endogenous optimal exercise policies. We then use these subjective prices under endogenous optimal exercise policies to price ESOs to firms.

Heron and Lie (2007) and Eikseth and Lindset (2011) studied ESOs backdating, Sen and Tumarkin (2014) looked at behavioural aspects of executives' ESOs exercise policies, Ittner et al. (2003) studied performance vested ESOs, and Palmon et al. (2008), executives' effort aversion.

The rest of the paper is organized as follows. Section 2 develops a theoretical ESO pricing model and prices European and American options. In particular, Subsection 2.1 presents the key results we need from the theory of constrained portfolio optimization; Subsection 2.2 looks at the pricing of NTNH European ESOs; Subsection 2.3 presents our results on American option pricing with block exercise; and Subsection 2.4 considers American option pricing with continuous partial exercise. In Section 3 we numerically price American ESOs under a continuous partial exercise policy; and Section 4 concludes.

2 ESOs pricing model

2.1 Constrained portfolio optimization

Cvitanić and Karatzas (1992) develop a duality technique to solve the constrained consumption/investment problems by transforming the original constrained problems into *auxiliary unconstrained problems*. This approach assumes the support function of the constraint is bounded below in the duality and existence proofs. However, in an ESO pricing problem settings, the aforementioned condition does not hold. To implement the Cvitanić and Karatzas (1992) approach in pricing ESOs, we relax the bounded below condition into a less strict one.

In the market $\mathcal{M}$ there is a traded bond whose price, $S_0(t)$, appreciates at a deterministic risk-free rate of interest, $r(t)$, thus evolving according the differential equation,

$$dS_0(t) = S_0(t)r(t)dt, \;\; S_0(0) = 1. \tag{1}$$

Uncertainty in the market is driven by a d dimensional standard Brownian Motion $W(t) = (W_1(t), \dots, W_d(t))^T$ in $\Re^d$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and we denote by $\{\mathcal{F}_t\}$ the $\mathbb{P}$-augmentation of the natural filtration $\mathcal{F}_t^W = \sigma(W(s); 0 \le s \le t)$, with time span $[0, T]$ for some finite $T > 0$.

Primary asset prices $S_i(t), i = 1, \dots, d$ follow the dynamics of

$$dS_i(t) = S_i(t)\left[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\right], \quad S_i(0) = s_i, i, j = 1,2, \dots, d. \tag{2}$$

Here $\sigma(t) \triangleq (\sigma_{ij}(t))$ is a $d \times d$ volatility matrix, and $b(t) \triangleq (b_i(t))$, is a $d \times 1$ drift rate vector. Let $\mathcal{K} \triangleq \{K(t,\omega); (t,\omega) \in [0,T] \times \Omega\}$ be a family of closed, convex, nonempty subsets of $\mathcal{R}^d$; and let $\delta(v(t)) \equiv \delta(v(t,\omega)|K(t,\omega)) \triangleq \{\sup_{\varrho \in K}(-\varrho^T v(t)): \Re^d \to \Re \cup \{+\infty\}; (t,\omega) \in [0,T] \times \Omega\}$ be the corresponding family of support functions, where $\widetilde{K}(t,\omega) \triangleq \{v(t) \in \Re^d; \delta(v(t)|K) < \infty\}$ is the effective domain of the support function and $v(t) \equiv v(t,\omega) = (v_1(t,\omega), \dots, v_d(t,\omega))^T$. Let $\mathcal{H}$ denote the Hilbert space of $\{\mathcal{F}_t\}$-progressively measurable processes v with values in $\Re^d$, and with the inner product $< v_1, v_2 > \triangleq \mathbb{E}\int_0^T (v_1(t))^T v_2(t)dt$. Without loss of generality, we assume that the ESO is written on $S_1(t)$.

Let $U: (0,\infty) \to \Re$ be a strictly increasing, strictly concave, utility function of class C^1, satisfying,

$$U'(0+) \triangleq \lim_{x \downarrow 0} U'(x) = \infty, \quad U'(\infty) \triangleq \lim_{x \to \infty} U'(x) = 0\,. \tag{3}$$

Let $x_t \in (0,\infty)$ be the initial, at $t = 0$, or realized, at $t > 0$, executive's wealth. Let the vector $\pi(t,\omega)$ represent the proportion of wealth invested, at time t, in each of the primary assets (henceforth, the portfolio process). $\mathcal{A}_0(x_t, t, T)$ is the set of admissible portfolio processes, ensuring that the portfolio wealth process, denoted by $X^{x_t,\pi}(s)$, is finite for $s \in [t,T]$ and satisfies $\mathbb{E}[U^-(X^{x_t,\pi}(T))|\mathcal{F}_t] < \infty$, where $U^-(x) \triangleq -min(U(x), 0)$.

We now define an auxiliary market $\mathcal{M}_v$. There, the risk-free rate is $r_v(t) = r(t) + \delta(v(t)), \forall\, t \in [0,T]$; the vector of drift rates of the primary assets is $b_v(t) = b(t) + v(t) + \delta(v(t))\mathbf{1}$, where $\mathbf{1} \triangleq (1\ 1 \dots 1)^T$, and $\mathcal{A}_v(x_t, t, T)$ is the admissible set of portfolio processes $\pi_v(t,\omega)$, ensuring that the portfolio wealth in this market, denoted by $X_v^{x_t,\pi_v}(s)$, is finite for $s \in [t,T]$ and satisfies $\mathbb{E}\left[U^-\left(X_v^{x_t,\pi_v}(T)\right)|\mathcal{F}_t\right] < \infty$.

It turns out that in the auxiliary market the constraints are not binding and, thus, can be ignored. In particular, the role of the vector $v(t)$ is to modify the original market, changing the expected returns of some assets in order to make them more (or less) attractive. So, for example, if one asset has a short-sale constraint, then $v(t)$ would increase the expected return on that asset until it is not optimal to short the asset. Proposition 1 below (see also Cvitanić and Karatzas, 1992, Proposition 8.3) then gives conditions under which the optimal unconstrained portfolio in the auxiliary market is also the optimal constrained portfolio in the original market.

Note that the total adjustment to the optimal portfolio expected return in the auxiliary market will be zero.

The *constrained optimization problem* in the original market $\mathcal{M}$ is to maximize, for all $0 < x < \infty$, the derived utility

$$J(x_t;\pi) \triangleq \mathbb{E}\big[U\big(X^{x_t,\pi}(T)\big)|\mathcal{F}_t\big], \qquad \forall\, t \in [0,T] \tag{4}$$

by choosing π over the class (here, ℓ is Lebesgue measure).

$$\mathcal{A}(x_t, K(t,\omega), t, T) \triangleq \{\pi \in \mathcal{A}_0(x_t,t,T);\ \pi(t,\omega) \in K(t,\omega)\ for\ \ell \otimes \mathbb{P} - a.e.(t,\omega)\}. \tag{5}$$

The unconstrained problem in the auxiliary market $\mathcal{M}_v$ (the *auxiliary unconstrained problem*) is to maximize, for all $0 < x_t < \infty$,

$$J_v(x_t;\pi) \triangleq \mathbb{E}\left[U\left(X_v^{x_t,\pi_v}(T)\right)|\mathcal{F}_t\right] \tag{6}$$

by choosing π over the class $\mathcal{A}_v(x_t,t,T)$.

The *dual problem of the auxiliary unconstrained problem* is to minimize

$$\tilde{J}(y_t;v) \triangleq \mathbb{E}\big[\widetilde{U}\big(y_t H_v(T)/H_v(t)\big)|\mathcal{F}_t\big] \tag{7}$$

(y_t is defined below) by choosing v over the class

$$\mathfrak{D}_t = \ \mathfrak{D}_t(K) \triangleq \left\{v \in \mathcal{H};\ \mathbb{E}\int_t^T \delta\big(v(s)\big)ds < \infty\right\}, \tag{8}$$

where $\widetilde{U}(y) \triangleq max_{x>0}[U(x) - xy] = U(I(y)) - yI(y)$, $I(\cdot)$ is the inverse function of $U'(\cdot)$, $H_v(t) \triangleq exp\left\{-\int_0^t r_v(s)\,ds\right\} exp\left\{-\int_0^t \theta_v(s)dW(s) - \frac{1}{2}\int_0^t \|\theta_v(s)\|^2 ds\right\}$, is the stochastic discount factor (SDF) in the auxiliary market $\mathcal{M}_v$, and $\theta_v(s) \triangleq \sigma^{-1}(t)[b_v(t) - r_v(t)]$. If we define $\gamma_v(t) \triangleq exp\left\{-\int_0^t r_v(s)ds\right\}$ (sometimes known as the reciprocal of the money market account) and $Z_v(t) \triangleq exp\left\{-\int_0^t \theta_v(s)dW(s) - \frac{1}{2}\int_0^t \|\theta_v(s)\|^2 ds\right\}$, then $H_v(t) = \gamma_v(t) \times Z_v(t)$. If $v = 0$, then $H_0(T) \triangleq H_v(T)|_{v=0}$ is the SDF in the original market $\mathcal{M}$. In Equation (7), $y_t = y \times H_v(t)$ where $y \equiv \mathcal{Y}_v(x)$ is the inverse function of $x = \mathcal{X}_v(y) \triangleq \mathbb{E}\big[H_v(T) \times I\big(yH_v(T)\big)\big]$. We define $x_t \triangleq \mathbb{E}\big[H_v(T)/H_v(t) \times I\big(yH_v(T)\big)|\mathcal{F}_t\big]$; then by the substitution $y_t = y \times H_v(t)$, we see that $y_t \equiv \mathcal{Y}_v(x_t)$ is the inverse function of $x_t = \mathcal{X}_v(y_t) \triangleq \mathbb{E}\big[H_v(T)/H_v(t) \times I\big(y_t H_v(T)/H_v(t)\big)|\mathcal{F}_t\big]$. We call the optimal solution $\hat{v}$; thus,

$$\hat{v} \triangleq \underset{v\in\mathfrak{D}_t}{\operatorname{argmin}}\, \tilde{J}(y_t;v) = \underset{v\in\mathfrak{D}_t}{\operatorname{argmin}}\, \mathbb{E}\big[\widetilde{U}\big(y_t H_v(T)/H_v(t)\big)|\mathcal{F}_t\big]. \tag{9}$$

As Cvitanić and Karatzas (1992) point out (p. 768), the dual approach "is of great importance here…because, as it turns out, it is far easier to prove existence of optimal policies in the dual, rather than in the primal, problem." We assume that the following conditions from Cvitanić and Karatzas (1992) are satisfied.

Let us state a few key properties of $\widetilde{K}$, δ, and U. The effective domain of the support function is the same for all (t, ω); that is,

$$\widetilde{K}(t, \omega) \equiv \widetilde{K}. \tag{10}$$

$$\delta\big(v(t)\big|K(t, \omega)\big), 0 \leq t \leq T, \text{ is } \{\mathcal{F}_t\} \text{-progressively measurable.} \tag{11}$$

$$\delta(v(t)|K(t, \omega)) \text{ is continuous on } \widetilde{K} \text{ for all } (t, \omega). \tag{12}$$

$$\text{For some } \alpha \in (0,1), \gamma \in (1, \infty), \text{ we have } \alpha U'(X) \geq U'(\gamma X), \forall X \in (0, \infty). \tag{13}$$

$$X \to XU'(X) \text{ is non-decreasing on } (0, \infty). \tag{14}$$

$$\forall y \in (0, \infty), \exists v \in \mathfrak{D} \text{ such that } \tilde{J}(y; v) < \infty. \tag{15}$$

We are now ready to state Proposition 1.

Proposition 1. For every $\{\mathcal{F}_t\} -$ progressively measurable process $v(t) \equiv v(t, \omega)$, $0 \leq t \leq T$, such that $v(t) \in \widetilde{K}(t, \omega)$ for $\ell \otimes \mathbb{P} - a.e.\,(t, \omega)$, assume that the following conditions hold:

$$\forall y \in (0, \infty), \lim_{\|v\| \to \infty} \tilde{J}(y; v) = \infty \tag{16}$$

and

$$\forall v \in (0, \infty), \lim_{y \downarrow 0} \tilde{J}(y; v) = \infty. \tag{17}$$

Then, when $\pi_{\hat{v}}(t, \omega) \in K(t, \omega)$ and $\delta\big(\hat{v}(t, \omega)\big) + \big(\pi_{\hat{v}}(t, \omega)\big)^T \hat{v}(t, \omega) = 0$ are satisfied, $\pi_{\hat{v}}(t, \omega)$ maximizes the *constrained optimization problem* in the original market $\mathcal{M}$ and the *unconstrained problem* in the auxiliary market, $\mathcal{M}_{\hat{v}}$. Also, $\hat{v}(t, \omega)$ minimizes the *dual problem of the auxiliary unconstrained problem* in the auxiliary market $\mathcal{M}_{\hat{v}}$.

Proof. The proof directly follows from Theorem 7.4, Proposition 8.3, Proposition 12.1, Proposition 12.2, Theorem 12.4, Theorem 13.1 and Proposition 13.2 in Cvitanić and Karatzas (1992).

Remark 1. If $K(t, \omega)$ contains the origin, then the support function of the constraint is bounded below and, clearly, Conditions (16) and (17) are satisfied. However, as we show in the following subsection, the NTNH constraints imply that $K(t, \omega)$ does not contain the origin. The existence of a minimum to Equation (7) can still be guaranteed, however, if these two conditions are first imposed then verified.

2.2 NTNH European ESO pricing

In this subsection, we show how constrained portfolio optimization can be used to price NTNH European ESOs. First, let us define some terms. There are three types of prices associated with ESOs: the *market price*, the *subjective price,* and the *objective price*.

Market price ($BS(t)$, $0 \leq t \leq T$) – the Black-Scholes price without taking the NTNH features into account. We call the corresponding SDF the *market SDF*.

Subjective price ($\hat{p}(t)$, $0 \leq t \leq T$) – the value of ESOs to executives (determined by executives' expected utility maximization). It is lower than the market price because of two effects of the NTNH constraints. First, for American options, executives' optimal exercise policies under NTNH constraints are suboptimal with respect to those without constraints. Second, even for European options, NTNH constraints restrict diversification, inducing subjective SDFs that apply discounts that are deeper than the discounts applied by market SDFs.

Objective price ($\tilde{p}(t), 0 \leq t \leq T$) –the cost of NTNH ESOs to firms (incurred upon executives' exercising them). This definition of the objective price was established by Ingersoll (2006) and is followed in the literature, see Leung and Sircar (2009), Carpenter et al. (2010), and Carmona et al. (2011). The objective price is lower than the market price because firms face executives' optimal exercise policies under NTNH constraints, which are suboptimal in absence of these constraints. Ordinarily, the objective price is higher than the subjective price because NTNH constraints do not apply to firms (the restricted diversification is irrelevant), thus firms use the market SDF. As pointed out in Ingersoll (2006) this holds for European ESOs, and for American ESOs with block exercise. Surprisingly, however, for American ESOs with continuous partial exercise, it is possible for the objective price to be (slightly) lower than the subjective price. We discuss this case in Remark 6 of Subsection 2.4 below.

The difference between the objective and subjective prices is the amount paid by firms but not appreciated by executives. We call it the *deadweight cost*. It measures the efficiency loss (enhancement) to firms in incentivizing their executives.

We assume that executives' portfolios consist of n shares of NTNH ESOs with exercise price k and maturity T,[3] and a sub-portfolio composed of primary assets only. We name the sub-portfolio *outside wealth* and denote its initial value by x_s and its value process by $X(t), t \geq s$. We denote the initial value of *total wealth* as $x_s^n \equiv x_s + nBS(t)$ and the value process of total wealth by $\mathbb{X}(t)$. We denote $x \equiv x_0$ and $x^n \equiv x_0^n$.[4]

In this subsection, we focus on pricing European NTNH ESOs (ignoring early exercise). The idea behind our pricing of NTNH derivatives is to replace the position of non-transferable derivatives with a replicating position composed of primary assets only. Because markets are complete, we can see firms that grant executives n option payoffs at time T, which we denote $nB(T)$, as equivalently granting executives, at time 0, a cash amount of $nBS(0)$ which they will use to dynamically replicate[5] (using self-financing strategies) the time T, option payoffs $nB(T)$.

[3] For simplicity and brevity, we often supress dependency on parameters such as T.

[4] While our framework does not restrict the composition of outside wealth, in the Section 3 simulation analysis we assume that executives' outside wealth is invested in the market index. Not only does this seem to be a natural general choice, it has been the choice in the literature (see, for example, Carpenter et al., 2010), and thus, facilitates comparison.

[5] According to the Black-Scholes formula, holding n shares of a non-transferable European options is equivalent to holding $n\Phi(d_1)$ shares of the firm's stock and $-n\Phi(d_2)ke^{-r(T-t)}$ dollars of the savings account, where k is the strike

Under the latter interpretation, executives must hold $n\Phi(d_1)$ shares of the firm's stock throughout the option's life, where this represents the Black-Scholes delta hedge for the n options. Because of non-hedgeability they must hold a non-negative position of the firm's stock in their outside wealth. Combining the option-replicating shares with the potential non-negative outside wealth holdings in the firm's stock, executives' total share holdings in the firm's stock can never be less than $n\Phi(d_1)$ shares. That is, the number of the firm's shares held by executives' belongs to the interval $[n\Phi(d_1), \infty)$; or, the fraction of the executives' firm's stock to their total wealth, belongs to the interval $[n\Phi(d_1)S_1(t)/\mathbb{X}(t), \infty)$. This allows us to frame the NTNH option pricing problems as constrained portfolio optimization ones. Taking all the other primary assets into account, the combined constraint for the portfolio process can be written as follows:

$$\forall t, \pi(t,\omega) \in K(t,\omega) = \left[\frac{n\Phi(d_1)S_1(t)}{\mathbb{X}(t)}, \infty\right) \times (-\infty, \infty)^{d-1}, \tag{18}$$

where $S_1(t)$ is the firm's stock price underlying the ESO. Then, we identify $\widetilde{K}$ and $\delta(v(t))$ as

$$\widetilde{K} = [0, \infty) \times \{0\}^{d-1}, \tag{19}$$

$$\delta(v(t)) \equiv -\frac{n\Phi(d_1)S_1(t)}{\mathbb{X}(t)} \times v_1(t) \text{ on } \widetilde{K}, \tag{20}$$

which also confirms the fulfillment of Conditions (10)-(12). We assume $v(t) \in \mathfrak{D}_t$, then $\mathbb{E}\int_0^T \delta\big(v(t)\big|K(t,\omega)\big)dt = \mathbb{E}\int_0^T [-\frac{n\Phi(d_1)S_1(t)}{\mathbb{X}(t)} \times v_1(t)]dt < \infty, \forall v(t) \in \widetilde{K}$.

We note that our use of the Black-Scholes options pricing formula implies that the volatility of the underlying asset is deterministic.

Assumption 1. $\forall t \in [0,T]$, and $\forall\, x_t \in (0,\infty)$, the indirect utility function, $M(x_t,t)$, is

$$M(x_t,t) \triangleq \underset{\pi\in\mathcal{A}(x_t,K(t,\omega),t,T)}{esssup} \mathbb{E}\big[U\big(X^{x_t,\pi}(T)\big)|\mathcal{F}_t\big]. \tag{21}$$

We denote the portfolio solution of Equation (21) under the constraint $K(t,\omega)$ of Equation (18) as π^*. We assume $M'(\cdot,t) > 0$ on $(0,\infty)$.

We have now developed sufficient mathematical structure to identify, in the following theorem, the subjective price of an NTNH European ESO, characterizing the SDF as the real-world marginal rate of substitution. Following Karatzas and Kou (1996), we set the subjective price to be the price at which executives are indifferent, in the utility sense, between holding their optimal portfolio without ESOs or marginally substituting a small fraction of portfolio wealth with ESOs (or more generally, they are indifferent between holding the optimal portfolio

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price of the ESOs. $\Phi(\cdot)$ is the standard normal cumulative density distribution function, $d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[ln\left(\frac{S_1}{k}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$, and $d_2 = d_1 - \sigma\sqrt{T-t}$.

with n ESOs and substituting a small fraction of wealth with another ESO).

Theorem 1. Under Assumption 1, the time t subjective price of NTNH *European ESOs* with terminal payoff $B(T)$ is uniquely determined by

$$\hat{p}(t) \equiv \hat{p}(x_t, n, t) = \frac{\mathbb{E}\left[B(T)U'\left(X^{x_t,\pi^*}(T) + nB(T)\right)|\mathcal{F}_t\right]}{M'(x_t + nBS(t), t)}, \forall x_t > 0, \tag{22}$$

where $X^{x_t,\pi^*}(T)$ represents the optimal outside wealth for the constrained problem.

Proof. See Appendix A.

Theorem 2. i) Under Assumption 1, the time t subjective price of NTNH *European ESOs* with terminal payoff $B(T)$ is uniquely determined by

$$\hat{p}(t) \equiv \hat{p}(x_t, n, t) = \frac{\mathbb{E}[H_{\hat{v}}(T)B(T)|\mathcal{F}_t]}{H_{\hat{v}}(t)} = \frac{\mathbb{E}^{\hat{v}}[\gamma_{\hat{v}}(T)B(T)|\mathcal{F}_t]}{\gamma_{\hat{v}}(t)}, \tag{23}$$

where $\mathbb{E}^{\hat{v}}(\cdot)$ is an expectation under the martingale measure in the auxiliary market $\mathcal{M}_{\hat{v}}$, for $\hat{v}$ as defined in Equation (9).

ii) The time t objective price of this NTNH European ESO is equal to the market price:

$$\tilde{p}(t) \equiv \tilde{p}(t) = BS(t) = \frac{\mathbb{E}[H_0(T)B(T)|\mathcal{F}_t]}{H_0(t)} = \frac{\mathbb{E}^0[\gamma_0(T)B(T)|\mathcal{F}_t]}{\gamma_0(t)}, \tag{24}$$

where $\mathbb{E}^0(\cdot)$ is expectation under the martingale measure in the auxiliary market $\mathcal{M}_0$ (where $\hat{v} = 0$).

Proof. See Appendix A.

Theorem 2 identifies the auxiliary markets $\mathcal{M}_{\hat{v}}$ and $\mathcal{M}_0$ as the "risk-neutral markets" corresponding to constrained and unconstrained pricing in the original market, respectively.

The number of ESO shares granted affects the subjective price in two ways. First, by affecting the initial total wealth.[6] Second, as n increases the constraint $K(t,\omega) = [n\Phi(d_1)S_1(t)/\mathbb{X}(t), \infty) \times (-\infty, \infty)^{d-1}$ becomes more binding, inducing a "more aggressive" SDF. In the following subsection, we show the effect of this dependence of the subjective price on the number of ESO shares and on optimal exercise policies for American ESOs (block exercise and a continuum of exercise ratios), which, in turn, affect the outstanding number of ESO shares, affecting ESO prices, and so on.

2.3 NTNH American ESO pricing with block exercise policy

The ESOs' exercise time τ: $(\Omega, \mathcal{F}, \mathbb{P}) \to [t_{vo}, T]$, is a stopping time with respect to the filtration $\{\mathcal{F}_t\}$, where the ESOs vest on date t_{vo}. Let Γ be the class of $\{\mathcal{F}_t\}$ stopping times. Before τ, the NTNH constraint is $K(t,\omega) = [n\Phi(d_1)S_1(t)/\mathbb{X}(t), \infty) \times (-\infty, \infty)^{d-1}$, and we denote the

[6] If the constraint is a closed, convex cone or the utility is logarithmic, then according to Cvitanić and Karatzas (1992), the initial wealth does not appear in the pricing formula. The non-hedgeability constraint is a closed, convex cone; however, the NTNH constraint is not.

corresponding SDF $H_f(t)$, where f is defined in Theorem 4 below. Because of the executives' general restriction from short selling their firm's stock, even after τ, although there are no NTNH ESOs in the portfolio anymore, the portfolio cannot have negative weights in the firm's stock. Hence, the constraint becomes $K(t,\omega) = [0,\infty) \times (-\infty,\infty)^{d-1}$, and we denote the corresponding SDF $H_\lambda(t)$ (see Remark 2). The post-exercise total wealth at τ is the sum of outside wealth $X^{x_t,\pi}(\tau)$ and the intrinsic value of ESOs $nB(\tau)$. Hence,

$$K(t,\omega) = \left[\frac{n\Phi(d_1)S_1(t)}{\mathbb{X}(t)}, \infty\right) \times (-\infty,\infty)^{d-1} 1_{\{t<\tau\}} + [0,\infty) \times (-\infty,\infty)^{d-1} 1_{\{t\geq\tau\}}. \tag{25}$$

We have now developed sufficient mathematical structure to identify, in the following theorem, the subjective price of an NTNH American ESO under block exercise policy, characterizing the SDF as the real-world marginal rate of substitution.

Theorem 3. Under a block exercise policy, the subjective price of an NTNH American ESO is uniquely determined by

$$\hat{p}(t) \equiv \hat{p}(x_t,n,t) = \frac{\mathbb{E}\left[M'\left(X^{x_t,\pi^*}(\tau^*) + nB(\tau^*),\tau^*\right)B(\tau^*)|\mathcal{F}_t\right]}{L'(x_t,n,t)}, \quad \forall x_t > 0, \tag{26}$$

where $\pi^* = \pi^*(s,\omega)$ is the solution to Problem (28) below for $s \in (t,\tau^*)$ but is the solution to Problem (27) below for $s \in (\tau,T)$, and $K(t,\omega)$ is as in Equation (25):

$$M(x_\tau,\tau) \triangleq \underset{\pi\in\mathcal{A}(x_\tau,K(\tau,\omega),\tau,T)}{esssup} \mathbb{E}\left[\left(U\left(X^{x_\tau,\pi}(T)\right)\middle|\mathcal{F}_\tau\right)\right], \tag{27}$$

and

$$L(x_t,n,t) \triangleq \underset{\tau\in\Gamma}{esssup} \underset{\pi\in\mathcal{A}(x_t,K(t,\omega),t,\tau)}{esssup} \mathbb{E}[M(X^{x_t,\pi}(\tau) + nB(\tau),\tau)|\mathcal{F}_t]\,. \tag{28}$$

Also, τ^* is the optimal exercise time; that is, the optimal stopping time in Problem (28).

Proof. See Appendix B.

We are now ready to develop a martingale pricing expression for an NTNH American ESO under block exercise.

Theorem 4. The subjective and objective prices of an NTNH American ESO under block exercise are uniquely determined as follows.

i) The subjective price is

$$\hat{p}(t) \equiv \hat{p}(x_t,n,t) = \frac{\mathbb{E}[H_f(\tau^*)B(\tau^*)|\mathcal{F}_t]}{H_f(t)} = \frac{\mathbb{E}^f[\gamma_f(\tau^*)B(\tau^*)|\mathcal{F}_t]}{\gamma_f(t)}, \tag{29}$$

where τ^* is the optimal exercise time identified by the optimal stopping time, Problem (28), and f minimizes the dual problem $\tilde{J}(y_t;v) \triangleq \mathbb{E}\left[\tilde{U}\left(y_tH_v(\tau^*)\right)\right]$ over $\mathfrak{D}_t$ with $K(t,\omega)$ as in Equation (25) for $t < \tau^*$.

ii) The objective price of this NTNH American ESO is

$$\tilde{p}(t) \equiv \tilde{p}(x_t, n, t) = \frac{\mathbb{E}[H_0(\tau^*)B(\tau^*)|\mathcal{F}_t]}{H_0(t)} = \frac{\mathbb{E}^0[\gamma_0(\tau^*)B(\tau^*)|\mathcal{F}_t]}{\gamma_0(t)}. \quad (30)$$

Proof. See Appendix B.

Remark 2. Equation (29) implies that the subjective price of an NTNH American ESO under block exercise is determined by $H_f(\tau^*)$, the SDF before the optimal exercise time τ^* with constraint $K(t,\omega) = [n\Phi(d_1)S_1(t)/\mathbb{X}(t), \infty) \times (-\infty, \infty)^{d-1}$. In contrast, because optimal stopping times are determined backward, τ^* is determined by $H_\lambda(t)$, the SDF after τ^*. Hence, the subjective price $\hat{p}(x_t, n, t)$ is determined by both the auxiliary market $\mathcal{M}_f$ before τ^* and the auxiliary market $\mathcal{M}_\lambda$ after τ^*.

Remark 3. Note that the subjective price $\hat{p}(x_t, n, t)$ is not equal to $\mathbb{E}[H_l(T)B(T)|\mathcal{F}_t]/H_l(t)$, where the auxiliary market $\mathcal{M}_l$ is defined as the one that encompasses both auxiliary markets $\mathcal{M}_f$ (before optimal exercise) and $\mathcal{M}_\lambda$ (after optimal exercise). The equality does not hold because it uses the terminal payoff rather than the one at exercise.

Remark 4. Further note that even after improving the previous pricing method, the subjective price $\hat{p}(x_t, n, t)$ is not equal to $esssup_\tau\{\mathbb{E}[H_f(\tau)B(\tau)|\mathcal{F}_t]\}/H_f(t)$. This is because executives with NTNH American ESOs do not simply maximize the subjective prices of their options. Under NTNH constraints, the optimal ESO exercise policy can be obtained only by maximizing the expected utility of the terminal wealth. As optimal exercise under NTNH constraints is a function of the executive's wealth relative to outside wealth, separation (between maximizing the option value and maximizing the expected utility) is lost. With no constraints, however, separation exists; thus, maximizing the subjective price of the options and maximizing the expected utility of total terminal wealth are equivalent.

2.4 NTNH American ESO pricing with continuous partial exercise policy

Leung and Sircar (2009) extended the block exercise policy into a discrete partial exercise one. We now determine the optimal exercise policy under the most general conditions: executives may exercise continuously in time and chose exercise rates from a continuum. We call this exercise policy *continuous partial exercise.*

Under continuous partial exercise, the dynamics of the total wealth process is

$$\begin{gathered} d\mathbb{X}(t) = \big(\mathbb{X}(t)r(t)\big)dt + \mathbb{X}(t)\big(\pi(t)\big)^T\sigma(t)dW_0(t) - \dot{n}(t)\big(BS(t) - B(t)\big)dt, \\ \mathbb{X}(0) = x + nBS(0), \qquad \dot{n}(t) \in \mathcal{N} \end{gathered} \quad (31)$$

where $W_0(t) \equiv W(t) + \int_0^t \theta(s)\,ds$, $\theta(t) \equiv \sigma^{-1}(t) \times [b(t) - r(t)\mathbf{1}]$, and the control variable, $\dot{n}(t) \equiv \dot{n}(t,\omega)$, is the time t optimal exercise rate that maximizes the expected utility of terminal wealth. The constant n is the number of ESO shares granted at time zero. The random exercise rate process, $\dot{n}(t)$, belongs to the set $\mathcal{N}(t)$, the collection of all feasible $\{\mathcal{F}_t\}$-progressively measurable exercise rates, where $\mathcal{N}(t) \triangleq \Big\{\dot{n}(t): \dot{n}(t) = 0 \text{ for } t < t_{vo},\ \dot{n}(t) \geq 0 \text{ for } t \geq t_{vo}, \text{and } \int_0^t \dot{n}(s)ds \leq n \text{ for } t \geq 0\Big\}$. We define $\widehat{N}(t) \triangleq \int_0^t \dot{n}(s)ds$ to be the accumulated number of ESOs that have been exercised at time t. The constraint of the portfolio process is

$$\pi(t) \in K(t, \omega) = \Big[\big(n - \widehat{N}(t)\big)\Phi(d_1)S_1(t)/\mathbb{X}(t), \infty\Big) \times (-\infty, \infty)^{d-1}. \tag{32}$$

The derived utility of an executive maximizing expected utility of terminal wealth by choosing the optimal portfolio and exercise rate processes is

$$\mathbb{J}\big(X^{*,\dot{n}^*}(t), \widehat{N}(t), S(t), t\big) = esssup_{(\pi,\dot{n})\in(\mathcal{A}(x_t,K(t,\omega),t,T),\mathcal{N})}\, \mathbb{E}\big[U\big(\mathbb{X}^{x_t,\pi}(T)\big)|\mathcal{F}_t\big], \tag{33}$$

where $K(t, \omega)$ is described in Equation (32).

For a given $\dot{n}(t)$, we know how to solve the portfolio optimization problem by employing the duality technique. As the indirect utility function is a monotone increasing function of initial wealth, the time t optimal portfolio process generates the ex-ante optimal initial wealth for the next period $t + \Delta t$; otherwise, the existence of another portfolio process, which leads to higher wealth, would contradict optimality. Hence, we can simplify the optimization problem by separating it into the following two stages:

$$\mathbb{J}\big(\mathbb{X}^{*,\dot{n}^*}(t), \widehat{N}(t), S(t), t\big) = \underset{\dot{n}\in\mathcal{N}(t)}{esssup}\ \underset{\pi\in\mathcal{A}(x_t,K(t,\omega),t,T)}{esssup}\ \mathbb{E}\big[U\big(\mathbb{X}^{x,\pi,\dot{n}}(T)\big)|\mathcal{F}_t\big], \tag{34}$$

where $\mathbb{X}^{*,\dot{n}}(t) \equiv \mathbb{X}^{x,\pi^*,\dot{n}}(t)$ is the total wealth process generated by the optimal portfolio process π^* for a given initial wealth x and under an exercise rate process $\dot{n}(t)$.

The following proposition identifies the first-order condition (FOC) necessary to determine the optimal exercise process.

Proposition 2. Assume the firm's stock pays no dividend. The FOC of the (viscosity) solution to the optimization problem in Equation (34) is

$$-\big(BS(t) - B(t)\big)\frac{\partial\mathbb{J}\big(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S(t),t\big)}{\partial\mathbb{X}^{*,\dot{n}}(t)} + \frac{\partial\mathbb{J}\big(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S(t),t\big)}{\partial\widehat{N}(t)} = 0. \tag{35}$$

Proof. See Appendix C.

It is well known that under a non-negative interest rate with constant strike prices, values of European call options, written on non-dividend-paying stocks, never fall below their intrinsic values. Thus, the corresponding American options, whose prices are never below their European counterparts, should never be exercised early (see, e.g., Musiela and Rutkowski, 2005, p. 215). However, under NTNH constraints, early exercise might become optimal. As executives exercise the NTNH ESOs, the newly received cash[7] is subject *only* to short sale constraints of the firm's stock, and the non-transferable constraints become irrelevant. Updating the

[7] We assume that upon the exercise of ESOs, executives receive the exercise value in cash. Under US tax laws, however, this cash is taxed as ordinary income. Only if the executives exercise at least two years from granting and hold onto the exercised shares at least one year, do capital gains tax rates apply.

corresponding SDF and auxiliary market definitions induces appropriate pricing. The optimal exercise rate is set so that the marginal utility increase due to partial elimination of the constraint exactly offsets the marginal utility decrease from the time value loss due to early exercise.

We are now ready to develop a martingale pricing expression for an NTNH American ESO under optimal continuous exercise using a continuum of exercise rates.

Theorem 5. i) The subjective price of the NTNH *American ESO* with continuous partial exercise policy is

$$\hat{p}(t) \equiv \hat{p}(x_t, t, T) = \mathbb{E}\left[\int_t^T H_{f_{\hat{N}^*}}(s)\dot{n}^*(s)B(s)ds \,|\mathcal{F}_t\right], \tag{36}$$

where $\dot{n}^*(t)$ solves Equation (35), and $v = f_{\hat{N}^*}(s)$ minimizes dual problem (7), under $K(t,\omega)$ described in Equation (32).

ii) The objective price of this NTNH American ESO is

$$\tilde{p}(t) \equiv \tilde{p}(x_t, t, T) = \mathbb{E}\left[\int_t^T H_0(s)\dot{n}^*(s)B(s)ds \,|\mathcal{F}_t\right]. \tag{37}$$

Proof. This follows immediately from the definition of $H_{f_{\hat{N}^*}}(s)$ and Theorem 2.

Remark 5. Our results, obviously, depend on π^*, and $\dot{n}^*$. While solving for π^*, we are able to use methods developed earlier (see Proposition 1). We now have to develop methods to solve for $\dot{n}^*$. Note that while both loss of option time value due to early exercise and an underlying stock paying dividend are wealth leakages to option holders, they are essentially different. To compensate for the latter, it is sufficient to deduct the present value of all future dividends from the stock price without affecting the SDF. However, exercising NTNH ESOs affects the portfolio process constraint and affects the corresponding SDF. Thus, we cannot model negative jumps due to the loss of option time value the way we model negative jumps in stock prices due to dividends.

At any time t, the indirect utility is the conditional expectation of the utility generated by the terminal total wealth under the optimal exercise rate process and optimal portfolio process. The common approach is to solve for the optimal controls backward. However, here, total wealth is determined by d geometric Brownian motions requiring many branches of the multinomial tree. In addition, the accumulated number of the firm's stock shares $\hat{N}(t)$ is another state variable. This makes the complexity of the problem computationally infeasible.

Moreover, working backward, at any time t, the time t total wealth is determined by the optimal portfolio process, $\pi^*(s)$, $0 \le s \le t$. Hence, using a binomial tree cannot solve the problem. Moreover, the FOC (35) is determined not only by the information given in the set $\{\mathcal{F}_s, s \in [0,t]\}$ but also by the future optimal portfolio process. Therefore, the forward-looking

Monte Carlo simulation cannot solve the problem either.

Under logarithmic preferences, however, the substitution and income effects offset each other, mathematically implying that the cross partial derivatives of the derived utility function are zero. Now solving the FOC (35) does not require future optimal knowledge of the optimal portfolio process π^*, in another manifestation of the so called myopia of logarithmic preferences. In this case, we can, thus, use a Monte Carlo simulation to price the ESO. We solve for $\dot{n}^*$, satisfying FOC (35), for each sample path.

Remark 6. The results in this section allow us to discuss the question of whether the objective price is always larger than the subjective price. In Subsection 2.2, we found that, for European ESOs, the subjective price (at t = 0) is equal to $\mathbb{E}^{\hat{v}}[\gamma_{\hat{v}}(T)B(T)]$,where the risk-free rate in the auxiliary market, $\mathcal{M}_v$, is $r_v(t) = r(t) + \delta(v(t)) \leq r(t)$ [see Equation (20)]. It is well known that, all else being equal, a lower risk-free rate results in a lower call price. Thus, $\mathbb{E}^{\hat{v}}[\gamma_{\hat{v}}(T)B(T)] \leq \mathbb{E}^0[\gamma_0(T)B(T)]$; that is, the subjective price is lower than the objective price. This is true if T replaced with a stopping time as well, so the result holds for European ESOs as well as American ESOs with block exercise. These results agree with Ingersoll (2006). On the other hand, for American options with continuous partial exercise, the subjective price is given by $\mathbb{E}\left[\int_0^T H_{f_{\hat{N}^*}}(s)\dot{n}^*(s)B(s)ds\right]$ and the objective price, by $\mathbb{E}\left[\int_0^T H_0(s)\dot{n}^*(s)B(s)ds\right]$. Surprisingly, it is possible to have $\mathbb{E}\left[\int_0^T H_{f_{\hat{N}^*}}(s)\dot{n}^*(s)B(s)ds\right] > \mathbb{E}\left[\int_0^T H_0(s)\dot{n}^*(s)B(s)ds\right]$ if $H_{f_{\hat{N}^*}}(s)$ and $\dot{n}^*(s)$ are highly correlated and $H_0(s)$ and $\dot{n}^*(s)$ have low correlation. For a proof of this result, see Appendix E. We show in our simulation experiment, in the following section, that this can happen with reasonable parameters. It appears that the exercise rate provides a sort of hedge for the executive that is not reflected in the objective price.

3 Simulation experiment

In this section we simulate the NTNH ESO prices and show that under logarithmic preferences, Conditions (16) and (17) are satisfied, and the FOC (35) is fully determined by the information available up to time t, not depending on future optimization outcomes. Thus, the forward-looking Monte Carlo simulation becomes useful. While we can price the ESOs for a large class of utility functions obeying Conditions (16) and (17), the assumption of logarithmic preferences, for which the substitution effect is equal to the income effect and the cross partials of the indirect utility function are zero, simulation times are manageable. For non-logarithmic preferences, FOC (35) can be solved by searching in high-dimensional state space, where supercomputing capacity would be required. Below we describe the simulation method, present the numerical results of NTNH ESO pricing, and perform a sensitivity analysis with respect to all model parameters. We also compare NTNH ESO prices under continuous and discrete partial exercise policies by changing exercise frequencies.

3.1 Solving for the optimal exercise rate

Executives are granted n_{option} shares of unvested NTNH ESOs and n_{stock} shares of unvested (restricted) stocks. The ESOs and the stocks vest on future dates t_{vo} and t_{vs}, respectively. For any $t \geq t_{vo}$, we call the total wealth before exercise $pre\_\mathbb{X}(t)$. After solving FOC[8] (35) for the optimal exercise rate $\dot{n}^*(t)$, we can write the total wealth after exercise as $post\_\mathbb{X}(t) = pre\_\mathbb{X}(t) - \dot{n}(t)dt(BS(t) - B(t))$, and the accumulated number of shares from exercised ESOs as $\widehat{N}(t)$. Upon exercise, the constraint $K(t,\omega)$ changes and, in turn, induces changes in $f_{\widehat{N}}(t)$, the optimal control of the dual problem, Equation (34). The updated $f_{\widehat{N}}(t)$ determines the corresponding SDF in the original market and the drift rate adjustments that convert the original market price processes to auxiliary market ones. We can now use $f_{\widehat{N}}(t)$ to solve for the new, after exercise, value of the optimal portfolio process, $\pi^*(t)$, making the appropriate primary asset weight adjustments. The changes in the primary asset prices, from $S(t)$ to $S(t+1)$, affect next period's wealth $pre\_\mathbb{X}(t+1)$, which then is adjusted to $post\_\mathbb{X}(t+1)$, after time $t+1$ exercise, and then further adjusted to $\pi^*(t+1)$, and so on. Figure 1 below describes this process. We calculate the following values for each time point: pre-exercise total wealth, $pre\_\mathbb{X}(t)$, optimal exercise rate, $\dot{n}^*(t)$, post-exercise total wealth, $post\_\mathbb{X}(t)$, updated constraints, $\widehat{N}(t-1), f_{\widehat{N}}(t-1)$, SDF, and the optimal portfolio process, $\pi^*(t)$.

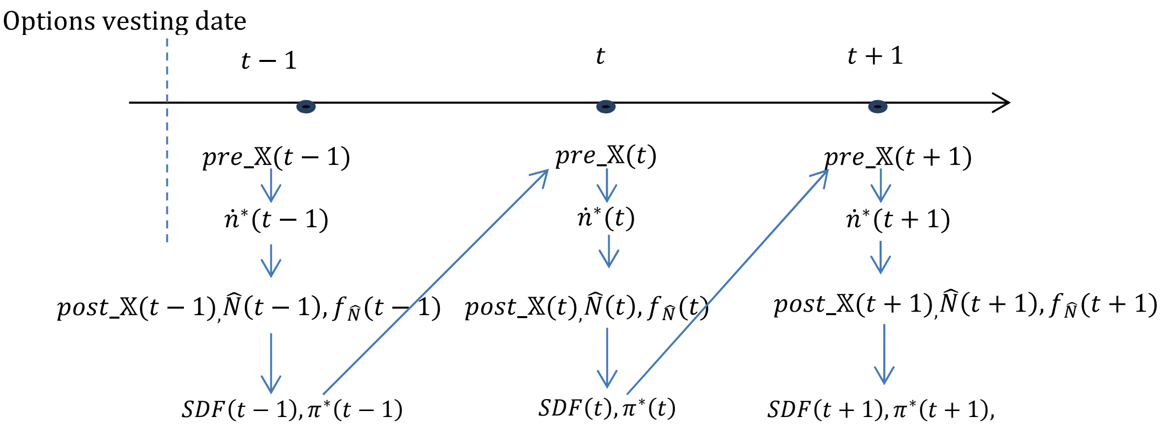


Figure 1. Sample path generating process for Monte Carlo simulation

The following Lemma states that under logarithmic preferences Conditions (16) and (17) are satisfied.

Lemma 1. Given $K(t,\omega) = [a,\infty) \times (-\infty,\infty)^{d-1}$, where $a \equiv a(t,\omega) > 0$, if $U(\cdot) = ln(\cdot)$, then for

[8] We actually solve Equation (38) below for the logarithmic preferences case of FOC (35), by linearly increasing the exercise rate $\dot{n}(t)$ to search for the turning points changing the sign of the left-hand side (LHS) of Equation (38) from positive to negative. Then, we find the values of the LHS of Equation (38) at the end points. The turning (end) points at which the expected utility [See Equation (D.1)] is the largest, are the optimal exercise rates $\dot{n}^*(t)$.

any $y \in (0,\infty)$, we have, $\lim_{\|v\|\to\infty} \tilde{J}(y;v) = \infty$, and for any $v \in (0,\infty)$, $\lim_{y\downarrow 0} \tilde{J}(y;v) = \infty$.

Proof. See Appendix D.

Finally, the proposition below identifies the logarithmic preferences case of FOC (35) that we use to solve for the optimal exercise rate, $\dot{n}^*(t)$.

Proposition 3. Under logarithmic preferences $U(\cdot) = ln(\cdot)$, which have an Arrow-Pratt relative risk-aversion coefficient of 1, the FOC (35) with respect to the optimal continuous partial exercise rate $\dot{n}(t)$ becomes

$$-\frac{BS(t)-B(t)}{pre\_\mathbb{X}(t)} + ln\left(post\_\mathbb{X}(t)\right) \times \frac{\partial\left(-\left(\frac{\left(n_{option}-\widehat{N}(t)\right)\Phi\left(d_1(t)\right)S_1(t)+n_{stock}1_{t<t_{vs}}S_1(t)}{post\_\mathbb{X}(t)}\right)f_{\widehat{N}}(t)+\frac{1}{2}\left\|\theta_{f_{\widehat{N}}}(t)\right\|^2\right)}{\partial\widehat{N}(t)}=0, \quad (38)$$

where $f_{\widehat{N}}(t) = \text{argmin}_{v\in\widetilde{K}(t,\omega)}\left[-2\frac{\left(n_{option}-\widehat{N}(t)\right)\Phi(d_1)S_1(t)+n_{stock}1_{t<t_{vs}}S_1(t)}{post\_\mathbb{X}(t)}v_1 + \|\sigma^{-1}(t)\times [b(t)-r(t)\mathbf{1}+v]\|^2\right]$,

$\theta_v(t) = \sigma^{-1}(t)\times[b(t)-r(t)\mathbf{1}+v(t)]$, and $\|\cdot\|$ is the Euclidean norm. Moreover, $f_{\widehat{N}}(t) = (f_{1,\widehat{N}}(t),0,\ldots 0)^{\mathrm{T}}$, where $f_{1,\widehat{N}}(t)$ is given by

$$f_{1,\widehat{N}}(t) = \frac{\left(n_{option}-\widehat{N}(s)\right)\Phi(d_1)S_1(s)+n_{stock}1_{t<t_{vs}}S_1(s)}{\hat{A}\times post\_\mathbb{X}(t)} - \frac{1}{\hat{A}}\hat{B},$$

where $\hat{A}$, $\hat{B}$, and $\hat{C}$ are constants given by

$$\hat{A} \equiv \left(\sigma_{11}^{-1}(s)\right)^2 + \left(\sigma_{21}^{-1}(s)\right)^2 + \cdots + \left(\sigma_{d1}^{-1}(s)\right)^2 > 0,$$

$$\hat{B} \equiv \theta_1(s)\sigma_{11}^{-1}(s) + \theta_2(s)\sigma_{21}^{-1}(s) + \cdots + \theta_d(s)\sigma_{d1}^{-1}(s),$$

$$\hat{C} = \left\{\left(\theta_1(s)\right)^2 + \left(\theta_2(s)\right)^2 + \cdots \left(\theta_d(s)\right)^2\right\} > 0,$$

and $\sigma_{ij}^{-1}(s)$ represents the entry in ith row and jth column of the inverse matrix $\sigma^{-1}(t)$.

Proof. The optimization expression for $f_{\widehat{N}}(t)$ is given by Equation (8.23) of Cvitanić and Karatzas (1992). We derive the explicit expression for $f_{1,\widehat{N}}(t)$ in Step 3 of the proof of Theorem 4 in Appendix B. For the rest of the proof, see Appendix F.

3.2 Simulation result and sensitivity analysis

In this section, we explicitly solve a family of examples of NTNH American call option prices under continuous partial exercise policies. While the general ESO pricing framework that we develop here does not restrict the composition of outside wealth, in the following simulations we assume that executives invest their outside wealth in the market index. Not only does this seem to be a natural choice, this has been the choice in the literature. This would facilitate comparison with, for example, Carpenter et al. (2010), who made this assumption and studied how the correlation between the underlying stock and the market affects exercise policy and options' costs.

We compare our results with plain vanilla Black-Scholes prices and study price sensitivity to spot prices, the interest rate, volatility, maturity, instantaneous correlation between primary assets' price processes, drift rates, vesting periods, and the initial endowments of unvested stocks, NTNH American ESOs, and cash. We define "incentives" as the delta of the subjective price $\hat{p}(t)$, i.e., the change in subjective price per unit of share price change, $\partial\hat{p}(t)/\partial S_1(t)$ (see Table 1). The deadweight cost is the discrepancy between objective price and subjective price $\tilde{p}(t) - \hat{p}(t)$ (see Table 2). And we define "efficiency" as the incentive adjusted by the deadweight-cost discount (premium), if deadweight cost positive (negative), $\partial\hat{p}(t)/\partial S_1(t) \times e^{-[\tilde{p}(t)-\hat{p}(t)]}$ (see Tables 1 to 3). In Tables 1 to 3, we present our results under two different assumptions regarding volatility. For each table, Panel A considers a high volatility regime (50% stock volatility, 30% index volatility) and Panel B, a low volatility regime (20% stock volatility, 10% index volatility).

The intuition behind the term incentives is straightforward, as it is the sensitivity of the value to the executive with respect to the share price. We call the difference between the objective price and the subjective one "deadweight cost," as it is a price that the firm pays but the executive does not benefit from. The term "efficiency" is intuitively appealing as it increases in incentives, decreases in deadweight cost, and gives deadweight cost adjusted incentives.

Executives as well as firms will be interested in the following comparative statics. [To simplify exposition and notation, we omit the designation of figure panels below if we refer to both panels. For example, Figure 2(a) means Figure 2.A(a) and Figure 2.B(a).]

1. The objective price is usually greater than the subjective price, and both prices move in the same direction when parameters change. However, surprisingly, we do see cases where the objective price is lower than the subjective price. This case is discussed in Remark 6, above, and is only possible for American ESOs with continuous partial exercise. It appears that the exercise rate provides a sort of hedge for the executive that is not reflected in the objective price. See Table 2 and Figure 2.
2. Figure 2(a) and Figure 2(c) demonstrate that NTNH ESOs induce strong incentives to improve performance. Not only do NTNH ESOs' subjective prices increase as a function of

stock prices, they also increase in the stock drift rates, forming a positive feedback cycle and further intensifying incentives.

3. Figure 2(g) demonstrates, as is well known, that ESOs incentivize the executives to increase the volatility of the firm's stock. It also demonstrates, however, that the marginal subjective price decreases as the firm's stock volatility increases. We find, see Figure 2(a) that the positive stimulation of stock price (return) effect dominates the risk taking side-effect of ESOs.
4. Longer maturity adds value to both subjective and objective prices. See Figure 2(f).
5. Counter-intuitively, a shorter vesting period might result in a lower subjective price, see Figure 2(l). If the option vesting period is shorter, executives exercise ESOs earlier and exercise more ESOs during the options' lives, See Figure 3(l). Because executives maximize the expected utility of final wealth, it could be the case that a shorter vesting period gives executives the opportunity for early ("over") exercise that would decrease the subjective price but would generate cash to invest in the outside wealth that would increase the derived utility.
6. Subjective and objective NTNH ESO prices decrease as the options become a larger proportion of their endowments of total initial wealth (of options, stocks, and cash). See Figure 2(m) to Figure 2(o).
7. Subjective and objective NTNH ESO prices are also affected by the drift rate and volatility of the market index. As the index's drift rate increases, the subjective and objective prices decrease. Similarly, as the index's volatility decreases, these prices tend to decrease; although, as the index volatility becomes larger than the stock volatility, this relationship can break down. See Table 2. Intuitively, as the index becomes more attractive, i.e., as its drift rate increases or its volatility decreases (all else being equal), the position in the ESO becomes less desirable. In fact, for the high volatility regime, the subjective price is more sensitive to the volatility of the index than it is to the volatility of the stock. See Figure 2(d) and Fig 2(h).
8. As the correlation between returns on the underlying stock and returns on the market index increases, the subjective and objective prices increase. For example, as the correlation approaches one, the call option on the stock effectively becomes a call option on the market index, which would be quite desirable. On the other hand, when the correlation is low, the ESO gives the executive a concentration of wealth in an asset that is "far from" the optimal market index. See Figure 2(b).
9. We argue that if the subjective price of the ESO increases (all else being equal), then the firm can grant fewer ESOs to the executives. So, for example, following the insights of the previous two points, if there is a decrease in the drift rate of the market index, then the

subjective price increases, and the firms can grant fewer ESOs. Similar conclusions hold for the other variables.

10. Efficiency decreases in vesting period, stock volatility, and the proportion of executives' option endowment of total wealth. See Table 2.
11. Executives, on average, exercise more shares of deeper in-the-money options. See Figure 3(a).
12. Lower stock drift rates induce executives to exercise NTNH ESOs sooner. See Figure 3(c).
13. A higher index drift rate induces an executive to exercise the NTNH ESOs more quickly. Intuitively, if the index is more "attractive" (all else being equal), the executive exercises the options in order to invest more money in the index. See Figure 3(d).
14. Low correlation between returns on the index and returns on the stock leads to the ESOs being exercised more quickly. See Figure 3(b). The rough intuition here is that a low correlation makes the ESO less "attractive" (all else being equal), causing executives to exercise them more quickly in order to invest in the index.
15. Under high volatility regime, higher risk-free rates induce higher exercise rates. See Figure 3.A(e). In contrast, under low volatility regimes, higher risk-free rates induce lower exercise rates. See Figure 3.B(e).
16. Under high volatility regime, shorter maturities induce executives to exercise shorter-term ESOs at a higher rate. However, granted with the same number of short term or long-term ESOs, executives granted with the long-term ones, eventually exercise more ESOs during the options life. See Figure 3.A(f). In contrast, under low volatility regime shorter maturities induce extremely low exercise rates and longer maturities induce higher ones. See Figure 3.B(f).
17. Higher stock volatility induces, on average, earlier exercise. See Figure 3.B(g).
18. Low index volatility induces, on average, earlier exercise. Intuitively, if the index is more attractive (e.g., has lower volatility), then executives prefer not to hold on to their ESOs. See Figure 3(h).
19. High strike price options are less likely to be in-the-money thus are less likely to be exercised. See Figure Figure 3.B(i). However, counter-intuitively, for a high volatility regime, high strike price ESOs are more likely to be exercised, see Figure 3.A(i). The reason is that a higher (lower) strike price induces a greater spread between the Black-Scholes price and the subjective price, see Figure 2.A(i), [Figure 2.B(i)], which can be understood as a greater *equivalent dividend yield* associated with the unconstrained plain vanilla call option. A higher dividend yield implies a higher rate of early exercise. Indeed, examining the relationship between spread and exercise rates, we find this result.
20. Both subjective and objective prices decrease and optimal exercise rate increase in the

index drift rate. See Figure 2(d) and Figure 3(d). The economic intuition is straightforward. A higher index drift rate makes switching from the ESOs to the index investment more attractive. As expected, the index price has no effect on ESO prices and optimal policies. The level of the index does not affect the dollar amount invested in the index, and so does not affect the subjective and objective prices. Only its expected rate of return, as expressed by its drift rate, (and its volatility and correlation) matters. See Figure 2(j) and Figure 3(j).

21. Lower proportions of option endowments out of initial total wealth, induce, on average, the exercise of fewer options. In this case, the subjective price is high, closer to the Black-Scholes price, and at that price the option would not be exercised early. See Figure 3(m) to Figure 3(o).
22. As the frequency of allowed exercise dates increases, the subjective and objective prices increase under a low volatility regime. Surprisingly, they decrease under a high volatility regime, an additional occurrence of the dominating effect of the ability to improve expected utility when the effect of the NTNH constraints is mitigated. See Table 3.

4 Conclusion

We identify a method for pricing non-transferable, non-hedgeable (NTNH) European and American call options under both block and continuous partial exercise. We implement this methodology to price executive stock options (ESOs) and run a sensitivity analysis that suggests efficiency-enhancing ESO policy implications. Our pricing methodology is simple and universal. It can be used to solve a category of pricing problems, including pensions, human capital, real estate, and intellectual property.

While some of the comparative statics results resemble those of vanilla options, the NTNH constraints induce a number of hard to predict, counter-intuitive results. A core issue is the incompleteness of the market with respect to the NTNH ESOs, which cause their values to be directly affected by executives' other wealth and by the index returns process. Executives face a continuous choice between holding on to the constrained NTNH ESOs wealth and substituting it with unconstrained wealth.

As expected, NTNH ESOs provide incentives for executives to increase stock prices. In addition, and contrary to the case of vanilla options, NTNH ESOs provide incentives to increase stock drift rates. Under high index and stock volatilities, ESOs incentivize executives to increase stock volatility. However, marginal volatility incentives decrease in volatility and are overwhelmed by incentives to increase stock prices and drift rates. Under low index and stock volatilities, on the other hand, a sufficient increase in stock volatility might result in negative marginal volatility incentives; that is, ESOs incentivize a reduction in stock volatility. Implications are that ESOs' incentives are not independent from the volatility regimes of the index and stock.

The larger the proportion of NTNH ESOs to other wealth, the heavier is the constraints burden, implying lower subjective prices and higher exercise rates. NTNH ESOs' subjective prices are not only directly sensitive to index returns, they could be more sensitive to index volatility and drift rates than to those of the firms' stock. In the same spirit, NTNH ESOs' subjective prices increase, and exercise rates decrease, in the correlation between firms' stock returns and the index returns. Moreover, the more attractive[9] (unattractive) investment in the index (firm's stock) is, the higher the exercise rate is.

Efficiency (dead weight cost adjusted incentive), interestingly, decreases in the proportion of executive's option endowment of total wealth. As intuition suggests it also decreases in vesting periods of both stock and options but only slightly under low volatility regime.

We also find intuitive results for the low volatility regime and counter-intuitive results for the high volatility one. i) Counter-intuitively, shorter vesting periods for ESOs and stocks could decrease the subjective price and increase exercise rates (when other wealth contributes more to the expected utility) under high volatility regime. This effect is minute under a low volatility regime. ii) High strike prices of NTNH ESOs are counter-intuitively more (intuitively less) likely to be exercised under a high (low) volatility regime. iii) Intuitively, the higher the frequency of allowed exercise dates, the higher are subjective prices under a low volatility regime. Counter-intuitively, however, under a high volatility regime, the higher the frequency of allowed exercise dates, the lower are subjective prices. The counter-intuitive results are driven by the substitution effect with respect to other wealth. The burden of the constraints may be represented by a dividend yield paid by the firm's stock. Higher dividend yields induce higher price reductions and higher exercise rates.

Future work might investigate executives hedging ESOs using primary assets that are correlated with the ESOs' underlying assets. While the benefit of such hedging is clear, there are costs, both in terms of lower performance relative to that of unrestricted portfolios and in terms of additional transaction costs. These costs clearly depend on the level of the executive's outside wealth relative to the ESOs' value, and an interesting problem is to identify the value of outside wealth that is required to mitigate the non-hedgeability consequences.

We could also price NTNH ESOs accounting for executives' job termination. This would be comparable to pricing default NTNH-contingent claims with the credit name being the executives' job termination. We could also apply our NTNH ESO valuation method to more complex ESO pricing; for example, NTNH reload ESOs. Upon reload, executives receive additional shares at market value equal to the current intrinsic value of (all) their ESOs and

[9] Higher drift rate, lower volatility.

additionally are re-granted new at-the-money NTNH ESOs of the same maturity as the original ones.

Table 1

Incentive of ESO

Panel A: High volatility regime (50% stock volatility, 30% index volatility)

Spot price (stock)	Black-Scholes price	Subjective price	Objective price	Incentive	Dead weight cost	Efficiency
8	5.82	4.75	5.07	0.78	0.32	0.56
10	7.59	6.30	6.76	0.79	0.46	0.50
12	9.41	7.92	8.51	0.81	0.59	0.45

Panel B: Low volatility regime (20% stock volatility, 10% index volatility)

Spot price (stock)	Black-Scholes price	Subjective price	Objective price	Incentive	Dead weight cost	Efficiency
8	3.51	3.38	3.36	0.74	-0.14	0.75
10	5.20	4.85	4.83	0.75	-0.01	0.76
12	7.00	6.50	6.34	0.83	-0.16	0.97

Panel A (B) of this table displays the Black-Scholes prices, subjective prices, and objective prices of NTNH ESO for both high (low) volatility benchmark regime. The total portfolio includes ESO, restricted firm's stock, and market index. The benchmark parameter values are as follows: for high volatility regime, Stock volatility=50%, Index volatility=30%; for low volatility regime, Stock volatility=20%, Index volatility=10%; All the other paramters for two regimes are the same: Maturity=15 years. Spot price of stock=$10; Spot price of index=$6; Strike price=$10; Correlation between stock and index=60%; Drift rate of stock=15%; Drift rate of index=8%; Risk-free rate=4%; Initial option endowment=200 shares; Initial stock endowment=200 shares; Initial cash endowment=$1000; Vesting period of firm's stock=1 year; Vesting period of option=2 years; Number of steps=30; Monte Carlo repeating times=10000. The Benchmark Black-Scholes price, subjective price, and objective price of NTNH ESO are 7.59, 6.49 and 6.95, respectively (see the middle row with Spot stock price being $10). By changing the spot price of stock to a lower (higher) level $8 ($12), we investigate the incentive of option $\partial\hat{p}(t)/\partial S(t)$, which is defined as the delta of the subjective price.

Table 2

Price sensitivity analysis and determinants of ESO's efficiency

Panel A: High volatility regime (50% stock volatility, 30% index volatility)

	Black-Scholes price	Subjective price	Objective price	Incentive	Dead weight cost	Efficiency
Correlation						
30%	7.59	5.82	6.49	0.79	0.67	0.40
60%	7.59	6.29	6.75	0.79	0.46	0.50
90%	7.59	7.33	7.33	0.79	0.00	0.79
Drift rate (stock)						
8%	7.59	2.96	4.06	0.79	1.10	0.26
15%	7.59	6.29	6.75	0.79	0.46	0.50
20%	7.59	7.28	7.39	0.79	0.11	0.71
Drift rate (index)						
6%	7.59	6.70	7.03	0.79	0.33	0.57
8%	7.59	6.29	6.75	0.79	0.46	0.50
20%	7.59	2.80	3.56	0.79	0.76	0.37
Risk -free rate						
2%	7.15	6.03	6.46	0.79	0.43	0.51
4%	7.59	6.29	6.75	0.79	0.46	0.50
6%	7.99	6.46	6.95	0.79	0.49	0.48
Maturity						
5	4.82	4.03	4.53	0.79	0.50	0.48
15	7.59	6.29	6.75	0.79	0.46	0.50
25	8.78	7.14	7.79	0.79	0.65	0.41
Volatility (stock)						
0.2	5.20	4.39	4.58	0.79	0.19	0.65
0.5	7.59	6.29	6.75	0.79	0.46	0.50
0.9	9.40	6.30	8.33	0.79	2.03	0.10
Volatility (index)						
0.1	7.59	2.75	3.90	0.79	1.15	0.25
0.3	7.59	6.29	6.75	0.79	0.46	0.50
0.6	7.59	6.69	7.03	0.79	0.33	0.57
Strike price						
8	7.89	6.79	7.26	0.79	0.47	0.49
10	7.59	6.29	6.75	0.79	0.46	0.50
12	7.33	5.83	6.29	0.79	0.46	0.50
Spot price (index)						
2	7.59	6.29	6.75	0.79	0.46	0.50
6	7.59	6.29	6.75	0.79	0.46	0.50
10	7.59	6.29	6.75	0.79	0.46	0.50
Vesting period						
0.5	7.59	6.48	6.75	0.79	0.27	0.60
1	7.59	6.29	6.75	0.79	0.46	0.50
1.5	7.59	6.10	6.75	0.79	0.65	0.41
Vesting period						
1	7.59	6.18	6.61	0.79	0.43	0.51
2	7.59	6.29	6.75	0.79	0.46	0.50
3	7.59	6.38	6.88	0.79	0.50	0.48
Initial						
20	7.59	7.35	7.59	0.79	0.24	0.62
200	7.59	6.29	6.75	0.79	0.46	0.50
2000	7.59	2.85	4.35	0.79	1.51	0.17
Initial						
20	7.59	4.51	5.23	0.79	0.72	0.38
200	7.59	6.29	6.75	0.79	0.46	0.50
2000	7.59	7.25	7.59	0.79	0.35	0.56
Initial						
500	7.59	5.84	6.42	0.79	0.58	0.44
1000	7.59	6.29	6.75	0.79	0.46	0.50
10000	7.59	7.59	7.59	0.79	0.00	0.79

Panel B: Low volatility regime (20% stock volatility, 10% index volatility)

	Black-Scholes price	Subjective price	Objective price	Incentive	Dead weight cost	Efficiency
Correlation						
30%	5.20	4.65	4.45	0.75	-0.20	0.91
60%	5.20	4.85	4.83	0.75	-0.01	0.76
90%	5.20	4.93	4.85	0.75	-0.08	0.81
Drift rate (stock)						
8%	5.20	2.67	3.16	0.75	0.49	0.46
15%	5.20	4.85	4.83	0.75	-0.01	0.76
20%	5.20	5.46	3.71	0.75	-1.75	4.30
Drift rate (index)						
6%	5.20	5.14	4.77	0.75	-0.37	1.08
8%	5.20	4.85	4.83	0.75	-0.01	0.76
20%	5.20	1.32	1.37	0.75	0.05	0.72
Risk -free rate						
2%	4.10	3.64	3.28	0.75	-0.35	1.07
4%	5.20	4.85	4.83	0.75	-0.01	0.76
6%	6.22	6.14	6.13	0.75	-0.00	0.75
Maturity						
5	2.67	2.69	2.29	0.75	-0.40	1.12
15	5.20	4.85	4.83	0.75	-0.01	0.76
25	6.79	7.36	5.20	0.75	-2.16	6.49
Volatility (stock)						
0.1	4.59	1.71	1.94	0.75	0.24	0.59
0.2	5.20	4.85	4.83	0.75	-0.01	0.76
0.5	7.59	2.88	3.95	0.75	1.07	0.26
Volatility (index)						
0.05	5.20	3.09	2.84	0.75	-0.24	0.96
0.1	5.20	4.85	4.83	0.75	-0.01	0.76
0.3	5.20	4.64	4.53	0.75	-0.10	0.83
Strike price						
8	5.97	5.57	5.43	0.75	-0.14	0.87
10	5.20	4.85	4.83	0.75	-0.01	0.76
12	4.54	4.46	4.31	0.75	-0.15	0.87
Spot price (index)						
2	5.20	4.85	4.83	0.75	-0.01	0.76
6	5.20	4.85	4.83	0.75	-0.01	0.76
10	5.20	4.85	4.83	0.75	-0.01	0.76
Vesting period (stock)						
0.5	5.20	4.85	4.84	0.75	-0.01	0.76
1	5.20	4.85	4.83	0.75	-0.01	0.76
1.5	5.20	4.84	4.84	0.75	-0.01	0.75
Vesting period (option)						
1	5.20	4.85	4.83	0.75	-0.01	0.76
2	5.20	4.85	4.83	0.75	-0.01	0.76
3	5.20	4.85	4.83	0.75	-0.01	0.76
Initial endowment (option)						
20	5.20	5.18	5.17	0.75	-0.02	0.76
200	5.20	4.85	4.83	0.75	-0.01	0.76
2000	5.20	3.48	3.59	0.75	0.11	0.67
Initial endowment (stock)						
20	5.20	4.28	4.34	0.75	0.06	0.70
200	5.20	4.85	4.83	0.75	-0.01	0.76
2000	5.20	5.16	5.15	0.75	-0.01	0.76
Initial endowment (cash)						
500	5.20	4.76	4.75	0.75	-0.01	0.76
1000	5.20	4.85	4.83	0.75	-0.01	0.76
10000	5.20	5.12	5.12	0.75	0.00	0.75

Panel A (B) of this table lists the results of ESO price sensitivity analysis for high (low) volatility regime; namely, how does parameter change affect the change of ESO prices? The table also displays all the determinants affecting the ESO's efficiency, which is defined as the deadweight cost adjusted incentive. The deadweight cost is the objective price net of the subjective price. Each sub-panel lists three levels (low, benchmark, high) of results from above to below. We use 0.79 (0.75), which is the benchmark incentive in Panel A (B) of Table 1, throughout the sensitivity analysis for high (low) volatility regime.

Table 3

Comparison between incentives with continuous partial exercise policies and discrete partial exercise policies

Panel A: High volatility regime (50% stock volatility, 30% index volatility)

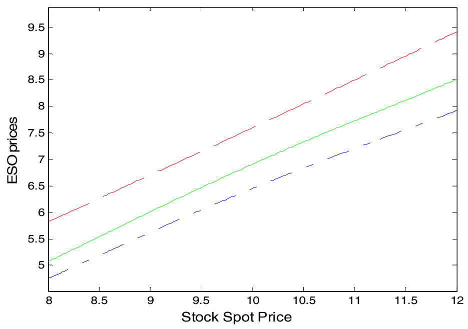
Spot price (stock)	Black-Scholes price	Subjective price			Objective price			Efficiency		
		$dt = 0.25$	$dt = 0.5$	$dt = 1$	$dt = 0.25$	$dt = 0.5$	$dt = 1$	$dt = 0.25$	$dt = 0.5$	$dt = 1$
8	5.82	4.73	4.75	4.82	5.03	5.07	5.13	0.58	0.56	0.56
10	7.59	6.29	6.30	6.36	6.72	6.76	6.80	0.52	0.50	0.51
12	9.41	7.91	7.92	7.98	8.47	8.51	8.55	0.46	0.45	0.46

Panel B: Low volatility regime (20% stock volatility, 10% index volatility)

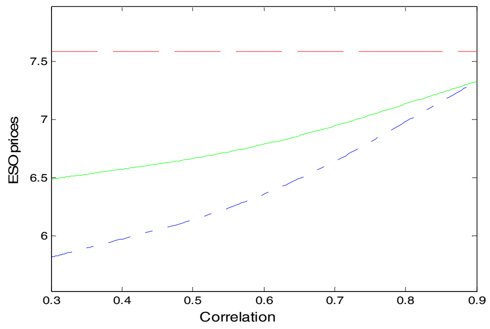
Spot price (stock)	Black-Scholes price	Subjective price			Objective price			Efficiency		
		$dt = 0.25$	$dt = 0.5$	$dt = 1$	$dt = 0.25$	$dt = 0.5$	$dt = 1$	$dt = 0.25$	$dt = 0.5$	$dt = 1$
8	3.51	3.46	3.38	3.41	3.43	3.36	3.28	0.76	0.75	0.76
10	5.20	4.94	4.85	4.74	4.93	4.83	4.72	0.78	0.76	0.72
12	7.00	6.55	6.50	6.23	6.43	6.34	6.27	0.91	0.97	0.72

This table compares the benchmark Black-Scholes price with the subjective and objective prices of NTNH ESO at different level of exercise frequency. This reflects the difference between the incentives under continuous partial exercise policies and discrete partial exercise policies. We employ the same parameter values as in Table 1, and set three levels (low, benchmark, high) of exercise frequency, dt $(= 0.25, 0.5, 1)$, where $dt = T/N, T(= 15$ years), is the ESO maturity, and $N(= 60, 30, 15)$ represents the Number of steps of Monte Carlo simulation for one sample path.

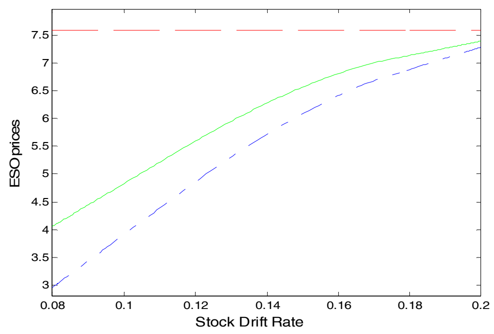
(a) Price sensitivity w.r.t spot price of stock



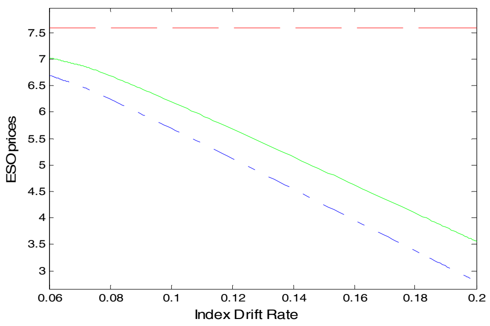
(b) Price sensitivity w.r.t correlation



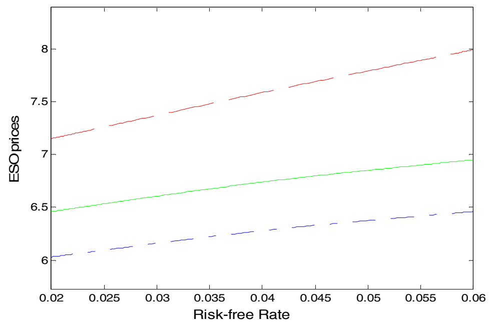
(c) Price sensitivity w.r.t drift rate of stock



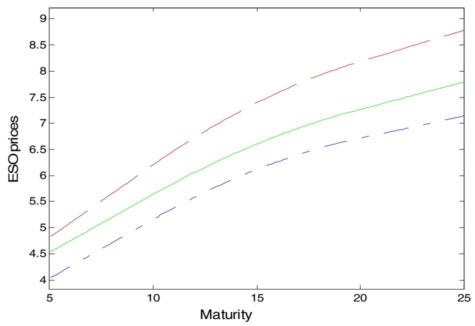
(d) Price sensitivity w.r.t drift rate of index



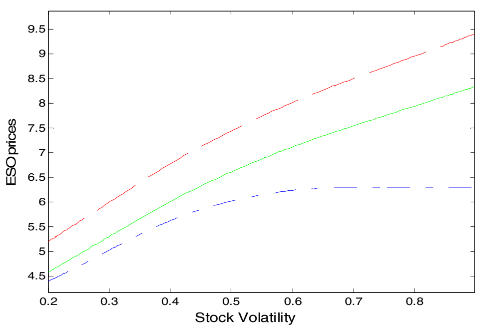
(e) Price sensitivity w.r.t risk-free rate



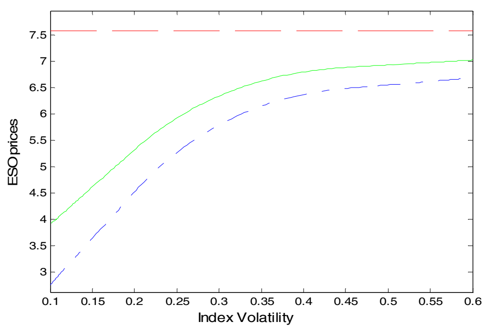
(f) Price sensitivity w.r.t maturity



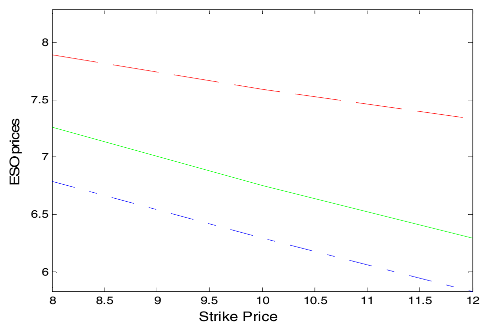
(g) Price sensitivity w.r.t stock volatility



(h) Price sensitivity w.r.t index volatility



(i) Price sensitivity w.r.t strike price



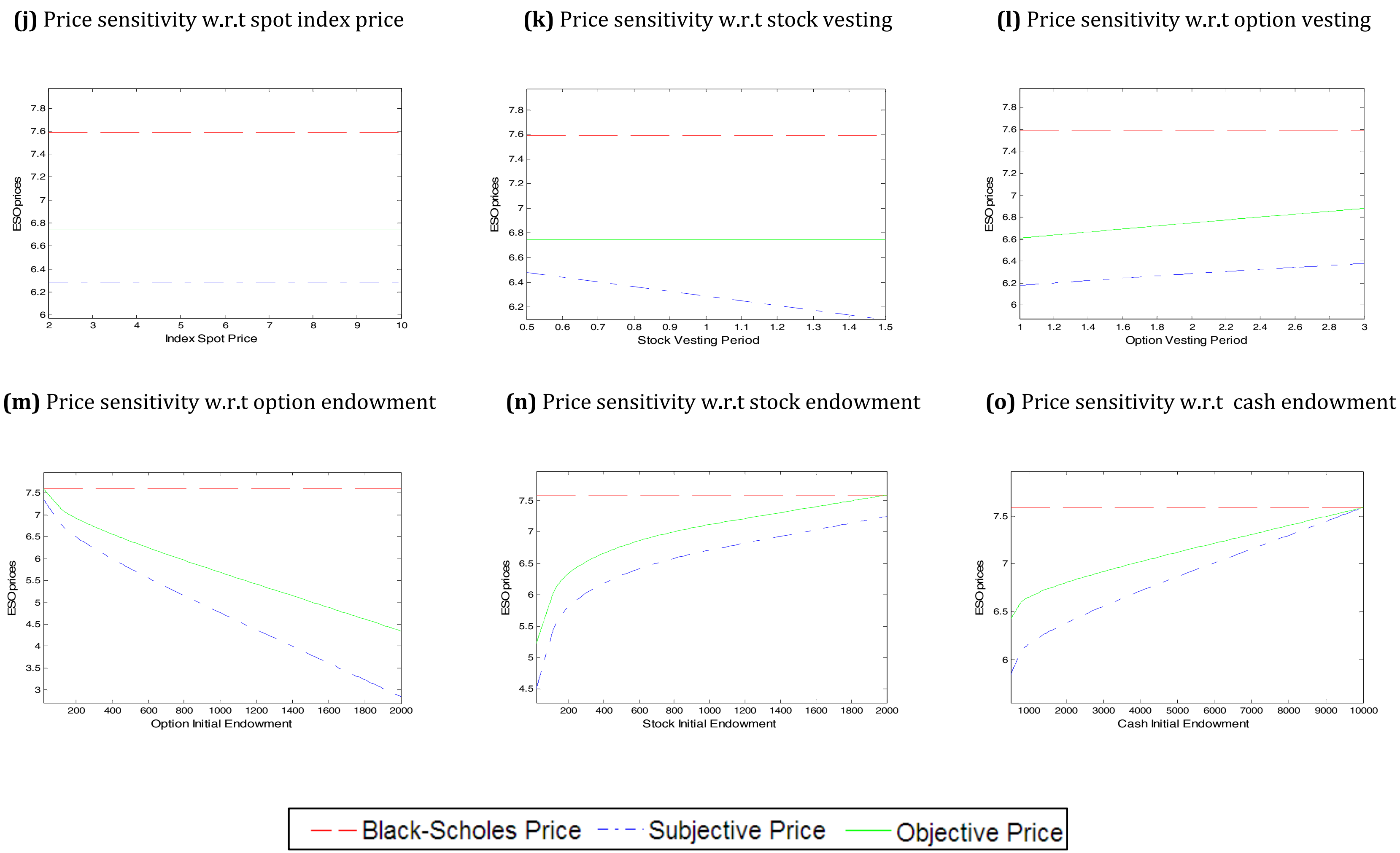
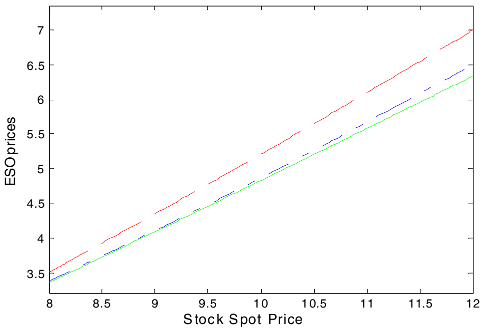
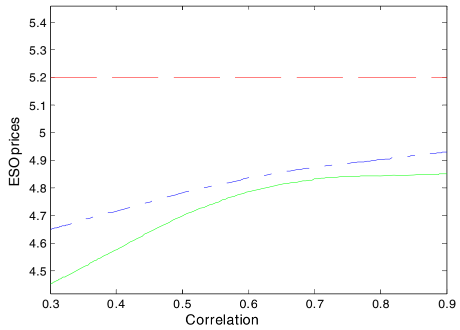


Figure 2. Price sensitivity. Panel A: High volatility regime (50% stock volatility, 30% index volatility). This figure displays the patterns of how parameters' changes affect ESO price changes (i.e., Black-Scholes, subjective, and objective prices). Each curve interpolates three parameter values specified in Panel A in Table 1 and Table 2, low level, benchmark level, and high level.

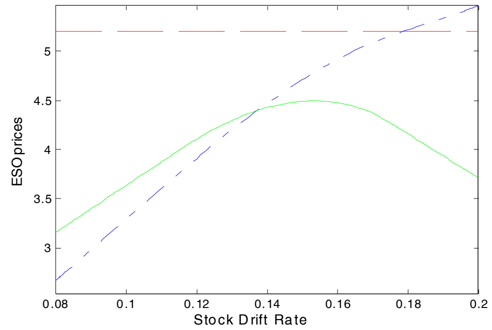
(a) Price sensitivity w.r.t spot price of stock



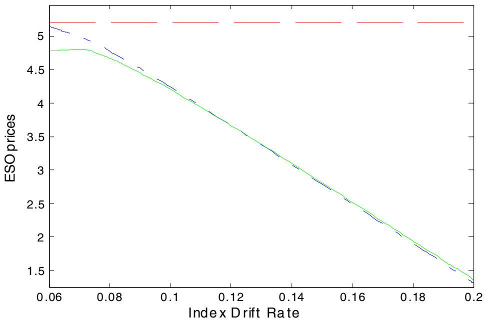
(b) Price sensitivity w.r.t correlation



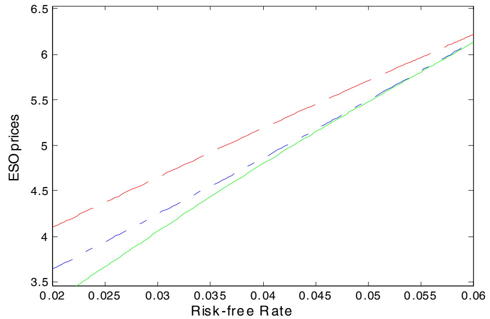
(c) Price sensitivity w.r.t drift rate of stock



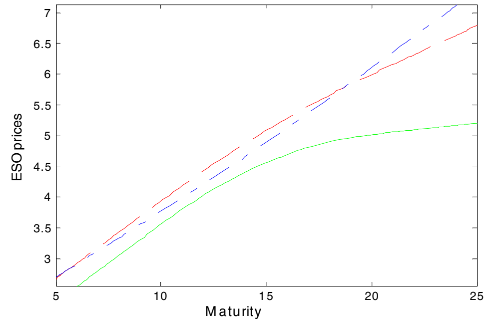
(d) Price sensitivity w.r.t drift rate of index



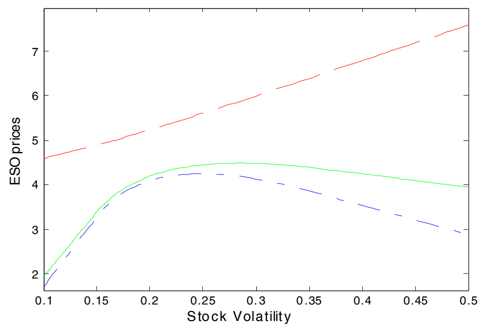
(e) Price sensitivity w.r.t risk-free rate



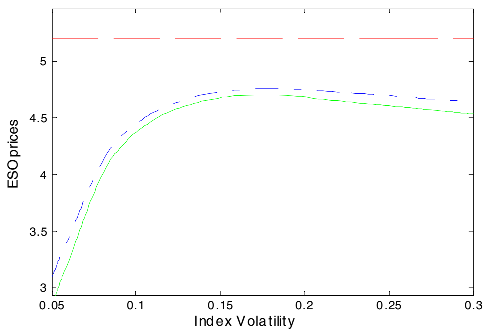
(f) Price sensitivity w.r.t maturity



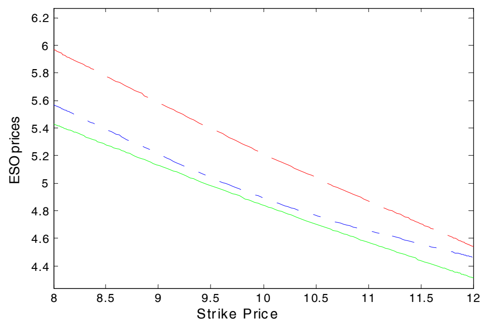
(g) Price sensitivity w.r.t stock volatility



(h) Price sensitivity w.r.t index volatility



(i) Price sensitivity w.r.t strike price



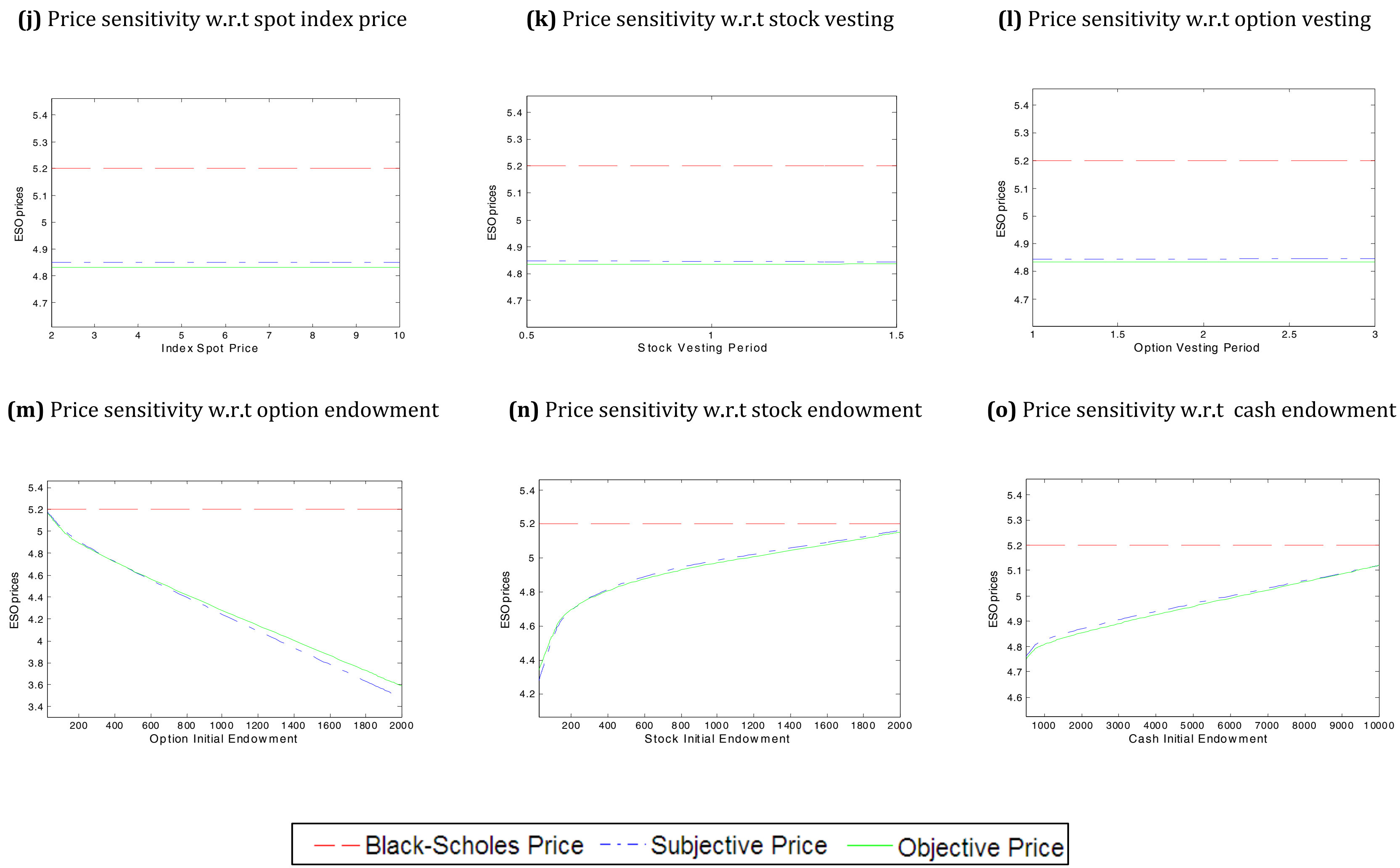
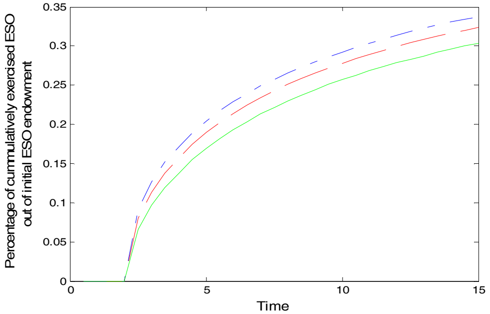
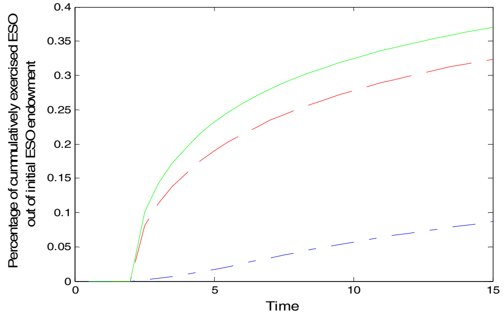


Figure 2. Price sensitivity. Panel B: Low volatility regime (20% stock volatility, 10% index volatility). This figure displays the patterns of how parameters' changes affect ESO price changes (i.e., Black-Scholes, subjective, and objective prices). Each curve interpolates three parameter values specified Panel B in Table 1 and Table 2, low level, benchmark level, and high level.

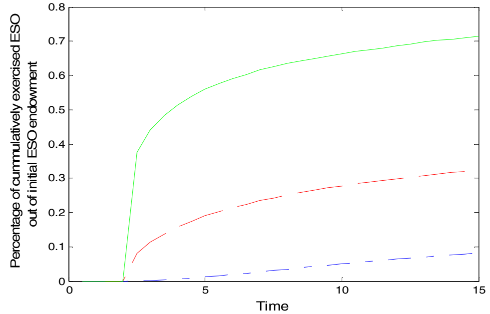
(a) Stock spot price



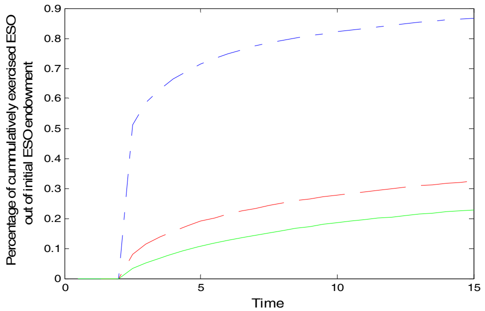
(b) Correlation



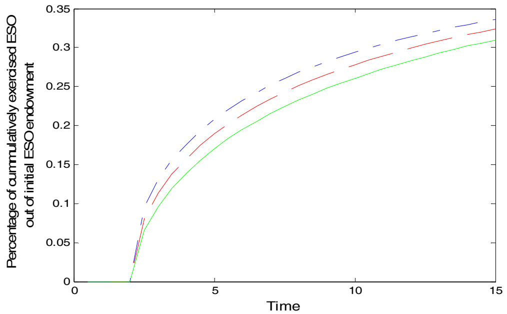
(c) Stock drift rate



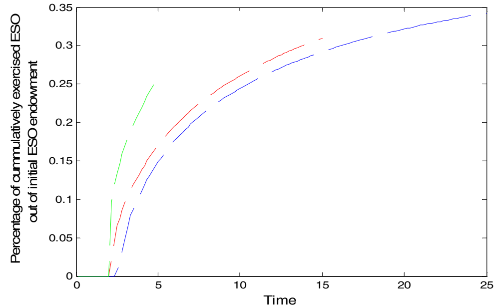
(d) Index drift rate



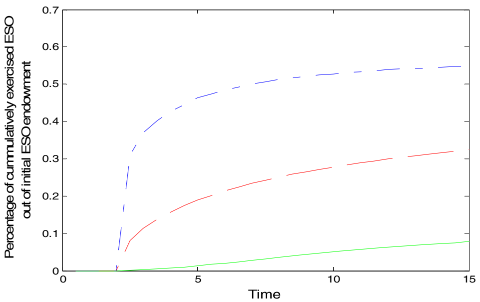
(e) Risk-free rate



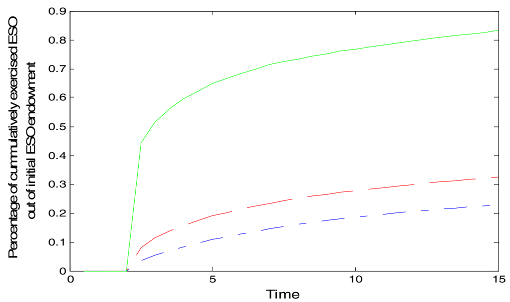
(f) Maturity



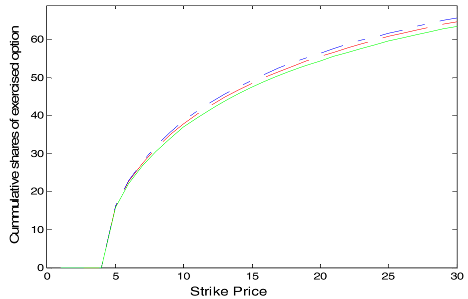
(g) Stock volatility



(h) Index volatility



(i) Strike price



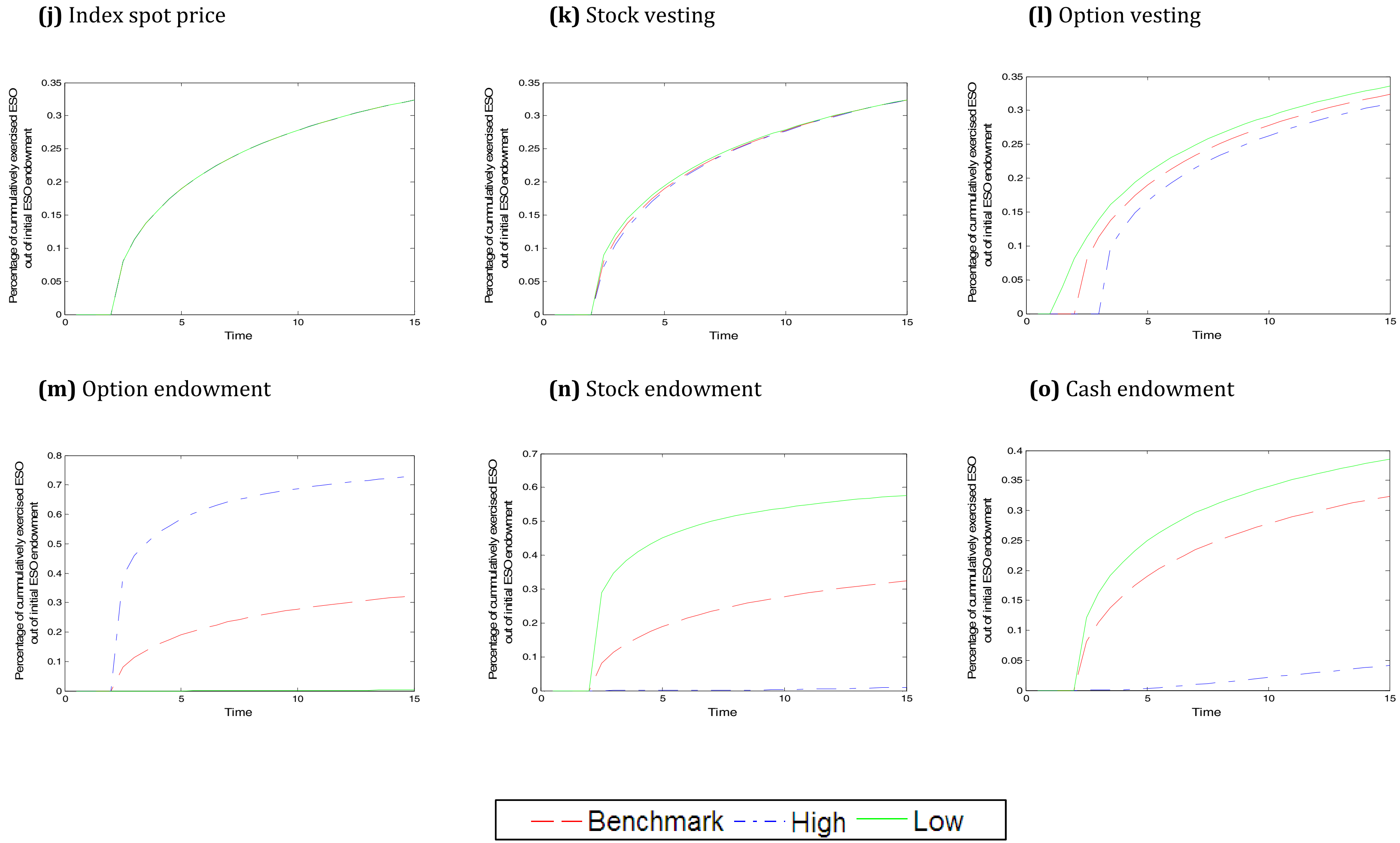
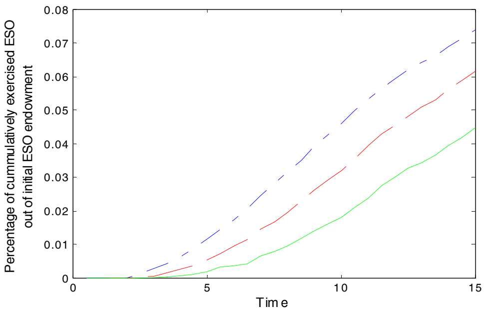
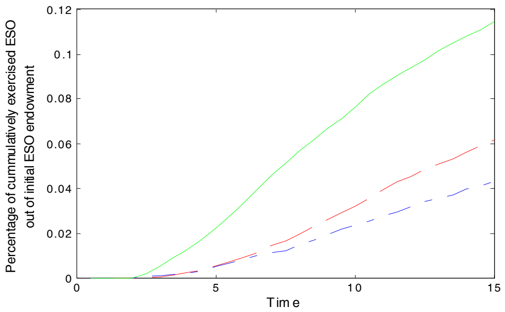


Figure 3. Exercise policies. Panel A: High volatility regime (50% stock volatility, 30% index volatility). This figure displays how different levels (Benchmark, High, Low) of the parameters affect the average exercise policies throughout the option's life. The *Y* axis represents the percentage of the averaged cumulative numbers of exercised options over 10,000 Monte Carlo sample paths out of the initial ESO endowment. The *X* axis represents running time in years with maturity of 15, except for in Figure 3.A(f), where maturity is the changing parameter.

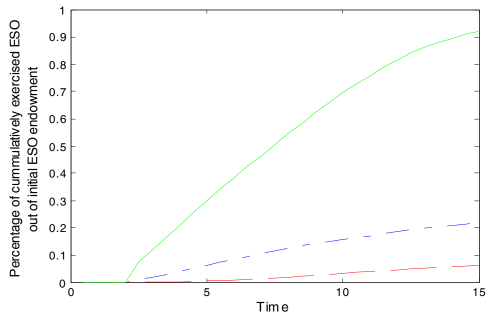
(a) Stock spot price



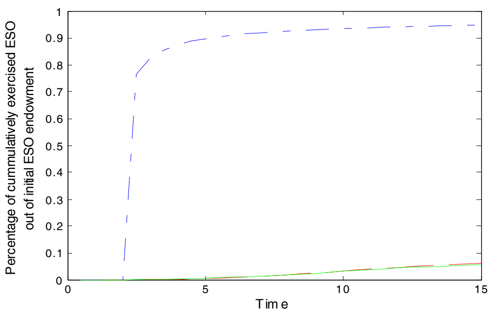
(b) Correlation



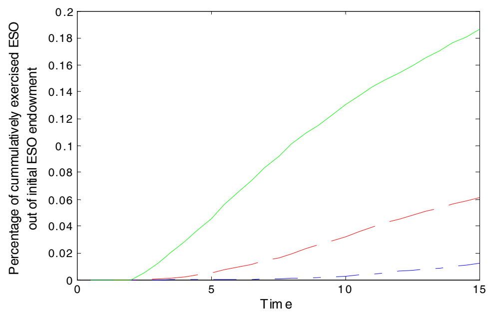
(c) Stock drift rate



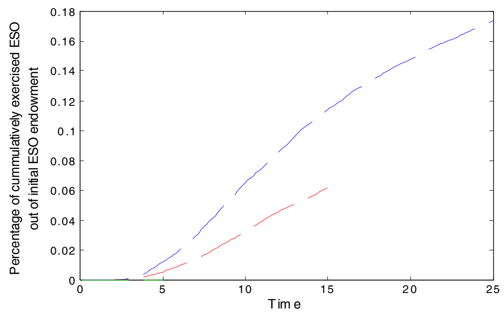
(d) Index drift rate



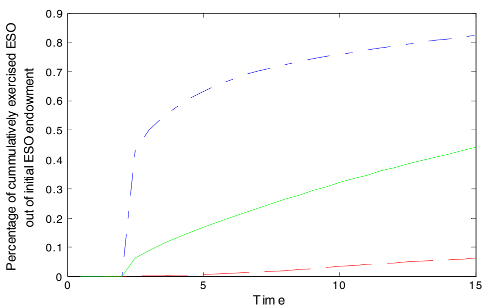
(e) Risk-free rate



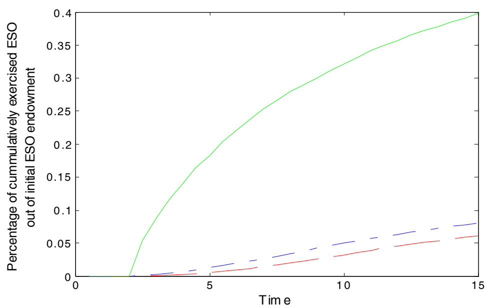
(f) Maturity



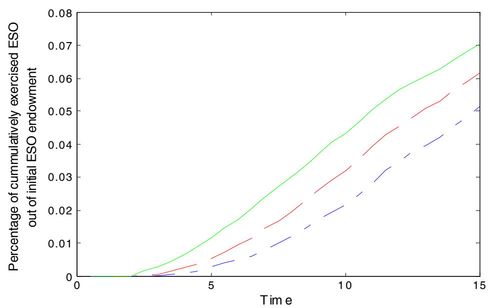
(g) Stock volatility



(h) Index volatility



(i) Strike price



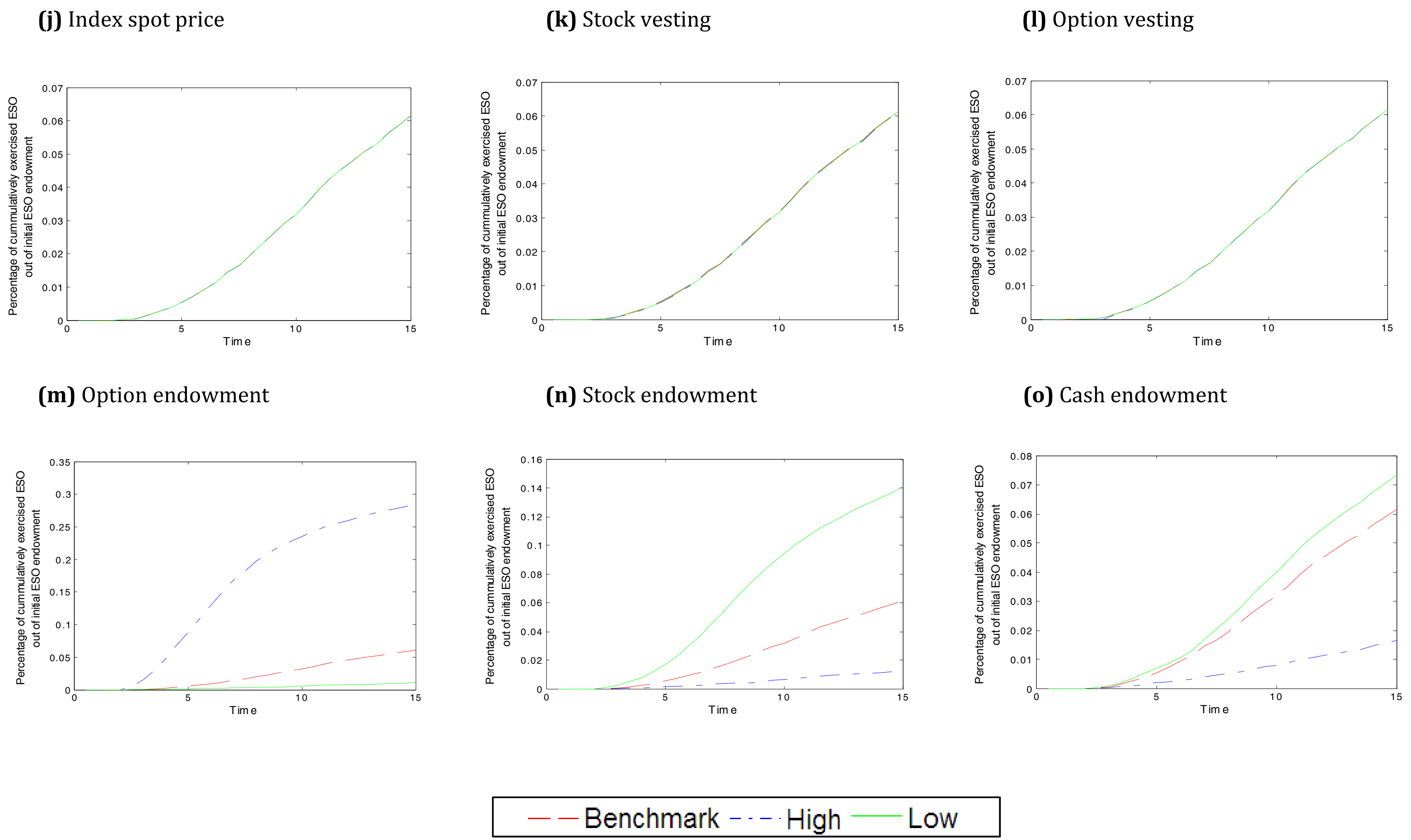


Figure 3. Exercise policies. Panel B: Low volatility regime (20% stock volatility, 10% index volatility). This figure displays how different levels (Benchmark, High, Low) of the parameters affect the average exercise policies throughout the option's life. The *Y* axis represents the percentage of the averaged cumulative numbers of exercised options over 10,000 Monte Carlo sample paths out of the initial ESO endowment. The *X* axis represents running time in years with maturity of 15, except for in Figure 4.B(f), where maturity is the changing parameter.

Appendix A. Proofs of Theorem 1 and Theorem 2

A.1. Proof of Theorem 1

The utility function $U\colon (0,\infty) \to \Re$ is strictly increasing, and strictly concave, of class C^1 and satisfies, $U'(0^+) \triangleq lim_{\varkappa \downarrow 0} U'(\varkappa) = \infty$, $U'(\infty) \triangleq lim_{\varkappa \to \infty} U'(\varkappa) = 0$.

The following inequality holds due to the concavity of the utility function:

$$U(\varkappa) + (\wp - \varkappa)U'(\varkappa) \geq U(\wp) \geq U(\varkappa) + (\wp - \varkappa)U'(\wp), \qquad \forall\, 0 < \varkappa < \wp < \infty. \tag{A.1}$$

For any initial outside wealth ς, we introduce the models,

$$M(\varsigma, 0) \triangleq \sup_{\pi \in \mathcal{A}(\varsigma, K(0,\omega), 0, T)} \mathbb{E}[U(X^{\varsigma,\pi}(T)], \qquad 0 < \varsigma < \infty \tag{A.2}$$

$$\breve{M}(\varsigma, n, 0) \triangleq \sup_{\pi \in \mathcal{A}(\varsigma, K(0,\omega), 0, T)} \mathbb{E}[U(X^{\varsigma,\pi}(T) + nB(T)], \qquad 0 < \varsigma < \infty \tag{A.3}$$

$$\Psi(\alpha, \hat{p}, \varsigma) \triangleq \sup_{\pi \in \mathcal{A}(\varsigma - \alpha, K(0,\omega), 0, T)} \mathbb{E}\left[U\left(X^{\varsigma - \alpha, \pi}(T) + \frac{\alpha}{\hat{p}} B(T)\right)\right], \qquad 0 < \varsigma < \infty, \tag{A.4}$$

where $\mathcal{A}(x, K(t,\omega), t, T)$ is defined in Equation (5). For any $\epsilon > 0$, write $\alpha = n\hat{p} - \epsilon$, then

$$\Psi(\alpha, \hat{p}, \varsigma) \geq \mathbb{E}\left[U\left(X_*^{\varsigma - \alpha}(T) + nB(T) - \tfrac{\epsilon}{\hat{p}} B(T)\right)\right],$$

where $X_*^{\varsigma-\alpha}(T) \equiv X^{\varsigma - \alpha, \pi^*}(T)$ and π^* solves $\breve{M}(\varsigma - \alpha, n, 0)$. Here we abandon $\frac{\epsilon}{\hat{p}}$ shares of the option in exchange for ϵ dollars in cash.

Let the initial outside wealth be $\varsigma = n\hat{p} + x$. Then, thanks to the first inequality in (A.1), we have

$$\begin{aligned}\mathbb{E}&\left[U\left(X_*^{\varsigma-\alpha}(T) + nB(T) - \frac{\epsilon}{\hat{p}} B(T)\right)\right] \\ &\geq \mathbb{E}\left[U\big(X_*^{\varsigma-\alpha}(T) + nB(T)\big) - \frac{\epsilon}{\hat{p}} B(T) U'\left(X_*^{\varsigma-\alpha}(T) + \frac{n\hat{p} - \epsilon}{\hat{p}} B(T)\right)\right].\end{aligned} \tag{A.5}$$

Since $\varsigma - \alpha = x + \epsilon \geq x$ and $\varsigma \mapsto X_*^{\varsigma}(T)$ is non-decreasing, we get

$$\Psi(\alpha, \hat{p}, \varsigma) \geq \mathbb{E}\big[U\big(X_*^{\varsigma-\alpha}(T) + nB(T)\big)\big] - \frac{\epsilon}{\hat{p}} \mathbb{E}\left[U'\left(X_*^{x}(T) + \frac{n\hat{p} - \epsilon}{\hat{p}} B(T)\right) B(T)\right].$$

By the definition of $\breve{M}(\varsigma - \alpha, n, 0)$,

$$\Psi(\alpha, \hat{p}, \varsigma) \geq \breve{M}(\varsigma - \alpha, n, 0) - \frac{\epsilon}{\hat{p}} \mathbb{E}\left[U'\left(X_*^{x}(T) + \frac{n\hat{p} - \epsilon}{\hat{p}} B(T)\right) B(T)\right].$$

Also, $\Psi(n\hat{p}, \hat{p}, x) = \breve{M}(\varsigma - n\hat{p}, n, 0)$. So,

$$-\Psi(\alpha, \hat{p}, \varsigma) \leq -\breve{M}(\varsigma - \alpha, n, 0) + \tfrac{\epsilon}{\hat{p}} \mathbb{E}\left[U'\left(X_*^{x}(T) + \tfrac{n\hat{p} - \epsilon}{\hat{p}} B(T)\right) B(T)\right], \text{ and}$$

$$\Psi(n\hat{p}, \hat{p}, x) - \Psi(\alpha, \hat{p}, \varsigma) \leq \breve{M}(\varsigma - n\hat{p}, n, 0) - \breve{M}(\varsigma - \alpha, n, 0) + \tfrac{\epsilon}{\hat{p}} \mathbb{E}\left[U'\left(X_*^{x}(T) + \tfrac{n\hat{p} - \epsilon}{\hat{p}} B(T)\right) B(T)\right].$$

So, for $n\hat{p} > \alpha$,

$$\frac{\Psi(n\hat{p},\hat{p},x)-\Psi(\alpha,\hat{p},\varsigma)}{n\hat{p}-\alpha} \leq \frac{\breve{M}(\varsigma-n\hat{p},n,0)-\breve{M}(\varsigma-\alpha,n,0)}{n\hat{p}-\alpha}+\frac{\epsilon}{\hat{p}(n\hat{p}-\alpha)}\mathbb{E}\left[U'\left(X_*^x(T)+\frac{n\hat{p}-\epsilon}{\hat{p}}B(T)\right)B(T)\right].$$

In general, $\limsup_{n\to\infty}(a_n+b_n)\leq \limsup_{n\to\infty}(a_n)+\limsup_{n\to\infty}(b_n)$.

Taking the *limsup* of both sides, it follows that

$$\limsup_{\alpha\uparrow n\hat{p}}\frac{\Psi(n\hat{p},\hat{p},x)-\Psi(\alpha,\hat{p},\varsigma)}{n\hat{p}-\alpha}\leq \limsup_{\alpha\uparrow n\hat{p}}\frac{\breve{M}(\varsigma-n\hat{p},n,0)-\breve{M}(\varsigma-\alpha,n,0)}{n\hat{p}-\alpha}$$
$$+\limsup_{\alpha\uparrow n\hat{p}}\frac{\epsilon}{\hat{p}(n\hat{p}-\alpha)}\mathbb{E}\left[U'\left(X_*^x(T)+\frac{n\hat{p}-\epsilon}{\hat{p}}B(T)\right)B(T)\right].$$

Now, using the fact that $\alpha=n\hat{p}-\epsilon$, we write this as

$$\limsup_{\epsilon\downarrow 0}\frac{\Psi(n\hat{p},\hat{p},x)-\Psi(n\hat{p}-\epsilon,\hat{p},\varsigma)}{\epsilon}$$
$$\leq \limsup_{\epsilon\downarrow 0}\frac{\breve{M}(\varsigma-n\hat{p},n,0)-\breve{M}(\varsigma-\alpha,n,0)}{\epsilon}+\limsup_{\epsilon\downarrow 0}\frac{\epsilon}{\hat{p}\epsilon}\mathbb{E}\left[U'\left(X_*^x(T)+\frac{n\hat{p}-\epsilon}{\hat{p}}B(T)\right)B(T)\right].$$

As $\epsilon\downarrow 0$, $X_*^x(T)+\frac{n\hat{p}-\epsilon}{\hat{p}}B(T)$ increases, and $U^{'}\left(X_*^x(T)+\frac{n\hat{p}-\epsilon}{\hat{p}}B(T)\right)$ decreases. Thus, by the Monotone Convergence Theorem

$$\limsup_{\epsilon\downarrow 0}\frac{\Psi(n\hat{p},\hat{p},x)-\Psi(n\hat{p}-\epsilon,\hat{p},\varsigma)}{\epsilon}\leq -\breve{M}'(\varsigma-n\hat{p},n,0)+\frac{1}{\hat{p}}\mathbb{E}\left[B(T)U'\left(X_*^x(T)+nB(T)\right)\right]$$
$$=-\breve{M}'(x,n,0)+\tfrac{1}{\hat{p}}\mathbb{E}\left[B(T)U'\left(X_*^x(T)+nB(T)\right)\right].$$

Now we consider the case, $\epsilon<0$, or equivalently, $n\hat{p}<\alpha$. From the second inequality in (A.1), we have

$$U\left(X_*^{\varsigma-\alpha}(T)+\frac{n\hat{p}-\epsilon}{\hat{p}}B(T)\right)\geq U\left(X_*^{\varsigma-\alpha}(T)+nB(T)\right)-\frac{\epsilon}{\hat{p}}B(T)U'\left(X_*^{\varsigma-\alpha}(T)+\frac{n\hat{p}-\epsilon}{\hat{p}}B(T)\right).$$

In this case, $\varsigma-\alpha=x+\epsilon\leq x$ and since $\varsigma\mapsto X_*^{\varsigma}(T)$ is non-decreasing, we get

$$U\left(X_*^{\varsigma-\alpha}(T)+\frac{n\hat{p}-\epsilon}{\hat{p}}B(T)\right)\geq U\left(X_*^{\varsigma-\alpha}(T)+nB(T)\right)-\frac{\epsilon}{\hat{p}}B(T)U'\left(X_*^{\varsigma}(T)+\frac{n\hat{p}-\epsilon}{\hat{p}}B(T)\right).$$

Taking expectations gives us the inequality in (A.5). Using the definitions of $\Psi(\alpha,\hat{p},\varsigma)$ and $\breve{M}(\varsigma-\alpha,n,0)$, and dividing by $n\hat{p}-\alpha<0$, it is not hard to see that

$$\frac{\Psi(n\hat{p},\hat{p},x)-\Psi(\alpha,\hat{p},\varsigma)}{n\hat{p}-\alpha}\geq \frac{\breve{M}(\varsigma-n\hat{p},n,0)-\breve{M}(\varsigma-\alpha,n,0)}{n\hat{p}-\alpha}+\frac{\epsilon}{\hat{p}(n\hat{p}-\alpha)}\mathbb{E}\left[U'\left(X_*^x(T)+\frac{n\hat{p}-\epsilon}{\hat{p}}B(T)\right)B(T)\right].$$

Again, in general, $\liminf_{n\to\infty}(a_n+b_n)\geq \liminf_{n\to\infty}(a_n)+\liminf_{n\to\infty}(b_n)$.

Taking the *liminf* of both sides, and using the fact that $\alpha=n\hat{p}-\epsilon$, it follows that

$$\liminf_{\epsilon \uparrow 0} \frac{\Psi(n\hat{p},\hat{p},x) - \Psi(n\hat{p}-\epsilon,\hat{p},\varsigma)}{\epsilon}$$

$$\geq \limsup_{\epsilon \uparrow 0} \frac{\breve{M}(\varsigma - n\hat{p}, n, 0) - \breve{M}(\varsigma - \alpha, n, 0)}{\epsilon} + \limsup_{\epsilon \uparrow 0} \frac{\epsilon}{\hat{p}\epsilon} \mathbb{E}\left[U'\left(X_*^x(T) + \frac{n\hat{p}-\epsilon}{\hat{p}} B(T)\right) B(T)\right]$$

$$= -\breve{M}'(x,n,0) + \tfrac{1}{\hat{p}}\mathbb{E}\big[B(T)U'\big(X_*^x(T) + nB(T)\big)\big].$$

Hence,

$$\limsup_{\alpha \downarrow n\hat{p}} \frac{\Psi(n\hat{p},\hat{p},x) - \Psi(\alpha,\hat{p},x)}{n\hat{p}-\alpha}$$

$$\leq -\breve{M}'(x,n,0) + \frac{1}{\hat{p}}\mathbb{E}\big[B(T)U'\big(X_*^x(T) + nB(T)\big)\big] \leq \liminf_{\alpha \uparrow n\hat{p}} \frac{\Psi(n\hat{p},\hat{p},x) - \Psi(\alpha,\hat{p},x)}{n\hat{p}-\alpha}.$$

If the derivative $\frac{\partial \Psi(\alpha,\hat{p},x)}{\partial \alpha}$ exists, then the fair subjective price of the NTNH ESO is set to be the solution of the equation, $\left.\frac{\partial \Psi(\alpha,\hat{p},x)}{\partial \alpha}\right|_{\alpha = n\hat{p}} = 0$.

This means that executives are indifferent, in the utility sense, between holding their current portfolio without ESOs and substituting a small fraction, α, of their portfolio with ESOs (or more generally, they are indifferent between holding the optimal portfolio with n ESOs and substituting a small fraction of wealth with another ESO). If Ψ is not differentiable, then the fair subjective price is set to be the *weak solution* (see definition 7.2 in Karatzas and Kou, 1996) of the above equation. In either case, the inequalities above show that $\hat{p}$ is determined by

$-\breve{M}'(x,n,0) + \tfrac{1}{\hat{p}}\mathbb{E}\big[B(T)U'\big(X_*^x(T) + nB(T)\big)\big] = 0.$

The subjective price of the ESO at time zero is

$$\hat{p}(x,n,0) \equiv \hat{p} = \frac{\mathbb{E}\big[B(T)U'\big(X_*^x(T) + nB(T)\big)\big]}{\breve{M}'(x,n,0)} = \frac{\mathbb{E}\big[B(T)U'\big(X_*^x(T) + nB(T)\big)\big]}{M'(x + nBS(0), 0)}.$$

Thanks to the Markov property, the subjective price of the ESO at time t is

$$\hat{p}(x_t,n,t) = \frac{\mathbb{E}\left[B(T)U'\left(X^{x_t,\pi^*}(T) + nB(T)\right)|\mathcal{F}_t\right]}{M'(x_t + nBS(t),\ t)}, \qquad \forall x_t > 0.$$

□

A.2. Proof of Theorem 2

Theorem 7.4 in Karatzas and Kou (1996) requires their Equation (7.24), which is based on the assumption that δ is bounded from below. Although we cannot assume that δ is bounded from below in this paper, it is a sufficient condition rather than a necessary one. This assumption is used to prove that the conditions $\forall y \in (0,\infty), \lim_{\|v\| \to \infty} \tilde{J}(y;v) = \infty$ and $\forall v \in (0,\infty), \lim_{y \downarrow 0} \tilde{J}(y;v) = \infty$ hold. Hence, we directly make these less strict assumptions, given in Equation (16) and Equation (17) respectively, so that the results in Theorem 7.4 still hold. We can then prove Theorem 2 directly:

$$\hat{p}(x_t,n,t) = \frac{\mathbb{E}[H_{\hat{v}}(T)B(T)|\mathcal{F}_t]}{H_{\hat{v}}(t)} = \frac{\mathbb{E}^{\hat{v}}[\gamma_{\hat{v}}(T)B(T)|\mathcal{F}_t]}{\gamma_{\hat{v}}(t)},$$

where the second equality is obtained by Abstract Bayes' theorem, and $\mathbb{E}^{\hat{v}}(\cdot)$ is expectation under the corresponding martingale measure with respect to the auxiliary market $\mathcal{M}_{\hat{v}}$. According to Theorem 9.1 of Cvitanić and Karatzas (1992), page 780, there exists a portfolio process satisfying the constraints, such that the initial $\hat{p}(x_t, n, t)$ will evolve to $B(T)$ at the terminal date.

Since the firm will repay the executive when the European call option is exercised at its maturity, and the NTNH constraints are not for the firm, the SDF for the objective price of the NTNH European ESO is simply given by the market SDF $H_0(T)$. Again, thanks to Abstract Bayes' theorem, the objective price of the NTNH ESO with terminal payoff $B(T)$ at time t is equal to the market price:

$$\tilde{p}(t) = p(t) = \frac{\mathbb{E}[H_0(T)B(T)|\mathcal{F}_t]}{H_0(t)} = \frac{\mathbb{E}^0[\gamma_0(T)B(T)|\mathcal{F}_t]}{\gamma_0(t)}.$$

□

Remark 7. The total initial wealth for the executive is x amount of cash and n shares of contingent claims remaining fixed in the portfolio. Those n shares can be replicated by a sub-portfolio of underlying assets. The outside initial wealth x can be invested freely as long as there is no negative position in the firm's stock. The combined non-transferable and non-hedgeable constraint is that the number of shares of the firm's stock in the portfolio has to be no less than n shares. Thus, we have translated the problem into a traditional constrained portfolio optimization problem. We can get the optimal portfolio strategy within the constraints and, correspondingly, the optimal terminal wealth under constraints. It is natural to define the price for the optimal terminal wealth $X_*^x(T) + nB(T)$ as the total initial wealth, which is

$$x + nBS(0) = \mathbb{E}^{\hat{v}}\left[\gamma_{\hat{v}}(T)\left(X^{x,\pi^*}(T) + nB(T)\right)\right].$$

From Theorem 2,

$$\hat{p}(x, n, t) = \frac{\mathbb{E}^{\hat{v}}[\gamma_{\hat{v}}(T)B(T)|\mathcal{F}_t]}{\gamma_{\hat{v}}(t)}.$$

Then the subjective price of outside wealth $\mathbb{E}^{\hat{v}}\left[\gamma_{\hat{v}}(T)X^{x,\pi^*}(T)\right]$ will be $x + nBS(0) - n\hat{p}(x, n, t)$.

Appendix B. Proofs of Theorem 3 and Theorem 4

B.1. Proof of Theorem 3

First, we prove that the indirect utility $M(\cdot, t)$ satisfies

$$M(\varkappa, t) + (\wp - \varkappa)M'(\varkappa, t) \geq M(\wp, t) \geq M(\varkappa, t) + (\wp - \varkappa)M'(\wp, t), \ \forall\, 0 < \varkappa < \wp < \infty. \quad \text{(B.1)}$$

According to Theorem 7.3 in Karatzas and Kou (1996), with their Assumption 6.2 being relaxed by our Assumptions (16) and (17) from Proposition 1, and assuming that the constraint set K is closed, convex and satisfies the regularity conditions, then for any $x > 0$, there exists a process $\hat{v} \equiv \hat{v}_x \in \mathfrak{D}$ and a portfolio process $\pi^* \in \mathcal{A}(x, K(0, \omega), 0, T)$ for the constrained portfolio optimization problem of $M(x,0) = sup_{(\pi,C)\in\mathcal{A}(x,K(0,\omega),0,T)}\ \mathbb{E}\left[U\left(X^{x,\pi}(T)\right)\right]$, $0 < x < \infty$, with corresponding terminal wealth $X^{x,\pi^*}(T) = I\left(\mathcal{Y}_{\hat{v}}(x)\gamma_{\hat{v}}(T)Z_{\hat{v}}(T)\right)$ $a.s.$, where $y \equiv \mathcal{Y}_{\hat{v}}(x)$ is the inverse function of $\mathcal{X}_{\hat{v}}(y) \triangleq \mathbb{E}[H_{\hat{v}}(T)I(yH_{\hat{v}}(T))]$, and $I(\cdot)$ is the inverse function of $U'(\cdot)$. π^* is optimal for the problem. For any t, set $y_t = y \times H_{\hat{v}}(t)$. We define $x_t \triangleq \mathbb{E}[H_{\hat{v}}(T)/H_{\hat{v}}(t) \times I(yH_{\hat{v}}(T))|\mathcal{F}_t]$, and with the substitution $y_t = y \times H_{\hat{v}}(t)$, y_t is the inverse function of $\mathcal{X}_{\hat{v}}(y_t) \triangleq \mathbb{E}[H_{\hat{v}}(T)/H_{\hat{v}}(t) \times I(y_t H_{\hat{v}}(T)/H_{\hat{v}}(t))|\mathcal{F}_t]$. The value function $M(\cdot,t)$ is continuously differentiable, and its derivative can be represented as $M'(x_t,t) \equiv M_{x_t}(x_t,t) = \mathcal{Y}_{\hat{v}}(x_t) > 0$, $\forall x_t > 0$. [See Karatzas, et al., 1987, Remark 4.5.]

Note that, according to Assumption 7.2 in Karatzas and Kou (1996), we have $\forall\pi \in \mathcal{A}(\varkappa, K(t,\omega), t, T)$, and $\forall\pi \in \mathcal{A}(\wp, K(t,\omega), t, T)$, $X^{\wp,\pi}(T)/X^{\varkappa,\pi}(T) = \wp/\varkappa$, which implies that

$$\frac{\partial X^{\varkappa,\pi^*}(T)}{\partial\varkappa} = \lim_{\varepsilon\to 0}\frac{X^{\varkappa+\varepsilon,\pi^*}(T) - X^{\varkappa,\pi^*}(T)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{\left(\frac{\varkappa+\varepsilon}{\varkappa} - 1\right)X^{\varkappa,\pi^*}(T)}{\varepsilon} = \frac{X^{\varkappa,\pi^*}(T)}{\varkappa}.$$

Also, the concavity of the utility function, $U(\cdot)$, gives us the inequality,

$$U(\varkappa) + (\wp - \varkappa)U'(\varkappa) \geq U(\wp) \geq U(\varkappa) + (\wp - \varkappa)U'(\wp),\ \ \forall\ 0 < \varkappa < \wp < \infty.$$

Therefore,

$$U(X^{\varkappa,\pi}(T) + \left(X^{\wp,\pi}(T) - X^{\varkappa,\pi}(T)\right)U'\left(X^{\varkappa,\pi}(T)\right) \geq U(X^{\varkappa,\pi}(T) + \left(X^{\wp,\pi}(T) - X^{\varkappa,\pi}(T)\right)U'\left(X^{\wp,\pi}(T)\right),$$

from which we can see that

$$\mathbb{E}\left(U(X^{\varkappa,\pi^*}(T)\right) + (\wp - \varkappa)\mathbb{E}\left(\frac{X^{\varkappa,\pi^*}(T)}{\varkappa}U'\left(X^{\varkappa,\pi^*}(T)\right)\right) \geq \mathbb{E}\left(U\left(X^{\wp,\pi^*}(T)\right)\right)$$
$$\geq \mathbb{E}\left(U(X^{\wp,\pi^*}(T)\right) + (\wp - \varkappa)\mathbb{E}\left(\frac{X^{\wp,\pi^*}(T)}{\wp}U'\left(X^{\wp,\pi^*}(T)\right)\right).$$

Note that

$$\frac{X^{\varkappa,\pi^*}(T)}{\varkappa}U'\left(X^{\varkappa,\pi^*}(T)\right) = \frac{\partial X^{\varkappa,\pi^*}(T)}{\partial\varkappa}U'\left(X^{\varkappa,\pi^*}(T)\right) = \frac{dU\left(X^{\varkappa,\pi^*}(T)\right)}{d\varkappa} = lim_{\varepsilon\to 0}\frac{U\left(X^{\varkappa+\varepsilon,\pi^*}(T)\right) - U\left(X^{\varkappa,\pi^*}(T)\right)}{\varepsilon},$$

since $U(\cdot)$ is a utility function, being strictly increasing and strictly concave of class C^1. Also, it is not hard to see that if $\varepsilon_2 > \varepsilon_1 > 0$, then $0 \leq \frac{U\left(X^{\varkappa+\varepsilon_1,\pi^*}(T)\right) - U\left(X^{\varkappa,\pi^*}(T)\right)}{\varepsilon_1} \leq \frac{U\left(X^{\varkappa+\varepsilon_2,\pi^*}(T)\right) - U\left(X^{\varkappa,\pi^*}(T)\right)}{\varepsilon_2}$.

Then, by the Monotone Convergence Theorem, we have,

$$\mathbb{E}\left(\lim_{\varepsilon\downarrow 0}\frac{U\left(X^{\varkappa+\varepsilon,\pi^*}(T)\right) - U\left(X^{\varkappa,\pi^*}(T)\right)}{\varepsilon}\right) = \lim_{\varepsilon\downarrow 0}\mathbb{E}\left(\frac{U\left(X^{\varkappa+\varepsilon,\pi^*}(T)\right) - U\left(X^{\varkappa,\pi^*}(T)\right)}{\varepsilon}\right).$$

On the other hand, if $\varepsilon_2 < \varepsilon_1 < 0$, then $0 \leq \frac{U\left(X^{\varkappa+\varepsilon_1,\pi^*}(T)\right) - U\left(X^{\varkappa,\pi^*}(T)\right)}{\varepsilon_1} \leq \frac{U\left(X^{\varkappa+\varepsilon_2,\pi^*}(T)\right) - U\left(X^{\varkappa,\pi^*}(T)\right)}{\varepsilon_2}$.

Again by the Monotone Convergence Theorem, we have

$$\mathbb{E}\left(\lim_{\varepsilon\uparrow 0}\frac{U\left(X^{\varkappa+\varepsilon,\pi^*}(T)\right)-U\left(X^{\varkappa,\pi^*}(T)\right)}{\varepsilon}\right)=\lim_{\varepsilon\uparrow 0}\mathbb{E}\left(\frac{U\left(X^{\varkappa+\varepsilon,\pi^*}(T)\right)-U\left(X^{\varkappa,\pi^*}(T)\right)}{\varepsilon}\right).$$

Then,

$$\mathbb{E}\left(\lim_{\varepsilon\to 0}\frac{U\left(X^{\varkappa+\varepsilon,\pi^*}(T)\right)-U\left(X^{\varkappa,\pi^*}(T)\right)}{\varepsilon}\right)=\lim_{\varepsilon\to 0}\mathbb{E}\left(\frac{U\left(X^{\varkappa+\varepsilon,\pi^*}(T)\right)-U\left(X^{\varkappa,\pi^*}(T)\right)}{\varepsilon}\right).$$

Hence,

$$\mathbb{E}\left(U(X^{\varkappa,\pi^*}(T)\right)+(\wp-\varkappa)\frac{\partial\mathbb{E}\left(U\left(X^{\varkappa,\pi^*}(T)\right)\right)}{\partial\varkappa}\geq\mathbb{E}\left(U\left(X^{\wp,\pi^*}(T)\right)\right)$$
$$\geq\mathbb{E}\left(U(X^{\varkappa,\pi^*}(T)\right)+(\wp-\varkappa)\frac{\partial\mathbb{E}\left(U\left(X^{\wp,\pi^*}(T)\right)\right)}{\partial\varkappa};$$

that is,

$M(\varkappa,t)+(\wp-\varkappa)M'(\varkappa,t)\geq M(\wp,t)\geq M(\varkappa,t)+(\wp-\varkappa)M'(\wp,t),\ \forall\ 0<\varkappa<\wp<\infty.$

So far, we have proved that the indirect utility function of the portfolio is concave. The indirect utility function of the portfolio that we define for pricing the NTNH ESOs is exactly the same except that the total initial wealth includes two parts, the net initial wealth x and n shares of non-transferable ESOs worth $nBS(0)$.

In our model, the n shares of the NTNH American ESOs are included in the total wealth portfolio, and the marginal ϵ/p share of the ESO is outside of the executive's total wealth; they are assumed shares for finding the indifference price. Since we are interested in the price of the ESO within the total wealth, we let the exercise time of the extra ϵ/p shares be the same as the former; otherwise it is a multi-option and multi-optimal exercise time problem, which we do not discuss in this paper.

We define

$$\Psi(\epsilon,\hat{p},x)\triangleq\sup_{\tau\in\Gamma}\ \sup_{\pi\in\mathcal{A}\left(x+\epsilon+nBS(0)-\frac{\epsilon}{\hat{p}}BS(0),K(0,\omega),0,\tau\right)}\mathbb{E}\left[M\left(X^{x+\epsilon,\pi}(\tau)+nB(\tau)-\frac{\epsilon}{\hat{p}}B(\tau),\tau\right)\right]$$
$$=\sup_{\tau\in\Gamma}\ \sup_{\pi\in\mathcal{A}\left(x+\epsilon+nBS(0)-\frac{\epsilon}{\hat{p}}BS(0),K(0,\omega),0,\tau\right)}\mathbb{E}\left[M\left(\mathbb{X}^{x+\epsilon+nBS(0)-\frac{\epsilon}{\hat{p}}BS(0),\pi}(\tau),\tau\right)\right].$$

Here, $X^{x+\epsilon,\pi}(\tau)$ represents the outside wealth, the wealth apart from the ESO, and $\mathbb{X}^{x+\epsilon+nBS(0)-\frac{\epsilon}{\hat{p}}BS(0),\pi}$ represents the total wealth that evolves from the initial total wealth $x+\epsilon+nBS(0)-\frac{\epsilon}{\hat{p}}BS(0)$. The stochastic constraint $K(t,\omega)$ in this model is $[(n-\epsilon/\hat{p})\Phi(d_1(t,\omega))S_1(t)/\mathbb{X}(t),\infty)\times(-\infty,\infty)^{d-1}$, which is a function of ϵ. In theory; there is no problem with having a stochastic constraint. This only makes the drift rate and the risk-free rate, $r_v(t)$, stochastic in the auxiliary market. Moreover, please be aware that the constraint $K(t,\omega)$ in Karatzas and Kou (1996) is not a function of ϵ, while it is a function of ϵ in this case.

Let τ be any stopping time, then using (B.1), we can write the following inequality, due to the concavity of the indirect utility function M:

$$M(\varkappa,\tau)+(\wp-\varkappa)M'(\varkappa,\tau)\geq M(\wp,\tau)\geq M(\varkappa,\tau)+(\wp-\varkappa)M'(\wp,\tau),\forall\ 0<\varkappa<\wp<\infty. \quad \text{(B.2)}$$

Define

$$\Psi(\epsilon,\hat{p},x)\triangleq \sup_{\tau\in\Gamma}\ \sup_{\pi\in\mathcal{A}\left(x+\epsilon+nBS(0)-\frac{\epsilon}{\hat{p}}BS(0),K(0,\omega),0,\tau\right)}\mathbb{E}\left[M\left(X^{x+\epsilon+nBS(0)-\frac{\epsilon}{\hat{p}}BS(0),\pi}(\tau),\tau\right)\right].$$

We next define

$$L(x,n,0)\triangleq \sup_{\tau\in\Gamma}\ \sup_{\pi\in\mathcal{A}\left(x+\epsilon+nBS(0)-\frac{\epsilon}{\hat{p}}BS(0),K(0,\omega),0,\tau\right)}\mathbb{E}[M(X^{x,\pi}(\tau)+nB(\tau),\tau)]. \quad \text{(B.3)}$$

Let τ^* and π^* solve Equation (B.3) and fix $\epsilon>0$. From the first inequality in (B.2), we have

$$M\left(X^{x+\epsilon,\pi^*}(\tau^*)+nB(\tau^*)-\frac{\epsilon}{\hat{p}}B(\tau^*),\tau\right)+\frac{\epsilon}{\hat{p}}B(\tau^*)M'\left(X^{x+\epsilon,\pi^*}(\tau^*)+nB(\tau^*)-\frac{\epsilon}{\hat{p}}B(\tau^*),\tau^*\right)$$
$$\geq M\left(X^{x+\epsilon,\pi^*}(\tau^*)+nB(\tau^*),\tau^*\right).$$

Because $\epsilon>0$ and X is non-decreasing, we get

$$M\left(X^{x+\epsilon,\pi^*}(\tau^*)+nB(\tau^*)-\frac{\epsilon}{\hat{p}}B(\tau^*),\tau^*\right)$$
$$\geq M\left(X^{x+\epsilon,\pi^*}(\tau^*)+nB(\tau^*),\tau^*\right)-\frac{\epsilon}{\hat{p}}B(\tau^*)M'\left(X^{x,\pi^*}(\tau^*)+nB(\tau^*)-\frac{\epsilon}{\hat{p}}B(\tau^*),\tau^*\right).$$

The rest of the proof is similar to the proof of Theorem 1. Thus, we have

$$\frac{\partial\Psi(0,\hat{p},x)}{\partial\epsilon}=L'(x,n,0)-\frac{1}{\hat{p}}\sup_{\tau\in\Gamma}\left\{\mathbb{E}\left[B(\tau)M'\left(X^{x,\pi^*}(\tau)+nB(\tau),\tau\right)\right]\right\}=0.$$

The fair price is $\hat{p}(x,n,0)=\frac{\mathbb{E}\left[M'\left(X^{x,\pi^*}(\tau^*)+nB(\tau^*),\tau^*\right)B(\tau^*)\right]}{L'(x,n,0)}$. By the Markov property,

$$\hat{p}(x_t,n,t)=\frac{\mathbb{E}\left[M'\left(X^{x_t,\pi^*}(\tau^*)+nB(\tau^*),\tau^*\right)B(\tau^*)|\mathcal{F}_t\right]}{L'(x_t,n,t)},\quad \forall x_t>0.$$

□

B.2. Proof of Theorem 4

Step 1. In this step we make some general comments about the drift of the total wealth process before and after the block exercise date. Assume that for any stopping time $\tau\leq T$ and for any $v,\grave{v}$, we have that

$$\mathcal{X}_{v,\tau,\grave{v}}(y)\triangleq\mathbb{E}\left[H_{\grave{v}}(\tau,T)H_v(0,\tau)I\left(yH_{\grave{v}}(\tau,T)H_v(0,\tau)\right)+n(BS(\tau)-B(\tau))H_v(0,\tau)\right]$$

is finite for every $y\in(0,\infty)$. Here, v and $\grave{v}$ indicate that the auxiliary market is $\mathcal{M}_v$ before the block exercise date, τ, and $\mathcal{M}_{\grave{v}}$ after date τ; in fact, let us write this market as $\mathcal{M}_{v,\tau,\grave{v}}$. Under this assumption, the function $\mathcal{X}_{v,\tau,\grave{v}}$: $(0,\infty)\to(0,\infty)$ is continuous and strictly decreasing, with $\mathcal{X}_{v,\tau,\grave{v}}(0+)=\infty$. We let $\mathcal{Y}_{v,\tau,\grave{v}}(\cdot)$ denote its inverse and introduce the following random variables.

For brevity, we denote $x^n=x+nBS(0)$, and we define,

$$\xi^{x^n}_{v,\tau,\grave{v}} \triangleq I\left(\mathcal{Y}_{v,\tau,\grave{v}}(x^n)H_{\grave{v}}(\tau,T)H_v(0,\tau)\right). \tag{B.4}$$

It is obvious from the above definition that

$$\mathbb{E}\left[\xi^{x^n}_{v,\tau,\grave{v}}H_{\grave{v}}(\tau,T)H_v(0,\tau) + n(BS(\tau) - B(\tau))H_v(0,\tau)\right] = x^n,$$

which guarantees that the initial outside wealth is x and the total initial wealth is x plus the Black-Scholes price of n shares of NTNH ESOs.

Please note that at the exercise time, the constraints are relaxed and there is a negative jump with respect to the total wealth process. (To be clear, there is an increase in what we call outside wealth, but a decrease in total wealth.) Holding n shares of non-transferable contingent claims is equivalent to holding the replicating portfolio. After being exercised, the shares become cash with the amount equal to the intrinsic value of the contingent claims and can be reinvested into any primary assets under the constraint of no shorting selling of the firm's stock. Then the NTNH constraint is relaxed. On the other hand, the amount of cash is equal to the intrinsic value of the contingent claim, which is lower than the market price of the replicating portfolio or the Black-Scholes price of the contingent claim. That means exercising the contingent claim will make the total wealth lose value immediately. Hence, there is a trade-off between those two. Finding the optimal exercise time involves balancing the benefit and loss, and maximizing the current expected indirect utility at the exercise time.

If the intrinsic value is equal to zero, early exercise will make the option holder worse off by losing time value in exchange for zero dollars of intrinsic value. Note that a non-zero intrinsic value could be reinvested without the non-transferability constraint.

In the auxiliary market, $\mathcal{M}_{v,\tau,\grave{v}}$, the dynamic of the total wealth process with block exercise is

$$d\mathbb{X}_{v,\tau,\grave{v}}(t) = \left(r(t)\mathbb{X}_{v,\tau,\grave{v}}(t)\right)dt + \mathbb{X}_{v,\tau,\grave{v}}(t)\left[(\delta(\grave{v}(t)) + \pi^*_{\grave{v}}(t)\grave{v}(t))1_{\{t>\tau\}} + (\delta(v(t)) + (\pi_v(t))^T v(t))1_{\{t<\tau\}}\right]dt + \mathbb{X}_{v,\tau,\grave{v}}(t)\pi_{v,\tau,\grave{v}}(t)\sigma(t)dW_0(t) - n\left(BS(t) - B(t)\right)1_{\{t=\tau\}}, \tag{B.5}$$

where $v(t,\omega) \in \widetilde{K}$, for $\mathcal{E}\otimes P - a.e.(t,\omega) \in [0,\tau] \times \Omega$, and $\grave{v}(t,\omega) \in \widetilde{K}$, for $\mathcal{E}\otimes P - a.e.(t,\omega) \in [\tau,T] \times \Omega$. Also, $\mathcal{E}$ represents the Effros Measure of t on the random closed set, e.g. $[0,\tau]$ and $[\tau,T]$ instead of the Lebesgue measure (See Molchanov, 2005, page 25, Definition 2.1). For any y, the adjustment of the drift rate $f(t,\omega)$ and $\lambda(t,\omega)$ are the optimal $v(t,\omega)$ and $\grave{v}(t,\omega)$, respectively, to obtain the following infimum for a given stopping time τ:

$$\inf_{v,\grave{v}\in\mathfrak{D}} \tilde{J}(y;v,\grave{v}) \triangleq \mathbb{E}\left[\widetilde{U}\left(yH_{\grave{v}}(\tau,T)H_v(0,\tau)\right)\right] - y\mathbb{E}\left[H_v(0,\tau)n\left(BS(\tau) - B(\tau)\right)\right],$$

where $\mathfrak{D}$ is determined by $K(t,\omega)$ of (25).

Step 2. We now prove the following lemma. (Cf. Proposition 8.3 of Cvitanić and Karatzas, 1992.)

Lemma B.1. For $\pi_\lambda(t,\omega) \in K(t,\omega)$ and $\pi_f(t,\omega) \in K(t,\omega)$, suppose

$$\delta(\lambda(t)) + (\pi_\lambda(t))^T \lambda(t) = 0, \tag{B.6}$$

$$\delta(f(t)) + \left(\pi_f(t)\right)^T f(t) = 0. \tag{B.7}$$

Let $\mathbb{X}_f(\cdot)$ and $\mathbb{X}_\lambda(\cdot)$ be the wealth dynamics in the auxiliary markets $\mathcal{M}_f$ and $\mathcal{M}_\lambda$, respectively. Then $\pi_f \in \mathcal{A}(x^n, K(0,\omega), 0, \tau\text{-})$ and $\pi_\lambda \in \mathcal{A}(\mathbb{X}_\lambda(\tau+), K(\tau+,\omega), \tau+, T)$, are the optimal portfolio processes for the constrained optimization problems (27) and (28) in the original market, and they satisfy, for any given $v, \grave{v} \in \mathfrak{D}$,

$$\mathbb{E}\left[U(\xi_{f,\tau,\lambda})\right] \le V_{v,\tau,\grave{v}}(x) \triangleq \underset{\tau\in\Gamma}{ess\ sup}\ \underset{\pi_v \in \mathcal{A}_v(x,0,\tau)}{ess\ sup}\ \mathbb{E}\left[M\left(X_v^{x,\pi_v}(\tau) + nB(\tau), \tau\right)\right],$$

where $M_{\grave{v}}(x_\tau, \tau) \triangleq esssup_{\pi_{\grave{v}} \in \mathcal{A}_{\grave{v}}(x_\tau,\tau,T)}\ \mathbb{E}\left[U(X_{\grave{v}}^{x_\tau,\pi_{\grave{v}}}(T))|\mathcal{F}_\tau\right]$.

Proof. For simplicity, we define $\pi_{v,\tau,\grave{v}} \triangleq \pi_v 1_{\{t<\tau\}} + \pi_{\grave{v}} 1_{\{t>\tau\}}$ and $\mathcal{M}_{v,\tau,\grave{v}} \triangleq \mathcal{M}_v 1_{\{t<\tau\}} + \mathcal{M}_{\grave{v}} 1_{\{t>\tau\}}$. By replacing v, $\grave{v}$ with f and λ, respectively, in (B.5), we have

$$\begin{aligned} d\mathbb{X}_{f,\tau,\lambda} &= \left(r(t)\mathbb{X}_{f,\tau,\lambda}(t)\right)dt \\ &\quad + \mathbb{X}_{f,\tau,\lambda}(t)\left[(\delta(\lambda(t)) + (\pi_\lambda(t))^T\lambda(t))1_{\{t>\tau\}} + \left(\delta(f(t)) + \left(\pi_f(t)\right)^T f(t)\right)1_{\{t<\tau\}}\right]dt \\ &\quad + \mathbb{X}_{f,\tau,\lambda}(t)\left(\pi_{f,\tau,\lambda}\right)^T(t)\sigma(t)dW_0(t) - n\left(BS(t) - B(t)\right)1_{\{t=\tau\}}, \\ X_{f,\tau,\lambda}(0) &= x^n,\ \mathbb{X}_{f,\tau,\lambda}(T) = \xi_{f,\tau,\lambda}^{x_n}(T), \end{aligned}$$

where $\xi_{f,\tau,\lambda}^{x_n} \triangleq I\left(\mathcal{Y}_{f,\tau,\lambda}(x)H_\lambda(\tau,T)H_f(0,\tau)\right)$, and $\mathcal{Y}_{f,\tau,\lambda}(x)$ is the inverse function of

$$\mathcal{X}_{f,\tau,\lambda}(y) = \mathbb{E}\left[H_\lambda(\tau,T)H_f(0,\tau)I\left(yH_\lambda(\tau,T)H_f(0,\tau)\right) + n(BS(\tau) - B(\tau))H_f(0,\tau)\right].$$

In different markets, original or auxiliary, the drift rate is different. The risk-free rate is different as well. Although the Black-Scholes price $BS(t)$ at any time t is independent of the drift rate at time t, it is a function of the risk-free rate at time t and the spot price at time t. Standing at any time before t, the spot price at time t is a function of the drift rate of the underlying asset. Hence, it is necessary for us to specify in which market the Black-Scholes price will be taken in the above pricing model. And $BS(t)$ in this research is the Black-Scholes price in the original market. When (B.6) and (B.7) are satisfied, the wealth dynamic in the auxiliary market $\mathcal{M}_{f,\tau,\lambda}$ is as follows:

$$d\mathbb{X}_{f,\tau,\lambda}(t) = \left(r(t)\mathbb{X}_{f,\tau,\lambda}(t)\right)dt + \mathbb{X}_{f,\tau,\lambda}(t)\left(\pi_{f,\tau,\lambda}(t)\right)^T\sigma(t)dW_0(t) - n\left(BS(t) - B(t)\right)1_{\{t=\tau\}},$$
$$\mathbb{X}(0) = x^n.$$

Compare it with the wealth dynamic in the original market,

$$d\,\mathbb{X}(t) = \left(\mathbb{X}(t)r(t)\right)dt + \mathbb{X}(t)\left(\pi(t)\right)^T\sigma(t)dW_0(t) - n\left(BS(t) - B(t)\right)1_{\{t=\tau\}},\quad \mathbb{X}(0) = x^n.$$

It is obvious that $\mathbb{X}_{f,\tau,\lambda}(t)$ is also a wealth process corresponding to $\pi_{f,\tau,\lambda}(t)$ in the original market $\mathcal{M}$. Furthermore, from this and the fact that $\pi_\lambda(t,\omega)$ and $\pi_f(t,\omega) \in K(t,\omega)$ of (25), we conclude that $\pi_f \in \mathcal{A}(x^n, K(0,\omega), 0, \tau\text{-})$, $\pi_\lambda \in \mathcal{A}(\mathbb{X}_{f,\tau,\lambda}(\tau+), K(\tau+,\omega), \tau+, T)$, and $V_{f,\tau,\lambda}(x) =$

$\mathbb{E}[U(\xi_{f,\tau,\lambda})] \le L(x,n,0)$, where $L(x,n,0)$ is defined in Equation (B.3). By the fact that $\left[\left(\delta(\lambda(t)) + (\pi_\lambda(t))^T \lambda(t)\right) 1_{\{t\ge\tau\}} + \left(\delta(f(t)) + (\pi_f(t))^T f(t)\right) 1_{\{t<\tau\}}\right] \ge 0$, for any given stopping time τ and for any given $v, \grave{v}$, we have $\mathbb{X}_{v,\tau,\grave{v}}(t) > \mathbb{X}(t) \ge 0, \forall 0 < t < T$. Hence,

$$\mathbb{E}\left(M\left(\mathbb{X}_{v,\tau,\grave{v}}(\tau),\tau\right)\right) > \mathbb{E}\left(M(\mathbb{X}(\tau),\tau)\right), \tag{B.8}$$

$$\mathbb{E}\left(U\left(\mathbb{X}_{v,\tau,\grave{v}}(T)\right)\middle|\mathcal{F}_\tau\right) > \mathbb{E}\left(U(\mathbb{X}(T))|\mathcal{F}_\tau\right). \tag{B.9}$$

Then, $\forall v, \grave{v} \in \mathfrak{D}$, from (B.8), we have $\mathcal{A}(x^n, K(0,\omega),\ 0, \tau\text{-}) \subset \mathcal{A}_v(x^n, 0, \tau\text{-})$, and from (B.9), we have $\mathcal{A}(x_{\tau^+}, K(\tau^+,\omega),\ \tau^+, T) \subset \mathcal{A}_{\grave{v}}\left((\mathbb{X}_{v,\tau,\grave{v}}(\tau^+), \tau^+, T\right)$ and furthermore, $L(x,n,0) \le V_{v,\tau,\grave{v}}(x)$. Because $V_{f,\tau,\lambda}(x)$ is a special case of $V_{v,\tau,\grave{v}}(x)$, it is trivial that $L(x,n,0) \le V_{f,\tau,\lambda}(x)$. Hence, we find that $L(x,n,0) = V_{f,\tau,\lambda}(x)$, which indicates that the optimal unconstrained portfolio process $\pi_{f,\tau,\lambda}(t)$ in the auxiliary market, $\mathcal{M}_{f,\tau,\lambda}$, is the optimal constrained portfolio process in the original market, $\mathcal{M}$.

Let $X_{v,\tau,\grave{v}}^{x^n,\pi_{f,\tau,\lambda}}$ be the wealth process with initial wealth, x^n, and portfolio process $\pi_{f,\tau,\lambda}$ in the market $\mathcal{M}_{v,\tau,\grave{v}}$. We have $\mathbb{X}_{v,\tau,\grave{v}}^{x^n,\pi_{f,\tau,\lambda}}(t) \ge \mathbb{X}_{f,\tau,\lambda}(t) > 0,\ \forall\ 0 < t < T$. Thus, $\pi_f \in \mathcal{A}_v(x_n, 0, \tau\text{-})$ before τ, and $\pi_\lambda \in \mathcal{A}_{\grave{v}}\left((\mathbb{X}_{v,\tau,\grave{v}}(\tau^+), \tau^+, T\right)$ after τ, which indicates that $V_{f,\tau,\lambda}(x) \le V_{v,\tau,\grave{v}}(x)$.

Step 3. In this step we derive the optimal portfolio for the auxiliary market with block exercise. However, to do this, we must confirm that $\mathbb{E}\left(U(\xi_{f,\tau,\lambda}^{x^n})\right)$ is well defined, where $\xi_{f,\tau,\lambda}^{x^n}$ was defined in Equation (B.4).

First, note that our problem is not quite standard. The proof of the theorem is based on the following logic. Assume we are able to solve the corresponding optimal portfolio process $\pi_{v,\tau,\grave{v}}$ for any auxiliary market $\mathcal{M}_{v,\tau,\grave{v}}$. If the auxiliary market is not arbitrarily chosen, but satisfies the condition

$$\left[\left(\delta(f(t)) + (\pi_f(t))^T f(t)\right) 1_{\{t<\tau\}} + (\delta(\lambda(t)) + (\pi_\lambda(t))^T \lambda(t)) 1_{\{t\ge\tau\}}\right] = 0,$$

then the optimal portfolio process for the unconstrained auxiliary market is the optimal constrained original market. But, do we know how to solve the optimal portfolio process in an unconstrained auxiliary market $\mathcal{M}_{f,\tau,\lambda}$? Because in this case, there is a negative jump at the optimal exercise time; that is, the problem is not exactly the same as the one in Cvitanić and Karatzas (1992).

To answer that question, we need to make sure $\mathbb{E}\left(U(\xi_{f,\tau,\lambda}^{x^n})\right)$ is well defined first, namely $\mathbb{E}[U^-(\xi_{f,\tau,\lambda}^{x^n})] < \infty$. We need to show that $H_\lambda(\tau,T)H_f(0,\tau)$ is integrable, but first let us find an explicit expression for $f(t)$. For log-utility, from Proposition 3, ignoring the stock vesting, we know that

$$f(s) = \underset{v \in \widetilde{K}(s,\omega)}{\operatorname{argmin}} \left[-2 \frac{\left(n - \widehat{N}(s)\right) \Phi(d_1) S_1(s)}{\mathbb{X}(s)} v_1 + \|\sigma^{-1}(s) \times [b(s) - r(s)\mathbf{1} + v]\|^2 \right].$$

Now, for $v = (v_1, 0, \dots 0)^T$,

$$\|\sigma^{-1}(s) \times [b(s) - r(s)\mathbf{1} + v]\|^2 = \|\theta(s) + \sigma^{-1}(s)v\|^2 = \hat{A}v_1(s)^2 + 2\hat{B}v_1(s) + \hat{C},$$

where

$$\hat{A} \equiv \left(\sigma_{11}^{-1}(s)\right)^2 + \left(\sigma_{21}^{-1}(s)\right)^2 + \dots + \left(\sigma_{d1}^{-1}(s)\right)^2 > 0,$$

$$\hat{B} \equiv \theta_1(s)\sigma_{11}^{-1}(s) + \theta_2(s)\sigma_{21}^{-1}(s) + \dots + \theta_d(s)\sigma_{d1}^{-1}(s),$$

$$\hat{C} = \left\{\left(\theta_1(s)\right)^2 + \left(\theta_2(s)\right)^2 + \cdots \left(\theta_d(s)\right)^2\right\} > 0.$$

From this we easily see that

$$f_1(s) = \frac{n\Phi(d_1)S_1(s)}{\hat{A} \times \mathbb{X}(s)} - \frac{1}{\hat{A}}\hat{B}.$$

Note that $f_1(s)$ is bounded. Next, we have

$$H_\lambda(\tau, T)H_f(0, \tau)$$
$$= exp\left\{-\int_\tau^T \theta_\lambda(s)dW(s) - \int_\tau^T r_\lambda(s) + \frac{1}{2}\|\theta_\lambda(s)\|^2 ds\right\} exp\left\{-\int_0^\tau \theta_f(s)dW(s) - \int_0^\tau r_f(s) + \frac{1}{2}\left\|\theta_f(s)\right\|^2 ds\right\}.$$

In this case, for $t < s \leq \tau$, $K(s,\omega) = [n\Phi(d_1)S_1(s)/\mathbb{X}(s), \infty) \times (-\infty, \infty)^{d-1}$, and $\delta(v(s)) = -[n\Phi(d_1)S_1(s)/\mathbb{X}(s)] \times f_1(s)$ on $\widetilde{K} = [0, \infty) \times \{0\}^{d-1}$; for $\tau < s < T$, $K(s,\omega) = [0, \infty) \times (-\infty, \infty)^{d-1}$, and $\delta(v(s)) = 0$, on $\widetilde{K} = [0, \infty) \times \{0\}^{d-1}$. Then, using the definitions of $r_\lambda(s)$, $\theta_\lambda(s)$, $r_f(s)$, and $\theta_f(s)$,

$H_\lambda(\tau, T)H_f(0, \tau) =$
$exp\left\{-\int_\tau^T r(s) + \frac{1}{2}\|\theta(s) + \sigma^{-1}(s)\lambda(s)\|^2 ds\right\} exp\left\{-\int_0^\tau r(s) - \frac{n\Phi(d_1)S_1(s)}{\mathbb{X}(s)} f_1(s) + \frac{1}{2}\|\theta(s) + \sigma^{-1}(s)f(s)\|^2 ds\right\} exp\left\{-\int_\tau^T \theta_\lambda(s)dW(s) - \int_0^\tau \theta_f(s)dW(s)\right\}$,

where f_1 is the first element of the vector f. Denote the ith row jth column element of inverse matrix of $\sigma(s)$ as $\sigma_{ij}^{-1}(s)$. Now, after some algebra, the dt term of $H_f(0, \tau)$ can be written

$$r(s) - \frac{n\Phi(d_1)S_1(s)}{\mathbb{X}(s)} f_1(s) + \frac{1}{2}\|\theta(s) + \sigma^{-1}(s)f(s)\|^2$$

$$= r(s) - \frac{n\Phi(d_1)S_1(s)}{\mathbb{X}(s)} f_1(s) + \frac{1}{2}\left[\hat{A}f_1(s)^2 + 2\hat{B}f_1(s) + \hat{C}\right]$$

$$= r(s) + \frac{1}{2}\hat{C} - \frac{1}{2\hat{A}}\left(\frac{n\Phi(d_1)S_1(s)}{\mathbb{X}(t)} - \hat{B}\right)^2.$$

The first thing to notice here is that this is clearly bounded, as $\frac{n\Phi(d_1)S_1(t)}{\mathbb{X}(t)}$ is bounded, as is $\theta_f(s)$, by the boundedness of $f_1(s)$. This shows that $\left(H_f(0,\tau)\right)^p$ is integrable for any p. A similar result holds for $H_\lambda(\tau,T)$.

According to the useful inequality $U(I(y)) \geq U(x) + y[I(y) - x]$, we have,

$$U\left(\xi_{f,\tau,\lambda}^{x^n}\right) \geq U(1) + \mathcal{Y}_{f,\tau,\lambda}(x^n)H_\lambda(\tau,T)H_f(0,\tau)\left[\xi_{f,\tau,\lambda}^{x^n} - 1\right],$$

$$-U\left(\xi_{f,\tau,\lambda}^{x^n}\right) \leq -U(1) - \mathcal{Y}_{f,\tau,\lambda}(x^n)H_\lambda(\tau,T)H_f(0,\tau)\left[\xi_{f,\tau,\lambda}^{x^n} - 1\right],$$

$$max\left(-U\left(\xi_{f,\tau,\lambda}^{x^n}\right),0\right) \leq max\left(-U(1) - \mathcal{Y}_{f,\tau,\lambda}(x^n)H_\lambda(\tau,T)H_f(0,\tau)\left[\xi_{f,\tau,\lambda}^{x^n} - 1\right],0\right).$$

Hence we have $\mathbb{E}\left(U^-\left(\xi_{f,\tau,\lambda}^{x^n}\right)\right) \leq |U(1)| + \mathcal{Y}_{f,\tau,\lambda}(x^n)\mathbb{E}\left(H_\lambda(\tau,T)H_f(0,\tau)\right) < \infty$. Here we note that $\mathcal{Y}_{f,\tau,\lambda}(x^n)$ is a monotonically continuous function and, hence, $\forall x^n > 0$ belongs to a closed interval, $\mathcal{Y}_{f,\tau,\lambda}(x^n)$ and is bounded.

We have now proved that $\mathbb{E}\left[U\left(\xi_{v,\tau,\grave{v}}^{x^n}\right)\right]$ is well defined, namely $\mathbb{E}\left[U^-\left(\xi_{f,\tau,\lambda}^{x^n}\right)\right] < \infty$, and so we are ready to prove that $\mathbb{E}\left[U\left(\xi_{v,\tau,\grave{v}}^{x^n}\right)\right]$ is optimal, that is $\mathbb{E}\left[U\left(\xi_{v,\tau,\grave{v}}^{x^n}\right)\right] \geq \mathbb{E}\left[U\left(\mathbb{X}^{x^n,\pi}(T)\right)\right]$. Now, consider any arbitrary portfolio process, $\pi(t) \in \mathcal{A}_v(x,0,\tau\text{-})$ for $t \in [0,\tau)$and $\pi(t) \in \mathcal{A}_{\grave{v}}(x_{\tau^+},\tau^+,T)$ for $t \in (\tau,T]$. Then, again using the inequality $U(I(y)) \geq U(x) + y[I(y) - x]$,

$$U\left(\xi_{v,\tau,\grave{v}}^{x^n}\right) \geq U\left(\mathbb{X}^{x^n,\pi}(T)\right) + \mathcal{Y}_{v,\tau,\grave{v}}(x^n)H_{\grave{v}}(\tau,T)H_v(0,\tau)\left[\xi_{v,\tau,\grave{v}}^{x^n} - \mathbb{X}^{x^n,\pi}(T)\right],$$

almost surely; therefore,

$$\begin{aligned}
&\mathbb{E}\left[U\left(\xi_{v,\tau,\grave{v}}^{x^n}\right)\right] \geq \mathbb{E}\left[U\left(\mathbb{X}^{x^n,\pi}(T)\right)\right] + \mathcal{Y}_{v,\tau,\grave{v}}(x^n)\mathbb{E}\left[H_{\grave{v}}(\tau,T)H_v(0,\tau)\left(\xi_{\grave{v}}^{x^n} - \mathbb{X}^{x^n,\pi}(T)\right)\right] \\
&= \mathbb{E}\left[U\left(\mathbb{X}^{x^n,\pi}(T)\right)\right] + \mathcal{Y}_{v,\tau,\grave{v}}(x^n) \\
&\qquad \times \mathbb{E}\Big[H_{\grave{v}}(\tau,T)H_v(0,\tau)\left(\xi_{\grave{v}}^{x^n} - \mathbb{X}^{x^n,\pi}(T)\right) + H_v(0,\tau)n(BS(\tau) - B(\tau)) \\
&\qquad - H_v(0,\tau)n(BS(\tau) - B(\tau))\Big] \\
&= \mathbb{E}\left[U\left(\mathbb{X}^{x^n,\pi}(T)\right)\right] + \mathcal{Y}_{v,\tau,\grave{v}}(x^n)\mathbb{E}\left[x^n - \left(H_{\grave{v}}(\tau,T)H_v(0,\tau)\mathbb{X}^{x^n,\pi}(T) + H_v(0,\tau)n(BS(\tau) - B(\tau))\right)\right] \\
&\geq \mathbb{E}\left[U\left(\mathbb{X}^{x^n,\pi}(T)\right)\right].
\end{aligned}$$

We can see that the above proof depends on the equality $\mathbb{E}\left[\xi_{\grave{v}}^{x^n}H_{\grave{v}}(\tau,T)H_v(0,\tau)+H_v(0,\tau)n(BS(\tau) - B(\tau))\right] = x^n$, which follows from the definition of $\xi_{\grave{v}}^{x^n}$, and the inequality $\mathbb{E}\left[H_{\grave{v}}(\tau,T)H_v(0,\tau)\mathbb{X}^{x^n,\pi}(T) + H_v(0,\tau)n(BS(\tau) - B(\tau))\right] \leq x^n$, (the budget constraint for π) in addition to the convexity of the direct and indirect utilities. Now, in the auxiliary market $\mathcal{M}_{v,\tau,\grave{v}}$, the dynamic of the wealth process is

$$d\mathbb{X}_{v,\tau,\grave{v}}(t) = \left(r(t)\mathbb{X}_{v,\tau,\grave{v}}(t)\right)dt$$
$$+ \mathbb{X}_{v,\tau,\grave{v}}(t)\left[(\delta(\grave{v}(t)) + \pi^*_{\grave{v}}(t)\grave{v}(t))1_{\{t>\tau\}} + (\delta(v(t)) + \pi^*_{v}(t)v(t))1_{\{t<\tau\}}\right]dt$$
$$+ \mathbb{X}_{v,\tau,\grave{v}}(t)\pi_{v,\tau,\grave{v}}{}^*(t)\sigma(t)dW_0(t) - \left(BS(t) - B(t)\right)1_{\{t=\tau\}}.$$

Without negative jumps, $-\left(BS(t) - B(t)\right)$, it is well-known that $H_{\grave{v}}(\tau,T)H_v(0,\tau)\mathbb{X}^{x^n,\pi}(T)$ is a super-martingale, hence by adding the negative jump, again it becomes a super-martingale.

Step 4. So far, we have studied, for an arbitrary exercise time τ, how to find the optimal portfolio process. The next question is how to determine the optimal exercise time τ^*.

Thanks to optimal stopping time theory, we obtain

$\tau^* = inf\{t \le u \le T\colon\ J(u, X_u) = M(X_u + nB(u), u)\}$.

Step 5. In the final step, we prove the equivalent martingale-based expression of the subjective an NTNH ESO price. We know that

$$\mathcal{X}_{f,\tau,\lambda}(y) \triangleq \mathbb{E}\left[H_\lambda(\tau,T)H_f(0,\tau)I\left(yH_\lambda(\tau,T)H_f(0,\tau)\right) + H_f(0,\tau)n(BS(\tau) - B(\tau))\right]$$

is finite, for every $y \in (0,\infty)$. Under this assumption, the function $\mathcal{X}_{f,\tau,\lambda}\colon\ (0,\infty) \to (0,\infty)$ is continuous and strictly decreasing, with $\mathcal{X}_{f,\tau,\lambda}(0+) = \infty$ and $\mathcal{X}_{f,\tau,\lambda}(\infty) = 0$. We let $\mathcal{Y}_{f,\tau,\lambda}(\cdot)$ denote its inverse and introduce the random variables

$\xi^{x^n}_{f,\tau,\lambda} \triangleq I\left(\mathcal{Y}_{f,\tau,\lambda}(x^n)H_\lambda(\tau,T)H_f(0,\tau)\right)$,

$U'(\xi^{x^n}_{f,\tau,\lambda}) \triangleq U'\left(I\left(\mathcal{Y}_{f,\tau,\lambda}(x^n)H_\lambda(\tau,T)H_f(0,\tau)\right)\right) = \mathcal{Y}_{f,\tau,\lambda}(x^n)H_\lambda(\tau,T)H_f(0,\tau)$.

We also know that $\xi^{x^n}_{f,\tau,\lambda} \triangleq \xi_\lambda^{X_*^{x^n}(\tau)} = I\left(\mathcal{Y}_\lambda\left(X_*^{x^n}(\tau) - n\left(BS(\tau) - B(\tau)\right)\right)H_\lambda(\tau,T)\right)$, where $X_*^{x^n}(\tau)$ denotes $X_{f,\tau,\lambda}^{x^n,\pi^*_{f,\tau,\lambda}}$, and $U'(\xi^{x^n}_{f,\tau,\lambda}) = U'\left(\xi_\lambda^{X_*^{x^n}(\tau)}\right) = \mathcal{Y}_\lambda\left(X_*^{x^n}(\tau) - n(BS(\tau) - B(\tau))\right)H_\lambda(\tau,T)$.

Therefore, $\mathcal{Y}_{f,\tau,\lambda}(x_n)H_\lambda(\tau,T)H_f(0,\tau) = \mathcal{Y}_\lambda\left(X_*^{x^n}(\tau) - n(BS(\tau) - B(\tau))\right)H_\lambda(\tau,T)$,

$$H_f(0,\tau) = \frac{\mathcal{Y}_\lambda\left(X_*^{x^n}(\tau) - n(BS(\tau) - B(\tau))\right)}{\mathcal{Y}_{f,\tau,\lambda}(x^n)} = \frac{M'\left(X^{x,\pi^*}(\tau) + nB(\tau), \tau\right)}{L'(x,n,0)},$$

since the payoff is the sum of the NTNH American ESOs and outside wealth. The second equality follows from the fact that $M'(x,t) \equiv M_x(x,t) = \mathcal{Y}_{\hat{v}}(x)$, in general, with a similar result for L. The price of the NTNH American Contingent Claim is defined as its expectation of the payoff at the optimal exercise time, discounted by the SDF. Hence, $\hat{p}(x,n,0) = \mathbb{E}\left[H_f(\tau^*)B(\tau^*)\right] = \mathbb{E}^f\left[\gamma_f(\tau^*)B(\tau^*)\right]$.

Due to the Markov property of stock price, we have

$$\hat{p}(x_t,n,t) = \frac{\mathbb{E}\left[H_f(\tau^*)B(\tau^*)|\mathcal{F}_t\right]}{H_f(t)} = \frac{\mathbb{E}^f\left[\gamma_f(\tau^*)B(\tau^*)|\mathcal{F}_t\right]}{\gamma_f(t)}.$$

□

Appendix C. Proof of Proposition 2

The primitive objects of a stochastic control problem are the set $Z \subset \Re^3$ of states $Z(t) \triangleq (\mathbb{X}(t), N(t), S_1(t))$, where $\mathbb{X}(t)$ is the total wealth process, $N(t)$ is the accumulated number of ESO shares that have been exercised, and $S_1(t)$ is the stock's price.

In the Merton Model, the total wealth is the only state variable. The reason the primary assets' price vector $S(t)$ is not a state variable is that the terminal wealth is sufficient to determine the utility, and terminal wealth is linear in $S(t)$. Also, the control variable portfolio process does not affect the stock price of the next period.

In our partial exercise model, as the stock price affects whether the option holder will exercise the option at any particular time in the post-vesting period before the expiration date, the stock price is necessarily included in the state variable set. In the case that the NTNH ESO is European, the stock price is still a necessary state variable because it determines whether the investor will exercise the option at the maturity. Additionally, the exercise payoff will determine the terminal wealth at the same time.

The state variable set also includes the cumulative number of shares of exercised options $\widehat{N}(t)$. On one hand, $n - \widehat{N}(t)$ is the maximum of number of shares that can be exercised, which determines the opportunity set of the control variable $\dot{n}(t)$. On the other hand, the control variable $\dot{n}(t)$ determines the cumulative number of shares of exercised options.

The dynamics of the system are as follows:

$$d\mathbb{X}(t) = \sum_{i=1}^{d} \pi^i_{f_{\widehat{N}}}(t)\,\mathbb{X}(t)\left\{b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\right\} + \left\{1 - \sum_{i=1}^{d} \pi^i_{f_{\widehat{N}}}(t)\right\}\mathbb{X}(t)r(t)dt \\ - \dot{n}(t)\big(BS(t) - B(t)\big)dt, \tag{C.1}$$

$$\mathbb{X}(0) = x^n \triangleq x + nBS(0),$$

$$dN(t) = \dot{n}(t)dt, \qquad N(0) = \int_0^{t=0} \dot{n}(s)ds = 0, \tag{C.2}$$

$$\frac{dS_1(t)}{S_1(t)} = b_1 dt + \sigma_{11} dW_1(t), S_1(0) = s_1. \tag{C.3}$$

The unique solution is taken as $\mathbb{X}^{*,\dot{n}}(t)$, where $\pi^*_{f_{\widehat{N}}}(t)$ is the optimal portfolio process given the constraint, and the drift adjustment $v(t)$ with respect to the auxiliary market $\mathcal{M}_v$ associated with the constraint $K(t,\omega)$ in (32) is $f_{\widehat{N}}(t)$.

Now, given any initial state $(\mathbb{X}(t), N(t), S_1(t))$ the utility of any admissible control $\dot{n}(t)$ is well defined as

$V^q(z) = \mathbb{E}\big[U(\mathbb{X}^{*,\dot{n}}_T)\big]$. The indirect utility at time $t = 0$ at initial state z is then $\mathbb{J}(z, 0) = sup_{\dot{n}(t)\in\mathcal{V}_{ad}[0,T]}\, \mathbb{E}\left[U(\mathbb{X}^{*,\dot{n}}_T)\right]$. We postulate that $\mathbb{J}(z, 0)$ is in $C^{2,1}(\mathcal{Z}, [0,T])$. According to the dynamic programming principle,

$$\mathbb{J}(z,0)=\mathbb{J}\left(\left(\mathbb{X}^{*,\dot{n}}(0),\widehat{N}(0),S_1(0)\right),0\right)=sup_{\dot{n}\in\mathcal{V}_{ad}[0,t]}\,\mathbb{E}\,[\mathbb{J}(Z_t,t))].$$

By Taylor's theorem, we have

$$\mathbb{J}(z,dt)=\mathbb{J}(z,0)+\sup_{\dot{n}\in\mathcal{V}_{ad}[0,T]}\mathbb{E}\{\mathbb{J}_Z(Z_t,t)g(\dot{n},Z_t)dt+\mathbb{J}_t(Z_t,t)dt$$
$$+\frac{1}{2}tr[h(\dot{n},Z_t)h(\dot{n},Z_t)'\mathbb{J}_{ZZ}(Z_t,t)]dZ\cdot dZ+O((dt)^2)\Big\}.$$

Hence we obtain the Bellman equation

$$0=\frac{\partial\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial t}dt+\frac{\partial\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial\mathbb{X}^{*,\dot{n}}(t)}d\mathbb{X}^{*,\dot{n}}(t)+\frac{\partial\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial\widehat{N}(t)}d\widehat{N}(t)+$$
$$\frac{\partial\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial S_1(t)}dS_1(t)+\frac{1}{2}\frac{\partial^2\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial^2\mathbb{X}^{*,\dot{n}}(t)}d\mathbb{X}^{*,\dot{n}}(t)d\mathbb{X}^{*,\dot{n}}(t)+$$
$$\frac{\partial^2\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial\mathbb{X}^{*,\dot{n}}(t)\partial\widehat{N}(t)}d\mathbb{X}^{*,\dot{n}}(t)d\widehat{N}(t)+\frac{1}{2}\frac{\partial^2\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial^2\widehat{N}(t)}d\widehat{N}(t)d\widehat{N}(t)+$$
$$\frac{1}{2}\frac{\partial^2\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial^2 S_1(t)}dS_1(t)dS_1(t)+\frac{\partial^2\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial\mathbb{X}^{*,\dot{n}}(t)\partial S_1(t)}d\mathbb{X}^{*,\dot{n}}(t)dS_1(t)+$$
$$\frac{\partial^2\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial\widehat{N}(t)\partial S_1(t)}d\widehat{N}(t)dS_1(t).$$

We know that

$$d\widehat{N}(t)=\dot{n}(t)dt,\ \ d\widehat{N}(t)d\widehat{N}(t)=(\dot{n}(t))^2(dt)^2\in o(dt),\ \ d\mathbb{X}^{*,\dot{n}}(t)d\widehat{N}(t)\in o(dt)$$
$$dS_1(t)d\widehat{N}(t)\in o(dt),\ dS_1(t)dS_1(t)=(b_1S_1dt+\sigma_{11}S_1dW_1)(b_1S_1dt+\sigma_{11}S_1dW_1)=\sigma_{11}^2S_1^2dt$$
$$d\mathbb{X}^{*,\dot{n}}(t)d\mathbb{X}^{*,\dot{n}}(t)=$$
$$\left(\textstyle\sum_{i=1}^d\pi_{f_{\widehat{N}}}^i(t)\,\mathbb{X}^{*,\dot{n}}(t)\{\sum_{j=1}^d\sigma_{ij}(t)dW_j(t)\}\right)\left(\sum_{i=1}^d\pi_{f_{\widehat{N}}}^i(t)\,\mathbb{X}^{*,\dot{n}}(t)\{\sum_{j=1}^d\sigma_{ij}(t)dW_j(t)\}\right)=$$
$$\textstyle\sum_{i=1}^d\sum_{j=1}^d\sum_{m=1}^d\sum_{n=1}^d\left\{\left(\mathbb{X}^{*,\dot{n}}(t)\right)^2\pi_{f_{\widehat{N}}}^i(t)\pi_{f_{\widehat{N}}}^m(t)\sigma_{ij}(t)\sigma_{mn}(t)dt\right\}=\left(\mathbb{X}^{*,\dot{n}}(t)\right)^2\pi'(t)\sigma'(t)\sigma(t)\pi(t)d$$
$$d\mathbb{X}^{*,\dot{n}}(t)dS_1(t)=\left(\textstyle\sum_{i=1}^d\pi_{f_{\widehat{N}}}^i(t)\,\mathbb{X}^{*,\dot{n}}(t)\{\sum_{j=1}^d\sigma_{ij}(t)dW_j(t)\}\right)(b_1S_1dt+\sigma_{11}S_1dW_1)=$$
$$\mathbb{X}^{*,\dot{n}}(t)\left\{\textstyle\sum_{i=1}^d\pi_{f_{\widehat{N}}}^i(t)\sigma_{i1}(t)\right\}(\sigma_{11}S_1dt).$$

Taking expectations on both sides and dividing both sides by dt, we get

$$0=sup_{\dot{n}\in\mathcal{V}_{ad}[0,T]}\Bigg\{\frac{\partial\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial t}+\frac{\partial\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial X^{*,\dot{n}}(t)}\Big(\mathbb{X}^{*,\dot{n}}(t)r(t)+$$
$$\textstyle\sum_{i=1}^d\pi_{f_N}^i(t)\,\mathbb{X}^{*,\dot{n}}(t)\big(b_i(t)-r(t)\big)-\dot{n}(t)\big(BS(t)-B(t)\big)\Big)+$$
$$\frac{\partial\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial\widehat{N}(t)}\dot{n}(t)+\frac{\partial J(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial S_1(t)}b_1S_1(t)+$$
$$\frac{1}{2}\frac{\partial^2\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial(\mathbb{X}^{*,\dot{n}}(t))^2}\left(\mathbb{X}^{*,\dot{n}}(t)\right)^2\left(\pi(t)\right)^T\left(\sigma(t)\right)^T\sigma(t)\pi(t)+$$
$$\frac{1}{2}\frac{\partial^2\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial(S_1(t))^2}\sigma_{11}^2S_1^2+\frac{\partial^2\mathbb{J}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t)}{\partial\mathbb{X}^{*,\dot{n}}(t)\partial S_1(t)}\mathbb{X}^{*,\dot{n}}(t)\{\textstyle\sum_{i=1}^d\pi_{f_N}^i(t)\sigma_{i1}(t)\}\sigma_{11}S_1\Bigg\}$$

This is the Hamilton-Jacobi-Bellman equation. We take the derivative with respect to the control variable $\dot{n}(t)$ and set it equal to zero. Then we obtain the FOC, which implies the optimal $\dot{n}(t)$:

$$-\big(BS(t)-B(t)\big)\frac{\partial\mathbb{J}\big(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t\big)}{\partial\mathbb{X}^{*,\dot{n}}(t)}+\frac{\partial\mathbb{J}\big(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S_1(t),t\big)}{\partial\widehat{N}(t)}=0$$

□

Appendix D. Proof of Lemma 1.

Suppose $U(\cdot)=ln(\cdot)$ and $K(t,\omega)=[a,\infty)\times(-\infty,\infty)^{d-1}$, where $a=a(t,\omega)>0$, so that $\delta(v(s))=-av_1(s)$ on $\widetilde{K}=[0,\infty)\times\{0\}^{d-1}$, and $\mathcal{Y}_v(x)\triangleq y=(T+1)/x$, where $v(s)\triangleq\big(v_1(s),\ldots,v_d(s)\big)$, and x is the initial wealth at time $t=0$. Then $\forall y=\mathcal{Y}_v(x)=(T+1)/x\in(0,\infty)$; note that

$$\tilde{J}(y;v)=\tilde{J}(\mathcal{Y}_v(x);v)=\mathbb{E}\big[\widetilde{U}\big(\mathcal{Y}_v(x)H_v(T)\big)\big]=-\mathbb{E}\big[1+ln\big(\mathcal{Y}_v(x)H_v(T)\big)\big]$$
$$=\mathbb{E}\left[-1-ln\left(\frac{T+1}{x}H_v(T)\right)\right]=-\left[1+ln\frac{T+1}{x}\right]+\mathbb{E}\left[ln\frac{1}{H_v(T)}\right]$$
$$=-[1+lny]+\mathbb{E}\int_0^T\left[r(s)+\delta\big(v(s)\big)+\tfrac{1}{2}\|\theta(s)+\sigma^{-1}(s)v(s)\|^2\right]ds.$$

To prove that $\lim_{\|v\|\to\infty}\tilde{J}(y;v)=\infty$, we need only to prove that $\lim_{\|v\|\to\infty}\left[r(s)+\delta\big(v(s)\big)+\tfrac{1}{2}\|\theta(s)+\sigma^{-1}(s)v(s)\|^2\right]=\infty$.

We have $\forall x\in(0,\infty)$,

$\lim_{\|v\|\to\infty}\left[r(s)+\delta\big(v(s)\big)+\tfrac{1}{2}\|\theta(s)+\sigma^{-1}(s)v(s)\|^2\right]=\lim_{\|v\|\to\infty}\left[r(s)-av(s)+\tfrac{1}{2}\|\theta(s)+\sigma^{-1}(s)v(s)\|^2\right]=\infty.$

On the other hand, for any fixed $v\in\widetilde{K}$, $\mathbb{E}\int_0^T\left[r(s)+\delta\big(v(s)\big)+\tfrac{1}{2}\|\theta(s)+\sigma^{-1}(s)v(s)\|^2\right]ds$ is a constant (i.e., independent of y) denoted as C; and so

$\lim_{y\downarrow 0}\tilde{J}(y;v)=\lim_{y\downarrow 0}(-[1+lny]+C)=\infty.$

□

Appendix E. Proof of Remark 6.

The key to understanding why subjective prices can be higher than objective prices with continuous partial exercise is the following observation. Consider random variables, X, Y, and n, where $n>0$. It is possible to have $\mathbb{E}[X]<\mathbb{E}[Y]$, yet $\mathbb{E}[nX]>\mathbb{E}[nY]$. This can hold if n and X are highly correlated and n and Y have low correlation. (A simple example, and a general condition under which this holds are available upon request.) In our case, we want to show that $\mathbb{E}\left[\int_0^T H_{f_{\widehat{N}^*}}(s)\dot{n}^*(s)B(s)ds\right]>\mathbb{E}\left[\int_0^T H_0(s)\dot{n}^*(s)B(s)ds\right]$ is possible. As shown in Step 3 of Appendix B, the drift term of $H_{f_{\widehat{N}^*}}(s)$ is given by $r(s)+\frac{1}{2}\hat{C}-\frac{1}{2\hat{A}}\left(\frac{n\Phi(d_1)S_1(s)}{\mathbb{X}(t)}-\hat{B}\right)^2$, for constants, $\hat{A}$, $\hat{B}$ and $\hat{C}$. for $\frac{n\Phi(d_1)S_1(s)}{\mathbb{X}(t)}>\hat{B}$, as S_1 increases, the above expression decreases, while for $\frac{n\Phi(d_1)S_1(s)}{\mathbb{X}(t)}>\hat{B}$, as S_1 increases, the above expression increases. Based on our parameter setting, under a high volatility regime (50% stock volatility, 30% index volatility), $\hat{B}=2.691$; under a low volatility regime (20% stock volatility, 10% index volatility), $\hat{B} = 0.014$. Hence, the chance that $\frac{n\Phi(d_1)S_1(s)}{\mathbb{X}(t)} > \hat{B}$, or equivalently, that $H_{f_{\hat{N}^*}}(s)$ is positively correlated with S_1, is greater under the low volatility regime. On the other hand, our Monte Carlo results suggest that as S_1 increases $\dot{n}(0, t_i)$ increases. So, $\dot{n}$ can be either positively or negatively correlated to $H_{f_{\hat{N}^*}}$.

Appendix F. Proof of Proposition 3

Without loss of generality, we assume that the current time is $t = 0$. The total initial wealth is $\mathbb{X}(0)$. If we define the initial marginal utility with respect to total initial wealth as y, then we have $\mathcal{X}_{f_{\hat{N}}}(y) = \mathbb{E}\left(H_{f_{\hat{N}}}(T) I\left(yH_{f_{\hat{N}}}(T)\right)\right)$.

Under the log utility assumption, we have $y = \mathcal{Y}_v(x) = 1/x^n$. Now we have simplified the left-hand side of the FOC, and next we work on the right-hand side. The relationship between total initial wealth and total initial marginal utility implies

$\mathbb{J}\left(\mathbb{X}^{*,\dot{n}}(t), \hat{N}(t), S_1(t), t\right) = \mathbb{E}\left(U\left(I\left(\mathcal{Y}_{f_{\hat{N}}}(x^n) H_{f_{\hat{N}}}(T)\right)\right)\right)$, that is,

$$\begin{aligned} \mathbb{J}\left(\mathbb{X}^{*,\dot{n}}(t), \hat{N}(t), S_1(t), t\right) &= \mathbb{E}\left(ln\left(\frac{x^n}{H_{f_{\hat{N}}}(T)}\right)\right) \\ &= ln(x^n) + \left(\int_0^t \left\{r(s) + \delta(f_{\hat{N}}(s)) + \frac{1}{2}\left\|\theta_{f_{\hat{N}}}(s)\right\|^2\right\} ds\right), \end{aligned} \tag{D.1}$$

Substituting the above results into the FOC, we obtain the following simplified FOC:

$$-\frac{BS(t) - B(t)}{\mathbb{X}(t)} + ln\left(\mathbb{X}(t)\right) \frac{\partial\left(-\frac{\left(n - \hat{N}(t)\right)\Phi\left(d_1(t)\right)S_1(t)}{\mathbb{X}(t)} f_{\hat{N}}(t) + \frac{1}{2}\left\|\theta_{f_{\hat{N}}}(t)\right\|^2\right)}{\partial \hat{N}(t)} = 0.$$

We rewrite it more explicitly by differentiating between the pre-exercise total wealth at time t and the after-exercise total wealth at time t. By considering the vesting period of the stock and option, respectively, then we have

$$-\frac{BS(t) - B(t)}{pre\_\mathbb{X}(t)} + ln\left(post\_\mathbb{X}(t)\right) \times \frac{\partial\left(-\left(\frac{\left(n_{option} - \hat{N}(t)\right)\Phi\left(d_1(t)\right)S_1(t) + n_{stock}1_{t<t_{vs}}S_1(t)}{post\_\mathbb{X}(t)}\right) f_{\hat{N}}(t) + \frac{1}{2}\left\|\theta_{f_{\hat{N}}}(t)\right\|^2\right)}{\partial \hat{N}(t)} = 0,$$

where $post\_\mathbb{X}(t) = pre\_\mathbb{X}(t) - \dot{n}(t)dt(BS(t) - B(t))$, and t is after the end of the vesting period for the ESOs . Also, $\hat{N}(t) \equiv 0$, before the end of the vesting period for the ESOs.

□

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