Instantaneous Squared VIX and VIX Derivatives

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Keywords: VIX; VXST; Instantaneous squared VIX; VIX futures; VIX option

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Abstract

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1 Introduction

The VIX options and futures market managed to grow extremely rapidly.[1] For example, the market size of VIX futures and options expand to be an average daily volume 158,580 and 567,460 in 2013 compared with 1,731 and 23,491 in 2006 since their introduction in 2004 and 2006 respectively.[2] Recently, on October 1, 2013, the Chicago Board Options Exchange (CBOE) introduced the Short-Term Volatility Index (VXST) which is 9-day VIX.[3] Subsequently, the CBOE Futures Exchange (CFE) launched VXST futures on February 13, 2014, and the CBOE launched VXST options on April 10, 2014. The new development of VIX derivatives indicates that the market for volatility trading is highly demanded by practitioners. It not only generates rich information beyond traditional equity index/stock derivatives but also pushes an urgent need for cutting-edge academic research on consistent pricing between VIX and VXST options. However, there is no commonly agreed framework within which we can both theoretically price VIX derivatives and SPX options simultaneously and empirically calibrate model parameters in an efficient way yet, not to mention study on VXST derivatives.

In this paper, we try to fill this gap by employing Luo and Zhang's (2012) newly-developed concept of instantaneous squared VIX (ISVIX), which is the sum of instantaneous Brownian and jump variances of SPX return.[4] Assuming that the ISVIX follows a mean-reverting jump-diffusion process with a stochastic long-term mean, we are able to derive analytical formulas for the VIX, VIX futures and options. Further, we use the informative

[1] Generally, the VIX refers to the 30-day future volatility. In this paper, the VIX includes volatility index with any arbitrary time-to-maturity. For simplicity, we denote VIX as the 30-day VIX and specify explicitly for VIX with other maturities when necessary.

[2] Data source: http://www.cboe.com/micro/vix/pricecharts.aspx. For an average VIX value 20, these volumes correspond to market values of 3.2 billion and 1.1 billion US dollars. Carr and Lee (2009) provide an overview of the market for volatility derivatives including variance swaps and VIX futures and options.

[3] For more information, please refer to: www.cboe.com/VXST. Note that the S&P 500 three-month volatility index under the ticker "VXV" was lunched on November 12, 2007 by the CBOE.

[4] The jump variance of SPX return is the second term of Equation (3), which is different from variance swap and will be further clarified when our model is introduced.

VIX term structure data to determine the mean-reverting speed and sequentially calibrate the volatility and jump parameters in the ISVIX process to market prices of VIX futures and options.

There is also a fast growing literature on VIX derivatives pricing. Up to now, these studies can be divided into three groups. The first group starts from classical option pricing theories in specifying underlying dynamics and derive VIX from the underlying. Due to fruitful option pricing development, this approach is widely used. In fact, Zhang and Zhu (2006) is the first study on the VIX futures by considering Heston (1993) model. Zhu and Zhang (2007) extend Zhang and Zhu (2006) to a model with time-varying long-term mean of variance. Later on, Lin (2007), Lu and Zhu (2010), Dupoyet, Daigler, and Chen (2011) and Zhu and Lian (2012) examine more complicated models for VIX futures. Meanwhile, Sepp (2008a, b), Albanese, Lo and Mijatovi'c (2009), Lin and Chang (2009, 2010)[5], Li (2010), Wang and Daigler (2011), Chung et al (2011), Branger and Volkert (2012), Song and Xiu (2012), Lian and Zhu (2013), Bardgett, Gourier and Leippold (2013), Chen and Poon (2013), Lo et al (2013), Romo (2014) and Papanicolaou and Sircar (2014) investigate various specifications to pricing VIX option. For example, Bardgett, Gourier and Leippold (2013) conduct a comprehensive analysis by combining time series of SPX, VIX and their options data and employing an accurate approximation and a powerful filter method in order to get estimation of totally 23 parameters at one time. They find that a stochastic central tendency of volatility and jumps in volatility are important in capturing volatility smiles in both the SPX and VIX markets and tail of variance risk-neutral distribution, respectively. Papanicolaou and Sircar (2014) propose a regime-switching augmented Heston model to capture VIX implied volatilities. The second group is more natural in that it directly model volatility/variance index. For example, Mencia and Sentana (2013) and

[5]Recently, Cheng et al (2012) prove that Lin and Chang's (2009, 2010) formula is not an exact solution of their pricing equation.

Carr and Madan (2013) start from the VIX while Madan and Pistorius (2014) consider the squared VIX. One drawback of this approach is that the tight link between VIX and its underlying is not considered and inconsistency may arise in pricing SPX options and replicating VIX index simultaneously. Bergomi (2008) and Cont and Kokholm (2013) overcome this drawback by modeling the dynamics of forward variance swap rate, which is equivalent to the squared VIX. In addition, some studies focus on modeling the dynamics of underlying volatility/variance, see e.g., Grunbichler and Longstaff (1996), Detemple and Osakwe (2000). Recently, Goard and Mazur (2013) consider VIX option pricing under the 3/2-model for underlying volatility. The third group are Huskaj and Nossman (2013) and Lin (2013), in which they specify exogenous dynamics for the VIX futures. In particular, Huskaj and Nossman (2013) investigate the normal inverse Gaussian process for term structure of VIX futures, while Lin (2013) defines a proxy of future VIX as a forward VIX squared normalized by the VIX futures and studies VIX option by considering various volatility functions for the proxy.

The purpose of this paper is to provide a unified theoretical framework to price VIX derivatives, including futures and options written on VIX, VXST and potentially other VIX with arbitrary time-to-maturity. We demonstrate that analytical formulas of VIX futures and options can be easily derived with the help of the ISVIX, and parameters can be sequentially calibrated by using VIX term structure and VIX option data. Our contributions are as follows. First, by building on the ISVIX, which is the sum of instantaneous Brownian and jump variances of SPX return, we maintain inherent linkage between SPX and VIX, thus introduce a consistent method to pricing both SPX option and VIX derivatives. Second, when modeling ISVIX and leaving underlying dynamics unspecified, we provide insights into pricing VIX derivatives by focusing on (instantaneous squared) VIX in the spirit of the second group studies. Calibration with the market data of VIX option implied volatility surface shows that our theory provides an efficient way of extracting the complete information from VIX derivatives market for the dynamics of underlying SPX. That is, ISVIX can be used to derive any VIXs with various maturities and estimate parameters by incorporating all information in VIX term structure, current 30-day VIX options and even other maturity VIX options that might be developed in the future. Third, we propose a sequential empirical method to efficiently calibrate parameters by using appropriate market data. As emphasized by Bardgett, Gourier and Leippold (2013) that it is important to combine both SPX and VIX option data to estimate parameters and risk premia. However, computational burden and difficulty in estimating jumps with short period data challenge this practice. More importantly, Duan and Yeh (2012) point out that it is crucial to use the VIX term structure in estimation in order to obtain reliable the risk-neutral volatility dynamic. In this sense, Bardgett, Gourier and Leippold (2013)'s one-step estimation without the VIX term structure may also suffer from insufficiency. We overcome this obstacle by proposing a sequential method to estimate parameters with the help of corresponding market data, which will be discussed in more detail in later section.

Specially, in terms of modelling, a stochastic central tendency is necessary to capture the dynamics of VIX term structure and it is nontrivial to derive VIX derivatives pricing formula in this setting. It is interesting to note that the importance of modeling the long-term mean of the variance as the second factor is well recognized in the recent volatility/variance derivatives literature, though Duffie, Pan and Singleton (2000) mention a stochastic volatility model with a central tendency and Bates (2000) allows the return variance to be the sum of the two independent factors which follow square root processes. For example, Zhang and Huang (2010) study the CBOE S&P 500 three-month variance futures and suggest that a floating long-term mean level of variance is a good choice for the variance futures pricing. Zhang, Shu and Brenner (2010) build a two-factor model for VIX futures, where the long term mean level of variance is treated as a pure Brownian motion. They find that the model produces good forecasts of the VIX futures prices. Egloff, Leippold and Wu (2010) show that the long-term mean factor is important to capture variance risk dynamics in variance swap markets.[6]

The rest of the paper is organized as follows. Section 2 discuss the design of our model setup. Section 3 derives analytical formulas for VIX futures and options. Section 4 introduces data, calibration procedure and the empirical results. Section 5 provides concluding remarks.

2 The design of model setup

In this section, we review the concept of ISVIX and main results of Luo and Zhang (2012), which our theory is built on.

It is an outstanding issue that how to design an efficient setup which can consistently modeling both SPX and VIX derivatives. The main reason is that the VIX is by construction a combination of SPX options. We build our model based on following observations. First, jumps in SPX index and volatility are well documented in the literature. Second, stochastic long-term mean of volatility is necessary to capture dynamics of volatility term structure. Third, it is important to derive an intuitive formula for VIX and sperate information in different markets in order to reduce computational burden in estimation. To deal with these concerns, we employ the model proposed by Luo and Zhang (2012), that is the following model for the SPX index, S_t, under the risk-neutral measure Q,

$$
\begin{aligned}
dS_t/S_{t^-} &= rdt + \sqrt{v_t}dB_{1,t}^Q + (e^x - 1)dN_t^x - \lambda_t^x E^Q(e^x - 1)dt, \\
V_t &\equiv v_t + 2\lambda_t^x E^Q(e^x - 1 - x), \\
dV_t &= \kappa(\theta_t - V_t)dt + \sigma_V \sqrt{V_t}dB_{2,t}^Q + ydN_t^y - \lambda^y E^Q(y)dt, \\
d\theta_t &= \sigma_\theta dB_{3,t}^Q
\end{aligned}
\tag{1}
$$

[6] While Adrian and Rosenberg (2008) and Christoffersen, Jacobs, Ornthanalai, and Wang (2008) investigate long-run and short-run volatility components, Christoffersen, Heston and Jacobs (2009) consider the two-factor square root process.

where r is the risk-free rate, $B^Q_{1,t}$ and $B^Q_{2,t}$ are two related Brownian motions. N^x_t is a pure jump processes with jump size x and jump intensity λ^x_t. κ is the mean-reverting speeds of V_t. θ_t is the long-term mean level of V_t. N^y_t is another pure jump processes with jump size y and jump intensity λ^y and $B^Q_{3,t}$ is another Brownian motion which is independent of $B^Q_{1,t}$ and $B^Q_{2,t}$.[7]

Then, the concept of instantaneous squared VIX can be introduced, that is,

Definition 1 *(Luo and Zhang, 2012) For the jump-diffusion model specified in (1), the instantaneous squared VIX (ISVIX), V_t, is given by*

$$V_t \equiv \lim_{\tau \to 0} \frac{2}{\tau} E^Q_t \left[\int_t^{t+\tau} \frac{dS_u}{S_u} - d(\ln S_u) \right], \tag{2}$$

$$= v_t + 2\lambda_t E^Q(e^x - 1 - x). \tag{3}$$

From (2), we can see that the ISVIX is the limit of squared VIX when maturity approaches zero. Because of the martingale specification for innovation parts in the dynamics of V_t, we can obtain an intuitive formula for the squared VIX, that is,

Proposition 1 *(Luo and Zhang, 2012) Under the previous jump-diffusion model, the squared VIX, at time t, with maturity τ_0, VIX^2_{t,τ_0}, can be expressed as*

$$VIX^2_{t,\tau_0} = \omega V_t + (1-\omega)\theta_t, \tag{4}$$

where

$$\omega \equiv \omega(\tau_0) = \frac{1 - e^{-\kappa\tau_0}}{\kappa\tau_0}. \tag{5}$$

Remark 1 τ_0 can be arbitrary time-to-maturity. Particularly, $\tau_0 = \frac{30}{365}$ and $\tau_0 = \frac{9}{365}$ correspond to the VIX and VXST respectively.

[7]At this point, we use constant jump intensity and martingale long-term mean to demonstrate our model, they can be relaxed very easily, but we do not want move too far as we want our model to be simple, yet useful.

3 Theory of VIX option pricing

In this section, we derive conditional probability density function of VIX and analytical formulas for VIX options and futures. Also, to illustrate accuracy of current model, we compare our VIX futures formula with the exact one studied in Zhang and Zhu (2006) for the Heston model. Further, we investigate sensitivity of VIX option implied volatility with respect to the parameters.

3.1 Conditional probability density function of VIX

For the VIX derivatives pricing, the key is to derive the joint moment generating function of (V_T, θ_T) conditional upon current observable variables (V_t, θ_t).

Lemma 1 *Under the previous jump-diffusion model, the conditional probability density function of VIX_T, $p(VIX_T|VIX_t)$, is given by*

$$p(VIX_T|VIX_t) \quad = \quad \frac{2VIX_T}{\pi}\int_0^\infty Re\left[e^{-i\phi VIX_T^2}\tilde{f}(i\phi;t,\tau,V_t,\theta_t)\right]d\phi_R, \tag{6}$$

where $\phi = \phi_R + \phi_I i$ is a complex variable and $\tilde{f}(\phi;t,\tau,V_t,\theta_t)$ is the moment generating function of VIX_T^2, which is given by

$$\begin{aligned}\tilde{f}(\phi;t,\tau,V_t,\theta_t) &= E^Q[e^{\phi VIX_T^2}|\mathcal{F}_t],\\ &= E^Q[e^{\phi\omega V_T+\phi(1-\omega)\theta_T}|\mathcal{F}_t],\\ &= f(\phi\omega,\phi(1-\omega);t,\tau,V_t,\theta_t),\end{aligned} \tag{7}$$

where $f(\phi,\psi;t,\tau,V_t,\theta_t)$ is the moment generating function of (V_T,θ_T) in the following

$$f(\phi,\psi;t,\tau,V_t,\theta_t) = e^{A_1(\phi,\psi;\tau)+A_2(\phi,\psi;\tau)V_t+A_3(\phi,\psi;\tau)\theta_t+A_4(\phi,\psi;\tau)}, \tag{8}$$

where

$$A_1(\phi,\psi;\tau) = \frac{-\lambda^y E^Q(y)}{\kappa}(A_3(\phi,\psi;\tau)-\psi)+\frac{1}{2}\sigma_\theta^2\int_0^\tau A_3^2(\phi,\psi;u)du, \quad (9)$$

$$A_2(\phi,\psi;\tau) = \frac{2\kappa\phi}{(2\kappa-\sigma_V^2\phi)e^{\kappa\tau}+\sigma_V^2\phi}, \quad (10)$$

$$A_3(\phi,\psi;\tau) = \psi-\frac{2\kappa}{\sigma_V^2}\ln\left[1+\frac{\sigma_V^2\phi}{2\kappa}(e^{-\kappa\tau}-1)\right], \quad (11)$$

$$A_4(\phi,\psi;\tau) = \int_0^\tau \lambda^y E^Q[e^{A_2(\phi,\psi;u)y}-1]du. \quad (12)$$

See Appendix A for proof.

Further, the characteristic function of VIX_T^2 is given by $\tilde{f}(i\phi;t,\tau,V_t,\theta_t)$ and the corresponding conditional density function of VIX_T^2, $p(VIX_T^2|VIX_t^2)$, is the (generalized) Fourier inversion of the characteristic function

$$p(VIX_T^2|VIX_t^2)=\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[e^{-i\phi VIX_T^2}\tilde{f}(i\phi;t,\tau,V_t,\theta_t)\right]d\phi_R, \quad (13)$$

3.2 VIX options and futures

The price of a European VIX call option, $c(t,T)$, at time t with maturity T and strike K is

$$\begin{aligned} c(t,T) &= e^{-r(T-t)}E^Q[(VIX_T-K)^+|\mathcal{F}_t], \\ &= e^{-r(T-t)}\int_0^\infty p(VIX_T|VIX_t)(VIX_T-K)^+dVIX_T, \\ &= \frac{e^{-r\tau}}{\pi}\int_0^\infty 2VIX_T\int_0^\infty \mathrm{Re}\left[e^{-i\phi VIX_T^2}\tilde{f}(i\phi;t,\tau,V_t,\theta_t)\right]d\phi_R(VIX_T-K)^+dVIX_T. \end{aligned} \quad (14)$$

Because Lewis (2000) and Poularikas (2000) derive

$$\int_0^\infty e^{-\xi z}(\sqrt{z}-K)^+dz=\frac{\sqrt{\pi}}{2}\frac{1-\mathrm{erf}(K\sqrt{\xi})}{(\sqrt{\xi})^3}, \quad (15)$$

where ξ is a complex Fourier transform variable with $\mathrm{Re}[\xi] > 0$, and $\mathrm{erf}(u)$ is a complex error function given by $\mathrm{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-v^2} dv$, we have

$$\begin{aligned} I_1 &= \int_0^\infty 2VIX_T e^{-i\phi VIX_T^2} (VIX_T - K)^+ dVIX_T, \\ &= \int_0^\infty e^{-i\phi VIX_T^2} (VIX_T - K)^+ dVIX_T^2, \\ &= \frac{\sqrt{\pi}}{2} \frac{1 - \mathrm{erf}\left(K\sqrt{i\phi}\right)}{(\sqrt{i\phi})^3}, \ \phi_I < 0. \end{aligned} \tag{16}$$

Proposition 2 *Therefore, the call option price becomes*

$$\begin{aligned} c(t,T) &= \frac{e^{-r\tau}}{\pi} \int_0^\infty Re\left[I_1 \cdot \tilde{f}(i\phi; t, \tau, V_t, \theta_t)\right] d\phi_R, \\ &= \frac{e^{-r\tau}}{2\sqrt{\pi}} \int_0^\infty Re\left[\frac{1 - erf\left(K\sqrt{i\phi}\right)}{(\sqrt{i\phi})^3} \tilde{f}(i\phi; t, \tau, V_t, \theta_t)\right] d\phi_R, \ \phi_I < 0. \end{aligned} \tag{17}$$

Remark 2 It is important to note that the ω is a function of τ_0, and Equation (17) can be applied to any τ_0. Thus, it provides a unified platform for us to study the consistency between options written on VIX, VXST and VIX with other maturities.

Remark 3 Lian and Zhu (2013) use similar methodology and derive a VIX option formula for a one-factor stochastic volatility model with simultaneous jumps in both index return and volatility process. However, our formula in (17) is based on a general two-factor stochastic volatility model with a time-varying central tendency θ_t. Further, we derive VIX option formula by using a simple relation between VIX and V_t and θ_t in (4), which enable us to obtain the two latent factors from observable VIX term structure data.

Corollary 1 *The VIX futures price can also be calculated from VIX option formula, that is*

$$F(t,T) = \frac{1}{2\sqrt{\pi}} \int_0^\infty Re\left[\frac{1}{(\sqrt{i\phi})^3} \tilde{f}(i\phi; t, \tau, V_t, \theta_t)\right] d\phi_R, \ \phi_I < 0. \tag{18}$$

Remark 4 In the following sections, we consider a special case, $y \sim \exp(\mu_y)$, to get closed-form solution for $A_4(\phi,\psi;\tau)$. We have

$$A_4(\phi,\psi;\tau) \;=\; \frac{2\lambda^y\mu_y}{2\kappa\mu_y-\sigma_V^2}\ln\left[1+\frac{(\sigma_V^2-2\mu_y\kappa)\phi}{2\kappa(1-\mu_y\phi)}(e^{-\kappa\tau}-1)\right]. \tag{19}$$

3.3 Property of the VIX futures and option pricing formulas

We discuss property of VIX futures and option pricing formulas in more detail in this section.

3.3.1 Comparison with Zhang and Zhu (2006)

First, we compare current VIX futures formula with the exact density function in the Heston model. In this case, we have $\lambda_t^x = \lambda^y = 0$, $V_t = v_t$ and $\theta_t = \theta$. To compare with the exact transit density function, we just compare it to the density function in our model. Now, the moment generating function of V_T becomes

$$f_1(\phi;t,\tau,V_t) \;\equiv\; E^Q[e^{\phi V_T}|\mathcal{F}_t], \tag{20}$$

$$\;=\; e^{A_1(\phi;\tau)+A_2(\phi;\tau)V_t}, \tag{21}$$

with

$$A_1(\phi;\tau) \;=\; \frac{-2\kappa\theta}{\sigma_V^2}\ln\left[1+\frac{\sigma_V^2\phi}{2\kappa}(e^{-\kappa\tau}-1)\right], \tag{22}$$

$$A_2(\phi;\tau) \;=\; \frac{2\kappa\phi}{(2\kappa-\sigma_V^2\phi)e^{\kappa\tau}+\sigma_V^2\phi}. \tag{23}$$

The corresponding conditional density function of V_T, $p_1(V_T|V_t)$, is

$$p_1(V_T|V_t) = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[e^{-i\phi V_T}f_1(i\phi;t,\tau,V_t,\theta_t)\right]d\phi_R. \tag{24}$$

Then, the VIX futures price is given by

$$F_1(t,T) \;=\; \frac{1}{2\sqrt{\pi}}\int_0^\infty \mathrm{Re}\left[\frac{1}{(\sqrt{i\phi})^3}e^{\phi(1-\omega)\theta}f_1(\phi\omega;t,\tau,V_t)\right]d\phi_R. \tag{25}$$

Table 1 provides comparison for VIX futures with maturities 15, 78, 169 and 260 days on March 1, 2005. Note that the physical measure parameters used in Zhang and Zhu (2006) are $\kappa^P = 5.7895$, $\theta^P = 0.0414$, $\sigma_V = 0.4868$ and the volatility risk premium is $\lambda = -0.8716$. In current VIX futures formula, we need to use risk-neutral parameters given by $\kappa = \kappa^P + \lambda$, $\theta = \frac{\kappa^P \theta^P}{\kappa^P + \lambda}$. In addition, the VIX level on March 1, 2005 is 12.04, which can be used to back out $V_t = 0.0071$. Table 1 shows that VIX futures prices are the same by using current VIX futures formula in (25) and the exact formula in Zhang and Zhu (2006).

3.3.2 Sensitivity of VIX option implied volatility

To guide parameter calibration in the next section, we conduct sensitivity analysis of VIX option implied volatility with respect to the four parameters, σ_V, λ^y, μ_y and σ_θ, for various combinations of maturity and moneyness. Figures 1-4 show the results.

When look vertically in the up-left part of Figure 1, we can see that implied volatility is sensitive to different values of σ_V and larger σ_V corresponds to higher implied volatility. The difference becomes less when moneyness goes up. Further, implied volatility curve is a increasing function of moneyness. When maturity increases to 60, 120 and 150 days, we find that the implied volatility curve becomes more and more flat and the difference caused by different σ_V becomes smaller. We observe similar pattern in Figures 2-3. However, Figure 4 indicates that the difference induced by different σ_θ becomes larger when maturity increases. Moreover, the implied volatility curve switches from upward sloping to downward sloping function of moneyness.

4 Calibration

In this section, we introduce data and calibrate the model to the average VIX futures term structure, ATM implied volatility term structure and implied volatility skew term structure with maturities 30, 60, 90, 120 and 150 days. By minimizing squared errors, we are able to get solutions of σ_V, λ^y, μ_y and σ_θ.

4.1 Data

The daily market data employed in this paper includes VIX futures from March 26, 2004 to May 28, 2010 and options from February 24, 2006 to May 28, 2010 provided by the CBOE. The VIX futures data contains trading date and maturity date; open, high, low and close of the futures price; settle price, which is used as the proxy of futures price on the trading date; trading volume and open interest. On each trading day, we calculate VIX futures prices with the standardized times-to-maturity of 30, 60, 90, 120 and 150 days by using linear interpolation. We treat VIX index as the VIX futures price with zero day to maturity. Table 2 provides descriptive statistics of VIX futures data. The VIX options data contains trading date and expiration date; type of options (call or put) and strike price; open, high, low and close prices; trading volume and open interests; last bid and last ask prices; underlying VIX index values. For each VIX options contract, last traded prices of VIX options are often unavailable, and reported last traded prices are often zero in the original dataset. We therefore use mid-price of closing bid and ask as the closing price of the options.

We notice that the expiration dates in the original VIX options dataset are not the true expiration dates. They are the third or fourth Saturdays of the months, while the VIX options expiration dates are Wednesdays. Therefore we need to transfer the original expiration dates to true expiration dates before we start to process the data. For example, if the original expiration dates are March 18, 2006 and April 22, 2006, we first find the third Friday in the next month of the original expiration dates, which are April 21, 2006 and May 19, 2006. We then subtract 30 days from these “third Fridays in the next month” and obtain some Wednesdays in the original months, and these Wednesdays, such as March

22, 2006 and April 19, 2006 for the example, are the true expiration dates. We obtain the days to maturity by calculating the number of calendar days between the trading date and the true expiration date, and we then divide it by 365 to annualize the time to maturity. For example, if the trading date is March 1, 2006, and true expiration dates are March 22, 2006 and April 19, 2006, then the days to maturity are 21 days and 49 days, and annualized time to maturity are 0.0575 and 0.1342 year.

We also use U.S. treasury daily yield curve rates during the same period above, provided by the U.S. Treasury Department. The U.S. treasury daily yield curve rates include treasury yields for 1 month, 3 months, 6 months, 1 year, 2 years, 3 years, 5 years, 7 years, 10 years, 20 years, and 30 years, on each trading day. We construct risk-free interest rates for different maturities on each trading day by using linear interpolation.

4.1.1 ATM implied volatility

Before calculating ATM implied volatility, we first introduce implied forward price and ATM implied forward price. The implied forward price is computed from the options market by using put-call parity, i.e.,

$$c_t(K) - p_t(K) = S_t - Ke^{-r(T-t)}. \tag{26}$$

The forward price at time t with maturity T is given by

$$F_{implied}(K) = S_t e^{r(T-t)} = (c_t(K) - p_t(K))e^{r(T-t)} + K. \tag{27}$$

Notice that the implied forward price could be a function of strike price K. We define the pair of call and put options with the same strike and the smallest absolute value of the price difference as the *ATM options*. The implied forward price computed from the pair of ATM call and put is called *ATM implied forward price*.

The implied volatility of an option is defined as the volatility that equates the Black-

Scholes option price with the market price of the option, i.e.,

$$c_t^{Market}(K, T-t) = c_t^{Black-Scholes}(\sigma; K, T-t) = F_t^T e^{-r(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2), \quad (28)$$

where F_t^T is the ATM implied forward price and

$$d_1 = \frac{\ln \frac{F_t^T}{K} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \; d_2 = d_1 - \sigma\sqrt{T-t}. \quad (29)$$

The *ATM implied volatility* is regarded as the implied volatility for the option with strike price being equal to the ATM implied forward price, i.e., $K = F_t^T$. We look for two strike prices, K_{lower} and K_{upper}, that are the nearest strike below and above the ATM implied forward price, and then calculate the ATM implied volatility by using linear interpolation with the implied volatilities at these two strike prices.

4.1.2 ATM implied volatility skew

On each trading day, for the VIX options with the same maturity, the *implied volatility skew* is defined as the ATM slope of the implied volatility as a function of strike price, that is

$$skew = \frac{\sigma_{implied}(K_{upper}) - \sigma_{implied}(K_{lower})}{K_{upper} - K_{lower}}. \quad (30)$$

We can calculate daily ATM implied volatility and ATM implied volatility skew over the sample period for maturities 30, 60, 90, 120 and 150 days. The average ATM implied volatility term structure and average ATM implied volatility skew term structure are means of corresponding daily values. That is, the average ATM implied volatility term structure are 0.7764, 0.695, 0.6312, 0.5923, 0.5612, and the average ATM implied volatility skew term structure are 0.0409, 0.0249, 0.0182, 0.0154, 0.0131.

4.2 Calibration

We do calibration in two steps. First, we use VIX term structure daily data from January 2, 1992 to May 28, 2010 to estimate parameter set $\{\kappa, V_t, \theta_t\}_{t=1,\ldots,T}$ over the sample period.

We employ an efficient procedure in Luo and Zhang (2012) and obtain $\kappa = 5.5282$. We average V_t and θ_t to get $\bar{V} = 0.0473$ and $\bar{\theta} = 0.0459$. Second, we obtain $\sigma_V = 0.7945$, $\lambda^y = 2.3316$, $\mu_y = 0.0128$ and $\sigma_\theta = 0.1013$ by solving the following minimization problem

$$\{\hat{\sigma_V}, \hat{\lambda^y}, \hat{\mu_y}, \hat{\sigma_\theta}\} = \arg\min \sum_{j=1}^{5} \left(\frac{IV_j - IV_j^{Mkt}}{\overline{IV}} \right)^2 + \left(\frac{Skew_j - Skew_j^{Mkt}}{\overline{Skew}} \right)^2, \tag{31}$$

where $j = 1, ..., 5$ indicate 30, 60, 90, 120 and 150 days, IV_j is averaged model-implied ATM implied volatility, $Skew_j$ is averaged model-implied ATM implied volatility skew, IV_j^{Mkt} and $Skew_j^{Mkt}$ are corresponding market-implied values, and $\overline{IV}$ and $\overline{Skew}$ are the mean of IV_j^{Mkt} and $Skew_j^{Mkt}$ respectively.

Table 3 provides pricing errors of VIX futures. Note that we consider subsample period and full sample period in Panels A and B, respectively. It can be seen that pricing errors are small for both sample periods.

Figures 5-7 show fitting performance of current VIX futures and options formulas by using estimated parameters. In particular, Figure 5 shows that model-implied average ATM implied volatility term structure is similar to market data. Further, Figure 6 means that our model can also track the shape of average ATM implied volatility skew term structure. Figure 7 plots six annual snapshot of VIX futures term structure from 2004 to 2009. It is clear that theoretical VIX futures values closely match market data and capture various shapes of VIX term structure at both low and high levels of volatility market conditions.

5 Concluding remarks

In this paper, we introduce a unified framework to price VIX derivatives, which includes futures and options written on VIX, VXST and VIX with any other maturity, by using the instantaneous squared VIX proposed in Luo and Zhang (2012). We also show that parameters can be efficiently estimated by a sequential method. It turns out that our model is simple yet powerful to capture various shapes of VIX futures term structure and fit term structures of average ATM implied volatility and its skew.

Note that, we do not compare different specifications and investigate their relative contributions. Instead, we focus on pointing out the importance of the ISVIX in pricing derivatives on VIX with various maturities in a unified model, and bridging the SPX and VIX derivatives markets. We demonstrate that analytical formulas for VIX options and futures can be obtained in a mean-reverting jump-diffusion process with a stochastic long-term mean. Further, we propose a sequential estimation method by using appropriate market information to determine relevant parameters. That is, VIX term structure data can be used to estimate the mean-reverting speed and VIX option implied volatility surface is employed to calibrate the volatility and jump parameters. By doing this, we actually decompose a complicated estimation task into several simpler tasks.

At last, it is worthwhile to mention that how to specify instantaneous Brownian variance and jump dynamics of SPX return, how to estimate these parameters by combining SPX options, and how to further extend ISVIX dynamics are important issues can be considered. Nevertheless, we leave them for future research.

A Solution of $f(\phi, \psi; t, \tau, V_t, \theta_t)$

Let $\tau \equiv T - t$ and

$$f(\phi, \psi; t, \tau, V_t, \theta_t) \equiv E^Q[e^{\phi V_T + \psi \theta_T} | \mathcal{F}_t], \tag{32}$$

be the joint moment generating function of (V_T, θ_T). Since $f(\phi, \psi; t, \tau, V_t, \theta_t)$ is a martingale, it satisfies backwards Kolmogorov equation

$$-f_\tau + [\kappa(\theta_t - V_t) - \lambda^y E^Q(y)] f_V + \frac{1}{2}\sigma_V^2 V f_{VV} + \frac{1}{2}\sigma_\theta^2 f_{\theta\theta} + \lambda^y E^Q[f(V + y) - f(V)] = 0, \tag{33}$$

with a boundary condition $f(\phi, \psi; t + \tau, 0, V_t, \theta_t) = e^{\phi V_T + \psi \theta_T}$. Assume a solution of with the following form

$$f(\phi, \psi; t, \tau, V_t, \theta_t) = e^{A_1(\phi,\psi;\tau) + A_2(\phi,\psi;\tau) V_t + A_3(\phi,\psi;\tau)\theta_t + A_4(\phi,\psi;\tau)}, \tag{34}$$

with the boundary conditions $A_1(\phi, \psi; 0) = A_4(\phi, \psi; 0) = 0$, $A_2(\phi, \psi; 0) = \phi$ and $A_3(\phi, \psi; 0) = \psi$. Substituting the derivatives to the above PIDE, we have

$$\begin{aligned} -\left[A_1^{'}(\tau) + A_2^{'}(\tau)V_t + A_3^{'}(\tau)\theta_t + A_4^{'}(\tau)\right] + [\kappa(\theta_t - V_t) - \lambda^y E^Q(y)]A_2(\tau) + \frac{1}{2}\sigma_V^2 A_2^2(\tau) V_t \\ + \frac{1}{2}\sigma_\theta^2 A_3^2(\tau) + \lambda^y E^Q[e^{A_2(\tau)y} - 1] = 0. \end{aligned} \tag{35}$$

We have the following ODEs

$$\begin{aligned} A_1^{'}(\tau) &= -\lambda^y E^Q(y) A_2(\tau) + \frac{1}{2}\sigma_\theta^2 A_3^2(\tau), \\ A_2^{'}(\tau) &= -\kappa A_2(\tau) + \frac{1}{2}\sigma_V^2 A_2^2(\tau), \\ A_3^{'}(\tau) &= \kappa A_2(\tau), \\ A_4^{'}(\tau) &= \lambda^y E^Q[e^{A_2(\tau)y} - 1]. \end{aligned}$$

We first solve Riccati equation of $B(\tau)$ and obtain

$$A_2(\phi, \psi; \tau) = \frac{2\kappa\phi}{(2\kappa - \sigma_V^2\phi)e^{\kappa\tau} + \sigma_V^2\phi}. \tag{36}$$

Then, $A_3(\tau)$ and $A_4(\tau)$ can be calculated as

$$A_3(\phi,\psi;\tau) = \psi - \frac{2\kappa}{\sigma_V^2}\ln\left[(1-\frac{\sigma_V^2\phi}{2\kappa}) + \frac{\sigma_V^2\phi}{2\kappa}e^{-\kappa\tau}\right], \tag{37}$$

$$A_4(\phi,\psi;\tau) = \int_0^\tau \lambda^y E^Q[e^{A_2(u)y} - 1]du, \tag{38}$$

and $A_1(\tau)$ is given by

$$\begin{aligned} A_1(\phi,\psi;\tau) &= -\lambda^y E^Q(y)\int_0^\tau A_2(u)du + \frac{1}{2}\sigma_\theta^2\int_0^\tau A_3^2(u)du, \\ &= \frac{-\lambda^y E^Q(y)}{\kappa}(A_3(\tau)-\psi) + \frac{1}{2}\sigma_\theta^2\int_0^\tau A_3^2(u)du. \end{aligned} \tag{39}$$

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Table 1: **Comparison with Zhang and Zhu (2006)**

This table provides VIX futures prices calculated by using the formula in (25) and the exact density function in the Heston model with maturities 15, 78, 169 and 260 days on March 1, 2005. Note that the physical measure parameters used in Zhang and Zhu (2006) are $\kappa^P = 5.7895$, $\theta^P = 0.0414$, $\sigma_V = 0.4868$ and the volatility risk premium is $\lambda = -0.8716$. In current VIX futures formula (25), we have to use risk-neutral parameters given by $\kappa = \kappa^P + \lambda$ and $\theta = \frac{\kappa^P \theta^P}{\kappa^P + \lambda}$. The VIX level on March 1, 2005 is 12.04, which can be used to back out $V_t = 0.0071$.

Maturities	15	78	169	260
Zhang and Zhu (2006)	141.7546	185.5667	204.9145	210.2486
Formula in (25)	141.7548	185.5668	204.9145	210.2485

Table 2: **Descriptive statistics of VIX futures data**

This table presents descriptive statistics of VIX futures. The sample period is 26/03/2004-28/05/2010.

	All Futures	$<$ 60 days	60-180 days	$>$ 180 days
Observations	10171	2676	4550	2945
Minimum	10.37	10.37	12.29	13.52
Mean	22.94	21.64	23.54	23.18
Maximum	66.23	66.23	59.77	45
Std	8.68	10.108	8.4	7.5
Skewness	1.14	1.58	0.9	0.91
Kurtosis	4.26	5.44	3.46	2.98

Table 3: **VIX futures pricing errors**

This table presents pricing errors for VIX futures with calibrated parameters $\sigma_V = 0.7945$, $\lambda^y = 2.3316$, $\mu_y = 0.0128$, $\sigma_\theta = 0.1013$.

	Errors	All Futures	<= 60 days	60-180 days	>= 180 days
Panel A:	26/03/2004-11/07/2008				
	RMSE	2.90	2.14	3.24	2.99
	MPE(%)	-15.98	-12.31	-18.55	-15.10
	MAE	2.74	1.95	3.18	2.77
Panel B:	26/03/2004-28/05/2010				
	RMSE	2.78	2.12	3.00	2.98
	MPE(%)	-13.02	-9.63	-14.94	-12.85
	MAE	2.54	1.87	2.84	2.66

Figure 1: The sensitivity of VIX option-implied volatility surface with respect to σ_V.

The implied volatility is defined as Black-Scholes implied volatility of VIX option. We use averaged V_t ($\bar{V} = 0.0473$) and θ_t ($\bar{\theta} = 0.0459$) as proxies of current values in computing the implied volatility. In sensitivity analysis, we allow σ_V fluctuates around the calibrated value $\sigma_V = 0.7945$ while keep other parameters to be the same as those obtained in the calibration section, that are $\lambda^y = 2.3316$, $\mu_y = 0.0128$, $\sigma_\theta = 0.1013$.

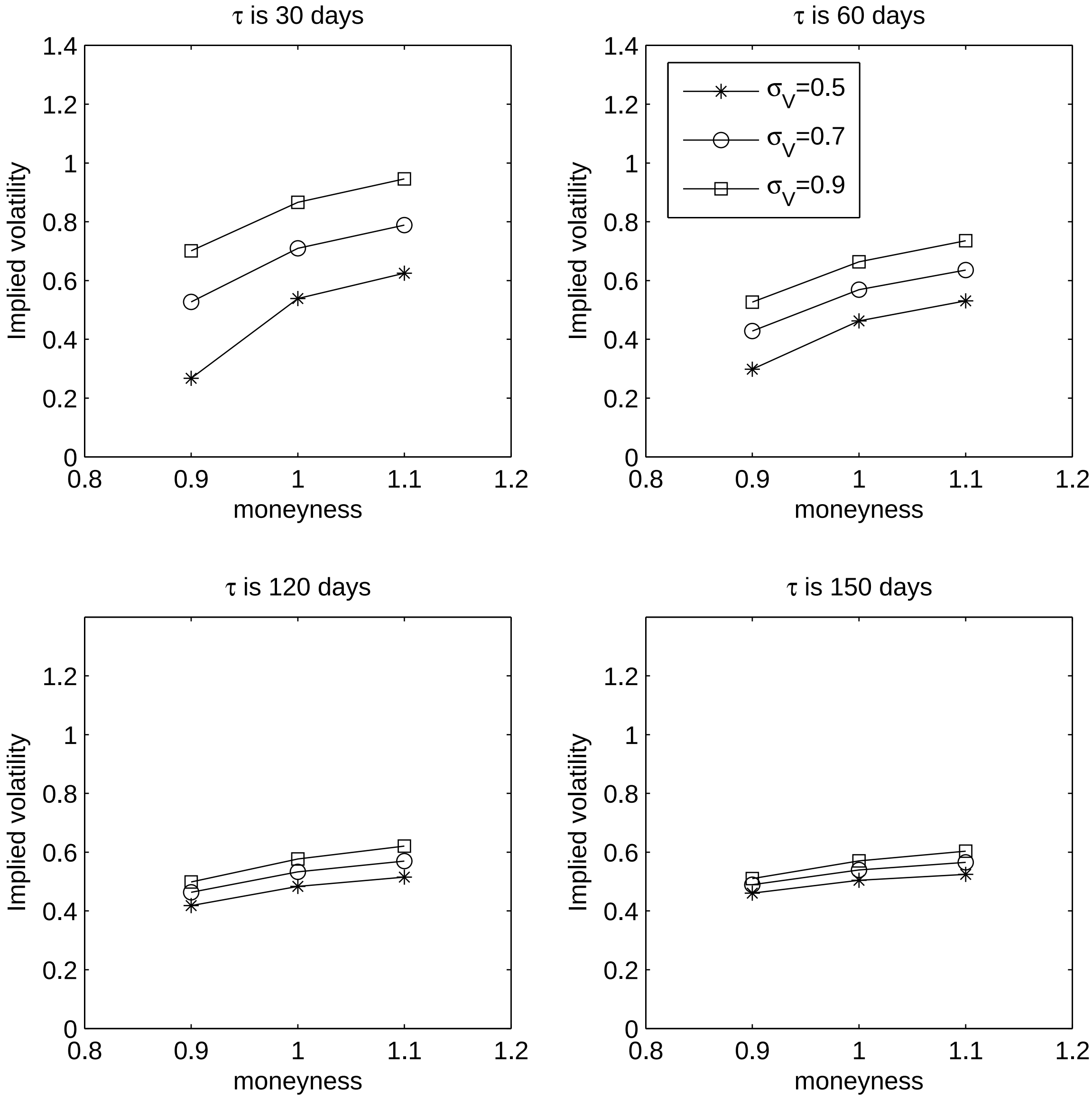

Figure 2: The sensitivity of VIX option-implied volatility surface with respect to λ^y.

The implied volatility is defined as Black-Scholes implied volatility of VIX option. We use averaged V_t ($\bar{V} = 0.0473$) and θ_t ($\bar{\theta} = 0.0459$) as proxies of current values in computing the implied volatility. In sensitivity analysis, we allow λ^y fluctuates around the calibrated value $\lambda^y = 2.3316$ while keep other parameters to be the same as those obtained in the calibration section, that are $\sigma_V = 0.7945$, $\mu_y = 0.0128$, $\sigma_\theta = 0.1013$.

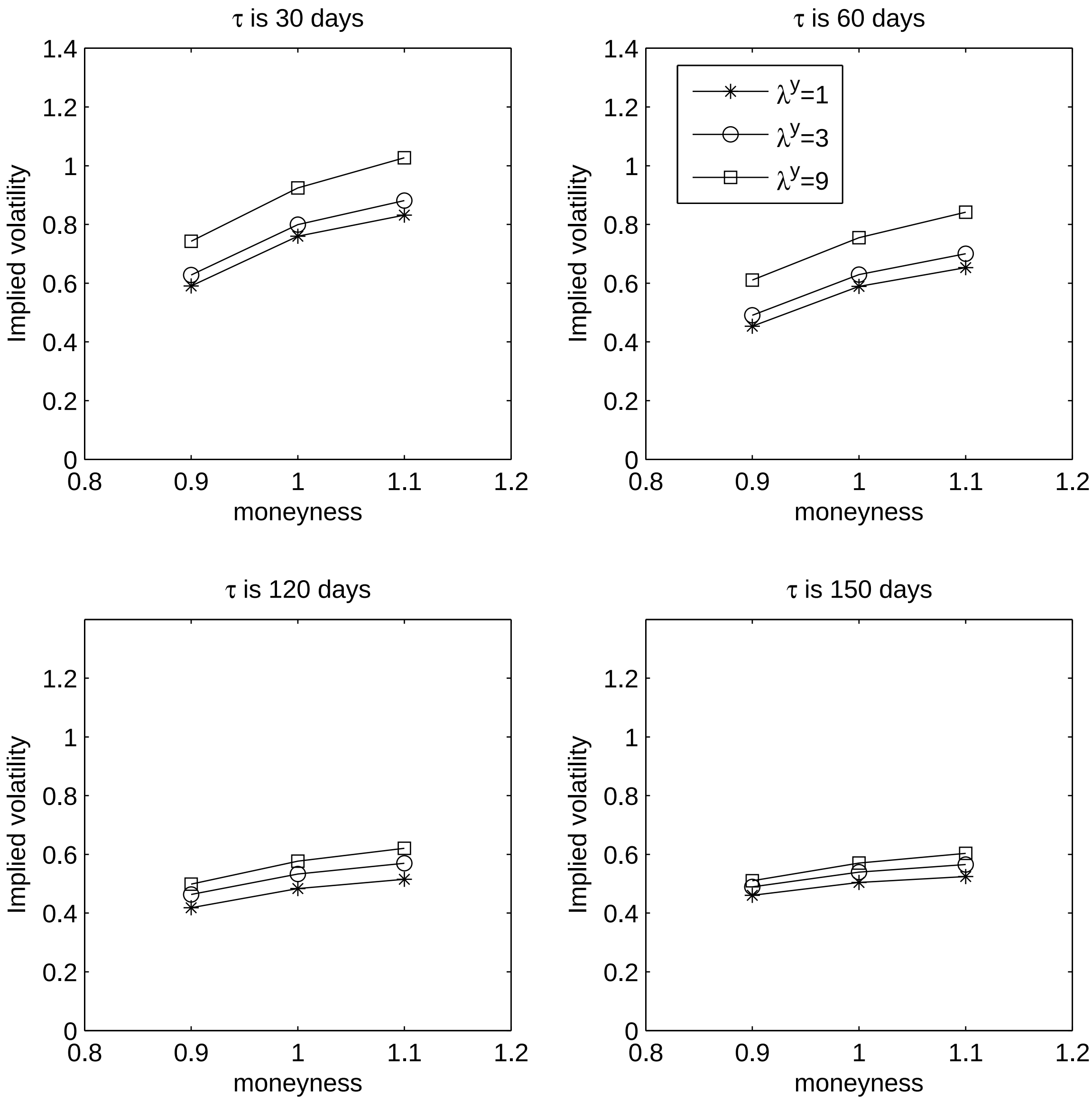

Figure 3: The sensitivity of VIX option-implied volatility surface with respect to μ_y.

The implied volatility is defined as Black-Scholes implied volatility of VIX option. We use averaged V_t ($\bar{V} = 0.0473$) and θ_t ($\bar{\theta} = 0.0459$) as proxies of current values in computing the implied volatility. In sensitivity analysis, we allow μ_y fluctuates around the calibrated value $\mu_y = 0.0128$ while keep other parameters to be the same as those obtained in the calibration section, that are $\sigma_V = 0.7945$. $\lambda^y = 2.3316$, $\sigma_\theta = 0.1013$.

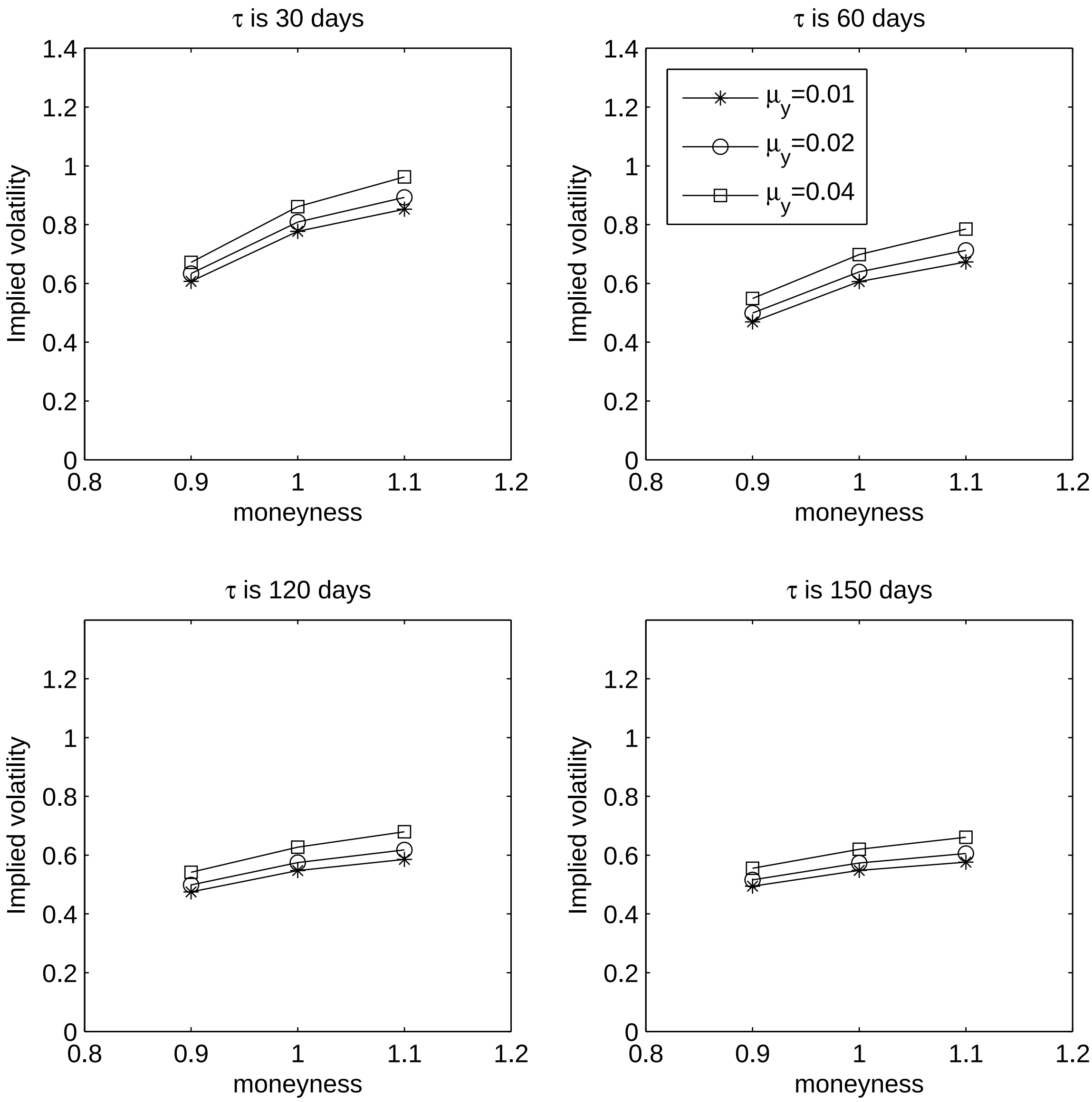

Figure 4: The sensitivity of VIX option-implied volatility surface with respect to σ_θ.

The implied volatility is defined as Black-Scholes implied volatility of VIX option. We use averaged V_t ($\bar{V} = 0.0473$) and θ_t ($\bar{\theta} = 0.0459$) as proxies of current values in computing the implied volatility. In sensitivity analysis, we allow σ_θ fluctuates around the calibrated $\sigma_\theta = 0.1013$ while keep the other parameters to be the same as those obtained in the calibration section, that are $\sigma_V = 0.7945$. $\lambda^y = 2.3316$, $\mu_y = 0.0128$.

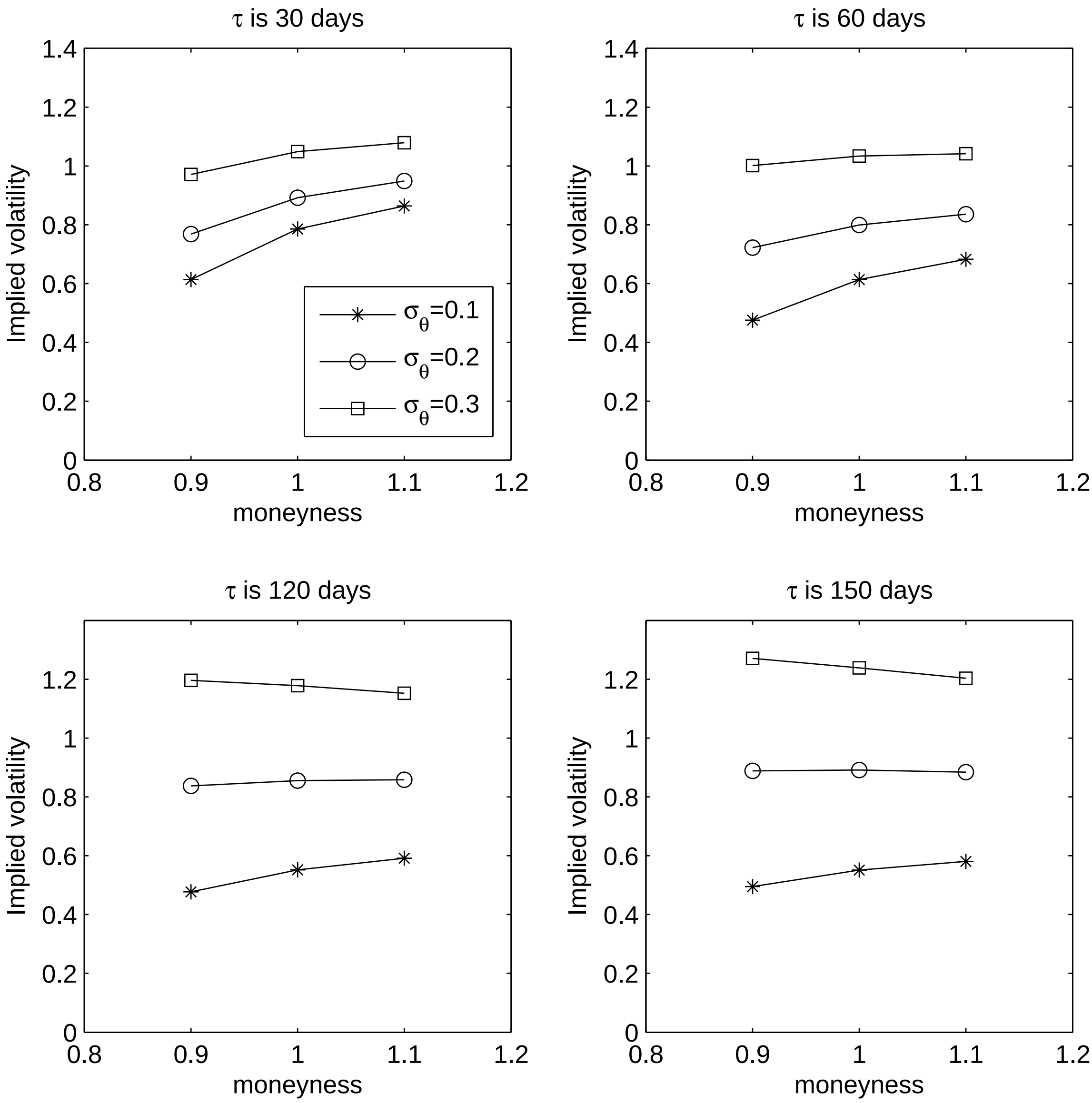

Figure 5: Average ATM implied volatility term structure.

Parameters are obtained from calibration exercise in Section 4.2, that are $\sigma_V = 0.7945$, $\lambda^y = 2.3316$, $\mu_y = 0.0128$, $\sigma_\theta = 0.1013$.

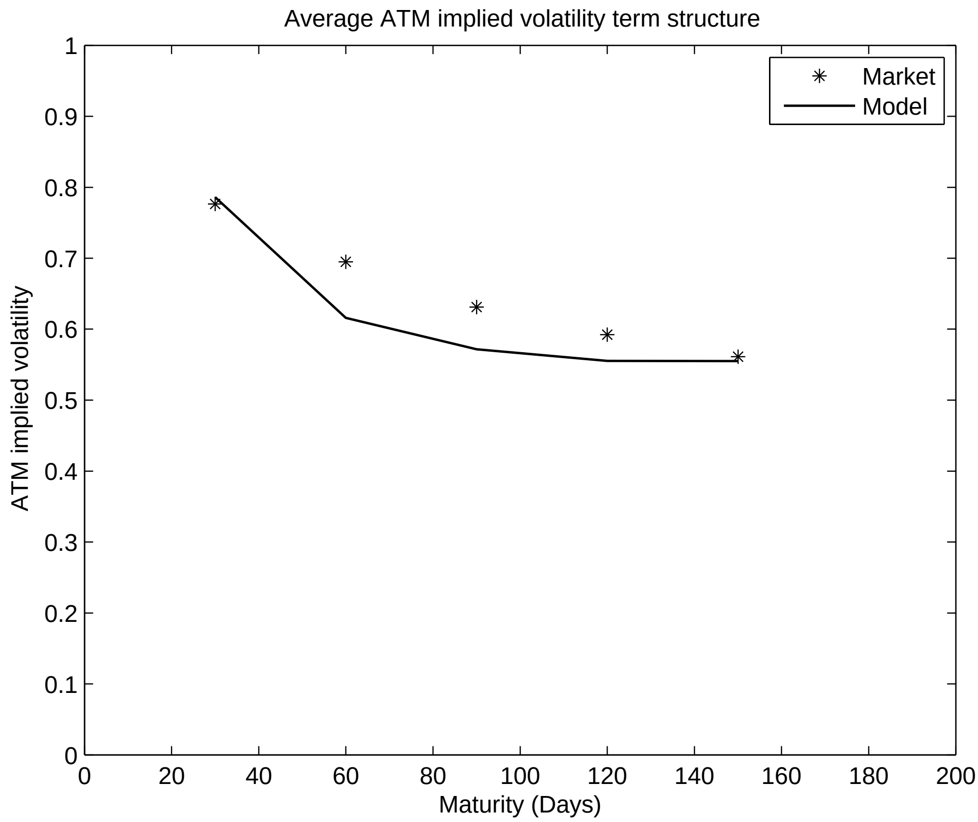

Figure 6: Average ATM implied volatility skew term structure.

Parameters are obtained from calibration exercise in Section 4.2, that are $\sigma_V = 0.7945$, $\lambda^y = 2.3316$, $\mu_y = 0.0128$, $\sigma_\theta = 0.1013$.

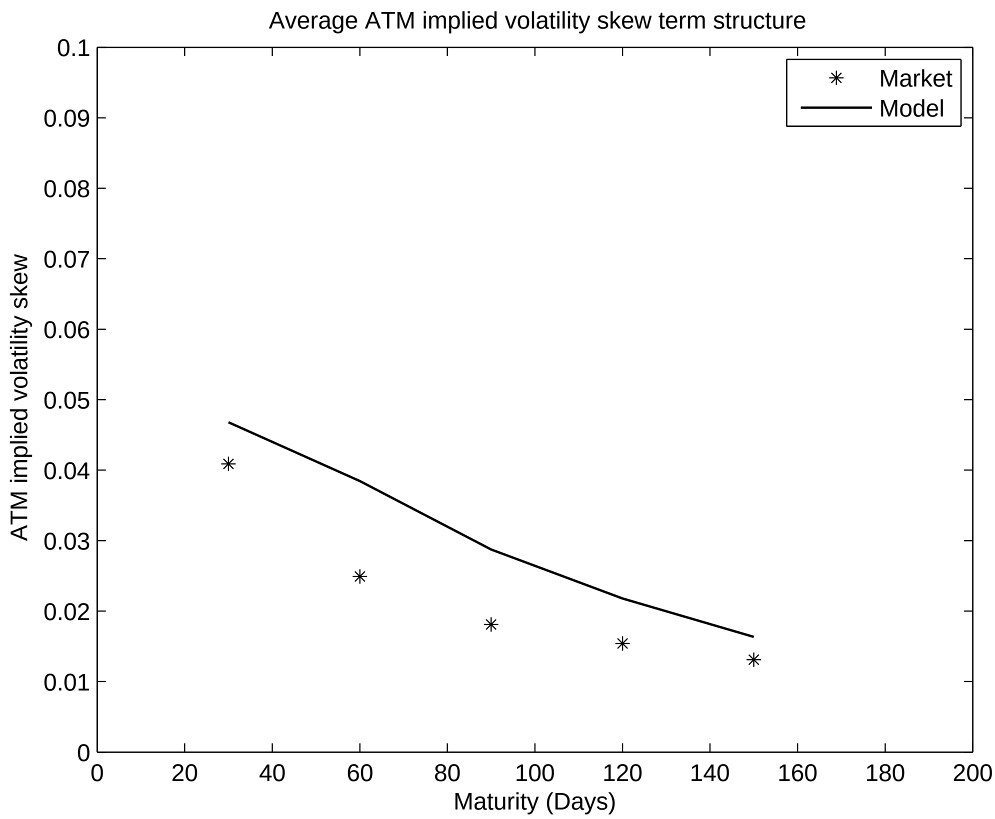

Figure 7: Annual snapshot of VIX futures term structure.

This table shows annual snapshot of VIX futures term structure in October from 2004 to 2009. Parameters are obtained from calibration exercise in Section 4.2, that are $\sigma_V = 0.7945$, $\lambda^y = 2.3316$, $\mu_y = 0.0128$, $\sigma_\theta = 0.1013$.