

Private Information, Securities Lending, and Asset Prices

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Abstract

We analyze the role of private information in the equity loan market in a rational–expectations model with endogenous short–sale constraints. We show that when a group of investors is inhibited from participating in the equity loan market, excess demand for short selling, driven by either privately informed investors or liquidity traders, allows equity lenders to charge a fee for lending their shares. We show that information asymmetry among investors, arising from private information, reduces the expected lending fee; reduces short–selling risk, measured by the volatility of the lending fee; and has an ambiguous effect on the expected return.

JEL Codes: G10, G11, G12

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1 Introduction

In the United States, short selling accounts for more than a quarter of trading volume in the stock market (see Diether et al. 2008). To short sell a security, an investor must borrow the security from the investor who owns it. A fee is typically charged for the security loan. According to securities finance data provider DataLend, the value of securities on loan globally has recently surpassed \$2 trillion, and security lenders earned over \$9 billion in 2016. In one of the few theoretical models of the equity lending market, Duffie et al. (2002) argue that search frictions give market power to stock lenders and, thereby, allow them to charge short sellers a fee for borrowing shares. Although the equity lending market in the United States is currently an over-the-counter market, and search frictions possibly play a role in determining lending fees, a centralized equity lending market on the floor of New York Stock Exchange existed in the 1920s and 1930s. Jones and Lamont (2002) analyze this centralized lending market and show that lenders were charging fees for lending their shares, which suggests that search frictions are not the only determinant of lending fees.

Given the interest in introducing a centralized marketplace for borrowing and lending securities, it is important to understand the determinants of equity lending fees. Considering that much prior empirical evidence suggests that short sellers are informed traders, this paper focuses on the roles that private information play in determining equity lending fees. The effects of private information on asset prices have long been of interest to scholars in finance and accounting. Nevertheless, the literature has been mainly silent about how private information affects prices in the equity lending market. Toward that end, we analyze a rational–expectations model of the equity lending market.

We show that when a group of investors is inhibited, whether because of legal/institutional requirements or heterogeneous preferences (or both), from participating in the lending market, trading by privately informed investors, along with trading by liquidity traders, generates endogenous short–sale constraints, leading lenders to charge a positive fee for lending their shares. We show that the equilibrium lending–fee curve is a hockey stick–shaped function

of firm fundamental, which is supported by the evidence in Kolasinski et al. (2013). We also show that both the *expected* lending fee and the *variance* of lending fee *decrease* as the private signal precision of informed investors increases. These findings suggest that, surprisingly, information, even when it is private, reduces both the tightness of short–sale constraints (measured as expected lending fee or the probability of a binding share–borrowing constraint) and the riskiness of short selling (measured by variance of the lending fee).

In our model, investors trade a risk–free asset and a risky asset (the stock). Investors who want to short the stock must pay a lending fee to borrow shares from investors who own the stock. In addition to liquidity traders, who trade for exogenous reasons, there are three types of risk–averse rational investors: (1) constrained uninformed, who cannot supply shares in the equity lending market and are long only; (2) unconstrained uninformed, who can supply shares in the equity lending market and are long only; and (3) informed, who observe private signals regarding the stock payoff and face no restriction on shorting or lending shares. We determine both the equilibrium stock price and the lending fee and examine how they are affected by information asymmetry among investors.

We show that the equilibrium lending fee curve is a hockey stick shape in the firm fundamental. When the (endogenous) borrowing demand is low (i.e., the firm’s fundamental is high), the fee curve is flat, and, thus, the fee is insensitive to shocks in demand for borrowing shares. However, when borrowing demand is high (i.e., the firm’s fundamental is low), the fee curve tends to become steep. Characterizing the kink point of the fee function helps us better understand what causes a stock to be “on special,” i.e., to be expensive to borrow. We show that the ownership structure, i.e., the composition of the three types of rational investors, the risk tolerance of investors, the variance of liquidity demand (which affects the precision of the price), and the private signal precision play important roles in determining specialness.

We also show that compared with our symmetric zero–information (i.e., zero–precision) benchmark, asymmetric information reduces both the expected lending fee and the variance

of lending fee. This finding is in stark contrast with the results presented in Duffie et al. (2002) and Blocher et al. (2013). A partial-equilibrium reasoning, which is consistent with the results presented in their papers, is that higher precision of private signals would lead to higher dispersion of information, and, therefore, higher dispersion of demand among informed investors. Thus, the lending fee might be expected to increase when informed investors obtain private signals. We show that this result can hold in our model, but only if investors ignore the information content of the price. If investors have rational expectations, asymmetric information *reduces* both the expected lending fee and the variance of lending fee.

Why is the expected lending fee lower compared with a symmetric zero-information benchmark? In our model, private information affects the lending fee through two opposing channels. The first channel is the *dispersion channel*, i.e., private signals generate differences of opinion among informed investors, as well as asymmetry of information between informed and uninformed investors. This disagreement among investors tends to increase the fee. The second channel is the *price-informativeness channel*, i.e., the (public) price signal aggregates and shares private information among investors, which reduces disagreement and uncertainty regarding the stock payoff and, therefore, reduces the fee. We show that in expectation, the price-informativeness channel prevails, resulting in an inverse relationship between the expected fee and the precision of private signals.

Because short sellers must borrow securities from investors who own them,¹ they face the risk that the lending fee can increase.² The literature has suggested measuring short-selling risk using variance of the lending fee, see Engelberg et al. (forthcoming). We characterize the variance of the lending fee and show that an increase in the precision of private information is associated with a decrease in the variance of the lending fee. The price-informativeness channel that explains the relationship between the expected fee and the precision of private

¹See, “Key points about Regulation SHO”, <https://www.sec.gov/investor/pubs/regsho.htm>.

²They also face the risk that the borrowed shares can be recalled, forcing a short seller to close his position prematurely.

signals also explains the inverse relationship between the variance of fee and precision of private signals.

We also study the effects of private information and the equity lending market on equilibrium asset prices. An important yet unsettled question in finance is how information asymmetry among investors affects asset prices. For example, Easley and O’Hara (2004) argue that when asymmetry of information between informed and uninformed investors is reduced by making some of the information of informed investors public, the expected return (cost of capital) declines. Lambert et al. (2012) call into question the finding of Easley and O’Hara (2004), and show that in perfectly competitive settings, only the average precision of information available in the market determines the expected return, and asymmetry of information does not affect the expected return. We show that the result in Lambert et al. (2012) only holds if short–sale constraints never bind. If these constraints can bind, the precision of information available to each group of investors has an impact on the expected return and the lending fee beyond the average precision. Moreover, we show that the expected rate of return is not always monotonic in private signal precision, and it can be either higher or lower with asymmetric information than it is in our symmetric zero–information (i.e., zero–precision) benchmark.

To the best of our knowledge, our paper is the first in the literature that analyzes the effects of private information on the lending fee and its moments. There are only a few theoretical models of the equity lending market, and none focus on the effects of information asymmetry. Duffie et al. (2002) present a model based on search frictions, in which pessimistic and optimistic traders (their numbers are exogenous, and invariant to price) who take short and long positions, are matched in succession. The resulting equilibrium price and fee are deterministic. Blocher et al. (2013) present a reduced–form model with exogenous asset demand and supply functions; therefore, the effects of information on the equilibrium depend on the conjectured effects on the demand and supply functions. Duffie (1996) models the prices and fees for two bonds with identical cash flows, but a trading friction for one bond

relative to the other one generates a borrowing fee.

Our model provides a number of empirical implications, as yet untested, on the effects of the dispersion and the quality/quantity of private information on a firm's stock price and the fee to borrow the shares. In addition, we propose a measure of breadth of ownership, similar in spirit to the one proposed in Chen et al. (2002), that directly measures the tightness of short-sale constraints and does not have the shortcomings of other measures, such as short interest, lending fee, and breadth of ownership, that are suggested in the literature. We provide testable hypotheses regarding how our measure of short-sale constraints is related to the stock price and the lending fee.

Our paper adds to two strands of the literature. First, we contribute to theoretical literature on the securities lending market (see, e.g., Duffie 1996; Krishnamurthy 2002; Duffie et al. 2002; Blocher et al. 2013; among others) by analyzing a rational-expectations model of the equity lending market and determining the effects of information asymmetry on the equilibrium lending fee, as well as the volatility of the fee (the fee is deterministic in Duffie 1996; Duffie et al. 2002; and Blocher et al. 2013). Second, a large body of literature studies the effects of exogenous short-sale constraints, i.e., constraints on investors' positions, on asset prices (see, e.g., Diamond and Verrecchia 1987; Hong and Stein 2003; Scheinkman and Xiong 2003; among others). In our model, the lending fee is endogenous, and, therefore, we can analyze the effects of the primitive parameters of the model on equilibrium fees and asset prices. The most important policy implication of the paper is that regulators should take into account the effects of information on both the asset market and the equity loan market when they impose regulations on corporate information disclosure. Our view is that such considerations were ignored when the Securities and Exchange Commission (SEC) adopted the Regulation Fair Disclosure (FD).

2 A Model of Asset Prices and Lending Fees

2.1 Setup of the Model

The model is static and has two dates. The financial market is populated by two categories of investors (rational investors and liquidity traders) who trade a risk-free asset and a risky asset (the stock). Short sellers must pay a lending fee to borrow shares from long investors.

Agents There is a continuum of rational investors on the unit interval, consisting of three types. Investors within each type are identical. A fraction λ^C of rational investors is long only, uninformed, and constrained, whether from legal/institutional requirements or heterogeneous preferences (or both), from supplying shares in the equity lending market.³ One might interpret these traders as mutual funds that are prohibited by their charters from short selling and participating in the equity loan market. We refer to these investors as uninformed constrained investors and apply the superscript C to their corresponding parameters. A fraction λ^U of rational investors is long only, uninformed, and may lend their shares for a fee of f per share. One might interpret these traders as mutual funds that are prohibited by their charters from short selling, but are permitted to participate in the equity lending market. We refer to these investors as uninformed unconstrained investors and apply the superscript U to their corresponding parameters. Uninformed rational investors do not observe private signals, but rather update their beliefs about the stock payoff by observing its price and lending fee. Finally, a fraction $\lambda^I (= 1 - \lambda^C - \lambda^U)$ of rational investors is informed, and they can be long or short. If they are long, they lend their shares for f per share, and if they are short, they pay f per borrowed share. Each informed investor observes a private

³As an example, Unit Investment Trusts (UITs) are not allowed by statute to lend securities. UITs include four well-known ETFs (DIA, SPY, MDY, and QQQ). Some mutual fund and pension fund managers are explicitly prohibited from short selling and lending securities. For instance, Almazan et al. (2004) document that about 30% of mutual funds are allowed to short sell, and only about 2% of those actually short sell. Evans et al. (2017) document that in 1997, 72% (60%) of active (index) mutual funds were allowed to lend securities, but only 14% (14%) of active (index) mutual funds were lending securities. In 1998, 86% (96%) of active (index) mutual funds were allowed to lend securities, but 43% (67%) of active (index) mutual funds were lending securities.

signal regarding the stock payoff and updates his belief about the stock by observing its price, its lending fee, and his private signal. We apply the superscript I to parameters of these informed investors. One might interpret these traders as hedge funds, which do not face any short-sale constraints, and are active in the equity lending market.

A second category of investors is the liquidity traders who trade for exogenous reasons. Liquidity traders can be long or short. If they are long, they lend their shares for f per share, and if they are short, they pay f per borrowed share.

Assets Two assets are traded in the market: a risk-free asset with a fixed gross return of r_f and a risky asset whose price at time $t = 1$ is p . Without loss of generality (by letting p denote the forward price of the stock), it is assumed that $r_f = 1$. The stock has a payoff of x per share realized at time $t = 2$, and its supply is assumed to be a constant $y > 0$. All investors have the prior belief that $x \sim N(\mu_x, \tau_x^{-1})$: that is, the stock payoff x is normally distributed with mean μ_x and precision (the reciprocal of the variance) τ_x .

Private Signals In the first period, before trading, informed investor j receives a private signal, denoted by x_j , about the stock payoff. Investor j 's signal is noisy but unbiased, and it has the following form:

$$x_j = x + \epsilon_j,$$

where the noise ϵ_j is independent across investors and independent of x , and $\epsilon_j \sim N(0, \tau_\epsilon^{-1})$, where τ_ϵ is the precision of ϵ_j .

Updating Beliefs and Portfolio Choice At $t = 1$, each informed investor j updates his belief using Bayes's law based on his private signal, the observed price of the stock, and the lending fee, and then chooses his portfolio holdings subject to a budget constraint. Let a_j^I denote the optimal number of shares chosen by informed investor j . At $t = 1$, each uninformed investor also updates his belief using Bayes's law based on the observed price of the stock and the lending fee. Let a^U (respectively, a^C) denote the optimal number of shares

chosen by a typical uninformed unconstrained (constrained) investor (investors within each uninformed group are identical). Liquidity investors trade based on exogenous reasons and, therefore, ignore the information available in the market. The aggregate demand of liquidity traders is denoted by z , where $z \sim N(0, \tau_z^{-1})$, and τ_z is the precision of z .

Each rational investor has exponential utility over his terminal wealth. The optimal portfolios of informed investor j , a typical uninformed unconstrained investor, and a typical informed constrained investor are, respectively,

$$\begin{aligned} a_j^I(p, f, x_j) &= \arg \max_a -\mathbb{E} [\exp(-\gamma^I a \{x - (p - f)\}) | p, f, x_j]; \\ a^U(p, f) &= \arg \max_{a \geq 0} -\mathbb{E} [\exp(-\gamma^U a \{x - (p - f)\}) | p, f]; \text{ and} \\ a^C(p, f) &= \arg \max_{a \geq 0} -\mathbb{E} [\exp(-\gamma^C a \{x - p\}) | p, f], \end{aligned}$$

where $\mathbb{E}[\cdot]$ is the expectation operator, and γ^i denotes the risk aversion coefficients of type $i \in \{I, U, C\}$. Uninformed investors (type U) lend any shares they hold, earning a fee of f per share; and informed investors (type I) lend shares, when long, earning f per share, and borrow shares, when short, paying f per share.

In the second period, each investor j 's portfolio payoff and utility are realized.

Equilibrium The equilibrium concept that is used is the standard rational-expectations equilibrium, in which investors maximize their expected utilities and update their beliefs using Bayes's law, and all markets clear. For each (x, z) , the equilibrium price p and equilibrium lending fee f (both are functions of x and z , although we omit the dependence in the notation) satisfy both an asset-market clearing condition and a loan-market clearing condition. Defining $\bar{a}^I(p, f, x) = \frac{1}{\lambda^I} \int_j \text{informed} a_j^I(p, f, x_j) dj$ as the average informed-investor demand, the asset-market clearing condition, including the liquidity demand z , is

$$\lambda^C a^C(p, f) + \lambda^U a^U(p, f) + \lambda^I \bar{a}^I(p, f, x) + z = y, \quad \text{all } x, z.$$

The equity loan market clears if the number of shares available for borrowing is at least as large as the number of shares actually borrowed (each share bought by a liquidity trader is available for lending, and each share sold short requires borrowing a share), i.e.,

$$\begin{aligned} \lambda^U a^U(p, f) + \lambda^I \bar{a}^I(p, f, x) + z &\geq 0, \quad f \geq 0 \\ f \cdot \{ \lambda^U a^U(p, f) + \lambda^I \bar{a}^I(p, f, x) + z \} &= 0, \quad \text{all } x, z. \end{aligned} \tag{1}$$

The equality in the second line says that the fee is zero if the share-borrowing constraint is slack.

In the next section, we show that a rational-expectations equilibrium exists and characterize the equilibrium price, p , and the lending fee, f .

2.2 Solution of the Model

We seek an equilibrium fee and price within the *monotonic-linear class*, defined as follows.⁴

Definition 1. *The function p and f are in the monotonic-linear class if p is some strictly monotonic function of v , and f is some function of v , where $v = x + \alpha z$ and $\alpha \in \mathbb{R} \setminus 0$.*

Define the *market participation rates*, ψ^I , ψ^C , and ψ^U , of the three investor types (informed, constrained, and unconstrained, respectively) as the ratios of the investor-type proportions to risk-aversion coefficients:

$$\psi^I = \frac{\lambda^I}{\gamma^I}, \quad \psi^C = \frac{\lambda^C}{\gamma^C}, \quad \psi^U = \frac{\lambda^U}{\gamma^U} = \frac{1 - \lambda^I - \lambda^C}{\gamma^U}.$$

Let τ^C , τ^U , and τ^I denote the precisions of information available to the investor types. Each precision is the sum of the prior precision, private-signal precision (if any), and price precision:

$$\tau^C = \tau^U = \tau_x + \frac{\tau_z}{\beta^2}, \quad \tau^I = \tau_x + \tau_\epsilon + \frac{\tau_z}{\beta^2}. \tag{2}$$

⁴We require α to be nonzero to exclude a perfectly revealing equilibrium.

Finally, define the total participation rate, ψ , and weighted average total precision, $\bar{\tau}$, as⁵

$$\psi = \psi^C + \psi^U + \psi^I, \quad \bar{\tau} = \frac{\psi^C + \psi^U}{\psi} \tau^U + \frac{\psi^I}{\psi} \tau^I.$$

The following proposition characterizes the equilibrium price of the stock in terms of the noisy signal v , and the two thresholds v_* and v^* ($v_* < v^*$), which are defined as

$$v_* = \mu_x - \left(\frac{\psi^I \tau^I + \psi^U \tau^U}{\psi^I \tau_\epsilon} \right) \frac{1}{\tau_x \psi^C} y, \quad v^* = \mu_x + \frac{\tau^U}{\tau_\epsilon} \frac{1}{\tau_x \psi^I} y, \quad (3)$$

Proposition 1 (equilibrium price). *The unique equilibrium price in the monotonic-linear class is the following convex function of v :*

$$p(v) = \mu_x + \begin{cases} \left(1 - \frac{\tau_x}{\bar{\tau}}\right) (v - \mu_x) - \frac{y}{\psi^C \tau^C} & \text{if } v < v_*, \\ \left(1 - \frac{\tau_x}{\bar{\tau}}\right) (v - \mu_x) - \frac{y}{\psi \bar{\tau}} & \text{if } v \in [v_*, v^*], \\ \left(1 - \frac{\tau_x}{\bar{\tau}}\right) (v - \mu_x) - \frac{y}{\psi^I \tau^I} & \text{if } v > v^*. \end{cases} \quad (4)$$

where $v = x + \beta z$ and $\beta = 1/(\psi^I \tau_\epsilon)$.

This proposition shows that the unique equilibrium price in the monotonic-linear class is piecewise linear and convex. Because the price is strictly increasing in v , the information in the price is equivalent to the information in the noisy payoff signal v , and, therefore, the precision of the price signal is determined solely by $\psi^I \tau_\epsilon$ (which determines β) and τ_z . We interpret v as the firm fundamental. In the region $v \leq v_*$, where the fundamental is low, short selling by informed investors exhausts the supply of available shares for shorting, and, therefore, the loan-market clearing condition binds. In the *nonbinding region*, $v_* < v < v^*$, the share-lending constraint is nonbinding, resulting in a zero fee; short selling by informed

⁵Note that

$$\bar{\tau} = \tau_x + \bar{\tau}_\epsilon + \tau_z (\psi \bar{\tau}_\epsilon)^2, \quad \text{where } \bar{\tau}_\epsilon = \frac{1}{\psi} \int_{i \in [0,1]} \frac{1}{\gamma^i} \tau_{\epsilon_i} dj = \frac{\psi^I}{\psi} \tau_\epsilon.$$

That is, $\bar{\tau}$ is the total precision in a homogeneous economy with only informed investors, each with precision $\bar{\tau}_\epsilon$ (the average overall private signal precision in the original economy).

investors is diminished; and the price is not sufficiently high for the short–sale constraint on the uninformed to bind. In the region $v \geq v_*$, where the fundamental is strong, the price is sufficiently high that the short–sale constraint faced by the uninformed investors binds, and the positive signal induces the informed investors to hold the entire supply.⁶

We next characterize the equilibrium lending fee.

Proposition 2 (equilibrium lending fee). *The unique equilibrium fee in the monotonic–linear class is the following convex function of v :*

$$f(v) = \begin{cases} -\frac{\tau_x}{\tau U} \left(\frac{\psi^I \tau \epsilon}{\psi^I \tau^I + \psi^U \tau^U} \right) (v - \mu_x) - \frac{y}{\psi^C \tau^U} & \text{if } v < v_*, \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

This proposition shows that the equilibrium lending fee curve is a hockey stick shape in firm fundamental. In particular, when the (endogenous) borrowing demand is low (firms fundamental is high), the fee curve is flat, and, thus, the fee is insensitive to shocks in demand for borrowing shares, whereas, when borrowing demand is high (firm fundamental is low), the fee curve tends to become steep. Christoffersen et al. (2007) analyze the market for corporate votes using the U.S. stock lending market and document that although the loan market volume increases significantly on the voting record date for annual shareholder meetings, the lending fees are mainly unresponsive to the higher demand for borrowing shares. The strongest supporting evidence of the hockey stick shape for the lending fee is the evidence in Kolasinski et al. (2013). They use data from 12 lenders and show that when demand is moderate, the lending fee is insensitive to shocks to demand for borrowing shares, and when the demand is high, the lending fee is highly sensitive to shocks to demand for borrowing shares. They argue that this finding is consistent with search frictions in the equity lending market. Nevertheless, the equity lending model with constant search frictions presented in Duffie et al. (2002) does not imply such a kinked fee curve.

⁶If we remove the short–selling prohibition on the unconstrained uninformed investors, the only change in the equilibrium is that the price slope in the region $v > v^*$ become shallower. The equilibrium fee is unchanged.

The comparative statics of the fee, examined in Section 3, depend on equilibrium demands, which we examine next. Define the aggregate demand quantities

$$A^C(v) = \lambda^C a^C, \quad A^U(v) = \lambda^U a^U, \quad A^I(v) = \lambda^I \bar{a}^I + z, \quad (6)$$

which represent, respectively, aggregate demand by the constrained and unconstrained uninformed and the aggregate demand by the informed investors plus liquidity traders.

Lemma 1 (equilibrium demands). *Equilibrium aggregate demands by the uninformed investors, and informed plus liquidity investors, are*

a) For $v \leq v_*$:

$$A^I(v) = \psi^I \tau_x \left(\frac{\tau_\epsilon \psi^U}{\psi^I \bar{\tau}^I + \psi^U \bar{\tau}^U} \right) (v - \mu_x), \quad A^U(v) = -A^I(v), \quad A^C(v) = y.$$

b) For $v_* < v < v^*$:

$$A^I(v) = (\psi^C + \psi^U) b (v - \mu_x) + \frac{\tau^I \psi^I}{\bar{\tau}} y \quad (7)$$

$$A^U(v) = -\psi^U b (v - \mu_x) + \frac{\psi^U \tau^U}{\bar{\tau}} y, \quad A^C(v) = \frac{\psi^C}{\psi^U} A^U(v),$$

where parameter b is the demand sensitivity to the firm fundamental and is defined as

$$b = \tau_x \frac{\bar{\tau}_\epsilon}{\bar{\tau}}, \quad (8)$$

and where $\bar{\tau}_\epsilon = \tau_\epsilon \psi^I / \psi$ denote the average signal precision over all investors.

c) For $v \geq v^*$:

$$A^I(v) = y, \quad A^U(v) = A^C(v) = 0.$$

This lemma shows that when the firm fundamental is low ($v \leq v_*$), the constrained uninformed investors, attracted by the low price, are holding the total supply of the shares;

informed investors (and liquidity traders) are short, because of the ‘pessimistic’ signal; and unconstrained uninformed investors are supplying all of the shares that are shorted. Therefore, the equilibrium fee equates the shares borrowed by informed and liquidity traders with the shares lent by the unconstrained uninformed investors, which is analogous to an asset–market clearing condition on $p - f$ with only these two classes of investors and a zero total supply. As v decreases within this region, the lendable shares (the shares held by the unconstrained) increases, matching the increase in shares shorted. When the fundamental is moderate ($v_* < v < v^*$), the aggregate position of the informed investors (together with liquidity traders) can be either short (when v is close to v_*) or long (when v is close to v^*), there is an excess supply of shares for borrowing, and, thus, the lending fee is zero. When the fundamental is strong ($v \geq v^*$), on aggregate, informed investors (together with liquidity traders) are long; and they hold the total supply of shares, there is an excess supply of lendable shares, and, therefore, the lending fee is zero. The high price induces the uninformed investors to take zero positions. (Recall that they are constrained from short selling.)

2.3 Properties of the Model

In our model, a positive lending fee arises when some investors do not supply their shares in the equity lending market. We next investigate what causes a stock to become expensive to borrow and how the parameters of the model affect the lending fee.

Precision of Private Signals and Tightness of Short–Sale Constraints We start with studying what causes a stock to become expensive to borrow. Lemma 1 shows that when a stock fundamental is above v_* , one can observe short–selling activities by some of the informed investors, but the loan–market clearing condition in equation (1) does not bind, and, therefore, the lending fee is zero. If the firm fundamental deteriorates sufficiently, the loan–market clearing condition binds, and, thus, the fee becomes positive. Therefore, the threshold v_* is one measure of the tightness of short–sale constraints. The lending fee is also a measure of short–sale constraints (see, e.g., Jones and Lamont 2002). The following

lemma characterizes how the precision of private signals affects the tightness of short-sale constraints.

Lemma 2 (nonmonotonic effect of precision on thresholds and the fee). *Define $\hat{\tau}_\epsilon = \frac{1}{\psi^I} \sqrt{\tau_x/\tau_z}$. The threshold v_* , defined in equation (3), and demand-sensitivity parameter b , defined in (8), are both hump shaped in private signal precision, τ_ϵ . Both v_* and b are increasing in τ_ϵ for $\tau_\epsilon < \hat{\tau}_\epsilon$ and decreasing in τ_ϵ toward zero for $\tau_\epsilon > \hat{\tau}_\epsilon$.⁷ For each $v < v_*$, $f(v)$ is strictly increasing in τ_ϵ for sufficiently small τ_ϵ , and, if $\tau_\epsilon > \hat{\tau}_\epsilon$, then $f(v)$ is strictly decreasing in τ_ϵ .*

This lemma, together with Lemma 1, implies that in the nonbinding region, $v \in (v_*, v^*)$, aggregate demands of the uninformed and the informed investors (plus liquidity traders) first diverge as precision increases and then converge. This pattern arises because the public price signal, which is proportional to τ_ϵ^2 , is of a lower order than the precision of the private signals when precision is small, but for large precision (for $\tau_\epsilon > \hat{\tau}_\epsilon$), the public signal begins to dominate, and demand sensitivities converge towards zero, and aggregate demands converge to the constants $\lim_{\tau_\epsilon \rightarrow \infty} A^i = \psi^i y / \psi$, $i \in \{I, U, C\}$.

The divergence and then the convergence of aggregate demands as precision increases explains why v_* is nonmonotonic in precision. When precision is low, an increase in precision makes the equilibrium demands of informed investors more signal sensitive and that of uninformed investors more price sensitive. An increase in the precision, for a pessimistic signal, increases the demand for short selling more than it increases the supply of shares held by the unconstrained uninformed investors, and, thus, v_* increases. (The positive fee region expands.) When precision is high, both the uninformed and informed investment choices are dominated by the public price signal, causing investments to converge and v_* to decrease (i.e., the positive-fee region contracts).

Why is the lending fee not monotonic in the precision of private signals? In our model, there are two opposing channels through which private information affects the lending fee.

⁷ $\mathbb{E}[A^I]$ is also hump shaped (with maximum at $\hat{\tau}_\epsilon$), and $\mathbb{E}[A^U]$ is u shaped (with minimum at $\hat{\tau}_\epsilon$).

As discussed earlier, the dispersion channel refers to the increase in the signal and price sensitivity of informed investors' trades caused by an increase in precision of private signals (which reduces the conditional uncertainty about the payoff). Higher precision combined with a strong signal (a high or a low value of v) generates heterogeneity between the trading of informed and uninformed investors. The price-informativeness channel refers to the trades becoming homogenous as the (public) price signal aggregates more precise private information as the precision of private signals increases. The market clearing condition requires the price to absorb the common component of private signals, although that information is only partially revealed because of random demand by liquidity traders. As precision increases, informed trading responds more aggressively (uniformly across the traders) to signals, but liquidity trading is unaffected, resulting in an equilibrium price that more precisely reveals the payoff. When precision is high, the price signal dominates the information sets of traders, resulting in similar trading strategies. Indeed, the following lemma shows that as the precision goes to infinity, the lending fee goes to zero, and the nonbinding region applies for all v .⁸

Lemma 3 (perfect signal limits). *The limiting values of the lending fee and thresholds are as follows:*⁹

$$\text{If } \tau_\epsilon \rightarrow \infty, \text{ then } v_* \rightarrow -\infty, \quad v^* \rightarrow \infty, \quad \text{and} \quad f(v) \rightarrow 0 \text{ all } v.$$

⁸An alternative interpretation is that private information, loosely speaking, is generating an adverse selection problem. The uninformed investors face the adverse selection problem that they may trade with (or lend shares to) those investors with private information when accommodating the liquidity traders' demands. In our setting, one cannot separate these two roles. However, if traders were to ignore the information content of the price (eliminating the price-informativeness role), then it is straightforward to argue that the adverse selection problem becomes more severe as the precision of private signals increases. In the limiting case of $\tau_z = 0$, the price signal is useless (liquidity trading demand is infinite), and an increase in signal precision unambiguously increases information asymmetry. This limiting case also corresponds to the case when investors ignore the information in the price. Because uninformed unconstrained investors do not know whether higher short-selling demand is driven by higher demand for short-selling by liquidity traders or by the negative information of informed investors, they charge a higher fee for lending their shares as the precision increases.

⁹Note that $\tau_\epsilon \rightarrow 0$ implies that $\beta \uparrow \infty$ and $\sigma_v \uparrow \infty$, and, therefore, the pointwise limit $f(v) \rightarrow 0$ does not imply the same for $\mathbb{E}[f(v)]$.

The role of the public price signal can be further demonstrated by examining how the volatility of the demand by liquidity traders affects the lending fee. Note that Propositions 1 and 2 show that τ_z affects the equilibrium price and fee solely through the price precision, $\tau_z (\psi^I \tau_\epsilon)^2$ (see equation (2)). An increase in τ_z (i.e., a reduction in the variance of liquidity traders' demand) makes the price more informative for any private signal precision τ_ϵ , and it causes the informed and uninformed investors' information sets and trading strategies to converge (the demand sensitivity b in equation (8) decreases), which reduces the equilibrium lending fee, as the following lemma shows.

Lemma 4 (role of liquidity traders). *For each v , the borrowing fee function $f(v)$ is monotonically decreasing in τ_z , converging to zero as $\tau_z \rightarrow \infty$.*

If traders were to ignore the information content of the price (eliminating the price-informativeness role), then private signals generate information asymmetry only,¹⁰ and the demand-sensitivity parameter b is strictly increasing in precision for all precision levels. The equilibrium is given in the next proposition.

Proposition 3 (equilibrium when investors ignore information in the price). *Suppose that each investor ignores the information in the price.*

a) *The unique (in the monotonic-linear class) equilibrium price, lending fee, and thresholds, respectively, satisfy equations (4), (5), and (3), after setting $\tau_z = 0$. That is, the equations hold after setting the price precision to zero in the total precision expressions:*

$$\tau^C = \tau^U = \tau_x, \quad \tau^I = \tau_x + \tau_\epsilon, \quad \bar{\tau} = \tau_x + \frac{\psi^I}{\psi} \tau_\epsilon.$$

b) *The threshold v_* , and, for each v , $f(v)$, are monotonically increasing in τ_ϵ .*

A comparison of Propositions 2 and 3 shows that fees are uniformly lower in the REE setting. The threshold v_* is strictly lower in the REE setting, as is the fee $f(v)$ for any

¹⁰In the limiting case of $\tau_z = 0$, the price signal is useless (liquidity trading demand is infinite), and an increase in signal precision unambiguously increases information asymmetry. This limiting case also corresponds to the case when investors ignore the information in the price.

$v < v_*$. In contrast to the REE setting, when information content of the price is ignored, v_* and the lending fee are monotonically increasing in the precision of private signals .

Composition of Investors’ Types and Tightness of Short–Sale Constraints Chen et al. (2002) measure breadth of ownership for a stock, B , as the ratio of the number of mutual funds that hold a long position in that stock to the total number of mutual funds in the sample in that quarter. They argue that breadth of ownership is a proxy for the tightness of short–sale constraints. When the breadth is low, the tightness of short–sale constraints is high, and, therefore, the stock is overpriced. They provide evidence that stocks with a decline in breadth of ownership underperform stocks with an increase in breadth. This finding is interpreted as short–sale constraints bias prices upward. Nagel (2005) shows that the main finding documented in Chen et al. (2002) does not hold in an extended time period sample, and, therefore, their result is not robust.

The breadth of ownership measure B is supposed to capture the fraction of long–only informed investors who have negative information about a stock and are sidelined from taking a short position because of short–sale constraints. However, it is not clear whether the theoretically motivated variable B can indeed measure the tightness of short–sale constraints as Chen et al. (2002) claims. In particular, if one introduces rational expectations in their model, the price would reveal the superior information of informed hedge funds, causing mutual funds to update their valuations and, thus, trade based on the information content of the price, resulting in identical trading strategies among the mutual funds.

Equations (3) and (5) show that the key investor characteristics that determine the fee function are the total precision that enters the price, $\psi^I \tau_\epsilon (= 1/\beta)$, the total participation rate of the unconstrained investors, $\psi^I + \psi^U$, and the participation rate of the constrained investors, ψ^C . The following lemma shows that greater participation by the constrained uninformed investors (i.e., an increase in ψ^C) or reduced participation by the unconstrained uninformed (i.e. a reduction in ψ^U) results in an expansion of the positive–fee region and an increase in the fee function. An increase in ψ^I has mixed effects: it increases the fee

when we hold the price precision constant (e.g., when $\tau_z = 0$), but also increases the price precision, which causes a reduction in the fee.

Lemma 5 (comparative statics with respect to unrestricted uninformed investors).

$$\frac{\partial v_*}{\partial \psi^U} < 0, \quad \frac{\partial v_*}{\partial \psi^C} > 0,$$

$$\frac{\partial f(v)}{\partial \psi^U} \leq 0, \quad \frac{\partial f(v)}{\partial \psi^C} \geq 0, \quad \text{for all } v.$$

(Both derivative are strictly negative for $v < v_*$.) Further,¹¹ if $\gamma^C = \gamma^U$ and constrained uninformed investors are replaced with unconstrained uninformed investors (i.e., λ^C is decreased while holding fixed $\lambda^C + \lambda^U$), then v_* strictly decreases and $f(v)$ decreases for all v (strictly, for $v < v_*$). Finally, $\lim_{\psi^C \rightarrow 0} f(v) = 0$.

This lemma also shows that the zero-fee region expands and the fee function decreases as constrained uninformed investors are replaced with unconstrained uninformed investors, that is, as more investors can lend their shares. This result suggests that a novel and useful empirical measure of the tightness of short-sale constraint is the fraction of mutual funds that are prohibited by charter from participating in the equity lending market. In particular, the change in tightness of short-sale constraints can be measured based on the change in the fraction of mutual funds that are holding a stock but cannot participate in the equity lending market because of the restriction in their charters.

What are the restrictions on participating in the equity lending market that investors face in practice? The SEC restricts any U.S. regulated fund (e.g., a mutual fund) to lend no more than one-third of the fund's total assets subject to any investment policies and restrictions disclosed in the fund's registration statement. Evans et al. (2017) document that although 85% of mutual funds are allowed to lend, only 42% actually do. As another example, Unit Investment Trusts (UITs) are not allowed by statute to lend securities. UITs include four

¹¹If $\gamma^C \neq \gamma^U$, then a change in λ^U and λ^C also changes the weighted average risk aversion of the uninformed, and, therefore, the price in the region $v \in [v_*, v^*]$ also changes.

well-known ETFs: DIA, SPY, MDY, and QQQ.

Investors may also have heterogeneous preferences regarding participation in the lending market as suggested by the chief investment officer for Fidelity Worldwide, Dominic Rossi:

The idea that we would lend the stock that we obviously like, otherwise we would not own it, to someone who is then going to short it does not really make much sense. It is not in the interests of our clients to have to foster that short selling, nor is it in the interests of the company in which we invest. We do a very limited amount related to dividends and I suspect even that practice will stop shortly.¹²

Why might investors have heterogeneous preferences regarding participation in the equity loan market? A factor that may hinder investors' participation is collateral risk. In a typical transaction, a lending agent lends securities on behalf of the lender (beneficial owner). In the United States, equity borrowers post cash in the amount of 102% of the value of loaned U.S. stocks and 105% of the value of non-U.S. stocks with the lending agent. The lender typically pays rebate interest to the equity borrower; therefore, the cash must be reinvested at a higher rate in order to make any net return on the collateral. Any gain on the collateral is shared between the lender and the lending agent, and any loss on the collateral is borne only by the lender.^{13,14} We have left modeling this aspect of the securities lending market as a topic of future research.

The literature often uses institutional ownership as a proxy for lending supply (see, e.g., D'Avolio 2002; Asquith et al. 2005). However, Lemma 5 suggests that more important than institutional ownership is the participation rate of unconstrained institutional owners, i.e., owners that lend securities in the equity loan market (changes in λ^C and λ^U have opposite

¹²Steve Johnson, "Fidelity eyes stock loans exit," April 6, 2013, *Financial Times*, <https://www.ft.com/content/487527f2-9de7-11e2-9ccc-00144feabdc0>.

¹³Louise Story, "Banks shared clients' profits, but not losses," *New York Times*, Oct. 17, 2010, <http://www.nytimes.com/2010/10/18/business/18advantage.html>.

¹⁴Another factor may be that in a less than perfectly competitive setting, investors may not lend their shares if they believe that there is a feedback effect from the financial markets to the real investment, so that short sellers can distort investment decisions, leading to a permanent decline in a stock price (see, e.g., Goldstein and Guembel 2008). Such decline in the stock price may dominate the lending fee that investors collect for lending their shares and, thus, may lead to lack of participation in the lending market.

effects on the tightness of short–sale constraints).¹⁵ Although the above finding may seem obvious, the literature has been using the fraction of mutual funds that are holding a stock as a measure of short–sale constraints, regardless of whether those mutual funds participate in the equity lending market. Given the finding of our model, it is not surprising that Nagel (2005) documents that the results presented in Chen et al. (2002) are not robust.

3 Private Information, Asset Prices, and Lending Fees

This section examines how information, investor participation rates, and price precision affect the expected rate of return, the probability of specialness, the expected lending fee, and the risk of short selling measured by the variance of the lending fee. We show that these lending–fee moments are all strictly decreasing in the precision of the informed investors’ private signals, and are all minimized when precision is zero (our symmetric zero–information benchmark). In addition, we show that expected return is not necessarily decreasing in signal precision.

3.1 Information Asymmetry and Moments of the Lending Fee

In our REE setting, price aggregates private signals, and the information in the price tends to align expectations and trading strategies as precision increases. As the following propositions show, this causes the probability of a binding short–sale constraint to decrease with precision.

Proposition 4 (probability of a binding stock–lending constraint). *The probability that the stock is special is*

$$P(v < v_*) = \mathbb{N}(\zeta_*), \tag{9}$$

where $\mathbb{N}(\cdot)$ is the normal distribution function; we define the normalized positive–fee threshold

¹⁵See Porras Prado et al. 2016 for evidence supporting this argument.

$\zeta_* = \frac{v_* - \mu_x}{\sigma_v}$, and the standard deviation of v is

$$\sigma_v = \sqrt{\frac{1}{\tau_x} + \beta^2 \frac{1}{\tau_z}}. \quad (10)$$

When $\tau_\epsilon = 0$, equation (9) holds for $\zeta_* = \zeta_0$ where we define

$$\zeta_0 = \lim_{\tau_\epsilon \rightarrow 0} \zeta_* = - \left(\frac{\psi^I + \psi^U}{\psi^C} \right) \sqrt{\tau_z} y. \quad (11)$$

The probability of specialness is strictly decreasing in τ_ϵ , τ_z , ψ^I , and ψ^U ; and strictly increasing in ψ^C .

When a short seller takes a position in a stock, in addition to the risk that the stock price may go up, he takes the risk that the lending fee can increase significantly, and, thus, the stock can become special. Beneish et al. (2015) documents that in the sample of U.S. firms during the time period of 2004–2014, approximately 15% of firms are special (have a DCBS value of 3 or higher). Proposition 4 shows that as the precision of private signals increases, the probability of specialness decreases. It also shows that the probability of specialness is decreasing in τ_z (a decrease in τ_z decreases the precision of the price signal), decreasing in the participation rates of both the informed and unrestricted uninformed (the former because of increased price precision, and the latter because of an increase in the supply of lendable shares), and decreasing the participation rate of the constrained uninformed (which reduces the supply of lendable shares).

We next characterize the equilibrium expected lending fee.

Proposition 5 (expected lending fee). *Suppose the precision of the private signals of informed investors is $\tau_\epsilon > 0$. The expected lending fee is¹⁶*

$$\mathbb{E}[f(v)] = - \frac{y}{\psi^C \tau^U} \left(\frac{\mathbb{N}'(\zeta_*)}{\zeta_*} + \mathbb{N}(\zeta_*) \right). \quad (12)$$

¹⁶Note that the $\frac{\mathbb{N}'(\zeta_*)}{\zeta_*} + \mathbb{N}(\zeta_*) < 0$ because of the well-known inequality $\mathbb{N}(-k) < \frac{\mathbb{N}'(k)}{k}$ for any $k > 0$, and the fact that $\zeta_* < 0$.

The expected lending fee is strictly decreasing in τ_ϵ , τ_z , ψ^I , and ψ^U ; and strictly increasing in ψ^C . If the precision of private signals is zero, i.e., $\tau_\epsilon = 0$, the expected lending fee also satisfies equation (12) with $\zeta_ = \zeta_0$ and $\tau^U = \tau_x$.*

The comparative statics for the expected fee are all in the same direction as in Proposition 4. In particular, as τ_ϵ increases, the expected fee decreases. Earlier we argued that private information affects the lending fee through two opposing channels. A dispersion channel, through which higher precision generates asymmetry in information and trading, which generally increases the fee, and a price-informativeness channel, through which the market-clearing price absorbs the common component of private signals, which reduces disagreement and, therefore, the fee. When determining the expected fee, another consideration is Jensen's inequality: the variance of the price signal v always decreases in signal precision, and the fee is a convex function of v . Therefore, even though the fee function is initially increasing in precision when precision is small, the fee function is sufficiently convex that the expected fee is always monotonically decreasing in signal precision.

Consistent with the conflicting effects of private signals on information asymmetry and the public signal, the proposition also shows that improving the quality of the price signal, whether through an increase in ψ^I (which results in more aggressive trading by the informed) or an increase in τ_z (which diminishes noise trading), reduces the expected fee. A higher expected fee results from either a decrease in ψ^U or an increase in ψ^C because the effect of either is to increase the fraction of long shares held by the constrained when $v < v_*$, reducing the shares available for lending.

The negative relationship between expected lending fee and precision is the opposite of the finding in the models presented in Duffie et al. (2002) and Blocher et al. (2013). The framework presented in Blocher et al. (2013) is a reduced-form model in which investors' demand functions are exogenously given. They model disagreement as opposite shifts in the exogenous long- and short-demand curves, and argue, "If there is very little disagreement, the equity lending constraint is likely to be slack, so the price to borrow shares will be zero.

... At some point, increasing disagreement causes short demand to be great enough that the price to borrow shares rises.” As we discussed earlier, this argument is in a partial equilibrium framework in which prices and optimal portfolios are not determined jointly. Such a partial equilibrium framework ignores the effect of information on price informativeness, which, our model shows, plays an essential role in determining the effect of information asymmetry on equilibrium fees and prices.

Even if investors ignore the information content of the price, the expected fee is not always increasing in the precision of private signals. Although Proposition 3 shows that the lending fee is monotonically increasing in precision of private signals, the convexity of the fee function combined with σ_v declining in precision imply that $\mathbb{E}[f(v)]$ is u shaped in precision: decreasing in precision for $\tau_\epsilon < [\tau_z (\psi^I + \psi^u) \psi^I]^{-1}$ and increasing in precision for $\tau_\epsilon > [\tau_z (\psi^I + \psi^u) \psi^I]^{-1}$.¹⁷

In practice, short sellers are concerned with not only the level of lending fees but also the variation of the lending fees over time, as the variation represents a source of risk for borrowing shares (see, e.g., D’Avolio 2002). We use the ex ante variance of the lending fee as a measure of risk for borrowing shares, and the following lemma shows that the variance of the lending fee is decreasing in precision.

Proposition 6 (variance of lending fee). *Suppose the precision of the private signals of informed investors is $\tau_\epsilon > 0$. The variance of the lending fee is*

$$\mathbb{V}[f(v)] = \left(\frac{y}{\psi^C \tau^U} \right)^2 \left(\frac{\mathbb{N}'(\zeta_*)}{\zeta_*} + \mathbb{N}(\zeta_*) + \frac{\mathbb{N}(\zeta_*)}{\zeta_*^2} - \left(\frac{\mathbb{N}'(\zeta_*)}{\zeta_*} + \mathbb{N}(\zeta_*) \right)^2 \right). \quad (13)$$

The variance of the lending fee is strictly decreasing τ_ϵ , τ_z , ψ^I , and ψ^U ; and strictly increasing in ψ^C . If the precision of private signals is zero, the variance of lending fee satisfies equation (13) with $\zeta_ = \zeta_0$ and $\tau^U = \tau_x$.¹⁸*

¹⁷The proof of this statement is available upon request from the authors.

¹⁸If each investor ignores the information in the price, then, the variance of fee is u shaped in precision, decreasing in precision for $\tau_\epsilon < [\tau_z (\psi^I + \psi^u) \psi^I]^{-1}$ and increasing in precision for $\tau_\epsilon > [\tau_z (\psi^I + \psi^u) \psi^I]^{-1}$. The variance of fee is hump-shaped in γ^I , increasing for $\gamma^I < \lambda^I \psi^u \tau_z \tau_\epsilon^2 / (\tau_x + \tau_\epsilon)$ and decreasing for

3.2 Information Asymmetry and the Expected Return

Many scholars have studied the effect of information asymmetry (see, e.g., Easley and O'Hara 2004; Kelly and Ljungqvist 2012; Lambert et al. 2012; among others), and disagreement among investors (see, e.g., Abel 1989; Anderson et al. 2005; David 2008; among others) on the expected rate of return with no consensus reached. Having characterized the effects of information asymmetry on the equity lending market, we next show that the expected return is nonmonotonic in private signal precision.

Proposition 7 (expected return). *Suppose the precision of the private signals of informed investors is $\tau_\epsilon \geq 0$. The expected return of the stock is¹⁹*

$$\mathbb{E}[R(v)] = \mu_x - \mathbb{E}[p(v)] = \frac{y - H(\tau_\epsilon)}{\psi\bar{\tau}}, \quad (14)$$

where (note that both braced terms are positive)

$$H(\tau_\epsilon) = \sqrt{\frac{\tau_x}{\tau_z}} \left(\frac{1}{\sqrt{\tau^U}} \{N'(\zeta_*) + \zeta_* N(\zeta_*)\} + \frac{\sqrt{\tau^U}}{\tau^I} \left(\frac{\psi^C + \psi^U}{\psi^I} \right) \{N'(\zeta^*) - \zeta^* N(-\zeta^*)\} \right), \quad (15)$$

and the normalized thresholds ζ_* and ζ^* are defined as

$$\zeta_* = \frac{v_* - \mu_x}{\sigma_v}, \quad \zeta^* = \frac{v^* - \mu_x}{\sigma_v}, \quad (16)$$

where the standard deviation of v is defined in equation (10). The function $H(\cdot)$ is strictly positive; strictly decreasing in τ_ϵ , τ_z , and ψ^I ; strictly increasing in ψ^C ; and satisfies $\lim_{\tau_\epsilon \rightarrow \infty} H(\tau_\epsilon) = 0$. Moreover, there exists some precision level $\tilde{\tau}_\epsilon > 0$ such that $\mathbb{E}[R(v)]$ is strictly decreasing in precision for all $\tau_\epsilon > \tilde{\tau}_\epsilon$.

If $\tau_\epsilon = 0$, then let $\zeta_* = \zeta_0$ (defined in equation (11)) and $\zeta^* = \sqrt{\tau_z}y$.

By curtailing short selling, the two constraints in the model, the inability of the un-

$\gamma^I > \lambda^I \psi^u \tau_z \tau_\epsilon^2 / (\tau_x + \tau_\epsilon)$.

¹⁹If all investors were unconstrained (allowed to both lend and short shares), then $\mathbb{E}[R] = y / (\psi\bar{\tau})$, which is monotonically decreasing in precision.

constrained investor to short sell, and the requirement that the informed borrow shares to short sell, both have the effect of supporting equilibrium prices, thereby reducing equilibrium expected returns. The *price-support function* H captures these effects. The first term in the expression (15) for H is caused by the lending constraint, and the second term by the short-sale constraint on the uninformed (both terms are positive).

An increase in private signal precision has conflicting effects on the expected return. In the absence of constraints, the function H equals zero, and an increase in private-signal precision, by reducing uncertainty about the payoff, increases the expected price and, therefore, reduces the expected return. However, the price-supporting effects of the constraints diminish with precision (exerting a positive influence on expected return) for two reasons: (1) by Jensen’s inequality, if we hold the convex function $p(v)$ fixed, higher precision implies smaller σ_v^2 and, therefore, a lower expected price; (2) as precision increases, the normalized thresholds widen (i.e., ζ_* decreases and ζ^* increases), reducing the probability of a binding constraint. When the constraints are more binding (low value of τ_z , ψ^I , or ψ^U ; or a high value of ψ^C), the effects of the constraints can dominate and expected return can increase with precision, particularly when precision is small. When precision is sufficiently large, however, the constraints have a minimal effect on prices and, therefore, the expected return is decreasing in precision.²⁰

In an influential paper, Easley and O’Hara (2004) study the effects of reducing the asymmetry of information between informed and uninformed investors by making some of the information of informed investors public. They conclude that the expected return (cost of capital) declines because of the reduction in information asymmetry. In a perfectly competitive setting, Lambert et al. (2012) show that the expected return is decreasing in the average precision of information available to investors, and is invariant to the degree of asymmetry. Lambert and Verrecchia (2015) show that this finding also holds in an imperfect competi-

²⁰The proof of Proposition 7 shows that $0 < H(\tau_\epsilon) < K/\tau_\epsilon$ for some constant $K > 0$, and, therefore, for large precision, the expected return expression in equation (14) is dominated by the term $y/(\psi\bar{\tau})$ (which is the expected return absent any short-selling or lending restrictions).

tion setting. Given these findings, Lambert et al. (2012) call into question the finding of Easley and O’Hara (2004), and state that “in a perfect competition setting, it makes no difference whether some investors have more information than others; a firm’s cost of capital is governed solely by the average precision of investors’ information.”

This criticism in our framework can be understood by supposing that all investors were unconstrained (allowed to both lend and short shares). In this case, the expected return of the stock would be $\mathbb{E}[R(v)] = y/(\psi\bar{\tau}) (= y/(\psi^I\tau_\epsilon))$, which is monotonically decreasing in the average precision of information available to investors. Thus, Lambert et al. (2012) associate the findings of Easley and O’Hara (2004) to an increase in average precision of information available to investors and not the change in information asymmetry. Lambert et al. (2012) further argue that “in perfect competition settings, information asymmetry does not give rise to a separate (or additional) risk factor, and there is no compensation for being less informed, as claimed in Easley and O’Hara (2004).”

In our model, the precision of the information in the price, as in the unconstrained setting, is captured by $\psi^I\tau_\epsilon (= 1/\beta)$. Once constraints are introduced, however, the fee function, the fee moments, and the expected return also depend on the asymmetry of information. The degree of asymmetry and the fraction of shares available for lending depend on the composition of the investors types. If shortsale constraints are sufficiently strong (e.g., ψ^C high), then the expected return can be initially increasing in precision. But for large precision, the price signal dominates, and the effects of the constraints (and asymmetry) diminish.

4 Empirical Implications

In this section, we briefly summarize the empirical implications of our model that we believe are novel to the literature on the equity lending market.

An extensive literature has studied the role of information on asset prices. Nevertheless, to the best of our knowledge, both the empirical and theoretical literature have been silent

about the role of information in securities lending markets. Understanding the role of private information in equity lending market is important because much prior empirical evidence suggests that short sellers are informed traders (see, e.g., Boehmer et al. 2008; Engelberg et al. 2012; among others).

One central question about short-sale constraints is their effects on asset prices. Given that the equity lending fee is an impediment to short selling, it is of importance to understand the determinants of the fee to have a better understanding of the effects of short-sale constraints on asset prices. In our model, a higher precision of private information is associated with a lower probability of being special (see Lemma 4). In other words, information, even when it is private and dispersed among only informed investors, reduces the likelihood that a stock becomes expensive to borrow. In addition, our model also suggests that a higher precision of private information is associated with a lower expected lending fee (see Proposition 5). These findings suggest that, surprisingly, private information reduces the constraints that short sellers face in the equity lending market.

Given that in practice short sellers must borrow securities from investors who own them, they face additional sources of risk in comparison to a typical long investor. In particular, the lending fee can increase substantially or even the borrowed shares can be recalled forcing a short seller to close his position prematurely. Engelberg et al. (forthcoming) provide evidence suggesting that short-selling risk measured by the variance of the lending fee is an important risk that short sellers face, and this risk affects the cross-section of asset prices. Our paper characterizes the variance of the lending fee in a general equilibrium setting. More important, our model suggests that an increase in the precision of private information is associated with a decrease in the variance of the lending fee.

Our model provides insights regarding the regulation of information disclosure in financial markets. For example, in the year 2000, the SEC adopted Regulation Fair Disclosure (FD), which mandated that publicly traded companies disclose material information to all investors and the same time. We are not aware of any article that has investigated the effects of Reg FD

on the equity lending market. This is surprising given that an extensive literature documents that short-sale constraints do affect asset prices, and hence, cost of capital. Our model suggests that if Reg FD has caused managers to reduce the quality and quantity of private disclosures, then the cost and riskiness of short selling will have increased.²¹ Moreover, in contrast to an argument used in the literature that such reduction in information disclosure will increase the expected return (cost of capital), our model shows that the reduction in information disclosure could reduce the expected return of a stock by making short-sale constraints tighter (see Miller 1977).

Our model also has interesting implications about the roles of liquidity/noise traders in both the asset market and the equity lending market. In the absence of equity-lending restrictions, the uncertainty in the noise traders' demand affects the expected return of a stock solely through the average precision of information available to the market participants. But our model shows that in the presence of lending restrictions, the variance of noise traders' demand, σ_z^2 , also affects asset prices through the equity lending market. Proposition 7 shows that the expected return of the stock can be decomposed into three terms: the first term is the expected return in the absence of any constraint, and two price-support terms (via H) that arise because of short-sale constraint on the uninformed and the share-borrowing constraint faced by the informed traders. Increasing the uncertainty of noise traders' demand affects all three terms: average precision deteriorates because the price signal becomes noisier; and both price-support terms increase because the short-sale constraints become tighter and the signal-variability increases (increasing the expectation of the convex price function). Our model also suggests that an increase in the uncertainty of noise traders' demand is associated with a higher probability of specialness, a higher expected fee, and higher short-selling risk (see Propositions 5 and 6). Overall, our model highlights the roles that noise traders play in the equity lending market and a channel through which they can affect asset prices. We leave the empirical tests of these implications as topics of future research.

²¹No consensus has been reached regarding the effect of Reg FD on information disclosure: see, for example, Bailey et al. (2003) and Sidhu et al. (2008).

5 Conclusions

We develop a model of asset prices in which short sellers must borrow shares in the equity lending market before short selling, and we examine the effect of information asymmetry on the lending fee. In contrast to conventional wisdom, we show that asymmetric information reduces both the expected lending fee and the variance of lending fee compared with a symmetric zero-information benchmark. Our findings suggest that information, even when it is private, reduces both the tightness of short-sale constraints (measured as expected lending fee or the probability of a binding short-sale constraint) and the riskiness of short selling (measured by variance of the lending fee).

We have interpreted an increase in precision of private signals as an increase in information asymmetry between informed and uninformed investors. An alternative interpretation is that an increase in precision of private information corresponds to an increase in the quality of information communicated by a firm to a subset of investors (see, e.g., Solomon and Soltes 2015; Bushee et al. 2016). Even under this interpretation, surprisingly, both the average lending fee and the variance of lending fee decrease as the quality of private information is improved.

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Appendix

A Proofs

Lemma A.1. *Let Y be a normal random variable with mean μ_Y and standard deviation σ_Y . Then*

$$\begin{aligned}\mathbb{E}[Y \cdot 1_{\{Y < 0\}}] &= \mu_Y \mathbb{N}\left(-\frac{\mu_Y}{\sigma_Y}\right) - \sigma_Y \mathbb{N}'\left(-\frac{\mu_Y}{\sigma_Y}\right), \\ \mathbb{E}[Y \cdot 1_{\{Y > 0\}}] &= \mu_Y \mathbb{N}\left(\frac{\mu_Y}{\sigma_Y}\right) + \sigma_Y \mathbb{N}'\left(\frac{\mu_Y}{\sigma_Y}\right),\end{aligned}$$

and

$$\mathbb{E}[Y^2 \cdot 1_{\{Y < 0\}}] = (\mu_Y^2 + \sigma_Y^2) \mathbb{N}\left(-\frac{\mu_Y}{\sigma_Y}\right) - \mu_Y \sigma_Y \mathbb{N}'\left(\frac{\mu_Y}{\sigma_Y}\right).$$

Proof. The proofs are omitted for the simplicity of the presentation. □

Lemma A.2. *The function $g(k)$, defined by*

$$g(k) = \mathbb{N}(-k) - \frac{\mathbb{N}'(k)}{k} + \frac{1}{k^2} \mathbb{N}(-k) - \left(\mathbb{N}(-k) - \frac{\mathbb{N}'(k)}{k}\right)^2, \quad (17)$$

is strictly decreasing in k for $k > 0$.

Proof. Note that $g(k) > 0$ with $\lim_{k \downarrow 0} g(k) = \infty$ and $\lim_{k \rightarrow \infty} g(k) = 0$. Using $\frac{\partial}{\partial k} \left(\mathbb{N}(-k) - \frac{\mathbb{N}'(k)}{k}\right) = \frac{\mathbb{N}'(k)}{k^2}$, we get

$$g'(k) = -\frac{2}{k^2} h(k) \quad \text{where} \quad h(k) = \frac{\mathbb{N}(-k)}{k} + \left(\mathbb{N}(-k) - \frac{\mathbb{N}'(k)}{k}\right) \mathbb{N}'(k). \quad (18)$$

Note that $\lim_{k \downarrow 0} h(k) = \infty$ and $\lim_{k \rightarrow \infty} h(k) = 0$. To show that $g'(k) < 0$ for all $k > 0$, it is, therefore, sufficient to show that $h(k)$ is monotonically decreasing in k (which implies $h(k) > 0$ all $k > 0$). Differentiating, we get

$$h'(k) = \frac{(\mathbb{N}'(k))^2 - \mathbb{N}(-k)}{k^2} - \left(\frac{1}{k} + k\mathbb{N}(-k) - \mathbb{N}'(k)\right) \mathbb{N}'(k).$$

The proof is completed by showing two inequalities:

$$J(k) \equiv (\mathbb{N}'(k))^2 - \mathbb{N}(-k) < 0 \quad \text{all } k > 0, \quad (19)$$

$$K(k) \equiv \frac{1}{k} + k\mathbb{N}(-k) - \mathbb{N}'(k) > 0 \quad \text{all } k > 0. \quad (20)$$

To prove inequality (19), $J(0) < 0$ and $\lim_{k \rightarrow \infty} J(k) = 0$ imply that it is sufficient to show that $J(\cdot)$ is monotonically increasing in k . Differentiating,

$$J'(k) = -2\mathbb{N}'(k)^2 k + \mathbb{N}'(-k) = [1 - 2k\mathbb{N}'(k)] \mathbb{N}'(k), \quad k > 0.$$

Note that $1 - 2k\mathbb{N}'(k)$ equals one at $k = 0$ and one for $k \rightarrow \infty$, and achieves its minimum for positive k at $k = 1$, at which point the function is positive, which proves

$$1 - 2k\mathbb{N}'(k) > 0 \quad \text{for } k > 0, \quad (21)$$

and, therefore, $J'(k) > 0$ for $k > 0$.

To prove the inequality (20), $\lim_{k \rightarrow 0} K(k) = \infty$ and $\lim_{k \rightarrow \infty} K(k) = 0$ imply that it is sufficient to show that K is decreasing. Differentiating, we get

$$K'(k) = \mathbb{N}(-k) - \frac{1}{k^2} < \frac{1}{k} \left(\mathbb{N}'(k) - \frac{1}{k} \right) < 0,$$

where the first inequality follows from $\mathbb{N}(-k) < \frac{\mathbb{N}'(k)}{k}$ for $k > 0$, and the second from equation (21). \square

Lemma A.3. *The normalized threshold ζ_* , defined in equation (16), is a) strictly decreasing in precision, with*

$$\frac{d\zeta_*}{d\tau_\epsilon} = - \left(\frac{\tau_x}{\tau^U \tau_\epsilon} + (\psi^I + \psi^U) \tau_z \psi^I \right) \frac{y}{\sigma_v \tau_x \psi^C}, \quad (22)$$

and the limits

$$\lim_{\tau_\epsilon \rightarrow 0} \zeta_* = - \left(\frac{\psi^I + \psi^U}{\psi^C} \right) \sqrt{\tau_z} y, \quad \lim_{\tau_\epsilon \rightarrow \infty} \zeta_* = -\infty.$$

b) strictly decreasing in ψ^U and ψ^I ; and is strictly increasing in ψ^C .

c) strictly decreasing τ_z .

Proof. a) Using the expressions for v_* in equation (3), we get

$$\zeta_* = - \left(\frac{\tau^U (\psi^U + \psi^I) + \tau_\epsilon \psi^I}{\tau_\epsilon \psi^I \sigma_v \tau_x \psi^C} \right) y. \quad (23)$$

Differentiate ζ_* , using

$$(\sigma_v \tau_\epsilon)^2 = \frac{\tau^U}{\tau_x \tau_z (\psi^I)^2} \quad (24)$$

and $\frac{\partial}{\partial \tau_\epsilon} (\sigma_v \tau_\epsilon) = \frac{1}{\tau_x \sigma_v}$, to get

$$\frac{\partial \zeta_*}{\partial \tau_\epsilon} = \left(- \frac{\psi^I + 2\tau_z (\psi^I)^2 \tau_\epsilon (\psi^I + \psi^U)}{\sigma_v \tau_\epsilon} + \frac{\tau_\epsilon \psi^I + \tau^U (\psi^I + \psi^U)}{(\sigma_v \tau_\epsilon)^2} \frac{1}{\tau_x \sigma_v} \right) \frac{y}{\psi^I \tau_x \psi^C}.$$

Substitute equation (24) and simplify to get

$$\frac{\partial \zeta_*}{\partial \tau_\epsilon} = \left(- \frac{1}{\tau_\epsilon} + \frac{\tau_\epsilon \tau_z (\psi^I)^2}{\tau^U} - (\psi^I + \psi^U) \tau_z \psi^I \right) \frac{y}{\sigma_v \tau_x \psi^C}.$$

Now substitute $\tau_\epsilon \tau_z (\psi^I)^2 = (\tau^U - \tau_x) / \tau_\epsilon$ and rearrange. The limits can be obtained from equation (25).

b) Substitute σ_v in equation (10) and v_* in equation (3) into the definition of ζ_* in equation (16) to get

$$\zeta_* = - \left(\frac{\tau_\epsilon \psi^I}{\sqrt{\tau^U}} + \sqrt{\tau^U} (\psi^I + \psi^U) \right) \frac{\sqrt{\tau_z}}{\sqrt{\tau_x} \psi^C} y, \quad (25)$$

which is obviously decreasing in ψ^U and increasing in ψ^C . Differentiating equation (25) with respect to ψ^I (note that τ^U depends on ψ^I via β) we get

$$\frac{d\zeta_*}{d\psi^I} = - \left\{ \frac{\tau_\epsilon}{\sqrt{\tau^U}} + \sqrt{\tau^U} + \frac{1}{\sqrt{\tau^U}} \left(\psi^U + \left(1 - \frac{\tau_\epsilon}{\tau^U} \right) \psi^I \right) \tau_z (\psi^I)^2 \tau_\epsilon \right\} \frac{\sqrt{\tau_z}}{\sqrt{\tau_x} \psi^C} y < 0.$$

c) Differentiate equation (25) with respect to τ_z :

$$\frac{d\zeta^*}{d\tau_z} = -\frac{1}{2} \left(-\frac{\tau_\epsilon}{\tau^U \sqrt{\tau^U}} \psi^I + \frac{1}{\sqrt{\tau^U}} (\psi^I + \psi^U) \right) \frac{\sqrt{\tau_z}}{\sqrt{\tau_x} \psi^C} y \frac{d\tau^U}{d\tau_z} + \frac{1}{2} \frac{\zeta^*}{\tau_z}$$

Using $\frac{d\tau^U}{d\tau_z} = (\tau^U - \tau_x) / \tau_z$, we get

$$2\tau_z \frac{d\zeta^*}{d\tau_z} = -\frac{1}{\tau^U} \left(-\frac{\tau_\epsilon}{\sqrt{\tau^U}} \psi^I + \sqrt{\tau^U} (\psi^I + \psi^U) \right) \frac{\sqrt{\tau_z}}{\sqrt{\tau_x} \psi^C} y (\tau^U - \tau_x) + \zeta^*$$

Negativity of the right side is equivalent to

$$-\left(-\frac{\tau_\epsilon}{\sqrt{\tau^U}} \psi^I + \sqrt{\tau^U} (\psi^I + \psi^U) \right) \left(1 - \frac{\tau_x}{\tau^U} \right) - \left(\frac{\tau_\epsilon}{\sqrt{\tau^U}} \psi^I + \sqrt{\tau^U} (\psi^I + \psi^U) \right) < 0,$$

or, equivalently

$$-\left(\sqrt{\tau^U} (\psi^I + \psi^U) \right) \left(2 - \frac{\tau_x}{\tau^U} \right) < 0,$$

which is obviously true. □

Lemma A.4. *The normalized threshold ζ^* , defined in equation (16), is*

a) strictly increasing in precision, with

$$\frac{d\zeta^*}{d\tau_\epsilon} = \frac{\tau_z \psi^I}{\sigma_v \tau_x} y, \quad (26)$$

and the limits

$$\lim_{\tau_\epsilon \rightarrow 0} \zeta^* = \sqrt{\tau_z} y, \quad \lim_{\tau_\epsilon \rightarrow \infty} \zeta^* = \infty;$$

b) strictly increasing in ψ^I , and invariant to ψ^C and ψ^U ;

c) strictly increasing in τ_z .

Proof. Using the expressions for v^* in equation (3), we get

$$\zeta^* = \frac{\tau^U}{\sigma_v \tau_\epsilon} \frac{1}{\tau_x \psi^I} y. \quad (27)$$

Substituting the expression for $\sigma_v \tau_\epsilon$ in equation (24) implies

$$\zeta^* = \sqrt{\frac{\tau_z \tau^U}{\tau_x}} y, \quad (28)$$

from which it is apparent that ζ^* is increasing in τ_ϵ , τ_z and ψ^I ; and ζ^* is invariant to ψ^C and ψ^U . From equation (28) we also get the limiting expressions.

Differentiating equation (28) and then substituting equation (27) implies

$$\frac{d}{d\tau_\epsilon} \zeta^* = \frac{1}{2} \frac{1}{\tau^U} \zeta^* \frac{d\tau^U}{d\tau_\epsilon} = \frac{1}{\tau^U} \zeta^* \tau_z (\psi^I)^2 \tau_\epsilon = \frac{\tau_z \psi^I}{\sigma_v \tau_x} y.$$

□

Proof of Propositions 1 and 2: The assumption that p is strictly monotonic in v , where $v = x + \beta z$, implies the following conditional moments for the uninformed and informed investors, respectively:

$$\mathbb{V}[x|p]^{-1} = \tau^U = \tau_x + \frac{\tau_z}{\beta^2}, \quad \mathbb{E}[x|p] = \frac{1}{\tau^U} \left(\tau_x \mu_x + \frac{\tau_z}{\beta^2} v \right), \quad (29)$$

$$\mathbb{V}[x|p, x_j]^{-1} = \tau^I = \tau_x + \tau_\epsilon + \frac{\tau_z}{\beta^2}, \quad \mathbb{E}[x|p, x_j] = \frac{1}{\tau^I} \left(\tau_x \mu_x + \tau_\epsilon (x + \epsilon_j) + \frac{\tau_z}{\beta^2} v \right). \quad (30)$$

Substituting $\tau_z/\beta^2 = \tau^U - \tau_x$, optimal aggregate uninformed demands (defined in equation (6)) are

$$\begin{aligned} A^C(v) &= \psi^C (\tau_x \mu_x + (\tau^U - \tau_x) v - \tau^U p(v))^+, \\ A^U(v) &= \psi^U (\tau_x \mu_x + (\tau^U - \tau_x) v - \tau^U (p(v) - f(v)))^+. \end{aligned} \quad (31)$$

Demand by each informed investor j is

$$a_j^I = \frac{1}{\gamma^I} \left(\tau_x \mu_x + \tau_\epsilon (x + \epsilon_j) + \frac{\tau_z}{\beta^2} v - \tau^I (p(v) - f(v)) \right); \quad (32)$$

therefore,

$$\int_{\text{informed}} a_j^I dj = \psi^I \left(\tau_x \mu_x + \tau_\epsilon x + \frac{\tau_z}{\beta^2} v - \tau^I (p - f) \right);$$

and (recalling the definition of $A^I(v)$ in equation (6))

$$A^I(v, z) = \psi^I ((\tau^I - \tau_x) (v - \mu_x) - \tau^I (p - f - \mu_x)) + (1 - \psi^I \tau_\epsilon \beta) z. \quad (33)$$

We first establish that $\beta = 1/(\psi^I \tau_\epsilon)$. The asset-market clearing condition $y = A^C(v) + A^U(v) + A^I(v, z)$ must hold for all $(v, z) \in \mathbb{R}^2$, and, therefore, the coefficient of z in equation (33) must be zero (i.e., $\beta = 1/(\psi^I \tau_\epsilon)$). The asset-market clearing condition, therefore, can be written:

$$y = A^C(v) + A^U(v) + A^I(v), \quad (34)$$

where

$$A^I(v) = \psi^I ((\tau^I - \tau_x) (v - \mu_x) - \tau^I (p - f - \mu_x)). \quad (35)$$

The loan-market clearing condition (1) is equivalent to

$$A^U(v) + A^I(v) \geq 0, \quad f(v) \geq 0, \quad f(v) (A^U(v) + A^I(v)) = 0. \quad (36)$$

Henceforth, by letting v and p represent deviations from μ_x , we can, without loss of generality, assume $\mu_x = 0$. Define the three price functions:

$$p^a(v) = \left(1 - \frac{\tau_x}{\tau^U}\right) v - \frac{y}{\psi^C \tau^C}, \quad p^b(v) = \left(1 - \frac{\tau_x}{\bar{\tau}}\right) v - \frac{y}{\psi^U \bar{\tau}}, \quad \text{and} \quad p^c(v) = \left(1 - \frac{\tau_x}{\tau^I}\right) v - \frac{y}{\psi^I \tau^I}.$$

Note that $p(v)$ in equation (4) equals $p^a(v)$ if $v < v_*$, $p^b(v)$ if $v \in [v_*, v^*]$, and $p^c(v)$ if $v > v^*$. It is straightforward to show that $p^a(v_*) = p^b(v_*)$ and $p^b(v^*) = p^c(v^*)$, and, therefore, $p(v)$ is a continuous function. Also, it is easy to verify geometrically that convexity $p(v)$ implies

$$p^a(v) > p^b(v) > p^c(v) \text{ for } v < v_*, \quad \text{and} \quad p^c(v) > p^b(v) > p^a(v) \text{ for } v > v^*.$$

a) Suppose $v \geq v^*$. Write the aggregate demand functions as

$$\begin{aligned} y &= \psi^C \tau^U \left(\left(\frac{\tau^U - \tau_x}{\tau^U} \right) v - p(v) \right)^+ + \psi^U \tau^U \left(\left(\frac{\tau^U - \tau_x}{\tau^U} \right) v - (p(v) - f(v)) \right)^+ \\ &\quad + \psi^I \tau^I \left(\left(\frac{\tau^I - \tau_x}{\tau^I} \right) v - (p(v) - f(v)) \right). \end{aligned} \quad (37)$$

The inequality $(\tau^I - \tau_x) / \tau^I > (\tau^U - \tau_x) / \tau^U$ implies that for any $v > 0$,

$$\left(\frac{\tau^I - \tau_x}{\tau^I} \right) v > p(v) - f(v) \quad (38)$$

(i.e., $A^I(v) > 0$), otherwise the right side of equation (37) is nonpositive, violating the asset-market clearing. The inequality $A^I(v) > 0$ implies $f(v) = 0$, and the simplified asset-market clearing condition:

$$y = (\psi^C + \psi^U) \tau^U \left(\left(\frac{\tau^U - \tau_x}{\tau^U} \right) v - p(v) \right)^+ + \psi^I \tau^I \left(\left(\frac{\tau^I - \tau_x}{\tau^I} \right) v - p(v) \right). \quad (39)$$

We next show that uninformed demand is zero. Suppose $A^U(v) > 0$ (equivalently, $A^C(v) > 0$). Then $f(v) = 0$, and the asset-market clearing condition (39) implies $p(v) = p^b(v)$. But $v > v^*$ and the price $p^b(v)$ imply $A^I(v) > y$ (note that $A^I(v) = y$ if the price were $p^c(v)$, and $p^c(v) > p^b(v)$ for $v > v^*$), which violates the asset-market clearing; and if $v = v^*$, then price $p^b(v)$ implies $A^U(v) = 0$, contradicting the supposition. Therefore, only informed investors can be long when $v \geq v^*$, and asset-market clearing implies $A^I(v) = y$, and, therefore, $p(v) = p^c(v)$ for $v \geq v^*$.

b) Suppose $v < v_*$. We first rule out $f(v) = 0$. A zero fee implies (from equation (36)) $A^U(v) + A^I(v) \geq 0$ and the asset-market clearing condition (39). If equilibrium uninformed demand were zero, then equation (39) would imply $p(v) = p^c(v)$, and if equilibrium demand were strictly positive, then equation (39) would imply $p(v) = p^b(v)$. But either possibility results in a contradiction because $A^C(v) > y$ at either price (in fact $A^C(v) = y$ if $p(v) = p^a(v)$, and $p^a(v) > p^b(v) > p^c(v)$ for $v < v_*$), which violates the asset-market clearing condition. Therefore, $f(v) > 0$, which implies (from (36)) $A^U(v) + A^I(v) = 0$, and the asset-market clearing condition (34) implies $A^C(v) = y$ and $p(v) = p^a(v)$ for $v < v_*$. We solve for $f(v)$ from $A^U(v) + A^I(v) = 0$, which implies

$$\frac{\psi^I + \psi^U}{\psi^I \tau^I + \psi^U \tau^U} \tau_x \mu_x + \left(1 - \tau_x \frac{\psi^I + \psi^U}{\psi^I \tau^I + \psi^U \tau^U} \right) v = p(v) - f(v),$$

and, therefore, the equation (5), which can also be written:

$$f(v) = \frac{y}{\psi^C \tau^U} \frac{v - v_*}{v_* - \mu_x} \text{ if } v < v_*. \quad (40)$$

c) Suppose $v \in [v_*, v^*]$. We first show $f(v) = 0$. Suppose $f(v) > 0$. As shown in part (b), $f(v) > 0$ implies $p(v) = p^a(v)$ and the fee expression (40). But $v \geq v_*$ implies $f(v) \leq 0$, a contradiction. Having established that $f(v) = 0$, we next show that $A^U(v) > 0$ (equivalently, $A^C(v) > 0$). If not, then asset-market clearing implies $p(v) = p^c(v)$ and $A^U(v) > 0$ (which follows because $p(v) = p^b(v)$ and $v < v^*$ imply $A^U(v) > 0$, and $p^c(v) < p^b(v)$ if $v < v^*$), which is a contradiction. Strictly positive uninformed demand and asset-market clearing imply $p(v) = p^b(v)$. □

Proof of Lemma 1: We begin with the aggregate demand formulas in equations (31) and (35).

a) For $v < v_*$, a positive fee implies the loan-market clearing condition $A^I(v) + A^U(v) = 0$, and the asset-market clearing condition $A^C(v) = y$. Substitute (from the price and fee equations (4) and (5)):

$$p(v) - f(v) - \mu_x = \left(1 - \tau_x \frac{\psi^I + \psi^U}{\psi^I \tau^I + \psi^U \tau^U} \right) (v - \mu_x), \quad \text{for } v < v_*$$

into equation (35) to get the expression for $A^I(v)$.

b) $A^I(v)$ is obtained by substituting $f = 0$ and the price expression for $v \in [v_*, v^*]$ in equation (4) into equation (35). A zero fee implies (from equation (31)) $A^C(v) = \frac{\psi^C}{\psi^U} A^U(v)$, and the asset-market clearing condition (34), therefore, implies $A^I(v) + \left(\frac{\psi^U + \psi^C}{\psi^U} \right) A^U(v) = y$ (which can be rearranged to obtain $A^U(v)$).

Note that we can write informed demand as

$$A^I(v) = b \left((\psi^C + \psi^U)(v - \mu_x) + \frac{\tau^U/\tau_\epsilon + 1}{\tau_x} y \right).$$

Recall that τ^U/τ_ϵ is u shaped in precision (with a minimum at $\hat{\tau}_\epsilon$), which implies that the zero of $A^I(v)$ is hump shaped (with a maximum $\hat{\tau}_\epsilon$). That is, the region $\{A^I(v) < 0\}$ first expands then contracts as the precision increases. Further, b being hump-shaped in precision implies that short positions are first larger in this region, and then they shrink.

c) As shown in the proof of Propositions 1 and 2, the short-sale constraint binds for the uninformed investors in this region, and, therefore, the informed investors hold the entire supply. \square

Proof of Lemma 2 Substitute $\beta = 1/(\psi^I \tau_\epsilon)$ into τ^U to get $\tau^U/\tau_\epsilon = \frac{\tau_x}{\tau_\epsilon} + \tau_z (\psi^I)^2 \tau_\epsilon$. Differentiating shows that τ^U/τ_ϵ is decreasing in τ_ϵ if $\tau_\epsilon < \hat{\tau}_\epsilon$ and increasing if $\tau_\epsilon > \hat{\tau}_\epsilon$. Apply this to (from equation (3))

$$v_* = \mu_x - \left(\psi^I + (\psi^I + \psi^U) \frac{\tau^U}{\tau_\epsilon} \right) \frac{1}{\tau_x \psi^I \psi^C} y. \quad (41)$$

The absolute fee slope can be written as

$$|f'(v)| = \frac{\tau_x}{\tau^U} \left(\frac{\psi^I}{\psi^I + (\psi^U + \psi^I) \frac{\tau^U}{\tau_\epsilon}} \right), \text{ for } v < v_*. \quad (42)$$

Obviously, τ^U is always increasing in precision, and τ^U/τ_ϵ is increasing in precision for $\tau_\epsilon > \hat{\tau}_\epsilon$. Together with v_* decreasing for $\tau_\epsilon > \hat{\tau}_\epsilon$, we find that f is decreasing for $\tau_\epsilon > \hat{\tau}_\epsilon$.

For the comparative statics of b , substitute the definitions of $\bar{\tau}_\epsilon$ and $\bar{\tau}$ into equation (8) to get $b = \tau_x \frac{\psi^I}{\psi(\tau^U/\tau_\epsilon) + \psi^I}$, which is decreasing in τ^U/τ_ϵ . \square

Proof of Lemma 3: The limits follow from the expressions for v^* in equation (3), and the expressions for v_* and $|f'(v)|$ in the proof of Lemma 2, together with $\lim_{\tau_\epsilon \rightarrow 0} \tau^U/\tau_\epsilon = \infty$ and $\lim_{\tau_\epsilon \rightarrow \infty} \tau^U/\tau_\epsilon = \infty$. \square

Proof of Lemma 4: From equation (41), the threshold v_* is decreasing in τ_z . Further, the absolute fee slope for $v < v_*$ is decreasing τ_z . The results follow from $f(v_*) = 0$. \square

Proof of Proposition 3: a) The proof of Propositions 1 and 2 applies after replacing the price precision τ_z/β^2 with zero (i.e., letting $\tau_z = 0$). The proof proceeds the same, and the results are the same after replacing the investor precision expressions in equation (2) with

$$\tau^U = \tau^C = \tau_x, \quad \tau^I = \tau_x + \tau_\epsilon$$

(again equivalent to letting $\tau_z = 0$).

b) The proof of Lemma 2 still holds except that when price is ignored, $\tau^U/\tau_\epsilon = \tau_x/\tau_\epsilon$, which is monotonically decreasing in τ_ϵ (consistent with letting $\tau_z = 0$, which yields $\hat{\tau}_\epsilon = \infty$). \square

Proof of Lemma 5: The results for v_* and v^* follow from equation (3). Note that v_* strictly declines (towards $-\infty$) as λ^C decreases toward zero.

The absolute fee slope (42) decreases as λ^U increases: combined with the decrease in v_* , this implies $f(v)$ is monotonically decreasing in λ^U , converging to zero for every v as λ^C goes to zero.

The price function for $v > v^*$ is invariant to λ^U (because only informed investors have nonzero positions in this region). Also, the slope $p'(v)$ in each range $\{v < v_*\}$ and $\{v \in [v_*, v^*]\}$ is invariant to λ^U (in the intermediate range, $p'(v)$ depends on $\lambda^U + \lambda^C$, which we are holding fixed). It follows that $p(v)$ is decreasing in the range $\{v < v_*\}$ because v_* is decreasing in λ^U (and $p'(v)$ is larger in the intermediate range than in the $\{v < v_*\}$ range).

From the expressions (3) and (5), it is easy to confirm that v_* and $f(v)$, in the region $v < v_*$, are strictly increasing in ψ^C and decreasing in ψ^U .

$$v_* = \mu_x - \left(\frac{\psi^I \tau^I + \psi^U \tau^U}{\psi^I \tau_\epsilon} \right) \frac{1}{\tau_x \psi^C} y.$$

v_* is decreasing in ψ^U , increasing in ψ^C . Fixing the price precision (e.g., if $\tau_z = 0$), we get

$$\left. \frac{d}{d\psi^I} v_* \right|_{\tau^U \text{ fixed}} = \frac{\psi^U}{(\psi^I)^2} \frac{\tau^U}{\tau_\epsilon \tau_x \psi^C} y > 0$$

and

$$\left. \frac{d}{d\psi^I} f(v) \right|_{\tau^U \text{ fixed}} = -\frac{\tau_x}{\tau^U} \left(\frac{\psi^U \tau^U \tau_\epsilon}{[\psi^I \tau_\epsilon + (\psi^I + \psi^U) \tau^U]^2} \right) (v - \mu_x), \quad v < v_*,$$

which is strictly positive when $v < v_*$. However, fixing ψ^I , we get that $dv_*/d\tau^U < 0$ and

$$\frac{d}{d\tau^U} f(v) = -\frac{1}{\tau^U} f(v) + \frac{\tau_x}{\tau^U} \left(\frac{(\psi^I + \psi^U) \psi^I \tau_\epsilon}{[\psi^I \tau_\epsilon + (\psi^I + \psi^U) \tau^U]^2} \right) (v - \mu_x) < 0, \quad v < v_*.$$

□

Proof of Proposition 4: Use

$$P(v < v_*) = P\left(\frac{v - \mu_x}{\sigma_v} < \frac{v_* - \mu_x}{\sigma_v}\right) = \mathbb{N}(\zeta_*),$$

together with Lemma A.3, which shows that ζ_* is strictly decreasing in τ_ϵ . The zero precision limit is from the same lemma. The monotone convergence theorem implies that the limit, $\mathbb{N}(\zeta_0)$, is the same as $P(z < z_*)$ in Proposition A.1. The comparative statics follow from Lemma A.3.

□

Proof of Proposition 5: Take the expectation of the expression (40) and apply Lemma A.1 to get

$$\mathbb{E}[f(v)] = -\frac{y}{\psi^C \tau^U} \left(\mathbb{N}\left(-\frac{\mu_x - v_*}{\sigma_v}\right) - \frac{\sigma_v}{\mu_x - v_*} \mathbb{N}'\left(-\frac{\mu_x - v_*}{\sigma_v}\right) \right),$$

and then use the definitions of ζ_* and ζ^* in equation (16). If $\tau_\epsilon = 0$, then $\mathbb{E}[f(v)]$ satisfies equation (14) with $\bar{\tau} = \tau_x$ and

$$H(0) = \sqrt{\frac{1}{\tau_z}} \left(\mathbb{N}'(\zeta_0) - \zeta_0 \mathbb{N}(\zeta_0) + \left(\frac{\psi^C + \psi^U}{\psi^I} \right) \{ \mathbb{N}'(\sqrt{\tau_z} y) - \sqrt{\tau_z} y \mathbb{N}(-\sqrt{\tau_z} y) \} \right),$$

where ζ_0 is defined in equation (11).

Differentiating equation (12) with respect to ζ_* (holding fixed $y/(\psi^C \tau^U)$) we get

$$\frac{\partial \mathbb{E}f(v)}{\partial \zeta_*} = \frac{y}{\psi^C \tau^U} \frac{1}{\zeta_*^2} \mathbb{N}'(\zeta_*) > 0. \quad (43)$$

Lemma A.3 shows that ζ_* is strictly decreasing in τ_ϵ , τ_z , ψ^I , and ψ^U . Also, $y/(\psi^C \tau^U)$ is decreasing in τ_ϵ , τ_z , and ψ^I , and is invariant to ψ^U . Therefore, $\mathbb{E}f(v)$ is strictly decreasing in τ_ϵ , τ_z , ψ^I , and ψ^U .

Differentiate equation(25) with respect to ψ^C to get $d\zeta_*/d\psi^C = -\zeta_*/\psi^C$, and then differentiate equation (12) with respect to ψ^C , and substitute (43) to get

$$\frac{\partial \mathbb{E}f(v)}{\partial \psi^C} = -\frac{1}{\psi^C} \left(\mathbb{E}f(v) + \frac{\partial \mathbb{E}f(v)}{\partial \zeta_*} \zeta_* \right) = \frac{1}{\psi^C} \frac{y}{\psi^C \tau^U} \mathbb{N}(\zeta_*) > 0.$$

Alternatively, the result for γ^C follows from the pointwise monotonicity of f in γ^C and the monotone convergence theorem. (Note that β and, therefore, σ_v are invariant γ^C .) \square

Proof of Proposition 6: We compute the expected squared fee by squaring equation (40), taking expectations, and applying Lemma A.1:

$$\begin{aligned}\mathbb{E}\left[f(v)^2\right] &= \left(\frac{y}{\psi^C \tau^U} \frac{1}{v_* - \mu_x}\right)^2 \mathbb{E}\left[(v - v_*)^2 \mathbf{1}_{\{v - v_* < 0\}}\right] \\ &= \left(\frac{y}{\psi^C \tau^U} \frac{1}{v_* - \mu_x}\right)^2 \left(\left((\mu_x - v_*)^2 + \sigma_v^2\right) \mathbb{N}\left(-\frac{\mu_x - v_*}{\sigma_v}\right) - (\mu_x - v_*) \sigma_v \mathbb{N}'\left(\frac{\mu_x - v_*}{\sigma_v}\right)\right),\end{aligned}$$

which is equivalent to

$$\mathbb{E}\left[f(v)^2\right] = \left(\frac{y}{\psi^C \tau^U}\right)^2 \left(\frac{\mathbb{N}'(\zeta_*)}{\zeta_*} + \mathbb{N}(\zeta_*) + \frac{\mathbb{N}(\zeta_*)}{\zeta_*^2}\right). \quad (44)$$

Now use $\mathbb{V}[f(v)] = \mathbb{E}\left[f(v)^2\right] - (\mathbb{E}[f(v)])^2$ and the expression for $\mathbb{E}[f(v)]$ in equation (12).

Lemma A.3 shows that ζ_* is strictly decreasing in τ_ϵ , τ_z , ψ^I , and ψ^U . Further, Lemma A.2 (with $k = -\zeta_*$) shows that $\mathbb{V}[(f(v))]$ is strictly increasing in ζ_* (holding fixed $y/(\psi^C \tau^U)$). Also, $y/(\psi^C \tau^U)$ is decreasing in τ_ϵ , τ_z , and ψ^I , and is invariant to ψ^U . Therefore, $\mathbb{V}[(f(v))]$ is strictly decreasing in τ_ϵ , τ_z , ψ^I , and ψ^U .

To show $\frac{d}{d\psi^C} \mathbb{V}[f(v)] > 0$, write the expression for $\mathbb{V}[f(v)]$ in equation (13) as

$$\mathbb{V}[f(v)] = \left(\frac{y}{\psi^C \tau^U}\right)^2 g(-\zeta_*),$$

and differentiate using equation (18) (in which the function h is defined), and $d\zeta_*/d\psi^C = -\zeta_*/\psi^C$, to get

$$\begin{aligned}\frac{d}{d\psi^C} \mathbb{V}[f(v)] &= \frac{-2}{\psi^C} \left(\frac{y}{\psi^C \tau^U}\right)^2 g(-\zeta_*) + \left(\frac{y}{\psi^C \tau^U}\right)^2 \frac{2}{\zeta_*^2} h(-\zeta_*) \frac{-\zeta_*}{\psi^C} \\ &= -\frac{2}{\psi^C} \left(\frac{y}{\psi^C \tau^U}\right)^2 \left\{g(-\zeta_*) + \frac{h(-\zeta_*)}{\zeta_*}\right\}.\end{aligned}$$

The term in braces is positive:

$$\begin{aligned}g(-\zeta_*) + \frac{h(-\zeta_*)}{\zeta_*} &= \frac{\mathbb{N}'(\zeta_*)}{\zeta_*} + \mathbb{N}(\zeta_*) - \mathbb{N}(\zeta_*)^2 - \frac{\mathbb{N}'(\zeta_*)}{\zeta_*} \mathbb{N}(\zeta_*) \\ &= \left(\frac{\mathbb{N}'(\zeta_*)}{\zeta_*} + \mathbb{N}(\zeta_*)\right) \mathbb{N}(-\zeta_*) < 0.\end{aligned}$$

\square

Proof of Proposition 7: Define

$$\Pi = (\psi^C + \psi^U + \psi^I) \tau^U + \psi^I \tau_\epsilon$$

and write the price expression (4) as

$$\begin{aligned}\Pi(p(v) - \mu_x) &= ((\psi^C + \psi^U + \psi^I) (\tau^U - \tau_x) + \psi^I \tau_\epsilon) (v - \mu_x) - y \\ &\quad - \frac{1}{\tau^U} \left(\tau_x \psi^I \tau_\epsilon (v - \mu_x) + y \frac{(\psi^U + \psi^I) \tau^U + \psi^I \tau_\epsilon}{\psi^C} \right) \mathbf{1}_{\{v < v_*\}} \\ &\quad + \frac{1}{\tau^I} \left(\tau_x (\psi^C + \psi^U) \tau_\epsilon (v - \mu_x) - y \frac{(\psi^C + \psi^U) \tau^U}{\psi^I} \right) \mathbf{1}_{\{v > v_*\}}.\end{aligned}$$

Take expectations and substitute

$$P(v < v_*) = \mathbb{N}\left(-\frac{\mu_x - v_*}{\sigma_v}\right), \quad \mathbb{E}[(v - \mu_x) 1_{\{v < v_*\}}] = -\sigma_v \mathbb{N}'\left(\frac{\mu_x - v_*}{\sigma_v}\right)$$

(the latter using Lemma A.1 and $\mathbb{E}[(v - \mu_x) 1_{\{v < v_*\}}] = \mathbb{E}[(v - v_*) 1_{\{v < v_*\}}] + (v_* - \mu_x) P(v < v_*)$), and

$$\mathbb{E}[(v - \mu_x) 1_{\{v > v_*\}}] = \sigma_v \mathbb{N}'\left(\frac{\mu_x - v_*}{\sigma_v}\right),$$

to get

$$\begin{aligned} \Pi(\mathbb{E}[p(v)] - \mu_x) &= -y - \frac{1}{\tau^U} \left(-\tau_x \psi^I \tau_\epsilon \sigma_v \mathbb{N}'\left(\frac{\mu_x - v_*}{\sigma_v}\right) + y \frac{(\psi^U + \psi^I) \tau^U + \psi^I \tau_\epsilon}{\psi^C} \mathbb{N}\left(-\frac{\mu_x - v_*}{\sigma_v}\right) \right) \\ &\quad + \frac{1}{\tau^I} \left(\tau_x (\psi^C + \psi^U) \tau_\epsilon \sigma_v \mathbb{N}'\left(\frac{\mu_x - v_*}{\sigma_v}\right) - y \frac{(\psi^C + \psi^U) \tau^U}{\psi^I} \mathbb{N}\left(\frac{\mu_x - v_*}{\sigma_v}\right) \right). \end{aligned}$$

From the definitions of ζ_* and ζ^* in equation (16) and their expressions in equation (23)), we get

$$\mathbb{E}[p(v)] - \mu_x = -\frac{y}{\Pi} + \frac{\tau_x (\sigma_v \tau_\epsilon)}{\Pi} \left(\frac{\psi^I}{\tau^U} (\mathbb{N}'(\zeta_*) + \zeta_* \mathbb{N}(\zeta_*)) + \left(\frac{\psi^C + \psi^U}{\tau^I} \right) (\mathbb{N}'(\zeta^*) - \zeta^* \mathbb{N}(-\zeta^*)) \right).$$

Now substitute for $(\sigma_v \tau_\epsilon)$ using equation (24) to get equation (14).

Monotonicity of $H(\cdot)$ is shown in Lemma A.6. Monotonicity of $\mathbb{E}[R(v)]$ for all precisions above some $\bar{\tau}_\epsilon > 0$ is shown in Lemma A.5.

The zero-precision results can be obtained by writing the price function in equation (50) as

$$\tau_x \{p(z) - \mu_x\} = \frac{1}{\psi} \left[z - y - \left(\frac{\psi^C z + (\psi^I + \psi^U) y}{\psi^C} \right) 1_{\{z < z_*\}} + \left(\frac{(\psi^C + \psi^U)(z - y)}{\psi^I} \right) 1_{\{z > z_*\}} \right],$$

and taking expectations using

$$P(z < z_*) = \mathbb{N}(\sqrt{\tau_z} z_*), \quad \mathbb{E}[z 1_{\{z < z_*\}}] = -\frac{1}{\sqrt{\tau_z}} \mathbb{N}'(\sqrt{\tau_z} z_*), \quad \mathbb{E}[z 1_{\{z > z_*\}}] = \frac{1}{\sqrt{\tau_z}} \mathbb{N}'(\sqrt{\tau_z} z_*).$$

Alternatively, take the limit of equation (14) as $\tau_\epsilon \rightarrow 0$ using the dominated convergence theorem, using the limits of ζ_* and ζ^* in Lemma A.3. \square

Lemma A.5. *There exists some precision level $\bar{\tau}_\epsilon > 0$ such that $\mathbb{E}[p(v)]$ is strictly increasing in precision for all $\tau_\epsilon > \bar{\tau}_\epsilon$.*

Proof. Differentiate equation (14) to get

$$\frac{\partial \mathbb{E}[p(v)]}{\partial \tau_\epsilon} = \frac{H'(\tau_\epsilon) + \{y - H(\tau_\epsilon)\} \frac{\Pi'(\tau_\epsilon)}{\Pi(\tau_\epsilon)}}{\Pi(\tau_\epsilon)}.$$

We show that the numerator is positive for sufficiently large τ_ϵ . First, we have the inequality

$$\frac{\Pi'(\tau_\epsilon)}{\Pi(\tau_\epsilon)} = \frac{(\psi^C + \psi^U + \psi^I) 2(\tau^U - \tau_x) / \tau_\epsilon + \psi^I}{(\psi^C + \psi^U + \psi^I) \tau^U + \psi^I \tau_\epsilon} = \frac{1}{\tau_\epsilon} \frac{2(\tau^U - \tau_x) + \frac{\psi^I}{(\psi^C + \psi^U + \psi^I)} \tau_\epsilon}{\tau^U + \frac{\psi^I}{(\psi^C + \psi^U + \psi^I)} \tau_\epsilon} \geq \frac{1}{\tau_\epsilon} \frac{\tau^U - \tau_x}{\tau^U}.$$

Lemma A.6 shows the inequality

$$H'(\tau_\epsilon) \geq -\frac{1}{\sqrt{\tau^U}} \frac{\tau_\epsilon}{\tau^U} J(\tau_\epsilon),$$

where J is some function declining in τ_ϵ for $\tau_\epsilon > 1$. Therefore, when τ_ϵ is sufficiently large that $y - H(\tau_\epsilon) \geq 0$ (recall that $H(\tau_\epsilon)$ is strictly decreasing in τ_ϵ towards zero) we get

$$\begin{aligned} \Pi(\tau_\epsilon) \frac{\partial \mathbb{E}[p(v)]}{\partial \tau_\epsilon} &\geq \frac{1}{\tau_\epsilon} \left(-\frac{1}{\sqrt{\tau^U}} \frac{\tau_\epsilon^2}{\tau^U} J(\tau_\epsilon) + \{y - H(\tau_\epsilon)\} \frac{\tau^U - \tau_x}{\tau^U} \right) \\ &= \frac{1}{\tau_\epsilon} \frac{(\tau^U - \tau_x)}{\tau^U} \left(y - \frac{1}{\sqrt{\tau^U}} \frac{1}{\tau_z (\psi^I)^2} J(\tau_\epsilon) - H(\tau_\epsilon) \right) \end{aligned}$$

(using $\tau_\epsilon^2 = (\tau^U - \tau_x) / (\tau_z (\psi^I)^2)$). Because both $J/\sqrt{\tau^U}$ and H decline monotonically towards zero, for $\tau_\epsilon > 1$, it follows that $\frac{\partial}{\partial \tau_\epsilon} \mathbb{E}p(v) > 0$ for all τ_ϵ above some critical level. \square

Lemma A.6. *The derivative of the function $H(\cdot)$ defined in equation (15) satisfies*

$$0 > H'(\tau_\epsilon) \geq -\frac{1}{\sqrt{\tau^U}} \frac{\tau_\epsilon}{\tau^U} J(\tau_\epsilon),$$

where $J(\cdot)$ is some strictly positive function that is declining in τ_ϵ for $\tau_\epsilon > 1$.

Proof. Differentiate $H(\cdot)$ in equation (15) using

$$\frac{\partial}{\partial \tau_\epsilon} \frac{\sqrt{\tau^U}}{\tau^I} = \frac{\tau_z (\psi^I)^2 \tau_\epsilon^2 - \tau^U (1 + \tau_z (\psi^I)^2 \tau_\epsilon)}{\sqrt{\tau^U} (\tau^I)^2} = -\frac{\tau_x + \tau^U \tau_z (\psi^I)^2 \tau_\epsilon}{\sqrt{\tau^U} (\tau^I)^2} \quad (45)$$

to get

$$\begin{aligned} -\sqrt{\tau^U} \sqrt{\frac{\tau_z}{\tau_x}} H'(\tau_\epsilon) &= \frac{\tau_z (\psi^I)^2 \tau_\epsilon}{\tau^U} G_*(\tau_\epsilon) + \frac{\tau_x + \tau^U \tau_z (\psi^I)^2 \tau_\epsilon}{(\tau^I)^2} \tau_\epsilon \left(\frac{\psi^C + \psi^U}{\psi^I} \right) G^*(\tau_\epsilon) \\ &\quad + N(|\zeta_*|) \left(\frac{\tau_x}{\tau^U \tau_\epsilon} + (\psi^I + \psi^U) \tau_z \psi^I \right) \frac{y}{\sigma_v \tau_x \psi^C} + \frac{\tau^U}{\tau^I} \left(\frac{\psi^C + \psi^U}{\psi^I} \right) N(-\zeta^*) \frac{\tau_z \psi^I}{\sigma_v \tau_x} y, \end{aligned} \quad (46)$$

where we define

$$G^*(\zeta^*) = N'(\zeta^*) - \zeta^* N(-\zeta^*), \quad G_*(\zeta_*) = N'(\zeta_*) + \zeta_* N(\zeta_*), \quad (47)$$

which are both positive-valued functions that are monotonically decreasing in τ_ϵ . The proof is completed by showing that each of the four right-hand terms on the right side of equation (46) is dominated by the product of τ_ϵ/τ^U (recall that τ_ϵ/τ^U is declining in τ_ϵ for $\tau_\epsilon > \hat{\tau}_\epsilon$) and some positive-valued function that is declining when $\tau_\epsilon > 1$:

1) The first term follows immediately from the properties of G_* .

2) We first have $\tau_\epsilon > 1$ implies $\frac{\tau_x}{(\tau^I)^2} < \frac{\tau_\epsilon}{\tau^U}$. We also have $\frac{\tau^U \tau_z (\psi^I)^2 \tau_\epsilon}{(\tau^I)^2} \leq \frac{\tau_\epsilon}{\tau^U} \tau_z (\psi^I)^2$. Finally, $G^*(\tau_\epsilon)$ is positive valued and declining in τ_ϵ .

3) Use $N(-|\zeta_*|) < N'(\zeta_*)/|\zeta_*|$ (from $N'(k) > kN(-k)$ for $k > 0$) and substitute $|\zeta_*|$ from equation (23) into the right-hand side to get

$$\begin{aligned} N(-|\zeta_*|) \left(\frac{\tau_x}{\tau^U \tau_\epsilon} + (\psi^I + \psi^U) \tau_z \psi^I \right) \frac{y}{\sigma_v \tau_x \psi^C} &\leq \frac{N'(\zeta_*) \tau_\epsilon \psi^I}{(\tau^U (\psi^U + \psi^I) + \tau_\epsilon \psi^I)} \left(\frac{\tau_x}{\tau^U \tau_\epsilon} + (\psi^I + \psi^U) \tau_z \psi^I \right) \\ &\leq \frac{\tau_\epsilon}{\tau^U} \frac{N'(\zeta_*) \psi^I}{\psi^U + \psi^I} \left(\frac{\tau_x}{\tau^U \tau_\epsilon} + (\psi^I + \psi^U) \tau_z \psi^I \right). \end{aligned}$$

Lemma A.3 shows that $|\zeta_*|$ is increasing (and, therefore, $N'(\zeta_*)$ is decreasing) in τ_ϵ .

4) Finally, use $N(-\zeta^*) < N'(\zeta^*)/\zeta^*$ and the expression for ζ^* in (23) to get

$$\frac{\tau^U}{\tau^I} \left(\frac{\psi^C + \psi^U}{\psi^I} \right) N(-\zeta^*) \frac{\tau_z \psi^I}{\sigma_v \tau_x} y \leq \frac{\tau_\epsilon}{\tau^U} (\psi^C + \psi^U) \psi^I N'(\zeta^*) \tau_z.$$

Lemma A.3 shows that ζ^* is increasing (and, therefore, $N'(\zeta^*)$ is decreasing) in τ_ϵ . \square

Lemma A.7. For each $\tau_\epsilon > 0$, $H(\tau_\epsilon)$ is decreasing in τ_z , decreasing in ψ^I , and increasing in ψ^C .

Proof. Using equation (47), write the function $H(\cdot)$ defined in equation (15) as

$$\frac{1}{\sqrt{\tau_x}} H(\tau_\epsilon) = \frac{1}{\sqrt{\tau_z \tau^U}} G_*(\zeta_*) + \frac{\sqrt{\tau^U/\tau_z}}{\tau^I \psi^I} (\psi^C + \psi^U) G^*(\zeta^*). \quad (48)$$

The function G_* is increasing in ζ_* , and Lemma A.3 shows that ζ_* is decreasing in τ_z and ψ^I . Further $\frac{1}{\sqrt{\tau_z \tau^U}}$ is also decreasing in τ_z and ψ^I . Therefore, positivity of G_* implies that the first term on the right side of equation (48) is decreasing in τ_z and ψ^I . The function G^* is decreasing in ζ^* , and Lemma A.4 shows that ζ^* is increasing in τ_z and ψ^I . Further,

$$\frac{\sqrt{\tau^U/\tau_z}}{\tau^I \psi^I} = \frac{\sqrt{\tau_x/\tau_z + (\psi^I \tau_\epsilon)^2}}{(\tau_x + \tau_z (\psi^I \tau_\epsilon)^2 + \tau_\epsilon) \psi^I} = \frac{\sqrt{\tau_x / [\tau_z (\psi^I)^2] + (\tau_\epsilon)^2}}{\tau_x + \tau_z (\psi^I \tau_\epsilon)^2 + \tau_\epsilon}$$

is decreasing in both τ_z and ψ^I . Therefore, positivity of G^* implies that the second term on the right side of equation (48) is also decreasing in τ_z and ψ^I . An increase in ψ^C increases ζ_* (from Lemma A.3) and decreases ζ^* (from Lemma A.4), and has no effect on the coefficients of G_* and G^* ; therefore, $H(\tau_\epsilon)$ increases. \square

Proposition A.1 (equilibrium with no private information). *Suppose the precision of the private signals of informed investors is zero. Define the two liquidity-trader-demand thresholds, z_* and z^* , as follows:*

$$z_* = - \left(\frac{\psi^I + \psi^U}{\psi^C} \right) y, \quad z^* = y. \quad (49)$$

There is a unique equilibrium price and lending fee. The price is a convex function of z and it satisfies

$$p(z) = \mu_x + \begin{cases} -\frac{1}{\tau_x} \frac{y}{\psi^C} & \text{if } z < z_*, \\ \frac{1}{\tau_x} \frac{z-y}{\psi} & \text{if } z \in [z_*, z^*], \\ \frac{1}{\tau_x} \frac{z-y}{\psi^I} & \text{if } z > z^*, \end{cases} \quad (50)$$

and the lending fee is the following convex function of z :

$$f(z) = \begin{cases} -\frac{z}{\tau_x (\psi^I + \psi^U)} - \frac{y}{\tau_x \psi^C} & \text{if } z < z_*, \\ 0 & \text{otherwise.} \end{cases} \quad (51)$$

Proof. With zero precision ($\tau_\epsilon = 0$), total investor precision are $\tau^U = \tau^C = \tau^I = \tau_x$, and, therefore, aggregate optimal demands are

$$A^C = \psi^C \tau_x (\mu_x - p)^+, \quad A^U = \psi^U \tau_x (\mu_x - (p - f))^+, \quad A^I = \psi^I \tau_x (\mu_x - (p - f)) + z.$$

The asset-market clearing condition is

$$A^C + A^U + A^I = y, \quad (52)$$

and the loan-market clearing condition (1) is equivalent to

$$A^U + A^I \geq 0, \quad f \geq 0, \quad f \{A^U + A^I\} = 0. \quad (53)$$

a) Suppose $z \geq y$. The asset-market clearing condition (52) is

$$\psi^C \tau_x (\mu_x - p)^+ + \psi^U \tau_x (\mu_x - (p - f))^+ + \psi^I \tau_x (\mu_x - (p - f)) = y - z,$$

where $z \geq y$ implies that the left side must nonpositive, and, therefore, $\mu_x \leq p - f$, and $A^U = A^C = 0$, and equation (52) simplifies to

$$\psi^I \tau_x (\mu_x - (p - f)) = y - z; \quad (54)$$

that is, $A^I = y (> 0)$, which implies that $f = 0$. Together, equation (54) and $f = 0$ give the price and fee expressions for $z \geq z^*$.

b) Suppose $z < z^*$. We first rule out $f = 0$. If $f = 0$, then asset-market clearing (52) (and $z < 0$) requires $\mu_x > 0$ and

$$\psi \tau_x (\mu_x - p) + z = y, \quad (55)$$

and the loan-market clearing condition (53) requires

$$(\psi^U + \psi^I) \tau_x (\mu_x - p) + z \geq 0. \quad (56)$$

Writing equation (55) as $\tau_x (\mu_x - p) = (y - z) / \psi$ and substituting into equation (56) yields

$$z \geq -\frac{(\psi^U + \psi^I)}{\psi^C} y,$$

which contradicts the supposition that $z < z^*$.

Therefore, $f > 0$, which implies (from equation (53)) $A^U + A^I = 0$, and the asset-market clearing condition (52) implies $A^C = y$. Together, these imply the price and fee expressions for $z < z^*$.

c) Suppose $z \in [z_*, z^*)$. We first show $f = 0$. Suppose $f > 0$, which implies $A^U + A^I = 0$ (from equation (53)) and $A^C = y$ (from equation (52)), and the fee

$$f = -\frac{z}{\tau_x (\psi^I + \psi^U)} - \frac{y}{\psi^C \tau_x}.$$

But $z \geq z^*$ implies $f \leq 0$, which is a contradiction. Having established that $f = 0$, the market-clearing condition is

$$(\psi^C + \psi^U) \tau_x (\mu_x - p)^+ + \psi^I \tau_x (\mu_x - p) + z = y, \quad (57)$$

together with $y - z > 0$ (because $z < z^*$) imply $\mu_x > p$, and, therefore, equation (57) is equivalent to $\psi \tau_x (\mu_x - p) + z = y$, which gives the price expression for the $z \in [z_*, z^*)$ region. \square