CMS derivatives pricing in the linear-rational Wishart model

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Abstract

This study derives analytical pricing formulas for the constant maturity (CMS) and related derivatives in a linear-rational multi-curve term structure model based on the Wishart process. In the proposed approach, using the affine property of the Wishart process and the linear rational structure of the model, it is shown how the CMS price can be efficiently approximated by using a polynomial function of the Wishart model parameters. This methodology is also applied to the pricing of other CMS derivatives such as the CMS spread option and leads to fast and accurate approximation results. Using a calibrated model on market data, the approximation results compared to a Monte Carlo approach confirm the accuracy of the method. The approach is in fact very versatile as it can also be applied to other interest rate derivatives such as the in-arrears swap or in-arrears cap.

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1 Introduction

Following the works of Rogers (1997) and Filipović et al. (2017), Da Fonseca et al. (2022) proposes a three-factor linear-rational multi-curve term structure model based on the Wishart process. Within this framework, the zero-coupon bond price, whose value depends on the overnight index swap (OIS) curve, and the spread between the Euribor and OIS curves are linear-rational functions of our first two factors given by the diagonal terms of a 2×2 Wishart process. The model enables a stochastic correlation – our third factor – between these two curves, as the Wishart process allows a non trivial correlation between its diagonal terms governed by its off-diagonal component.

In this paper, we proceed further to developing a pricing scheme for exotic interest rate derivatives. The pricing of exotic interest rate derivatives is of importance in financial modeling. The Constant Maturity Swaps (CMS) is one of those products that draw a significant amount of research. Some works on this topic are but are not limited to Hanton and Henrard (2012), Antonov and Arneguy (2009). In Antonov and Arneguy (2009) the authors analyzed the CMS option and CMS spread option using the LMM with stochastic volatility. An approximation the product price is provided using the markovian projection technique. While in Hanton and Henrard (2012) the authors used the HJM framework to derive an approximation pricing scheme of the exotic derivatives such as CMS, CMS spread option, and swaption with CMS trigger. However, no paper to our knowledge has used the linear-rational interest rate model to price the exotic product, such as the CMS and CMS derivatives. This paper covers this gap by proposing a pricing and approximation scheme for these products. We also consider the commonly traded in-arrears swap and in-arrears caps that can be efficiently approximated the proposed framework.

In Da Fonseca et al. (2022) we derived a pricing formula for swaptions whose numerical cost is at par with caps and floors. The approximation for the swaption price used in Da Fonseca et al. (2022), by adjusting Collin-Dufresne and Goldstein (2002)'s methodology, is very versatile as it also applies to the CMS and CMS spread option product. In this paper, we follow a similar approach and show how more exotic interest rate derivatives such as the constant maturity swap (CMS), CMS spread option, and the in-arrears swap can be priced in this framework. The approximation crucially relies on the affine property of the Wishart process. Using the calibrated model, we show that the approximation formulas, for the CMS, CMS spread options and the in-arrears swap, are very accurate. These results convincingly demonstrate the performance of the linear-rational model based on the Wishart process as a modeling tool for a wide range of interest rate derivatives.

The structure of the paper is as follows. Section 2 recalls the model and its main analytical properties. Section 3, provides the approximation for interest rate derivatives, such as CMS, CMS spread option as well as in-arrears swap and cap. Section 4 provides the numerical analysis of the proposed approach and illustrates how well the approach performs in practice. Section 5 concludes the paper while proofs and tables are gathered in the appendix.

2 A multi-curve model

2.1 The OIS and Euribor-OIS term structure curves

In this section, we recall the Wishart based multi-curve model as proposed in Da Fonseca et al. (2022). Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ we denote by $\mathbb{E}\left[\,\cdot\,\right]$ (resp. $\mathbb{E}_t\left[\,\cdot\,\right] := \mathbb{E}\left[\,\cdot\,|\mathcal{F}_t\right]$) the expectation (resp. conditional expectation) under the probability measure $\mathbb{P}$. The Wishart process, proposed in Bru (1991) and introduced in finance in Gouriéroux and Sufana (2010), satisfies the matrix stochastic differential equation

$$dx_t = (\omega + mx_t + x_t m^\top)dt + \sqrt{x_t}dw_t\sigma + \sigma^\top dw_t^\top \sqrt{x_t}\,, \tag{1}$$

where x_t is an $n \times n$ matrix that belongs to the set of positive definite matrices denoted $\mathbb{S}_n^{++}$, m, σ belong to the set of $n \times n$ real matrices denoted $\mathsf{M}(n)$, $\{w_t; t \geq 0\}$ is a matrix Brownian motion of dimension $n \times n$ (*i.e.*, a matrix of n^2 independent scalar Brownian motions) under the probability measure $\mathbb{P}$ and $\cdot^\top$ stands for the matrix transposition.[1] The matrix $\omega \in \mathbb{S}_n^{++}$ satisfies

[1]By definition, w_t is an $(n \times n)$ matrix Brownian motion if and only if $\forall u, v \in \mathbb{R}^n$, $(w_t u, w_t v)$ is a vector Brownian motion with covariance structure $\mathrm{cov}_t\left[dw_t u, dw_t v\right] = u^\top v I_n dt$ with I_n the $n \times n$ identity matrix.

certain constraints involving $\sigma^\top \sigma$ to ensure the positiveness of the matrix process x_t. Note that the transpositions in Eq. (1) are necessary to preserve the symmetry of the solution. The quantity $\sqrt{x_t}$ is well defined since $x_t \in \mathbb{S}_n^{++}$. The matrix m is such that $\{\Re(\lambda_i^m) < 0; i = 1, \dots, n\}$ where $\lambda_i^m \in Spec(m)$ for $i = 1, \dots, n$ and $Spec(m)$ is the spectrum of the matrix m while $\Re(\cdot)$ stands for the real part. The matrix σ belongs to $\mathsf{GL}_n(\mathbb{R})$ the general linear group over $\mathbb{R}$ (*i.e.*, the set of real invertible matrices). Thanks to the invariance of the law of the Brownian motion to rotations and the polar decomposition of σ, we can assume that $\sigma \in \mathbb{S}_n^{++}$.

We consider a two-curve model based on a 2×2 Wishart process (*i.e.*, $n = 2$), where first, we define a pricing kernel as: [2]

$$\zeta_t := e^{-\alpha t}(1 + x_{11,t}), \tag{2}$$

with $\alpha \in \mathbb{R}_+$ and $(x_{11,t})_{t \geq 0}$ is the $(1,1)^{th}$ element of a Wishart process $(x_t)_{t\geq 0}$. According to Rogers (1997), the pricing kernel can be rewritten as follows. Define the positive function $f : \mathbb{S}_2^{++} \to \mathbb{R}^+$ such that $f(x) := 1 + \mathrm{tr}[e_{11}x]$. Define $g(x) := (\alpha - \mathcal{G})f(x)$, which is a positive function for sufficiently large α (*i.e.*, $\alpha > \mathrm{tr}[\omega]$).
The pricing kernel allows us to compute the time t value of a collateralized zero-coupon bond with maturity T, denoted $P(t,T)$, that is given by

$$\begin{aligned} P(t,T) &:= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right] = \mathbb{E}_t\left[\frac{\zeta_T}{\zeta_t}\right], & (3)\\ &= e^{-\alpha(T-t)}\frac{1 + \mathbb{E}_t[x_{11,T}]}{1 + x_{11,t}}, & (4) \end{aligned}$$

with $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ the (conditional) expectation under the risk neutral probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which zero-coupon bond prices are martingale.

The expectation in Eq. (4) can be explicitly computed thanks to the affine property of the Wishart process and for the particular specification adopted here is very simple as the following proposition shows.

[2] Note that Filipović et al. (2017) suggests to consider a function of the form $e^{-\alpha t}(a_0 + a_1 x_{11,t})$ with $a_0 > 0$ and $a_1 > 0$ but for identification reasons, clearly explained in Filipović et al. (2017, Theorem 5), one needs to impose $a_0 = 1$ and $a_1 = 1$.

Proposition 2.1. *Under the assumption that the matrix m in Eq.* (1) *is diagonal, the zero-coupon bond is given by*

$$P(t,T) = e^{-\alpha(T-t)} \frac{b_1(T-t) + a_1(T-t)x_{11,t}}{1 + x_{11,t}}\,, \tag{5}$$

with $b_1(t) := 1 + \frac{\omega_{11}}{2m_{11}}(e^{2m_{11}t} - 1)$ *and* $a_1(t) := e^{2m_{11}t}$.

The discount factor $P(T, T+\delta)$ is related to the time T overnight indexed swap (OIS) rate with maturity $T+\delta$ by the formula

$$\mathrm{OIS}(T, T+\delta) = \frac{1}{\delta}\frac{1 - P(T,T+\delta)}{P(T,T+\delta)}. \tag{6}$$

The above formula holds for an OIS with maturity less than one year.

Additionally we consider the Euribor rate $L(T, T+\delta)$, which is the rate at time T for the period $[T, T+\delta]$. Let us denote by $\mathrm{Spread}(T, T+\delta)$, the spread between the Euribor and OIS rates, this is the difference between $L(T, T+\delta)$ and $\mathrm{OIS}(T, T+\delta)$ and is often called the Euribor-OIS spread. Before the global financial crisis, the spread was negligible but after the crisis it widened significantly and a multi-curve interest rate model aims at taking into account that spread and its stochastic evolution. The Euribor-OIS spread is defined by:

$$\mathrm{Spread}(T, T+\delta) := L(T, T+\delta) - \mathrm{OIS}(T, T+\delta)\,, \tag{7}$$

$$= L(T, T+\delta) - \frac{1}{\delta}\left(\frac{1}{P(T,T+\delta)} - 1\right). \tag{8}$$

Similar to the approach in the appendix of Filipović et al. (2017), we specify for the time T deflated value of the Euribor-OIS spread payment at time $T+\delta$ a linear functional of the stochastic process $(x_{22,t})$, the $(2,2)^{th}$ element of the Wishart process. More precisely, the deflated time-T value of the Euribor-OIS spread time-$T+\delta$ payment is defined as[3]

$$\zeta_T P(T, T+\delta)\delta \mathrm{Spread}(T, T+\delta) = e^{-\alpha T} x_{22,T}. \tag{9}$$

Once the deflated value of the Euribor-OIS spread payment at a future date is specified, its expectation gives the value of the spread as a (linear-rational) function of the process as shown in the next proposition.

[3]Filipović et al. (2017, online appendix) suggest to specify the right hand side of Eq. (9) as $e^{-\alpha T}(1 + x_{22,T})$ but we found that specification rather inconvenient as the left hand side of Eq. (9) can be arbitrarily small, if for example the spread is small, and as $x_{22,t}$ is a positive process it can lead to calibration problems. In fact, the specification Eq. (9) matches the one of Rogers (1997, Example 3.7).

Proposition 2.2. *The time-t value of the Euribor-OIS spread payment set at time T and made at time $T+\delta$, simply called the (time-t value) Euribor-OIS spread, is given by:*

$$A(t,T,T+\delta) = \frac{1}{\zeta_t}\mathbb{E}_t\left[\zeta_T P(T,T+\delta)\delta \mathrm{Spread}(T,T+\delta)\right] , \tag{10}$$

$$= \frac{1}{\zeta_t}\mathbb{E}_t\left[e^{-\alpha T} x_{22,T}\right] , \tag{11}$$

$$= e^{-\alpha(T-t)}\frac{b_2(T-t)+a_2(T-t)x_{22,t}}{1+x_{11,t}} , \tag{12}$$

with $b_2(t) := \frac{\omega_{22}}{2m_{22}}(e^{2m_{22}t}-1)$ and $a_2(t) := e^{2m_{22}t}$ if we assume that m is diagonal.

2.2 CMS and CMS spread option pricing

The vanilla swaption proved to be surprisingly simple to value in the linear-rational Wishart model and a natural question is whether other exotic products can be also easily priced in that framework. Looking at the interest rates derivatives actively traded on the market, the CMS is certainly the most obvious choice to consider. Following the academic literature (*e.g.*, Brigo and Mercurio 2006, Chapter 13.7) we now recall the characteristics of that product.

Consider a CMS with tenor dates $T_0, \cdots, T_{n_1}$, with $T_j - T_{j-1} = \delta$. The two legs of the CMS have the same payment dates $T_1, \cdots, T_{n_1}$. At a payment date T_{j+1}, with $j = 0, \ldots, n_1 - 1$, one leg pays the Euribor rate resetting at time T_j plus a fixed spread K, while the other leg pays the swap rate $S_{T_j}^{T_{j,0},T_{j,n_s}}$, which is the swap rate with tenor structure and payment dates $T_{j,l} = T_j + l\delta_s$ with $l = 0, \ldots, n_s$ for the floating leg and $t_{j,k} = T_j + k\Delta_s$ for $k = 0, \ldots, m_s$ for the fixed leg and $T_{j,n_s} = t_{j,m_s}$. We suppose that δ and δ_s are equal so that there is no need to introduce another factor (or several factors) to handle the two tenor structures. In practice δ, δ_s and Δ_s are different but we stress the fact that all the computations below can be performed for that more general case without additional significant difficulty.

Proposition 2.3. *The time-t value of the CMS receiving the Euribor (plus a fixed rate K) leg* *and paying the swap leg is therefore given by*

$$\Pi_t^{\text{CMS}} = P(t,T_0) - P(t,T_{n_1}) + \sum_{j=1}^{n_1} A(t,T_{j-1},T_j) + \delta K \sum_{j=1}^{n_1} P(t,T_j) \\ - \sum_{j=0}^{n_1-1} \delta \mathbb{E}_t \left[\frac{\zeta_{T_{j+1}}}{\zeta_t} S_{T_j}^{T_j,T_{j,n_s}} \right], \tag{13}$$

with

$$\mathbb{E}_t \left[\frac{\zeta_{T_{j+1}}}{\zeta_t} S_{T_j}^{T_j,T_{j,n_s}} \right] = \frac{e^{-\alpha(T_{j+1}-t)}}{1+x_{11,t}} \mathbb{E}_t \left[\frac{c_0 + c_1 x_{11,T_j} + c_2 x_{22,T_j} + c_{12} x_{11,T_j} x_{22,T_j} + c_{11} x_{11,T_j}^2}{\mu_0 + \mu_1 x_{11,T_j}} \right], \tag{14}$$

and c_0, c_1, c_2, c_{12}, c_{11}, μ_0 and μ_1 defined in Eqs. (60-66) in the appendix.

To compute the value of the CMS, the expectation Eq. (14) needs to be evaluated but its simple structure, a rational function, combined with the affine property of the Wishart process enable an explicit computation thanks to the following well known remark.

Remark 2.4. *Suppose that we know the moment generating function of the vector (X,Y), that is $G(z_1,z_2) = \mathbb{E}\left[e^{z_1 X + z_2 Y}\right]$. To compute $\mathbb{E}\left[\frac{X}{Y}\right]$, the relation $1/y = \int_0^{+\infty} e^{-sy} ds$ leads to $\mathbb{E}\left[\frac{X}{Y}\right] = \int_0^{+\infty} \mathbb{E}\left[X e^{-sY}\right] ds$, and using the propriety of the moment generating function, we get $\mathbb{E}\left[\frac{X}{Y}\right] = \int_0^{+\infty} \partial_{z_1} \mathbb{E}\left[e^{z_1 X - sY}\right] ds|_{z_1=0}$. As $\mathbb{E}\left[e^{z_1 X - sY}\right]$ is known and can possibly be derived explicitly with respect to z_1, we obtain a quasi closed form for the expectation of the ratio of the two random variables.*

Proposition 2.5. *The integral representation of the ratio of two random variables combined with the moment generating function of the Wishart process (Da Fonseca et al. (2022)) give explicit expressions for the expectations:*

$$\mathbb{E}_t \left[\frac{1}{\mu_0 + \mu_1 x_{11,T_j}} \right], \quad \mathbb{E}_t \left[\frac{x_{22,T_j}}{\mu_0 + \mu_1 x_{11,T_j}} \right], \tag{15}$$

which are sufficient to compute Eq. (14)*.*

The CMS naturally serves as an underlying for interest rate derivatives but instead of the standard call/put on an CMS rate what is frequently found is the CMS spread single-option which involves two CMS rates as its name suggests. Let us denote by $\Pi_t^{\text{CmsSpSO}}(T_1, n_{s_1}, n_{s_2}, K)$

the t-value of a CMS spread call option with single expiration date T_1 and strike K. It is an option whose value at time T_1 is based on the difference between the spot swap rate $S_{T_1}^{T_{1,0},T_{1,n_{s_1}}}$, starting at time T_1 and ending at time $T_{1,n_{s_1}} > T_1$, and the spot swap rate $S_{T_1}^{T_{1,0},T_{1,n_{s_2}}}$ starting at time T_1 and ending at time $T_{1,n_{s_2}} > T_1$.

The underlying swap $S_{T_1}^{T_{1,0},T_{1,n_{s_1}}}$ floating leg's tenor and payment dates are $T_{1,l} = T_1 + l\delta_s$ with $l = 0, \ldots, n_{s_1}$ whilst the fixed leg's tenor and payment dates are $t_{1,k} = t_1 + k\Delta_s$ with $k = 0, \ldots, m_{s_1}$ and we further have that $T_{1,0} = T_1$, $t_{1,0} = t_1 = T_1$ and $T_{1,n_{s_1}} = t_{1,m_{s_1}}$ which imply that both legs start and end at the same time. The swap $S_{T_1}^{T_{1,0},T_{1,n_{s_2}}}$ is defined similarly.

Those two CMS rates are the underlyings of the CMS spread single-option whose pricing formula is presented in the next proposition.

Proposition 2.6. *The option time-t value, denoted Π_t^{CmsSpSO} for simplicity, is given by:*

$$\Pi_t^{\mathrm{CmsSpSO}} = \mathbb{E}_t\left[\frac{\zeta_{T_1}}{\zeta_t}\left(S_{T_1}^{T_{1,0},T_{1,n_{s_1}}} - S_{T_1}^{T_{1,0},T_{1,n_{s_2}}} - K\right)_+\right], \tag{16}$$

$$= \frac{e^{-\alpha(T_1-t)}}{1+x_{11,t}}\mathbb{E}_t\left[\left(g^1(x_{11,T_1}, x_{22,T_1}) - g^2(x_{11,T_1}, x_{22,T_1}) - K(1+x_{11,T_1})\right)_+\right], \tag{17}$$

with

$$g^i(x_{11,T_1}, x_{22,T_1}) = \frac{c_0^i + c_1^i x_{11,T_1} + c_2^i x_{22,T_1} + c_{12}^i x_{11,T_1} x_{22,T_1} + c_{11}^i x_{11,T_1}^2}{\mu_0^i + \mu_1^i x_{11,T_1}}, \tag{18}$$

and for $i \in \{1,2\}$ with c_0^i, c_1^i, c_2^i, c_{12}^i, c_{11}^i μ_0^i and μ_1^i for $i \in \{1,2\}$ given in the appendix.

Similar to caplets (or floorlets) that are not traded individually but as a component of a cap (floor), the CMS spread single-option is traded through a CMS spread multi-option which is just a portfolio of CMS spread single-options and is defined as follows. Let $\Pi_t^{\mathrm{CmsSpMO}}(T_1, T_{n_1}, n_{s_1}, n_{s_2}, K)$ be the t-value of the multi CMS spread call option with exercise dates $T_1, \cdots, T_{n_1}$, with $T_j - T_{j-1} = \delta$ and strike K. It is a sum of CMS spread single call option with maturity dates $T_1, \ldots, T_{n_1}$. All the options' two underlying swaps have the same tenor structures. Using the previous definition, the option time t-value denoted Π_t^{CmsSpMO}, for simplicity and when no confusion is possible, is given by:

$$\Pi_t^{\mathrm{CmsSpMO}} = \sum_{j=1}^{n_1} \Pi_t^{\mathrm{CmsSpSO}}(T_j, n_{s_1}, n_{s_2}, K) \tag{19}$$

Unfortunately, the pricing formula of the CMS spread single-option Eq. (17) is not as simple as the swaption pricing formula. Notice, however, that it only involves $(x_{11,T}, x_{22,T})$, the marginal distribution of the process at time T and not the process path from t to T. In the Bru case, the marginal distribution of the process can be expressed, when the parameter β is an integer, as the square of a matrix Gaussian distribution is therefore computable by Monte Carlo very efficiently. When β is not an integer, Ahdida and Alfonsi (2013) derived an exact and fast simulation algorithm. Still, having accurate price approximations for these products is of interest and the following section shows that such approximations are available thanks to the affine property of the Wishart process.

3 Interest rate derivative approximations

3.1 Price approximation approach

In Collin-Dufresne and Goldstein (2002), the authors propose an approximation of the swaption price by approximating the density of a coupon bearing bond.Their result crucially relies on the affine property of the process driving the interest rates.

In the approach adopted here, the affine property of the Wishart process is used to derive an approximation in the spirit of Collin-Dufresne and Goldstein (2002) for the interest rate derivatives and option pricing. Notice that the affine property is used in two different ways. In Collin-Dufresne and Goldstein (2002), the authors use the fact that the expected value of the exponential of an affine variable is exponential affine whereas here we use the fact that the expected value of a polynomial function of a given order of an affine process can be expressed as a polynomial function of the process, in order words the set of polynomials is stable for the infinitesimal generator of the Wishart process.

To establish the approximation, we need the two following lemmas.

Lemma 3.1. *Let $y(t)$ a function solution of the ordinary differential equation*

$$\frac{dy(t)}{dt} = \kappa y(t) + \sum_{i=1}^{l} \bar{a}_i + \bar{b}_i e^{\kappa_i t}\,, \tag{20}$$

with $\kappa \neq \kappa_i \; \forall i \in \{1,\ldots,l\}$ and $\bar{a}_i, \bar{b}_i \; \forall i \in \{1,\ldots,l\}$ some constants. Then it can be integrated to

$$y(t) = \bar{c} + \sum_{i=1}^{l+1} \bar{d}_i e^{\kappa_i t}\,, \tag{21}$$

with $\kappa_{l+1} = \kappa$ and

$$\bar{c} = -\sum_{i=1}^{l} \frac{\bar{a}_i}{\kappa}\,, \tag{22}$$

$$\bar{d}_i = \frac{\bar{b}_i}{\kappa_i - \kappa}\,, \quad i = 1,\ldots,l\,, \tag{23}$$

$$\bar{d}_{l+1} = y(0) + \sum_{i=1}^{l} \frac{\bar{a}_i}{\kappa} - \sum_{i=1}^{l} \frac{\bar{b}_i}{\kappa_i - \kappa}\,. \tag{24}$$

For notational convenience, it is useful to introduce $\bar{\sigma} = \sigma^2$ with $(\sigma_{11}^2 + \sigma_{12}^2) = (\sigma^2)_{11} = \bar{\sigma}_{11}$, $(\sigma_{12}^2 + \sigma_{22}^2) = (\sigma^2)_{22} = \bar{\sigma}_{22}$ and $(\sigma_{11}\sigma_{12} + \sigma_{12}\sigma_{22}) = (\sigma^2)_{12} = \bar{\sigma}_{12}$. The following lemma shows that the affine property of the Wishart process implies a simple expression for the expected value of a polynomial function of the process.

Lemma 3.2. *Let us denote $g(t,i,k,j) = \mathbb{E}[x_{11,t}^i x_{12,t}^k x_{22,t}^j]$ where $(x_{11,t}, x_{12,t}, x_{22,t})$ are the components of a 2×2 Wishart process. Then using Itô's Lemma we get*

$$\frac{dg(t,i,k,j)}{dt} = (i2m_{11} + k(m_{11} + m_{22}) + 2jm_{22})\, g(t,i,k,j) \tag{25}$$

$$+ (i\omega_{11} + 2ik\bar{\sigma}_{11} + 2i(i-1)\bar{\sigma}_{11})\, g(t,i-1,k,j) \tag{26}$$

$$+ (k\omega_{12} + k(k-1)\bar{\sigma}_{12} + 2ik\bar{\sigma}_{12} + 2jk\bar{\sigma}_{12})\, g(t,i,k-1,j) \tag{27}$$

$$+ (j\omega_{22} + 2j(j-1)\bar{\sigma}_{22} + 2jk\bar{\sigma}_{22})\, g(t,i,k,j-1) \tag{28}$$

$$+ \frac{k(k-1)}{2}\bar{\sigma}_{22} g(t,i+1,k-2,j) \tag{29}$$

$$+ \frac{k(k-1)}{2}\bar{\sigma}_{11} g(t,i,k-2,j+1) \tag{30}$$

$$+ 4ij\bar{\sigma}_{12} g(t,i-1,k+1,j-1)\,. \tag{31}$$

Notice that Eqs. (26–31) involve polynomials with degree lower or equal to $i+k+j-1$ whilst Eq. (25) involves a polynomial of degree $i+k+j$, it is a consequence of the affine property of the Wishart process. As $g(t,1,0,0)$, $g(t,0,1,0)$ and $g(t,0,0,1)$ can be written in the form $\bar{a}_0+\bar{b}_0e^{\kappa t}$ with suitable $\bar{a}_0,\bar{b}_0$ and κ coefficients then we deduce by recurrence that $g(t,i,k,j)$ solves an ODE of the form Eq. (20) and therefore Lemma 3.1 applies. Notice that the condition $\kappa \neq \kappa_i\ \forall i \in \{1,\ldots,l\}$ of Lemma 3.1 is satisfied as $m_{11}<0$ and $m_{22}<0$.

Similar to the case of the swaption derived in Da Fonseca et al. (2022), let us consider an interest rate option whose price is given by $\mathbb{E}\left[(Y_{T_0})_+\right]$ where $Y_{T_0} = B(T_0) + A_1(T)x_{11,T_0} + A_{12}(T_0)x_{12,T_0} + A_2(T_0)x_{22,T_0}$.

To apply Collin-Dufresne and Goldstein (2002)'s approximation approach, one needs to compute the q^{th} moment of Y_{T_0} that is simply given by

$$\mathbb{E}[Y_{T_0}^q] = \sum_{l_0+l_1+l_{12}+l_2=q} \binom{q}{l_0,l_1,l_{12}l_2} B(T_0)^{l_0}A_1(T_0)^{l_1}A_{12}(T_0)^{l_{12}}A_2(T_0)^{l_2}\mathbb{E}[x_{11,T_0}^{l_1}x_{12,T_0}^{l_{12}}x_{22,T_0}^{l_2}]\,. \tag{32}$$

As $\mathbb{E}[x_{11,T_0}^{l_1}x_{12,T_0}^{l_{12}}x_{22,T_0}^{l_2}] = g(T_0,l_1,l_{12},l_2)$ is known thanks to Lemma 3.2, the moments are also known.

Starting from Collin-Dufresne and Goldstein (2002, Eq. 17), which presents an expansion of the density of Y_{T_0}, given by

$$\frac{1}{\sqrt{2\pi c_2}}e^{-\frac{(y-c_1)^2}{2c_2}}\left(\sum_{j\geq 0}\gamma_j(y-c_1)^j\right), \tag{33}$$

the expectation $\mathbb{E}\left[(Y_{T_0})_+\right]$ can be expressed as

$$\mathbb{E}_t\left[(Y_{T_0})_+\right] \sim \sum_{j\geq 0}\gamma_j\int_0^{+\infty}\frac{1}{\sqrt{2\pi c_2}}y(y-c_1)^je^{-\frac{(y-c_1)^2}{2c_2}}dy\,, \tag{34}$$

$$= \sum_{j\geq 0}\gamma_j\sqrt{c_2}c_2^{j/2}\int_{\frac{-c_1}{\sqrt{c_2}}}^{+\infty}z^{j+1}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\,, \tag{35}$$

$$+\sum_{j\geq 0}\gamma_jc_1c_2^{j/2}\int_{\frac{-c_1}{\sqrt{c_2}}}^{+\infty}z^j\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\,, \tag{36}$$

$$= \sum_{j\geq 0}\gamma_j\lambda_{j+1} + c_1\sum_{j\geq 0}\gamma_j\lambda_j\,, \tag{37}$$

where $\{\gamma_j; j \in \mathbb{N}\}$ are related to the cumulants through Eqs. (B.18–B.25) in Collin-Dufresne and Goldstein (2002), $\{c_j; j \in \mathbb{N}\}$ are the cumulants of the variable Y_{T_0} that are related to the moments of that variable through Eqs. (A.1–A.7) in Collin-Dufresne and Goldstein (2002) whilst $\{\lambda_j; j \in \mathbb{N}\}$ are related to the normal density/cumulative distribution (see Collin-Dufresne and Goldstein, 2002, Eqs. B.10–B.17).

3.2 CMS and CMS derivative approximations

To evaluate a CMS, one needs to compute the expectation given in Eq. (17), it can be done exactly thanks to Proposition 2.5 but it requires a one or two dimensional integration depending on whether the Bru condition (*i.e.*, $\omega = \beta\sigma^2$) is satisfied or not. For standard interest rate models, such as exponential affine models, there is no closed form solution for that expectation and one needs to rely on some approximations, see Brigo and Mercurio (2006) and Hanton and Henrard (2012). It is useful to notice that Eq. (17) is the expectation of a ratio of two polynomials of the Wishart process and a series expansion of the denominator enables us to rewrite the problem as an expectation of a series of the Wishart process that when truncated leads to an expectation of a polynomial function of the Wishart process. The moments of the Wishart process being known, thanks to Lemma 3.2, we obtain an approximation of the expectation as the following proposition shows.

Proposition 3.3. *The time-t value expectation*

$$I = \mathbb{E}_t\left[\frac{c_0 + c_1 x_{11,T_j} + c_2 x_{22,T_j} + c_{12} x_{11,T_j} x_{22,T_j} + c_{11} x_{11,T_j}^2}{\mu_0 + \mu_1 x_{11,T_j}}\right], \tag{38}$$

can be approximated by

$$\begin{aligned} I(M) = {} & \frac{c_0}{\mu_0}\sum_{m=0}^{M}(-1)^m\left(\frac{\mu_1}{\mu_0}\right)^m \mathbb{E}_t\left[x_{11,T_j}^m\right] + \frac{c_1}{\mu_0}\sum_{m=0}^{M}(-1)^m\left(\frac{\mu_1}{\mu_0}\right)^m \mathbb{E}_t\left[x_{11,T_j}^{m+1}\right] \\ & + \frac{c_2}{\mu_0}\sum_{m=0}^{M}(-1)^m\left(\frac{\mu_1}{\mu_0}\right)^m \mathbb{E}_t\left[x_{11,T_j}^m x_{22,T_j}\right] + \frac{c_{12}}{\mu_0}\sum_{m=0}^{M}(-1)^m\left(\frac{\mu_1}{\mu_0}\right)^m \mathbb{E}_t\left[x_{11,T_j}^{m+1} x_{22,T_j}\right] \\ & + \frac{c_{11}}{\mu_0}\sum_{m=0}^{M}(-1)^m\left(\frac{\mu_1}{\mu_0}\right)^m \mathbb{E}_t\left[x_{11,T_j}^{m+2}\right], \end{aligned} \tag{39}$$

where M is the truncation order of the series $1/(1+(\mu_1/\mu_0)x)$ and the constants c_0, c_1, c_2, c_{12}, c_{11}, μ_0 and μ_1 are those of Proposition 2.3 while the expectations in Eq. (39) *are given by Lemma 3.2. Notice that $0<\mu_1/\mu_0<1$ by construction.*

In the Collin-Dufresne and Goldstein (2002) swaption price approximation, the key ingredient is the set of moments of the random variable whose law, which is unknown, is needed to compute the expectation associated option price. In the CMS option case, the underlying random variable is the discounted swap rate that is given by a ratio of polynomial functions of the Wishart process. The ratio can be expanded as a series, by performing a series expansion of the denominator, and after a truncation it gives an approximation of the variable of interest by a polynomial function of the Wishart process. The moments of the Wishart process being known, the different moments of the CMS option underlying variable are known and therefore Collin-Dufresne and Goldstein (2002) can be readily applied. To illustrate the method, the CMS option spread is used with the following proposition providing the details.

Proposition 3.4. *Consider the CMS spread option of Proposition 2.6, and define the expectation in Eq.* (17) *by $\mathbb{E}_t\left[(Y_T)_+\right]$ where $Y_T=g^1(x_{11,T_1},x_{22,T_1})-g^2(x_{11,T_1},x_{22,T_1})-K(1+x_{11,T_1})$ with $g^1(.,.)$ and $g^2(.,.)$ defined by Eq.* (18)*. Define the approximation Y_T^M of order M of Y_T by*

$$Y_T^M=\sum_{i=0}^{2M+4}\zeta_i y_i\,, \tag{40}$$

with $\{\zeta_i;i=0,\dots,2M+4\}$ constants given in the proof while $y_i=x_{11,T}^i$ for $i=0,\dots M+2$ and $y_i=x_{22,T}x_{11,T}^{i-M-2}$ for $i=M+3,\dots,2M+4$. The q^{th} moment of Y_T can be approximated by the q^{th} moment of Y_T^M given by

$$\mathbb{E}\left[(Y_T^M)^q\right]=\sum_{k_0+\ldots+k_{2M+4}=q}\binom{q}{k_0,\dots,k_{2M+4}}\prod_{j=0}^{2M+4}\zeta_j^{k_j}\mathbb{E}\left[\prod_{l=0}^{2M+4}y_l^{k_l}\right]. \tag{41}$$

The expectation $\mathbb{E}\left[\prod_{l=0}^{2M+4}y_l^{k_l}\right]$ is known thanks to Lemmas 3.1 and 3.2, so the q^{th} moment of Y_T^M is known.

Notice that to compute the moment of order q of Y_T^M, we need the moments of order $q(M+2)$ of x_T.

3.3 Other derivatives approximations

In-Arrears Swaps Let us consider an interest rate swap starting at T_0 and maturing at T_n where the floating leg payment dates are $T_1, \cdots, T_n$, with $T_j - T_{j-1} = \delta$, $j = 1, \ldots, n$, the fixed leg payment rate K and the fixed leg payment dates are $t_1, \cdots, t_m = T_n$, $t_i - t_{i-1} = \Delta$ for $i = 1, \ldots, m$ and $t_0 = T_0$. The floating leg is fixing in-arrears, and pays $\delta L(T_j, T_{j+1})$ at time T_j (for $j = 1, \ldots, n$). The time-t value of the floating leg payment at time T, denoted by $C(t, T, T+\delta)$ is given by

$$C(t,T,T+\delta) = \frac{1}{\zeta_t}\mathbb{E}_t\left[\zeta_T \delta L(T,T+\delta)\right], \tag{42}$$

$$= \frac{e^{-\alpha(T-t)}}{1+x_{11,t}} I_1 + \frac{e^{-\alpha(T-t)}}{1+x_{11,t}} I_2 - P(t,T), \tag{43}$$

where $I_1 = \mathbb{E}_t\left[e^{\alpha\delta}\frac{x_{22,T}(1+x_{11,T})}{b_1(\delta)+a_1(\delta)x_{11,T}}\right]$ and $I_2 = \mathbb{E}_t\left[e^{\alpha\delta}\frac{(1+x_{11,T})^2}{b_1(\delta)+a_1(\delta)x_{11,T}}\right]$.

Similar to the technique used for the CMS, we proceed with the expansion of the integrand in the calculation of I_1 and I_2. The resulting approximation at order M for I_1 and I_2 is:

$$I_1(M) = \frac{e^{\alpha\delta}}{b_1(\delta)}\sum_{m=0}^{M}(-1)^m\left(\frac{a_1(\delta)}{b_1(\delta)}\right)^m \mathbb{E}_t\left[x_{11,t}^m x_{22,t}\right] + \frac{e^{\alpha\delta}}{b_1(\delta)}\sum_{m=0}^{M}(-1)^m\left(\frac{a_1(\delta)}{b_1(\delta)}\right)^m \mathbb{E}_t\left[x_{11,t}^{m+1} x_{22,t}\right], \tag{44}$$

$$I_2(M) = \frac{e^{\alpha\delta}}{b_1(\delta)}\sum_{m=0}^{M}(-1)^m\left(\frac{a_1(\delta)}{b_1(\delta)}\right)^m \left(\mathbb{E}_t\left[x_{11,t}^m\right] + 2\mathbb{E}_t\left[x_{11,t}^{m+1}\right] + \mathbb{E}_t\left[x_{11,t}^{m+2}\right]\right). \tag{45}$$

From this, we obtain the time-t value Π_t^{IARS} of the in-arrears swap given by

$$\Pi_t^{IARS} = \sum_{j=1}^{n} C(t,T_j,T_{j+1}) + \Delta K \sum_{j=1}^{m} P(t,t_j), \tag{46}$$

$$= \sum_{j=1}^{n} \frac{e^{-\alpha(T_j-t)}}{1+x_{11,t}}(I_1+I_2) - P(t,T_j) + \Delta K \sum_{j=1}^{m} P(t,t_j). \tag{47}$$

We see that the technique used for the CMS can be applied here as well, as the expectation in the Eq. (44) Eq. (45) are known.

In-Arrears Cap We follow Brigo and Mercurio (2006, section 13.2) and consider the in-arrears cap, resetting and paying at time $T_1, \cdots, T_n$, with $T_j - T_{j-1} = \delta$ for $j = 1, \ldots, n$, with the strike K. At time T_j, the in-arrears caplet pays $\delta(L(T_j, T_j+\delta)-K)_+$. The time-t value of the time T payment, denoted by $C(t,T,T+\delta)$ is given by:

$$C(t,T,T+\delta) = \frac{1}{\zeta_t}\mathbb{E}_t\left[\zeta_T \delta(L(T,T+\delta)-K)_+\right], \tag{48}$$

$$= \frac{1}{\zeta_t}\mathbb{E}_t\left[\left(\frac{c_0 + c_1 x_{11,T} + c_2 x_{22,T} + c_3 x_{11,T}^2 + c_4 x_{11,T} x_{22,T}}{b_1(\delta) + a_1(\delta) x_{11,T}} - \delta e^{-\alpha T} K (1 + x_{11,T})\right)_+\right], \tag{49}$$

with

$$c_0 = e^{-\alpha T}(e^{\alpha\delta} - b_1(\delta), \tag{50}$$

$$c_1 = e^{-\alpha T}(2e^{\alpha\delta} - b_1(\delta) - a_1(\delta)), \tag{51}$$

$$c_2 = e^{-\alpha T}(e^{\alpha\delta}), \tag{52}$$

$$c_3 = e^{-\alpha T}(e^{\alpha\delta} - a_1(\delta)), \tag{53}$$

$$c_4 = e^{-\alpha(T-\delta)}. \tag{54}$$

This can be computed similarly to the calculation for the CMS call option.

4 Numeric results and analysis

Once the model is calibrated on liquid products such as swaptions as described in Da Fonseca et al. (2022), it can be used to price exotic derivatives. We focus on the CMS and CMS spread options as these are important products for which section 3.2 provides price approximations that we now evaluate. For the model parameters, we consider those of Table I while for the product parameters in Eq. (14) we take $T_j = 1Y$, $2Y$, $3Y$ or $5Y$, the swap tenor is equal to $1Y$, $5Y$ or $7Y$ and $\delta_s = \Delta_s = 0.5$. Regarding the truncation level M in Proposition 3.3, we consider $M = 3$, $M = 5$, and $M = 7$. We benchmark the approximation given by Eq. (39) with a Monte-Carlo method with 50000 paths and a time discretisation of 250 days per year. The results reported in Table II confirm the quality of the approximation as evidenced by the small discrepancy between the two methods. Not surprisingly, the error decreases with the truncation level M in Eq. (39) and deteriorates with the maturity of the CMS (everything else being equal). It's important to notice also that the error for $M = 5$ and $M = 7$ are very similar. Hence the results indicate the convergence around $M = 5$.

[Insert Table I here]

[Insert Table II here]

Following these encouraging results, we consider the CMS spread option of Proposition 2.6 using the approximation of the underlying variable given by Proposition 3.4 and the Collin-Dufresne and Goldstein (2002) approximation formula Eq. (37) where we restrict to the first three moments/cumulants. The two underlying swaps have a tenor of 1Y and 5Y, respectively, while $\delta_s = \Delta_s = 0.5$. For the CMS spread option maturities, we take $1Y$ and $5Y$. While for the underlying option, we consider option with maturities $6M$, $1Y$, $2Y$, $3Y$, $4Y$, and $5Y$; and with strike offset $-100bps$, $-50bps$, $0bps$, $50bps$, $100bps$. For the CMS spread option approximation, we consider $M = 3$, $M = 4$, $M = 5$, and $M = 10$. As for the CMS, we compare the price approximation with a Monte-Carlo method with 50000 paths and a time discretisation of 250 days per year and report in Table III the absolute error between these two prices expressed in percent. The table confirms the accuracy of the approximation for all the parameters selected. Taking into account the importance of the CMS and CMS spread option, it shows an interesting property of the linear-rational Wishart model as it enables a better integration between the swaption market, which is used to calibrate the model, and the exotic interest rate derivative market, in this case the CMS and CMS spread option market, as these products can be priced easily using a polynomial approximation that is accurate.

[Insert Table III here]

We continue with the analysis of the numerical results of the in-arrears swap. The table IV shows the comparison between the approximation of the term I_1 and I_2 given respectively by Eq. (44) and Eq. (45). We provide the result for payment dates T=1Y, 2Y, 3Y, and 5Y and for different $M = 3, 4, 5$. The approximation is compared to the Monte carlo result for 50000 paths and a time discretisation of 250 days per year. As reported in the Table IV, we see the approximation is quite accurate, and that this approximation provides a fast, robust and accurate approach for pricing in arrears swap in the Wishart based Linear ration model.

[Insert Table IV here]

5 Conclusion

We develop the pricing formulas for exotic interest rate derivatives in the linear-rational Wishart model. The products covered in this paper are mainly the constant maturity swap (CMS), the CMS spread options, the in-arrears swap, and cap. Using the affine property of the Wishart process, we develop an approximation in the spirit of Collin-Dufresne and Goldstein (2002) that is accurate and simple to implement. This technique proves to be rather generic and applies to the CMS and CMS spread options and also to other products. The numerical results, using the calibrated model on actual market data attest to this. Indeed the approximation of the price of CMS and CMS spread options using the approximation formulas proves to be very accurate. Overall, the results underline the ability of the linear-rational model based on the Wishart process's to encompass interest rate dependencies, the pricing of exotics derivatives and also fit well for Collin-Dufresne and Goldstein (2002) type of pricing approximation.

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A Appendix

Table I: Calibrated parameters

α	x_{11}	x_{12}	x_{22}	ω_{11}	ω_{12}	ω_{22}	m_{11}	m_{22}	σ_{11}	σ_{12}	σ_{22}
0.024	0.125	-0.0112	0.00574	0.130	0.00179	0.0466	-0.375	-0.181	0.05	0.024	0.047

Note: The model parameters provided in this table are the result of the model calibration on the EUR data.

Table II: CMS approximation

Maturity 1Y				
Tenor	$M = 3$	$M = 4$	$M = 5$	$M = 7$
1Y	0.0296	0.0606	0.1134	0.0880
5Y	0.0190	0.0190	0.0287	0.0117
7Y	0.0283	0.0018	0.0194	0.0016
Maturity 2Y				
Tenor	$M = 3$	$M = 4$	$M = 5$	$M = 7$
1Y	0.0828	0.1154	0.1951	0.0354
5Y	0.0728	0.0081	0.1570	0.0025
7Y	0.0066	0.0164	0.0100	0.0149
Maturity 3Y				
Tenor	$M = 3$	$M = 4$	$M = 5$	$M = 7$
1Y	0.1574	0.1384	0.0165	0.0412
5Y	0.1452	0.0393	0.0192	0.0188
7Y	0.0336	0.0160	0.0116	0.0661
Maturity 5Y				
Tenor	$M = 3$	$M = 4$	$M = 5$	$M = 7$
1Y	0.1823	0.2030	0.0651	0.0846
5Y	0.0377	0.0645	0.0285	0.0610
7Y	0.0296	0.0160	0.0292	0.0465

Note: Absolute error expressed in % between the approximation $I(M)$ given by Eq. (39) and I given by the expectation on the right hand side of Eq. (14) or Eq. (38) computed using a Monte-Carlo method. The model parameters are those of Table I, the CMS maturity is $T_j = 1Y$, $2Y$, $3Y$ or $5Y$, the swap tenor is equal to $1Y$, $5Y$ or $7Y$ and $\delta_s = \Delta_s = 0.5$. The Monte-Carlo method is based on 50000 paths and a daily discretisation of the time interval.

Table III: CMS spread option approximation

Strike Offset	$M=3$	$M=4$	$M=5$	$M=10$
Maturity 6M				
-100 bps	0.0066	0.0085	0.0083	0.0083
-50 bps	0.0054	0.0035	0.0037	0.0037
0 bps	0.0029	0.0010	0.0011	0.0011
50 bps	0.0060	0.0080	0.0078	0.0078
100 bps	0.0090	0.0069	0.0071	0.0071
Maturity 1Y				
-100 bps	0.0190	0.0237	0.0233	0.0233
-50 bps	0.0070	0.0021	0.0026	0.0026
0 bps	0.0550	0.0500	0.0505	0.0505
50 bps	0.0491	0.0438	0.0443	0.0443
100 bps	0.0241	0.0187	0.0192	0.0192
Maturity 2Y				
-100 bps	0.0735	0.0874	0.0859	0.0860
-50 bps	0.0269	0.0415	0.0399	0.0401
0 bps	0.0577	0.0422	0.0439	0.0437
50 bps	0.0884	0.1048	0.1030	0.1032
100 bps	0.1035	0.1209	0.1190	0.1192
Maturity 3Y				
-100 bps	0.0762	0.0493	0.0524	0.0520
-50 bps	0.1160	0.0873	0.0905	0.0902
0 bps	0.2806	0.3111	0.3077	0.3080
50 bps	0.0301	0.0627	0.0591	0.0595
100 bps	0.2608	0.2954	0.2916	0.2920
Maturity 4Y				
-100 bps	0.2780	0.3169	0.3125	0.3130
-50 bps	0.0675	0.1087	0.1041	0.1046
0 bps	0.2724	0.2286	0.2335	0.2330
50 bps	0.0972	0.1432	0.1381	0.1386
100 bps	0.0426	0.0912	0.0859	0.0865
Maturity 5Y				
-100 bps	0.1103	0.1581	0.1528	0.1533
-50 bps	0.1873	0.1368	0.1424	0.1418
0 bps	0.0911	0.1442	0.1384	0.1390
50 bps	0.9066	0.8502	0.8563	0.8557
100 bps	0.3628	0.3040	0.3102	0.3096

Note: Absolute error expressed in % between the CMS spread option price approximation given by Proposition 3.4 and the Monte-Carlo price given by Proposition 2.6. The two underlying rates are the swaps with tenors 1Y and 5Y with fixed and floating legs such that $\delta_s = \Delta_s = 0.5$. The model parameters are those of Table I, the CMS spread option maturities are $6M$, $1Y$, $2Y$, $3Y$ or $5Y$. The strikes of the options are equal to the at the money strikes plus the strike offset. The Monte-Carlo method is based on 50000 paths and a daily discretisation of the time interval.

Table IV: In-Arrears Swap approximation

	I_1			I_2		
Tenor	$M=3$	$M=4$	$M=5$	$M=3$	$M=4$	$M=5$
1Y	0.2517	0.2647	0.2632	0.0261	0.0127	0.0142
2Y	0.0181	0.0367	0.0344	0.0227	0.0037	0.0061
3Y	0.0302	0.0520	0.0491	0.0090	0.0132	0.0103
5Y	0.0592	0.0351	0.0384	0.0255	0.0011	0.0044

Note: Absolute error expressed in % between the in-arrears swap coupon component I_1, I_2 (given by Eq. (44) Eq. (45)) and the corresponding Monte-Carlo price. The in-arrears Libor coupon dates considered in this analysis are for tenors 1Y, 2Y, 3Y and 5Y with the floating legs such that $\delta = 0.5$. The model parameters are those of Table I. The values for M are 3, 4, and 5 while the Monte-Carlo method used, is based on 50000 paths and a daily discretization of the time interval.

A.1 Proofs

Proof of Proposition 2.3. It is known from Da Fonseca et al. (2022) that the swap rate is given by

$$S_{T_j}^{T_{j,0},T_{j,n_s}} = \frac{P(T_j,T_j) - P(T_j,T_{j,n_s}) + \sum_{l=1}^{n_s} A(T_j,T_{j,l-1},T_{j,l})}{\Delta_s \sum_{k=1}^{m_s} P(T_j,t_{j,k})} \,. \tag{55}$$

The time t-value (with $t \leq T_0$) of the leg that pays the Euribor rate plus a fixed rate K is given by

$$P(t,T_0) - P(t,T_{n_1}) + \sum_{j=1}^{n_1} A(t,T_{j-1},T_j) + \delta K \sum_{j=1}^{n_1} P(t,T_j) \,, \tag{56}$$

while the time t-value (with $t \leq T_0$) of the leg that pays the swap rate is given by

$$\mathbb{E}_t \left[\sum_{j=0}^{n_1-1} \frac{\zeta_{T_{j+1}}}{\zeta_t} \delta S_{T_j}^{T_j,T_{j,n_s}} \right] \,, \tag{57}$$

and taking into account Eq. (55), it leads to evaluate

$$\mathbb{E}_t \left[\frac{\zeta_{T_{j+1}}}{\zeta_t} \frac{P(T_j,T_j) - P(T_j,T_{j,n_s}) + \sum_{l=1}^{n_s} A(T_j,T_{j,l-1},T_{j,l})}{\Delta_s \sum_{k=1}^{m_s} P(t_i,t_{i,k})} \right] . \tag{58}$$

Taking into account Eq. (5), Eq. (12) and $\mathbb{E}_{T_j}\left[e^{-\alpha T_{j+1}} \zeta_{T_{j+1}}\right] = e^{-\alpha T_{j+1}}(b_1(\delta_s) + a_1(\delta_s) x_{11,T_j})$, the above expectation is equal to

$$\frac{e^{-\alpha(T_{j+1}-t)}}{1 + x_{11,t}} \mathbb{E}_t \left[\frac{c_0 + c_1 x_{11,T_j} + c_2 x_{22,T_j} + c_{12} x_{11,T_j} x_{22,T_j} + c_{11} x_{11,T_j}^2}{\mu_0 + \mu_1 x_{11,T_j}} \right] , \tag{59}$$

with

$$c_0 = b_1(\delta_s) \left(b_1(T_j - T_j) - e^{-\alpha(T_{j,n_s} - T_j)} b_1(T_{j,n_s} - T_j) + \sum_{l=1}^{n_s} e^{-\alpha(T_{j,l-1} - T_j)} b_2(T_{j,l-1} - T_j) \right) , \tag{60}$$

$$c_1 = b_1(\delta_s) \left(a_1(T_j - T_j) - e^{-\alpha(T_{j,n_s} - T_j)} a_1(T_{j,n_s} - T_j) \right) + c_0 \frac{a_1(\delta_s)}{b_1(\delta_s)} \,, \tag{61}$$

$$c_2 = b_1(\delta_s) \sum_{l=1}^{n_s} e^{-\alpha(T_{j,l-1} - T_j)} a_2(T_{j,l-1} - T_j) \,, \tag{62}$$

$$c_{12} = a_1(\delta_s) \sum_{l=1}^{n_s} e^{-\alpha(T_{j,l-1} - T_j)} a_2(T_{j,l-1} - T_j) \,, \tag{63}$$

$$c_{11} = a_1(\delta_s) \left(a_1(T_j - T_j) - e^{-\alpha(T_{j,n_s} - T_j)} a_1(T_{j,n_s} - T_j) \right) , \tag{64}$$

$$\mu_0 = \Delta_s \sum_{k=1}^{m_s} e^{-\alpha(t_{j,k} - t_i)} b_1(t_{i,k} - t_i) \,, \tag{65}$$

$$\mu_1 = \Delta_s \sum_{k=1}^{m_s} e^{-\alpha(t_{j,k} - t_i)} a_1(t_{i,k} - t_i) \,, \tag{66}$$

which is the announced result. □

Proof of Proposition 2.5. The expectation Eq. (14) can be expressed

$$I = \left(\frac{c_1}{\mu_1} - \frac{c_{11}}{\mu_1}\frac{\mu_0}{\mu_1}\right) + \frac{c_{11}}{\mu_1}\cdot \mathbb{E}_t\left[x_{11,T_j}\right] + \frac{c_{12}}{\mu_1}\cdot \mathbb{E}_t\left[x_{22,T_j}\right] \tag{67}$$

$$+ \left(c_0 - c_1\frac{\mu_0}{\mu_1} + c_{11}\left(\frac{\mu_0}{\mu_1}\right)^2\right)\cdot \mathbb{E}_t\left[\frac{1}{\mu_0+\mu_1 x_{11,T_j}}\right] \tag{68}$$

$$+ \left(c_2 - c_{12}\frac{\mu_0}{\mu_1}\right)\cdot \mathbb{E}_t\left[\frac{x_{22,T_j}}{\mu_0+\mu_1 x_{11,T_j}}\right], \tag{69}$$

$$= \left(\frac{c_1 - c_{11}}{\mu_1} - \frac{c_{11}}{\mu_1}\frac{\mu_0}{\mu_1}\right) + \frac{c_{11}}{\mu_1}\cdot\left[b_1\left(T_j - t\right) + a_1\left(T_j - t\right)x_{11,t}\right] \tag{70}$$

$$+ \frac{c_{12}}{\mu_1}\cdot\left[b_2\left(T_j - t\right) + a_1\left(T_j - t\right)x_{22,t}\right] \tag{71}$$

$$+ \left(c_0 - c_1\frac{\mu_0}{\mu_1} + c_{11}\left(\frac{\mu_0}{\mu_1}\right)^2\right)\cdot \mathbb{E}_t\left[\frac{1}{\mu_0+\mu_1 x_{11,T_j}}\right] \tag{72}$$

$$+ \left(c_2 - c_{12}\frac{\mu_0}{\mu_1}\right)\cdot \mathbb{E}_t\left[\frac{x_{22,T_j}}{\mu_0+\mu_1 x_{11,T_j}}\right], \tag{73}$$

which leaves us with only two expectations to evaluate. Remark 2.4 allows the computation of the expectations

$$\mathbb{E}_t\left[\frac{1}{\mu_0+\mu_1 x_{11,T_j}}\right] = \int_0^{+\infty} e^{-s\mu_0}\Phi(\tau,\theta_1,0,x_t)ds\,, \tag{74}$$

$$\mathbb{E}_t\left[\frac{x_{22,T_j}}{\mu_0+\mu_1 x_{11,T_j}}\right] = \int_0^{+\infty} e^{-s\mu_0}\partial_z\Phi(\tau,\theta_2,0,x_t)ds|_{z=0}\,, \tag{75}$$

with $\tau = T_j - t$ and

$$\theta_1 = \mu_1 s e_{11}\,, \tag{76}$$

$$\theta_2 = z e_{22} - \mu_1 s e_{11}\,. \tag{77}$$

For Eq. (74), integrating the moment generating function leads to the result. For Eq. (75), one needs to differentiate the moment generating function of the Wishart process which can be explicitly carried out. From the Riccati equation Eq.(11) in Da Fonseca et al. (2022), (if we drop the dependency of the matrices A_{ij} on t), if θ_2 is given by Eq. (77) then the computation of Eq. (75) leads to

$$\frac{d}{dz}a(t,\theta_2,0) = -(\theta_2 A_{12} + A_{22})^{-1}e_{22}A_{12}a(t) + (\theta_2 A_{12} + A_{22})^{-1}e_{22}A_{11}\,. \tag{78}$$

We might denote the above derivative as $d_z a(t,\theta_2,0)$. Under the hypothesis that $\omega = \beta\sigma^2$, consider $z \to \mathrm{tr}\left[\log(\theta_2 A_{12} + A_{22})\right]$, denote $c = \theta_2 A_{12} + A_{22}$ and $l = c - I_n$ then using the Taylor expansions for $\ln(I_n + X)$ and $(I_n + X)^{-1}$ we get $\frac{d}{dz}\mathrm{tr}[\ln c] = \frac{d}{dz}\mathrm{tr}[\ln(I_n + l)] = \mathrm{tr}[\frac{d}{dz}\{l - l^2/2 + \ldots\}] = \mathrm{tr}[\frac{dl}{dz}\{I_n - l + l^2 - \ldots\}] = \mathrm{tr}[\frac{dc}{dz}c^{-1}]$ (thanks to $\mathrm{tr}[\frac{dl}{dz}l] = \mathrm{tr}[l\frac{dl}{dz}]$) and as a result

$$\frac{d}{dz}b(t,\theta_2,0) = -\frac{\beta}{2}\mathrm{tr}\left[e_{22}A_{12}(\theta_2 A_{12} + A_{22})^{-1}\right]\,. \tag{79}$$

We might denote the above derivative as $d_z b(t,\theta_2,0)$. Combining these two derivatives, we get the expression for $\partial_z\Phi(T-t,\theta_2,0,x_t) = \left(\mathrm{tr}[d_z a(t,\theta_2,0)x_t] + d_z b(t,\theta_2,0)\right)\Phi(T-t,\theta_2,0,x_t)$ that is involved in Eq. (75) and when evaluated at $z = 0$ then $\theta_2 = -\mu_1 s e_{11}$ and it leads to:

$$\mathbb{E}_t\left[\frac{x_{22,T_j}}{\mu_0+\mu_1 x_{11,T_j}}\right] = \int_0^{+\infty} e^{-s\mu_0}\left(\mathrm{tr}[d_z a(\tau,\theta_2,0)x_t] + d_z b(\tau,\theta_2,0)\right)\Phi(\tau,\theta_2,0,x_t)ds|_{z=0}\,, \tag{80}$$

with $\tau = T_j - t$. As a result, the expectations Eq. (15) are known up to an integration of dimension one. In the non Bru case (*i.e.*, $\omega \neq \beta\sigma^2$) then $b(T_j - t) = \int_t^{T_j}\mathrm{tr}[\omega a(u)]du$, the derivative of $b(T_j - t)$ can also be computed but it involves another integration. $\square$

Proof of Proposition 2.6. Starting from Eq. (16) and replacing $S_{T_1}^{T_{1,0},T_{1,n_{s_1}}}$ and $S_{T_1}^{T_{1,0},T_{1,n_{s_2}}}$ with Eq. (55) leads to the announced results after performing computations similar to those of Proposition 2.3 but with

$$c_0^i = \left(b_1(T_1 - T_1) - e^{-\alpha(T_{1,n_{s_i}} - T_1)} b_1(T_{1,n_{s_i}} - T_1) + \sum_{l=1}^{n_{s_i}} e^{-\alpha(T_{1,l-1} - T_1)} b_2(T_{1,l-1} - T_1) \right) , \tag{81}$$

$$c_1^i = \left(a_1(T_1 - T_1) - e^{-\alpha(T_{1,n_{s_i}} - T_1)} a_1(T_{1,n_{s_i}} - T_1) \right) + c_0^i \, , \tag{82}$$

$$c_2^i = \sum_{l=1}^{n_{s_i}} e^{-\alpha(T_{1,l-1} - T_1)} a_2(T_{1,l-1} - T_1) \, , \tag{83}$$

$$c_{12}^i = \sum_{l=1}^{n_{s_i}} e^{-\alpha(T_{1,l-1} - T_1)} a_2(T_{1,l-1} - T_1) \, , \tag{84}$$

$$c_{11}^i = \left(a_1(T_1 - T_1) - e^{-\alpha(T_{1,n_{s_i}} - T_1)} a_1(T_{1,n_{s_i}} - T_1) \right) , \tag{85}$$

$$\mu_0^i = \Delta_s \sum_{k=1}^{m_{s_i}} e^{-\alpha(t_{1,k} - t_1)} b_1(t_{1,k} - t_1) \, , \tag{86}$$

$$\mu_1^i = \Delta_s \sum_{k=1}^{m_{s_i}} e^{-\alpha(t_{1,k} - t_1)} a_1(t_{1,k} - t_1) \, . \tag{87}$$

□

Proof of Proposition 3.4. Using the series expansion $1/(1 + (\mu_1/\mu_0)x) = \sum_{i=0}^{+\infty} (-1)^i (\frac{\mu_1}{\mu_0})^i x^i$ and truncating it at the order M gives an approximation of Y_T by Y_T^M defined by

$$Y_T^M = \sum_{m=0}^{M+2} v_m x_{11,T}^m + \sum_{m=0}^{M+1} u_m x_{22,T} x_{11,T}^m \, , \tag{88}$$

where $v_0 = v_0^1 - v_0^2 - K$, $v_1 = v_1^1 - v_1^2 - K$, $\{v_m = v_m^1 - v_m^2 \, , \; m = 2, \ldots, M\}$, $v_{M+1} = v_{M+1}^1 - v_{M+1}^2$, $v_{M+2} = v_{M+1}^1 - v_{M+1}^2$, $u_0 = u_0^1 - u_0^2$, $u_1 = u_1^1 - u_1^2$, $\{u_m = u_m^1 - u_m^2 \, , \; m = 2, \ldots, M\}$, $u_{M+1} = u_{M+1}^1 - u_{M+1}^2$ with for $i \in \{1, 2\}$

$$v_0^i = \frac{c_0^i}{\mu_0^i} - K \, , \tag{89}$$

$$v_1^i = -\frac{\mu_1^i}{\mu_0^i} \frac{c_0^i}{\mu_0^i} + \frac{c_1^i}{\mu_0^i} - K \, , \tag{90}$$

$$v_m^i = (-1)^m \left(\frac{\mu_1^i}{\mu_0^i} \right)^m \frac{c_0^i}{\mu_0^i} + (-1)^{m-1} \left(\frac{\mu_1^i}{\mu_0^i} \right)^{m-1} \frac{c_1^i}{\mu_0^i} + (-1)^{m-2} \left(\frac{\mu_1^i}{\mu_0^i} \right)^{m-2} \frac{c_{11}^i}{\mu_0^i} \, , \; m = 2, \ldots, M \tag{91}$$

$$v_{M+1}^i = (-1)^M \left(\frac{\mu_1^i}{\mu_0^i} \right)^M \frac{c_1^i}{\mu_0^i} + (-1)^{M-1} \left(\frac{\mu_1^i}{\mu_0^i} \right)^{M-1} \frac{c_{11}^i}{\mu_0^i} \, , \tag{92}$$

$$v_{M+2}^i = (-1)^M \left(\frac{\mu_1^i}{\mu_0^i} \right)^M \frac{c_{11}^i}{\mu_0^i} \, , \tag{93}$$

$$u_0^i = \frac{c_2^i}{\mu_0^i} \, , \tag{94}$$

$$u_1^i = (-1) \left(\frac{\mu_1^i}{\mu_0^i} \right) \frac{c_2^i}{\mu_0^i} + \frac{c_{12}^i}{\mu_0^i} \, , \tag{95}$$

$$u_m^i = (-1)^m \left(\frac{\mu_1^i}{\mu_0^i} \right)^m \frac{c_2^i}{\mu_0^i} + (-1)^{m-1} \left(\frac{\mu_1^i}{\mu_0^i} \right)^{m-1} \frac{c_{12}^i}{\mu_0^i} \, , \; m = 2, \ldots, M \tag{96}$$

$$u_{M+1}^i = (-1)^M \left(\frac{\mu_1^i}{\mu_0^i} \right)^M \frac{c_{12}^i}{\mu_0^i} \, . \tag{97}$$

Rewrite Eq. (88) as

$$Y_T^M = \sum_{i=0}^{2M+4} \zeta_i y_i \,, \tag{98}$$

with the first $M+2$ terms given by $v_m x_{11,T}^m$ and the last $M+2$ terms given by $u_m x_{22,T} x_{11,T}^m$. Using the standard multinomial expansion we get

$$(Y_T^M)^q = \left(\sum_{i=0}^{2M+4} \zeta_i y_i \right)^q = \sum_{k_0+\ldots+k_{2M+4}=n} \binom{q}{k_0, \ldots, k_{2M+4}} \prod_{j=0}^{2M+4} (\zeta_j y_i)^{k_j} \ , \tag{99}$$

and therefore taking the expectation of Eq. (99) leads to the announced results. □