Jump Risk: A Cubic-Variation Approach

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Abstract

We identify jump risk in the S&P 500 index by using the sum of cubed returns (i.e., realized cubic variation). We further detect a jump event if the absolute value of the jump risk is higher than a given threshold. Both option-implied and past time-series information are used to forecast the future one-month jump risk and jump events. Our empirical results show that the option-implied information, which is extracted from the VIX and SKEW indices, coupled with past diffusive variance, is more efficient in forecasting future jump risk than is time-series information. We also find that the jump-size risk premium induces a significant negative relation between the realized cubic variation and its risk-neutral expectation. In addition, the past realized variance information outperforms the option-implied information in forecasting future jump events.

1 Introduction

Jump risk is an important risk factor. It signals instability and crashes of the financial markets. However, the identification methods of jump risk are not unified, and the understanding of the dynamics of jumps is far from complete. We propose a simple and intuitive approach to detect jump risk and jump events, and we forecast these in the S&P 500 index (SPX) with option-implied and lagged time-series information.

Jumps play an important role in financial modelling; see, among others, Press (1967), Merton (1976), Jarrow and Rosenfeld (1984), Pan (2002), Yan (2011), Chen, Joslin and Tran (2012) and Branger, Kraft and Meinerding (2016). The discontinuous jumps are different from diffusion in risk management and asset allocation; see, for instance, Liu, Longstaff and Pan (2003) and Liu, Pan and Wang (2005). Therefore, disentangling jumps from diffusion is a necessity for decision makers. In the literature, the jump component can be identified through different methods. Barndorff-Nielsen and Shephard (2004) use the standardized difference between the realized variance and the realized bi-power variation. Jiang and Oomen (2008) use the standardized difference between the swap variance and realized variance. Lee and Mykland (2008) use the standardized logarithmic return. In contrast, we propose a simpler way to detect jumps by using the sum of cubed daily returns, that is, realized cubic variation.[2]

The option market reflects participants' expectations about the underlying stock market, and option-implied volatility has shown forecasting ability superior to that of historical volatility; see, for instance, Christensen and Prabhala (1998) and Jiang

[2]The cubic variation with respect to a given partition is the sum of absolute cubed returns $\sum_{i=1}^{n} |\delta R_{t_i}|^3$ (see Nourdin, 2008), where δR_{t_i} denotes the continuously compounded return. However, in finance, the cubic variation without the absolute value is more meaningful. This is because the limit of its expectation is the third cumulant of the long-horizon returns with independent and identically distributed sub-period returns when the norm of partition approaches zero. Thus, in this paper, we drop the absolute value and use the sum of cubed returns as the realized cubic variation.

and Tian (2005). Since the launch of SKEW in 2011 and VIX in 1993, such public option-implied information has been easily obtainable from the Chicago Board Options Exchange (CBOE) website. SKEW and VIX track the risk-neutral skewness and volatility of the future 30-day returns of SPX, respectively. Zhen and Zhang (2014) show that SKEW contains both jump and variance information. Hence, in this paper, we use the option-implied third cumulant extracted from SKEW and VIX to forecast the future realized cubic variation of the SPX logarithmic returns, and compare its informational efficiency with the past realized cubic variation.[3] The past diffusive variance is used to control the impact of the return-correlation variance embedded in the implied third cumulant. We find that the option-implied information coupled with past diffusive variance is more efficient in forecasting future SPX jump risk, measured by realized cubic variation over a future month using daily returns. We also find that the realized cubic variation is negatively correlated with its risk-neutral expectation. Furthermore, we detect monthly jump events if the realized cubic variation exceeds a given threshold, and explore its property using both option-implied and time-series information. We find that the past realized variance shows the best performance in forecasting future jump events.

Our results shed some light on the specifications of modelling SPX jump risk, including jump size, jump intensity and the existence of jumps in variance. The price-jump-size risk premium explains the negative correlation between the realized cubic variation and its risk-neutral expectation. The price and variance co-jumps make the past diffusive variance contain information for future jump events. Our

[3]The nth cumulant κ_n of a random variable X can be derived through the cumulant-generating function $K(t)$, which is the natural logarithm of the moment-generating function, that is, $K(t) = \log E(e^{tX}) = \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!}$. Only the second and third cumulants are the same as the second and third central moments, respectively.

empirical findings suggest that the jump intensity is time-varying, and there exist variance jumps which happen contemporaneously with price jumps.

The remainder of this paper is organized as follows. Section 2 defines jump risk and proposes a jump-detecting approach. Section 3 shows the properties of the realized cubic variation in a parsimonious jump-diffusion model. Section 4 discusses the empirical methodology. Section 5 presents the results. Section 6 concludes. Proofs are provided in the Appendix.

2 Jump Risk and Jump Events

In the literature, Bandi and Renò (2016) adopt a jump-detecting approach with the same logic as that proposed by Lee and Mykland (2008), Jiang and Yao (2013) use the variance swap approach developed in Jiang and Oomen (2008) for jump identification, and Todorov (2010) applies Barndorff-Nielsen and Shephard's (2006) method to separate the quadratic variation of jumps. These articles either use return or difference of variance measured in different ways to capture jumps. However, in a jump-diffusion setup, both return and variance contain diffusion terms. By raising the power of moments to three, in this paper, we propose a new jump-risk barometer based on the sum of cubed returns, which solely captures the jump term.

Given a partition $\mathcal{P}$: $t = t_0 < t_1 < \cdots < t_n = T$ of the interval $[t, T]$, the annualized realized cubic variation, RCV, is defined as

$$RCV_{t,T} = \frac{1}{T-t}\sum_{i=0}^{n-1}(\delta R_{t_i})^3, \tag{1}$$

where t is the starting time, T is the ending time, n is the number of sub-intervals of the partition, $R_{t_i} = \ln \frac{S_{t_i}}{S_{t_0}}$ denotes the cumulative logarithmic return from time t_0 to time t_i $(i = 1, 2, \cdots, n)$, δ denotes the difference operator in a sub-interval, and δR_{t_i}

denotes the continuously compounded return over $[t_i, t_{i+1}]$.[4] Suppose the partition is equally divided, then the length of each sub-interval is given by $h = \frac{T-t}{n}$. The RCV is a natural filter for jumps, as the diffusion term is of order $h^{3/2}$ and the jump term is of order h in jump-diffusion models, which will be used in the following section. Thus, we use RCV as a measure of jump risk.

We apply our jump risk barometer RCV defined in Equation (1) to the SPX to capture its daily jumps in one month. The daily close SPX data are obtained from the Bloomberg Professional Service. The sample period is from 2 January 1990 to 31 December 2015. Table 1 reports the monthly RCV values that are greater than five basis points ($5/10^4$). The monthly RCV (or the analogous future term RCV^f) is computed with the daily data over the past (future) month (30 calendar days, which comprises 22 trading days at most) on the fourth Wednesday, which has fewest holidays among all weekdays, and the calculation period is matched with the 30-day SKEW and VIX. We choose the third trading day following the monthly option expirations (third Friday) to mitigate the interpolation errors in the 30-day VIX and SKEW indices. If SKEW or VIX is not available on the fourth Wednesday, we first use the following Thursday instead, then the preceding Tuesday and then the nearest trading day when both SKEW and VIX are available.

Furthermore, noting that a jump event is accompanied by an extremely large RCV value in magnitude, we identify the existence of jumps if the absolute value of RCV is greater than five basis points, that is,

$$J_t = \mathbb{1}_{|RCV_t| > 5/10^4}, \tag{2}$$

[4]As the calculation period is fixed in this paper, we omit one subscript, for example, RCV_t or RCV_t^f, where the former is computed with daily returns over the past month and the latter is over the future month.

where $\mathbb{1}$ denotes an indicator function.[5] We determine the cutoff value by the rule of thumb. Suppose that the volatility is 0.2, and the daily increment of a Brownian motion is $\sqrt{1/252}$ and keeps positive (or negative) for the whole year, then $|RCV| = 252 \times (0.2/\sqrt{252})^3 \approx 5/10^4$. However, it is extremely unlikely for the increment of a Brownian motion to stay positive (or negative) for a whole month. Thus, if there is no jumps in price, the RCV is not likely to reach five basis points. After applying our jump-detecting approach, we identify 39 jump months (12.5%) out of 312 total months.

Table 1 reports the jump months with extreme RCV values and the corresponding financial events. We split the jump months into six groups: 1990–1991, 1997–1998, late 1999–2002, 2007–2008, 2011 and 2015, which are early 1990s recession, Asian financial crisis, dot-com bubble burst, 2008 sub-prime mortgage crisis, 2011 stock market fall and 2015 stock market selloff, respectively. Overall, the jump size could be positive or negative. We identified three and four jump months for the first two groups, respectively. There are 13 jumps during the dot-com bubble, including the peak in March 2000, the 9/11 attacks in 2001 and the stock market downturn in 2002. During the 2008 financial crisis, the SPX jumped four times before the consecutive nine-month run of monthly jumps starting from the bankruptcy of large financial institutions, for example, the collapse of Lehman Brothers on 15 September 2008. The SPX also jumped for five consecutive months starting from late July 2011 and only jumped once in 2015.

[5]If RCV is calculated with future returns, then we put a superscript on RCV and J, which are RCV^f and J^f, respectively.

3 A Jump-Diffusion Model

The Poisson mixture of normal distributions has been widely used in the literature to model security prices (see, e.g., Press, 1967; Merton, 1976; Jarrow and Rosenfeld, 1984; Pan, 2002; Yan, 2011; Chen, Joslin and Tran 2012; Branger, Kraft and Meinerding, 2016). In this section, we adopt a jump-diffusion model to explain the meaning of the realized cubic variation RCV defined in Equation (1).

In a model with stochastic jump intensity and co-jumps in prices and variance, under the physical measure P, assume the price of the underlying asset follows a jump diffusion process, and the diffusive variance and jump intensity follow mean-reverting processes modelled by

$$\frac{dS_t}{S_t} = \mu_t dt + \sqrt{v_t} dB_t^S + (e^x - 1)dN_t - E(e^x - 1)\lambda_t dt, \tag{3}$$

$$dv_t = \kappa_1(\theta_{1,t} - v_t)dt + \sigma_v \sqrt{v_t} dB_t^v + y dN_t - \mu_y \lambda_t dt, \tag{4}$$

$$d\lambda_t = \kappa_2(\theta_2 - \lambda_t)dt + \sigma_\lambda \sqrt{\lambda_t} dB_t^\lambda, \tag{5}$$

where S_t is the price of the underlying asset, μ_t is the risky rate of return,κ_1 and κ_2 denote the mean-reverting speeds of the diffusive variance v_t and jump intensity λ_t, respectively, $\theta_{1,t}$ and θ_2 denote their long-term mean levels, respectively, B_t^S and B_t^v denote two Brownian motions with the correlation coefficient $corr(dB_t^S, dB_t^v) = \rho$, N_t is a Poisson counter with stochastic intensity λ_t: $\text{Prob}(dN_t = 1) = \lambda_t dt$, $\text{Prob}(dN_t = 0) = 1 - \lambda_t dt$, and given the arrival of a jump, the price jump size x is normally distributed with mean μ_x and variance σ_x^2, and the variance jump size y is exponentially distributed with mean μ_y. The risky rate of return is given by

$$\mu_t = r + \eta^s v_t + [E(e^x - 1)\lambda_t - E^Q(e^x - 1)\lambda_t^Q], \tag{6}$$

where the superscript Q denotes the risk-neutral measure, E^Q means the expectation in Q-measure, λ_t^Q is the jump intensity in Q-measure, r denotes the risk-free rate, $\eta^s v_t$ is the risk premium associated with the diffusive shock in prices, and $E(e^x - 1)\lambda_t - E^Q(e^x - 1)\lambda_t^Q$ is the risk premium associated with jumps in prices. The long-term mean level of diffusive variance is given by

$$\theta_{1,t} = \theta_1 + \frac{\mu_y}{\kappa_1}(\lambda_t - \lambda_t^Q). \tag{7}$$

For simplicity, we assume that x and y are independent. The discontinuous Poisson-driven jump component is independent of the continuous diffusive terms, and the Brownian motion B_t^λ is independent of B_t^S and B_t^v. The random jump sizes x and y are assumed to be independent of B_t^S, B_t^v, B_t^λ and N_t, and independent across different jump times. The co-jumps in prices and variance have been broadly accepted in the literature (see, e.g., Eraker, Johannes and Polson 2003; Broadie, Chernov and Johannes, 2007; Bandi and Renò, 2016). The model described by Equations (3)-(5) belongs to the affine jump-diffusion class in Duffie, Pan and Singleton (2000). In particular, we set the jump intensity λ_t to be stochastic to accommodate its time-varying feature.

Proposition 1 *Under the model setup described in Equations (3)-(5), as the norm of the partition approaches zero, in P-measure, the time-t conditional expectation of the realized cubic variation defined in Equation (1) is given by*

$$\lim_{||\mathcal{P}||\to 0} E_t(RCV_{t,T}) = E(x^3)\frac{1}{T-t}\int_t^T E_t(\lambda_u)du, \tag{8}$$

where $E(x^3) = 3\mu_x\sigma_x^2 + \mu_x^3$.

Proof. See Appendix A.

To link the physical processes with the risk-neutral ones, we propose a pricing kernel, π, which is a combination of the pricing kernels adopted by Pan (2002) and Liu, Pan and Wang (2005), of the following form:

$$\pi_t = e^{-rt}\pi_t^D \pi_t^J, \tag{9}$$

where π_t^D and π_t^J are given by

$$\pi_t^D = e^{-\int_0^t \zeta_s^{(1)} dW_s^{(1)} - \frac{1}{2}\int_0^t (\zeta_s^{(1)})^2 ds - \int_0^t \zeta_s^{(2)} dW_s^{(2)} - \frac{1}{2}\int_0^t (\zeta_s^{(2)})^2 ds - \int_0^t \zeta_s^{(3)} dW_s^{(3)} - \frac{1}{2}\int_0^t (\zeta_s^{(3)})^2 ds}, \tag{10}$$

$$\pi_t^J = e^{\int_0^t (g+hx-h\mu_x-\frac{1}{2}h^2\sigma_x^2)dN_s - (e^g-1)\int_0^t \lambda_s ds}, \tag{11}$$

$W_t^{(1)}$, $W_t^{(2)}$ and $W_t^{(3)}$ denote three independent Wiener processes, given by

$$W_t^{(1)} = B_t^S, \quad W_t^{(2)} = \frac{B_t^v - \rho B_t^S}{\sqrt{1-\rho^2}}, \quad W_t^{(3)} = B_t^\lambda, \tag{12}$$

the market prices ζ of diffusive shocks in prices, volatility and jump intensity are defined as

$$\zeta_t^{(1)} = \eta^s \sqrt{v_t}, \quad \zeta_t^{(2)} = -\frac{\rho\eta^s + \frac{\eta^v}{\sigma_v}}{\sqrt{1-\rho^2}}\sqrt{v_t}, \quad \zeta_t^{(3)} = -\frac{\eta^\lambda}{\sigma_\lambda}\sqrt{\lambda_t}. \tag{13}$$

Proposition 2 *The state price density π_t defined in Equations (9)-(13) converts the physical processes described by Equations (3)-(5) to the risk-neutral processes, given in the following equations:*

$$\frac{dS_t}{S_t} = rdt + \sqrt{v_t}dB_t^S(Q) + (e^x - 1)dN_t - E^Q(e^x-1)\lambda_t^Q dt, \tag{14}$$

$$dv_t = \kappa_1^Q(\theta_1^Q - v_t)dt + \sigma_v\sqrt{v_t}dB_t^v(Q) + ydN_t - \mu_y\lambda_t^Q dt, \tag{15}$$

$$d\lambda_t^Q = \kappa_2^Q(\theta_2^Q - \lambda_t^Q)dt + \sigma_\lambda^Q\sqrt{\lambda_t^Q}dB_t^\lambda(Q), \tag{16}$$

where $\kappa_1^Q = \kappa_1 - \eta^v$, $\theta_1^Q = \frac{\kappa_1\theta_1}{\kappa_1^Q}$, $\lambda_t^Q = e^g\lambda_t$, $\kappa_2^Q = \kappa_2 - \eta^\lambda$, $\theta_2^Q = e^g\frac{\kappa_2\theta_2}{\kappa_2^Q}$, $\sigma_\lambda^Q = e^{g/2}\sigma_\lambda$. Under the risk-neutral measure Q, the price jump size x is normally distributed with *mean $\mu_x + h\sigma_x^2$ and variance σ_x^2, the distribution of the variance jump size y is not changed, and the three Brownian motions $B_t^S(Q)$, $B_t^v(Q)$ and $B_t^\lambda(Q)$ are given by*

$$dB_t^S(Q) = dW_t^{(1)} + \zeta_t^{(1)} dt, \tag{17}$$

$$dB_t^v(Q) = \rho(dW_t^{(1)} + \zeta_t^{(1)} dt) + \sqrt{1-\rho^2}(dW_t^{(2)} + \zeta_t^{(2)} dt), \tag{18}$$

$$dB_t^\lambda(Q) = dW_t^{(3)} + \zeta_t^{(3)} dt. \tag{19}$$

Proof. See Appendix B.

Proposition 3 *Under the model setup in Equations (14)-(16), when the time to maturity $T-t$ is small and $R_T = \ln \frac{S_T}{S_t}$, the annualized implied third cumulant $ITC_{t,T}$, defined as*

$$ITC_t = \frac{1}{T-t} E_t^Q [R_T - E_t^Q(R_T)]^3, \tag{20}$$

is approximated as shown in the following equation:

$$ITC_{t,T} \approx \int_t^T \frac{E^Q(x^3)E_t^Q(\lambda_u^Q) + 3A_1(\kappa_1^Q, u)[\rho\sigma_v E_t^Q(v_u) + E^Q(xy)E_t^Q(\lambda_u^Q)]}{T-t} du, \tag{21}$$

where $A_1(\kappa_1^Q, u) = \frac{1-e^{-\kappa_1^Q(T-u)}}{\kappa_1^Q}$.

Proof. See Appendix C.[6]

Remark 3.1. Equation (21) shows that the implied third cumulant is mainly determined by three components: the jumps in prices, $\int_t^T \frac{E^Q(x^3)E_t^Q(\lambda_u^Q)}{T-t} du$, the return-correlated variance, $3\rho\sigma_v \int_t^T \frac{A_1(\kappa_1^Q,u)E_t^Q(v_u)}{T-t} du$, and the co-jumps in price and variance, $3E^Q(xy) \int_t^T \frac{A_1(\kappa_1^Q,u)E_t^Q(\lambda_u^Q)}{T-t} du$. The price jump component is also the risk-neutral expectation of the realized cubic variation.

[6] The exact formula is also provided in the Appendix.

Remark 3.2. Since the physical processes Equations (3)-(5) and risk-neutral processes Equations (14)-(16) are of the same structure with different parameters, the risk-neutral expected cubic variation is similar to that in Equation (8). We assume that the daily partition is sufficient to measure the limit of the expected RCV. For brevity, we set

$$\begin{aligned}
&\Psi^P = E(x^3), \quad \Psi^Q = E^Q(x^3), \quad \Phi^Q = \mu_x^Q \mu_y, \quad \mu_x^Q = \mu_x + h\sigma_x^2, \\
&\Lambda_t^P = \frac{1}{T-t}\int_t^T E_t(\lambda_u)du = \omega_1^P \lambda_t + (1-\omega_1^P)\theta_2, \quad \omega_1^P = \frac{1-e^{\kappa_2(T-t)}}{\kappa_2(T-t)}, \\
&\Lambda_t^Q = \frac{1}{T-t}\int_t^T E_t^Q(\lambda_u^Q)du = \omega_1^Q \lambda_t^Q + (1-\omega_1^Q)\theta_2^Q, \quad \omega_1^Q = \frac{1-e^{\kappa_2^Q(T-t)}}{\kappa_2^Q(T-t)}, \\
&\Gamma_t^Q = 3\int_t^T \frac{1-e^{-\kappa_1^Q(T-u)}}{\kappa_1^Q(T-t)} E_t^Q(\lambda_u^Q)du = \omega_2^Q \lambda_t^Q + \omega_3^Q \theta_2^Q, \\
&\omega_2^Q = 3\int_t^T \frac{1-e^{-\kappa_1^Q(T-u)}}{\kappa_1^Q(T-t)} e^{-\kappa_2^Q(u-t)}du, \quad \omega_3^Q = 3\int_t^T \frac{1-e^{-\kappa_1^Q(T-u)}}{\kappa_1^Q(T-t)}(1-e^{-\kappa_2^Q(u-t)})du,
\end{aligned}$$

then the covariance of the realized cubic variation and its risk-neutral expectation is given by

$$cov(RCV_{t,T}^f, \Psi^Q\Lambda_t^Q) = cov(\Psi^P\Lambda_t^P, \Psi^Q\Lambda_t^Q) = \Psi^P\Psi^Q e^g \omega_1^P \omega_1^Q var(\lambda_t),$$

the covariance of the realized cubic variation and the co-jump component in the third cumulant ITC is given by

$$cov(RCV_{t,T}^f, \Phi^Q\Gamma_t^Q) \;=\; cov(\Psi^P\Lambda_t^P, \Phi^Q\Gamma_t^Q) = \Psi^P\Phi^Q e^g \omega_1^P \omega_2^Q var(\lambda_t),$$

where every element in $\{\mu_y, e^g, \omega_1^P, \omega_1^Q, \omega_2^Q, var(\lambda_t)\}$ is positive, $\Psi^P = \mu_x(3\sigma_x^2 + \mu_x^2)$, $\Psi^Q = \mu_x^Q[3(\sigma_x)^2 + (\mu_x^Q)^2]$, and the covariance of the realized cubic variation and the variance component in ITC is zero. Note that $var(\lambda_t)$ is positive only when λ_t is stochastic. Therefore, the sign of the correlation between the future realized cubic variation RCV^f and the implied third cumulant ITC is the same as that of RCV^f and its risk-neutral expectation. Both of them are determined by the signs of the physical and risk-neutral expectations of the price jump size x, which are μ_x and $\mu_x + h\sigma_x^2$, respectively. We define the *price-jump-size risk premium* as the absolute difference of the expected price jump sizes under the physical and risk-neutral measures, that is, $h\sigma_x^2$.

4 Methodology

As option-implied information reflects market participants' expectation about future returns' movements. It is widely believed to be more efficient than historical time-series information in forecasting; see, for instance, Christensen and Prabhala (1998) and Jiang and Tian (2005) on forecasting future realized volatility. Therefore, in this paper, we use option-implied information to forecast the future jump risk, which is measured by the future realized cubic variation.

4.1 Predictor Variables

Zhen and Zhang (2014) show that the CBOE SKEW is a nested outcome of jumps and return-correlated variance and it could be useful in forecasting future jump risk. We extract the annualized option-implied third cumulant, defined in Equation (20), from the CBOE VIX and SKEW indices by using the following approximation formula

$$ITC_t \approx \frac{1}{T-t} \frac{SKEW_{t,T} - 100}{-10} \Big[\Big(\frac{VIX_{t,T}}{100} \Big)^2 (T-t) \Big]^{3/2}, \tag{22}$$

where the risk-neutral variance is approximated by scaled VIX square, that is,

$$RNV_t \approx \Big(\frac{VIX_{t,T}}{100} \Big)^2, \tag{23}$$

time t is the current date, T is the expiry date and the time to maturity $T - t = \frac{1}{12}$ is fixed. The CBOE SKEW is the scaled skewness of the risk-neutral distribution of the SPX log-returns, defined as,

$$SKEW_{t,T} = 100 - 10 \times Sk_{t,T}, \tag{24}$$

where $Sk_{t,T}$ denotes the risk-neutral skewness, which is a dimensionless statistic and given by the third cumulant divided by the variance to the power of $\frac{3}{2}$, that is,

$$Sk_{t,T} = \frac{E_t^Q[R_T - E_t^Q(R_T)]^3}{[E_t^Q[R_T - E_t^Q(R_T)]^2]^{3/2}}. \tag{25}$$

The VIX is defined as

$$VIX_{t,T} = 100\sqrt{\frac{E_t^Q[\sum^{\mathcal{P}} 2(e^{\delta R} - 1 - \delta R)]}{T-t}}, \tag{26}$$

where $\mathcal{P}$ denotes a partition of the interval $[t, T]$, and Q denotes the risk-neutral measure. In Q-measure, Zhang et al. (2016) show that the main contribution of the risk-neutral variance, $E_t^Q[R_T - E_t^Q(R_T)]^2$, where $R_T = \sum^{\mathcal{P}}(\delta R)$, comes from the variance swap rate, $E_t^Q[\sum^{\mathcal{P}}(\delta R)^2]$, which is the risk-neutral expected value of the floating leg of a variance swap. Carr and Wu (2009) use the scaled VIX square to approximate the variance swap rate and state that the approximation error is of the order of cubed returns, which is negligible compared with squared returns. Hence, the difference between the risk-neutral variance and the scaled VIX square is neglected when calculating the option-implied third cumulant.

We obtain the 30-day VIX and SKEW indices data from the CBOE website. Figure 1 shows the dynamics of SK and ITC for the sample period 2 January 1990 to 31 December 2015. As the skewness is erratic over the entire sample period, whereas there are several spikes in ITC, one could not visually tell the 2008 financial crisis from Sk but rather from ITC. These spikes in ITC correspond to the early 1990s recession, 1997-1998 Asian crisis, dot-com bubble, 2008 financial crisis, 2010 flash crash, 2011 stock market fall and 2015 stock market selloff (see Table 2). The large price jumps are usually accompanied with volatility jumps (see Bandi and Renò, 2016). This phenomenon could make the skewness rather small during a financial crisis when the jump components in the third cumulant, which are shown in Remark 4.3.1, is diluted by the jumps in variance, and consequently leads skewness to be a poor indicator of market jump risk. Hence, we choose ITC rather than SK as a predictor variable.

Nevertheless, the implied third cumulant ITC_t in Equation (20) is a noisy predictor for the realized cubic variation, RCV_t, which is caused solely by jumps in prices.[7] To control the impact of the return-correlated diffusive variance, we use the realized bi-power variation proposed by Barndorff-Nielsen and Shephard (2004), given by

$$RBV_t = \frac{\pi}{2(t-s)} \sum_{i=0}^{n-2} |\delta R_{t_i}||\delta R_{t_{i+1}}|, \tag{27}$$

where n is the number of sub-intervals of the partition of $[s, t]$, s is a past time and t is the current time. Barndorff-Nielsen and Shephard (2004) show that the realized bi-power variation RBV_t converges in probability to the annualized integrated diffusive variance and is robust to rare price jumps, whose impact is contaminated in the risk-neutral variance defined in Equation (23) and the realized variance defined as

$$RV_t = \frac{1}{(t-s)} \sum_{i=0}^{n-1} (\delta R_{t_i})^2. \tag{28}$$

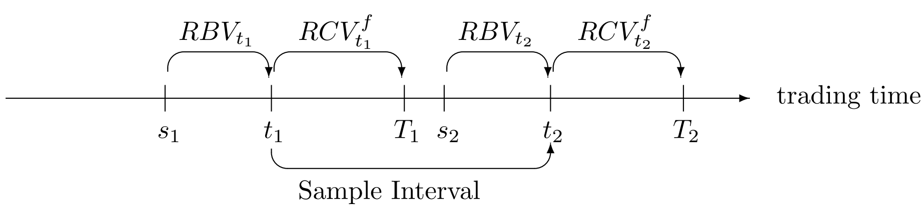
Thus, we use RBV, rather than the other two variance proxies, RNV and RV, as a control variable.

[7]This fact can be seen in Neuberger (2012). For simplicity, we assume that the logarithmic price $\ln S_u$ is a martingale. The third cumulant comprises two components: the sums of cubed returns and the covariances of return and change of conditional variance, that is, $E_t[R_T^3] = E_t[\sum_{u=t}^{T}((\delta R_u)^3 + 3\delta R_u \delta V_u)]$, where $V_u = var_u(R_T)$ is the time-u conditional variance of R_T. The realized variance is also an unbiased estimator of the second cumulant, that is, $E_t[\sum_{u=t}^{T}(\delta R_u)^2] = E_t[R_T^2]$. See Zhen and Zhang (2014) for the case when R_T is not a martingale.

4.2 Regression Analyses

We use the future realized cubic variation RCV^f as a barometer for the market jump risk, as it captures only the jumps in prices. The predictor variables are the implied third cumulant ITC defined in Equation (22) and the realized bi-power variation RBV defined in Equation (27). The calculation timeline for RBV and RCV^f is shown in the following figure. The calculation period for RBV is the past 30 calendar days, whereas RCV^f is calculated with the future 30 calendar days.

This graph shows the calculation timeline for the realized cubic variation RCV_t in Equation (1) and the realized bipower variation RBV_t in Equation (27). Let t_1 and t_2 denote two consecutive monthly observation dates. The realized cubic variation $RCV^f_{t_1}$ ($RCV^f_{t_2}$) is calculated with future logarithmic returns until time T_1 (T_2), and the realized bipower variation RBV_{t_1} (RBV_{t_2}) is constructed with past absolute logarithmic returns starting from time s_1 (s_2).



The following regression is used to investigate the forecasting power of ITC:

$$RCV^f_t = \alpha + \beta_1 ITC_t + \beta_2 RBV_t + \varepsilon_t, \tag{29}$$

where RBV controls for the diffusive variance term embedded in ITC. To deal with the multi-collinearity in Equation (29), we can first regress ITC on RBV, extract the residuals to be $ITCO$ (orthogonal ITC), and then replace ITC with $ITCO$. However, this orthogonalization does not help, as the estimated ITC slope and residuals in Equation (29) are not changed (Mitchell, 1991).[8] For comparison, we also run the

[8]This fact could also be seen in Section 7 in Fama and French (2015).

univariate regression using ITC or RBV. The realized variance (RV) and risk-neutral variance (RNV) are used as alternatives to RBV for a robustness check, and one-month lagged RCV is used to replace ITC to explore the mean-reverting behaviour of the jump risk.

Furthermore, we use a monotonic transform of the implied third cumulant ITC as a predictor variable to forecast the future jump events J^f. The response variable is binary with two outcomes, zero or more than one jump, and the number of jumps is assumed to follow a Poisson distribution; thus we use the binomial regression to forecast the jump event defined in Equation (2) with the complementary log-log link function as follows:

$$P(J_t^f = 1|ITC_t) = \phi_B(\alpha + \beta \ln(|ITC_t|)), \quad \phi_B(x) = 1 - e^{-e^x}, \tag{30}$$

where e^x represents jump intensity. Note that we implicitly assume that the jump intensity is time-varying by using this regression. The difference between our binomial regression and the logistic regression lies in the link function. The latter has been used by Kumar, Moorthy and Perraudin (2003) and Berger and Pukthuanthong (2012). Nevertheless, the complementary log-log link function is more appropriate in this paper as the occurrence of jumps is assumed to be controlled by a Poisson process. The difference between Equations (29) and (30) is that the former focuses on the total jump risk which is a compound outcome of jump size and jump intensity, whereas only the jump intensity matters in the latter regression. We also replace ITC with the realized bi-power variation RBV, the realized variance RV, the risk-neutral variance RNV, the lagged cubic variation RCV and the lagged jump events $1 + J$ to compare their predictive performances.

5 Empirical Results

Table 3 provides the summary statistics of the realized cubic variation (RCV, RCV^f), the implied third cumulant (ITC), the realized bi-power variation (RBV), the realized variance (RV) and the risk-neutral variance (RNV). All of these variables are stationary, as indicated by an augmented Dickey-Fuller test. The statistics of RCV and RCV^f are similar. The average future RCV^f only accounts for 1.4% of the average ITC. The small proportion confirms that ITC contains a diffusive variance term, and also indicates that the price-jump-size risk premium might be large in magnitude. The average RBV of 0.0285 is less than the average RV of 0.0324 because the former is robust to the jumps and simply captures the integrated diffusive variance. The high average RNV of 0.0440 implies the existence of the variance risk premium. The implied third cumulant ITC and RCV^f are also negatively correlated, which indicates that there exists a price-jump-size risk premium. Moreover, ITC, which includes both jumps and variance terms, is negatively correlated with the three different measures of variance (RBV, RV and RNV) due to the leverage effect, which is the negative correlation between return and its diffusive variance.

Table 4 reports the forecasting performance of both univariate and bivariate linear regressions. The baseline regression in Equation (29) shows the highest predictive power with R-squared 25.3%, followed by the univariate regression of one-month lagged RCV with R-squared 9.3%. This result shows that the option-implied information is indeed more efficient than the historical time-series information in forecasting future jump risk, as measured by the future realized cubic variation. We split the sample into two sub-periods, from January 1990 to December 2002, which includes the Asian financial crisis and dot-com bubble, and from January 2003 to December

2015, which includes the 2008 financial crisis. Our empirical results show that the predictive power mainly comes from the second sub-sample with R-squared 38.6%. This result is not surprising, as the 2008 financial crisis comprises nine consecutive jump months in our sample and it is broadly considered to be the worst financial crisis since the Great Depression of the 1930s. As jumps are rare events, we split the sample into only two sub-periods, which comprise 20 and 19 jump months, respectively. The findings regarding informational efficiency do not alter in the second period. However, for the first period, the time-series information shows a mild predictive power with R-squared 3.1%. The univariate regression of ITC exhibits a lower R-squared 1.8%, due to the noisy component caused by the return-correlated variance. The univariate regression of RBV also exhibits a minor predictive power with R-squared 3.7%. In addition, the significantly negative slope of ITC in the baseline regression specified in Equation (29) indicates that there exists a price-jump-size risk premium, which induces the opposite signs of the physical and risk-neutral expected jump sizes. The predicted and observed values in the baseline regression are shown in Figure 2.

Table 5 presents the results of the binomial regressions with the complementary log-log link function. Three pseudo R-squared values, Efron, Macfadden and Adjusted Count, are reported for the goodness-of-fit evaluation. The realized variance (RV) shows the best forecasting performance with Adjusted Count R-squared 51.28%, whereas the one-month lagged jump indicator exhibits the worst performance with zero Adjusted Count R-squared. These results suggest that the past realized variance information outperforms the option-implied information in forecasting future jump events. The realized bi-power variation also exhibit some predictive power with Adjusted Count R-squared 25.64%, which implies the existence of co-jumps in price and variance so that the diffusive variance contains price jump information. Moreover, the significantly positive coefficient β implies that an increase of volatility or absolute cubic variation signals a higher possibility of market jumps, that is, a higher jump intensity. The predicted and observed values in the binomial regression specified in Equation (30) against the predictor $\ln(|ITC|)$ (upper panel) and against the observation date (lower panel) are shown in Figure 3.

6 Conclusion

In this paper, we identify SPX jump risk by the sum of cubed daily returns, that is, realized cubic variation, and forecast it with option-implied third cumulant and lagged time-series information. We find that the option-implied information coupled with past diffusive variance is more efficient in forecasting future jump risk than is the time-series information. The price-jump-size risk premium induces a negative relation between the realized cubic variation and its risk-neutral expectation. Moreover, we detect a jump event when the realized cubic variation is larger than a given threshold in magnitude. Both the time-series and option-implied data have predictive power and the former outperforms the latter in forecasting future jump events. Our empirical findings suggest that the jump intensity is time-varying, and there exist variance jumps which happen contemporaneously with price jumps. Our jump-detecting approach could be easily applied to individual stocks or other financial markets. Further research could look at the intraday pattern of the realized cubic variation.

Appendix

A Proof of Proposition 1

Supposing the partition is equidistant, the unit length is h and the number of observations is n, we split the time-t_i unit logarithmic return into three parts, that is,

$$\delta R_{t_i} = A_{t_i} + M^c_{t_i} + M^d_{t_i},$$

where the drift is $A_{t_i} = \mu_{t_i} h$, and the continuous and discontinuous martingales are respectively given by

$$M^c_{t_i} = \sqrt{v_{t_i}}\delta B^S_{t_i}, \quad \delta B^S_{t_i} = B^S_{t_i+h} - B^S_{t_i},$$
$$M^d_{t_i} = x\delta N_{t_i} - \mu_x \lambda_{t_i} h, \quad \delta N_{t_i} = N_{t_i+h} - N_{t_i}.$$

In P-measure, the time-t_i conditional third cumulants of the terms related with $M^c_{t_i}$ are zero, that is,

$$E_{t_i}[(M^c_{t_i})^3] = E_{t_i}[(M^c_{t_i})^2 M^d_{t_i}] = E_{t_i}[M^c_{t_i}(M^d_{t_i})^2] = 0,$$

and the conditional third cumulant of $\{M^d_{t_i}\}$ is as follows

$$\begin{aligned}
& E_{t_i}[(M^d_{t_i})^3] = E_{t_i}[(x\delta N_{t_i} - \mu_x \lambda_{t_i} h)^3] \\
= \;& E_{t_i} E_{t_i}\{[(x-\mu_x)\delta N_t + \mu_x(\delta N_{t_i} - \lambda_{t_i} h)]^3 | \delta N_{t_i}\} \\
= \;& E_{t_i}\{E_t[(x-\mu_x)^3]\delta N_t + 3\mu_x E_t[(x-\mu_x)^2](\delta N_{t_i} - \lambda_{t_i} h)^2 + \mu_x^3(\delta N_{t_i} - \lambda_{t_i} h)^3\} \\
= \;& 3\sigma_x^2 \mu_x E_{t_i}(\delta N_{t_i} - \lambda_{t_i} h)^2 + \mu_x^3 E_{t_i}(\delta N_{t_i} - \lambda_{t_i} h)^3 = E(x^3)\lambda_{t_i} h.
\end{aligned}$$

Thus, we have

$$E_{t_0}\Big[\sum_{i=0}^{n-1}(\delta R_{t_i})^3\Big] = \sum_{i=0}^{n-1} E_{t_0}\Big[E_{t_i}[(\delta R_{t_i})^3]\Big] = \sum_{i=0}^{n-1} E_{t_0}\Big[E(x^3)\lambda_{t_i} h + O(h^2)\Big],$$

where $O(h^2)$ means in the order of h^2 and it equals $3\mu_{t_i}(v_{t_i}+E(x^2)\lambda_{t_i})h^2+\mu_{t_i}^3 h^3$. As $h\to 0$, by the definition of Riemann integral, we obtain Equation (8). This completes the proof.

B Proof of Proposition 2

Given a state price density

$$\pi_t = e^{-rt}\exp\Big(-\int_0^t \zeta_s dW_s - \frac{1}{2}\int_0^t \zeta_s^2 ds\Big),$$

the corresponding Radon-Nikodym derivative is

$$\Big(\frac{dQ}{dP}\Big)_t = \exp\Big(-\int_0^t \zeta_s dW_s - \frac{1}{2}\int_0^t \zeta_s^2 ds\Big),$$

where W_s is a 1-dimensional Brownian motion, Q denotes the risk-neutral measure and P denotes the physical measure. Define $B_t(Q)=W_t+\int_0^t \zeta_s ds$. Calculating its moment-generating function in risk-neutral measure gives

$$\begin{aligned} E^Q[e^{cB_t(Q)}] &= E^P\Big[\exp\Big(c(W_t+\int_0^t \zeta_s ds)\Big)\exp\Big(-\int_0^t \zeta_s dW_s - \frac{1}{2}\int_0^t \zeta_s^2 ds\Big)\Big] \\ &= E^P\Big[\exp\Big(\int_0^t (c-\zeta_s)dW_s + c\int_0^t \zeta_s ds - \frac{1}{2}\int_0^t \zeta_s^2 ds\Big)\Big] = \exp\Big(\frac{1}{2}c^2 t\Big). \end{aligned}$$

Therefore, $B_t(Q)$ is a Brownian motion under the risk-neutral measure. It is straightforward to generalize the proof to the case of state price density with n-dimensional independent Brownian motions. Rewriting the three risk-neutral Brownian motions $B_t^S(Q)$, $B_t^V(Q)$ and $B_t^\Lambda(Q)$ in a differentiation form gives Equations (17)-(19).

Under the risk-neutral measure, the jump intensity of the Poisson counter N_t is $e^g\lambda_t$ and the jump size $x\sim N(\mu_x+h\sigma_x^2,\sigma_x^2)$. This can be seen through the following conditional moment generating function of the compound Poisson process $\int_0^t x dN_u$,

$$
\begin{aligned}
& E^Q[e^{c\int_0^t x dN_u}|\lambda_{u\in[0,t]}] \\
= \; & E^P\Big[\exp\Big(\int_0^t [g+(c+h)x - h\mu_x - \frac{1}{2}h^2\sigma_x^2]dN_u - (e^g-1)\int_0^t \lambda_u du\Big)\Big|\lambda_{u\in[0,t]}\Big] \\
= \; & \exp\Big((e^{c(\mu_x+h\sigma_x^2)+\frac{1}{2}c^2\sigma_x^2}-1)\int_0^t e^g\lambda_u du\Big) = \exp\Big((\psi_x^Q(c)-1)\int_0^t \lambda_u^Q du\Big),
\end{aligned}
$$

where $\psi_x^Q(c)$ denotes the moment generating function of x and λ_u^Q denotes the jump intensity at time u under Q-measure. This completes the proof.

C Proof of Proposition 3

Applying Ito's Lemma to Equation (14) gives

$$
d\ln S_t = \Big[r - \frac{1}{2}v_t - \lambda_t^Q E^Q(e^x - 1 - x)\Big]dt + \sqrt{v_t}dB_t^S(Q) + xdN_t - \lambda_t^Q\mu_x^Q dt. \quad (31)
$$

The long-horizon return from current time t, to a future time, T, is defined by

$$
R_t^T \equiv \ln\frac{S_T}{S_t} = \int_t^T \Big[r - \frac{1}{2}v_u - \lambda_u^Q\mu_{f(x)}^Q\Big]du + \sqrt{v_u}dB_u^S(Q) + xdN_u - \lambda_u^Q\mu_x^Q du, \quad (32)
$$

where $f(x) = e^x - 1 - x$ and $\mu_{f(x)}^Q$ denotes its mean under Q-measure. The conditional expectation at time t is then given by

$$
E_t^Q(R_t^T) = \int_t^T \Big[r - \frac{1}{2}E_t^Q(v_u) - E_t^Q(\lambda_u^Q)\mu_{f(x)}^Q\Big]du. \quad (33)
$$

Adopting the notations in Zhang et al. (2016), we have

$$
X_T \equiv \int_t^T \sqrt{v_u}dB_u^S(Q), \qquad Y_T \equiv \int_t^T [v_u - E_t^Q(v_u)]du.
$$

and we introduce another two integrals

$$
I_T \equiv \int_t^T (xdN_u - \mu_x^Q\lambda_u^Q du), \qquad Z_T \equiv \int_t^T [\lambda_u^Q - E_t^Q(\lambda_u^Q)]du.
$$

With those notations as well as X_T and Y_T, subtracting (33) from (32) yields

$$R_t^T - E_t^Q(R_t^T) = X_T - \frac{1}{2}Y_T + I_T - \mu^Q_{f(x)} Z_T. \tag{34}$$

The diffusive variance is modelled by

$$dv_t = \kappa_1^Q(\theta_1^Q - v_t)dt + \sigma_v\sqrt{v_t}dB_t^v(Q) + ydN_t - \mu_y\lambda_t^Q dt, \tag{35}$$

where $B_t^v(Q)$ and $B_t^S(Q)$ have correlation coefficient ρ. Converting Equation (35) into the stochastic integral form yields

$$v_s = E_t^Q(v_s) + \sigma_v\int_t^s e^{-\kappa_1^Q(s-u)}\sqrt{v_u}dB_u^v(Q) + \int_t^s e^{-\kappa_1^Q(s-u)}(ydN_u - \mu_y\lambda_u^Q du)\tag{36}$$

where $E_t^Q(v_s) = \theta_1^Q + (v_t - \theta_1^Q)e^{-\kappa_1^Q(s-t)}$ is the conditional expected diffusive variance. Substituting Equation (36) and interchanging the order of integrations gives

$$Y_T = \sigma_v\int_t^T A_1(\kappa_1^Q, u)\sqrt{v_u}dB_u^v + \int_t^T A_1(\kappa_1^Q, u)(ydN_u - \mu_y\lambda_u^Q du),$$

where $A_1(\kappa_1^Q, u) = \frac{1-e^{-\kappa_1^Q(T-u)}}{\kappa_1^Q}$. We introduce new martingale processes, Y_s^H and Y_s^J, as follows

$$Y_s^H \equiv \sigma_v\int_t^s A_1(\kappa_1^Q, u)\sqrt{v_u}dB_u^v, \quad Y_s^J \equiv \int_t^s A_1(\kappa_1^Q, u)(ydN_u - \mu_y\lambda_u^Q du),$$

and at time T, we have $Y_T = Y_T^H + Y_T^J$. Hence, Equation (34) becomes

$$R_t^T - E_t^Q(R_t^T) = X_T - \frac{1}{2}Y_T^H + I_T - \frac{1}{2}Y_T^J - \mu^Q_{f(x)} Z_T. \tag{37}$$

The jump intensity follows a square root process modelled by

$$d\lambda_t^Q = \kappa_2^Q(\theta_2^Q - \lambda_t^Q)dt + \sigma_\lambda^Q\sqrt{\lambda_t^Q}dB_t^\lambda(Q),$$

where $B_t^\lambda(Q)$ is independent of $B_t^S(Q)$, $B_t^v(Q)$ and N_t. Similarly, we obtain

$$\lambda_s^Q = E_t^Q(\lambda_s^Q) + \sigma_\lambda^Q\int_t^s e^{-\kappa_2^Q(s-u)}\sqrt{\lambda_u^Q}dB_u^\lambda(Q),$$

$$Z_T = \sigma_\lambda^Q\int_t^T A_1(\kappa_2^Q, u)\sqrt{\lambda_u^Q}dB_t^\lambda(Q),$$

where $E_t^Q(\lambda_s^Q) = \theta_2^Q + (\lambda_t^Q - \theta_2^Q)e^{-\kappa_2^Q(s-t)}$ is the conditional expected jump intensity. Noting the independency of $\{B_t^S(Q), B_t^v(Q)\}$ and $\{B_t^\lambda(Q), N_t\}$, the co-third cumulant $E_t^Q[(X_T - \frac{1}{2}Y_T^H)(I_T - \frac{1}{2}Y_T^J - \mu_{f(x)}^Q Z_t)^2]$ is zero. Hence, the third cumulant of R_t^T is as follows

$$E_t^Q[R_t^T - E_t^Q(R_t^T)]^3 = TC^H + TC^J + TC^C, \tag{38}$$

where TC^H, TC^J and TC^C denote the third cumulant of that in the Heston model, the third cumulant of jump terms and the co-third cumulant of diffusion and jump terms, respectively, that is,

$$\begin{aligned} TC^H &= E_t^Q\Big[\Big(X_T - \frac{1}{2}Y_T^H\Big)^3\Big], \\ TC^J &= E_t^Q\Big[\Big(I_T - \frac{1}{2}Y_T^J - \mu_{f(x)}^Q Z_t\Big)^3\Big], \\ TC^C &= 3E_t^Q\Big[\Big(X_T - \frac{1}{2}Y_T^H\Big)^2\Big(I_T - \frac{1}{2}Y_T^J - \mu_{f(x)}^Q Z_t\Big)\Big]. \end{aligned}$$

The Heston-type third cumulant TC^H is given by Zhang et al. (2016), as follows

$$TC^H = E_t^Q[X_T^3] - \frac{3}{2}E_t^Q[X_T^2 Y_T^H] + \frac{3}{4}E_t^Q[X_T(Y_T^H)^2] - \frac{1}{8}E_t^Q[(Y_T^H)^3], \tag{39}$$

where the third and co-third cumulants of X_T and Y_T^H are given by

$$E_t^Q[X_T^3] = 3\rho\sigma_v \int_t^T A_1(\kappa_1^Q, u)\ E_t^Q(v_u)du, \tag{40}$$

$$E_t^Q[X_T^2 Y_T^H] = \sigma_v^2 \int_t^T A_2(\kappa_1^Q, u)\ E_t^Q(v_u)du, \tag{41}$$

$$E_t^Q[X_T(Y_T^H)^2] = \rho\sigma_v^3 \int_t^T A_3(\kappa_1^Q, u)\ E_t^Q(v_u)du, \tag{42}$$

$$E_t^Q[(Y_T^H)^3] = 3\sigma_v^4 \int_t^T A_4(\kappa_1^Q, u)\ E_t^Q(v_u)du, \tag{43}$$

where the weights are given by

$$\begin{aligned}
A_1(\kappa_1^Q, u) &= \frac{1 - e^{-\kappa_1^Q \tau^*}}{\kappa_1^Q}, \qquad \tau^* = T - u, \\
A_2(\kappa_1^Q, u) &= \Big(\frac{1 - e^{-\kappa_1^Q \tau^*}}{\kappa_1^Q}\Big)^2 + 2\rho^2 \frac{1 - e^{-\kappa_1^Q \tau^*} - \kappa_1^Q \tau^* e^{-\kappa_1^Q \tau^*}}{(\kappa_1^Q)^2}, \\
A_3(\kappa_1^Q, u) &= 2\frac{1 - e^{-\kappa_1^Q \tau^*} - \kappa_1^Q \tau^* e^{-\kappa_1^Q \tau^*}}{(\kappa_1^Q)^2} \frac{1 - e^{-\kappa_1^Q \tau^*}}{\kappa_1^Q} + \frac{1 - e^{-2\kappa_1^Q \tau^*} - 2\kappa_1^Q \tau^* e^{-\kappa_1^Q \tau^*}}{(\kappa_1^Q)^3}, \\
A_4(\kappa_1^Q, u) &= \frac{1 - e^{-2\kappa_1^Q \tau^*} - 2\kappa_1^Q \tau^* e^{-\kappa_1^Q \tau^*}}{(\kappa_1^Q)^3} \frac{1 - e^{-\kappa_1^Q \tau^*}}{\kappa_1^Q}.
\end{aligned}$$

The co-third cumulant $E_t^Q[(I_T - \frac{1}{2}Y_T^J)Z_T^2] = E_t^Q[Z_T^2 E_t^Q[(I_T - \frac{1}{2}Y_T^J)|\lambda_{u\in[t,T]}]]$ is zero. Therefore, the third cumulant of jump terms is given by

$$\begin{aligned}
TC^J &= E_t^Q[I_T^3] - \frac{3}{2}E_t^Q[I_T^2 Y_T^J] + \frac{3}{4}E_t^Q[I_T (Y_T^J)^2] - \frac{1}{8}E_t^Q[(Y_T^J)^3] \\
&\quad -3\mu_{f(x)}^Q E_t^Q\Big[\Big(I_T - \frac{1}{2}Y_T^J\Big)^2 Z_T\Big] - (\mu_{f(x)}^Q)^3 E_t^Q[Z_T^3], \qquad (44)
\end{aligned}$$

where the (co-) variance and (co-) third cumulants of I_T and Y_T^J are given by

$$E_t^Q(I_T^3) = E^Q(x^3) \int_t^T E_t^Q(\lambda_u^Q) du, \qquad (45)$$

$$E_t^Q[I_T^2 Y_T^J] = E^Q(x^2 y) \int_t^T A_1(\kappa_1^Q, u) E_t^Q(\lambda_u^Q) du, \qquad (46)$$

$$E_t^Q[I_T (Y_T^J)^2] = E^Q(xy^2) \int_t^T A_1(\kappa_1^Q, u)^2 E_t^Q(\lambda_u^Q) du, \qquad (47)$$

$$E_t^Q[(Y_T^J)^3] = E^Q(y^3) \int_t^T A_1(\kappa_1^Q, u)^3 E_t^Q(\lambda_u^Q) du. \qquad (48)$$

We obtain a similar result for $E_t^Q(Z_T^3)$ to that in the Heston model, that is, the third cumulant of Z_T is given by

$$E_t(Z_T^3) = 3(\sigma_\lambda^Q)^4 \int_t^T A_4(\kappa_2^Q, u)\ E_t^Q(\lambda_u^Q) du, \qquad (49)$$

and the co-third cumulant of $I_T - \frac{1}{2}Y_T^J$ and Z_t is given by

$$\begin{aligned} & E_t^Q\Big[\Big(I_T - \frac{1}{2}Y_T^J\Big)^2 Z_T\Big] = E_t^Q\Big[Z_T E_t^Q\Big[\Big(I_T - \frac{1}{2}Y_T^J\Big)^2\Big|\lambda_{u\in[t,T]}\Big]\Big] \\ = & \; E_t^Q\Big[\int_t^T (\lambda_u^Q - E_t^Q(\lambda_u^Q))du \int_t^T E^Q\Big[\Big(x - \frac{1}{2}yA_1(\kappa_1^Q, s)\Big)^2\Big]\lambda_s^Q ds\Big] \\ = & \; E_t^Q\Big[\sigma_\lambda^Q \int_t^T A_1(\kappa_2^Q, u)\sqrt{\lambda_u^Q} dB_u^\lambda(Q) \\ & \times \int_t^T E^Q\Big[\Big(x - \frac{1}{2}yA_1(\kappa_1^Q, s)\Big)^2\Big]\sigma_\lambda^Q \int_t^s e^{-\kappa_2^Q(s-u)}\sqrt{\lambda_u^Q} dB_u^\lambda(Q) ds\Big] \\ = & \; (\sigma_\lambda^Q)^2 \int_t^T A_1(\kappa_2^Q, u) \int_u^T E^Q\Big[\Big(x - \frac{1}{2}yA_1(\kappa_1^Q, s)\Big)^2\Big] e^{-\kappa_2^Q(s-u)} ds E_t^Q(\lambda_u^Q) du. \end{aligned} \tag{50}$$

Given that $B_t^\lambda(Q)$ is independent of $\{B_t^v(Q), N_t\}$, $E_t^Q[(X_T - \frac{1}{2}Y_T^H)^2 Z_t]$ is zero. Thus, the co-third cumulant of diffusion and jump terms is given by

$$\begin{aligned} TC^C \; = \; & 3E_t^Q[X_T^2 I_T] - 3E_t^Q[X_T Y_T^H I_T] + \frac{3}{4}E_t^Q[(Y_T^H)^2 I_T] \\ & -\frac{3}{2}E_t^Q[X_T^2 Y_T^J] + \frac{3}{2}E_t^Q[X_T Y_T^H Y_T^J] - \frac{3}{8}E_t^Q[(Y_T^H)^2 Y_T^J], \end{aligned} \tag{51}$$

where the co-third cumulants of X_T, Y_T^H and I_T or Y_T^J are given by

$$E_t^Q[X_T^2 I_T] = E^Q(xy) \int_t^T A_1(\kappa_1^Q, u) E_t^Q(\lambda_u^Q) du, \tag{52}$$

$$E_t^Q[X_T Y_T^H I_T] = \rho\sigma_v E^Q(xy) \int_t^T A_5(\kappa_1^Q, u) E_t^Q(\lambda_u^Q) du, \tag{53}$$

$$E_t^Q[(Y_T^H)^2 I_T] = \sigma_v^2 E^Q(xy) \int_t^T A_6(\kappa_1^Q, u) E_t^Q(\lambda_u^Q) du, \tag{54}$$

$$E_t^Q[X_T^2 Y_T^J] = E^Q(y^2) \int_t^T A_1(\kappa_1^Q, u)^2 E_t^Q(\lambda_u) du, \tag{55}$$

$$E_t^Q[X_T Y_T^H Y_T^J] = \rho\sigma_v E^Q(y^2) \int_t^T A_7(\kappa_1^Q, u) E_t^Q(\lambda_u^Q) du, \tag{56}$$

$$E_t^Q[(Y_T^H)^2 Y_T^J] = \sigma_v^2 E^Q(y^2) \int_t^T A_4(\kappa_1^Q, u) E_t^Q(\lambda_u^Q) du, \tag{57}$$

and the weights are given by

$$
\begin{aligned}
A_5(\kappa_1^Q, u) &= \frac{1 - e^{-\kappa_1^Q \tau^*} - \kappa_1^Q \tau^* e^{-\kappa_1^Q \tau^*}}{(\kappa_1^Q)^2}, \\
A_6(\kappa_1^Q, u) &= \frac{1 - e^{-2\kappa_1^Q \tau^*} - 2\kappa_1^Q \tau^* e^{-\kappa_1^Q \tau^*}}{(\kappa_1^Q)^3}, \\
A_7(\kappa_1^Q, u) &= \frac{1 - e^{-\kappa_1^Q \tau^*} - \kappa_1^Q \tau^* e^{-\kappa_1^Q \tau^*}}{(\kappa_1^Q)^2} \frac{1 - e^{-\kappa_1^Q \tau^*}}{\kappa_1^Q}.
\end{aligned}
$$

Noting that $E_t^Q[X_T^3]$, $E_t^Q(I_T^3)$ and $3E_t^Q[X_T^2 I_T]$ are the main terms, and all the other terms are related with drift in Equation (31), which is negligible for short-term returns, we obtain the approximate formula in Equation (21). This completes the proof.

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Table 1: Monthly Jump Risk

This table shows the monthly future jump risk, which is measured by the realized cubic variation, RCV^f, calculated with daily logarithmic returns over one future month (30 calendar days) starting from the fourth Wednesday (Data: January 1990 to December 2015). The calculation period is matched with the 30-day SKEW and VIX. If SKEW or VIX is not available on the fourth Wednesday, we first use the following Thursday instead, then the preceding Tuesday and then the nearest trading day when both SKEW and VIX are available. We report the starting date of the sample month when the absolute value of RCV^f is greater than five basis points (0.0005), and the corresponding financial events.

Starting Date	$RCV^f * 10^3$	Events
25-Jul-1990	-1.0848	
26-Dec-1990	0.5169	Early 1990s Recession
23-Oct-1991	-0.6255	
27-Aug-1997	0.5576	
22-Oct-1997	-2.6807	1997-98 Asian Financial Crash
22-Jul-1998	-0.6891	
26-Aug-1998	-2.5441	
27-Oct-1999	0.6269	
26-Jan-2000	-0.6422	
23-Feb-2000	1.6277	(Dot-com Bubble Peak)
22-Mar-2000	-2.0994	
24-May-2000	0.5274	
27-Dec-2000	0.9524	
28-Feb-2001	-1.5214	Dot-Com Bubble
28-Mar-2001	1.3092	
22-Aug-2001	-2.3063	(September 11 attacks)
24-Apr-2002	0.5164	
24-Jul-2002	2.2815	
28-Aug-2002	-1.7283	(2002 Stock Market Downturn)
25-Sep-2002	2.1081	
26-Dec-2007	-0.7618	
27-Feb-2008	0.8634	
26-Mar-2008	0.5968	
28-May-2008	-0.7273	
27-Aug-2008	-2.3516	(Lehman Brothers Bankruptcy)
24-Sep-2008	-13.7539	
22-Oct-2008	5.9481	2007-08 Financial Crisis
26-Nov-2008	-7.1596	
24-Dec-2008	-1.5327	
28-Jan-2009	-3.1088	
25-Feb-2009	5.3737	
25-Mar-2009	-0.5206	
22-Apr-2009	0.7018	
27-Jul-2011	-4.9485	
24-Aug-2011	-0.5853	
28-Sep-2011	0.7754	2011 Stock Market Fall
26-Oct-2011	-0.7287	
23-Nov-2011	1.3889	
22-Jul-2015	-0.5125	2015 Stock Market Selloff

Table 2: Implied Third Cumulant

This table shows the extreme values of the implied third cumulant, ITC (Data: January 1990 to December 2015 from the CBOE website). The monthly sample is selected on the fourth Wednesday with several exceptions. If SKEW or VIX is not available on the fourth Wednesday, we first use the following Thursday instead, then the preceding Tuesday and then the nearest trading day when both SKEW and VIX are available. We report the date when the value of ITC is less than -0.01, and the corresponding financial events.

Starting Date	ITC	Events
24-Jan-1990	-0.0136	Early 1990s Recession
22-Aug-1990	-0.0103	
24-Oct-1990	-0.0106	
26-Dec-1990	-0.0111	
25-Nov-1997	-0.0172	1997-98 Asian Financial Crash
24-Dec-1997	-0.0140	
26-Aug-1998	-0.0219	
23-Sep-1998	-0.0226	
28-Oct-1998	-0.0249	
27-Jan-1999	-0.0123	
26-May-1999	-0.0110	
22-Nov-2000	-0.0103	Dot-com Bubble
26-Sep-2001	-0.0183	
24-Oct-2001	-0.0171	
24-Jul-2002	-0.0185	
28-Aug-2002	-0.0163	
25-Sep-2002	-0.0165	
23-Oct-2002	-0.0149	
24-Sep-2008	-0.0146	2008 Financial Crisis
22-Oct-2008	-0.1886	
26-Nov-2008	-0.0636	
24-Dec-2008	-0.0307	
28-Jan-2009	-0.0403	
25-Feb-2009	-0.0385	
25-Mar-2009	-0.0310	
22-Apr-2009	-0.0203	
27-May-2009	-0.0196	
24-Jun-2009	-0.0120	
28-Oct-2009	-0.0115	
26-May-2010	-0.0266	2010 Flash Crash
23-Jun-2010	-0.0112	
24-Aug-2011	-0.0312	2011 stock market fall
28-Sep-2011	-0.0364	
26-Oct-2011	-0.0174	
23-Nov-2011	-0.0190	
26-Aug-2015	-0.0196	2015 stock market selloff
23-Sep-2015	-0.0115	

Table 3: Summary Statistics

This table presents the summary statistics of the realized cubic variation (RCV), the implied third cumulant (ITC), the realized bi-power variation (RBV), the realized variance (RV) and the risk-neutral variance (RNV). The realized variables, RCV (or RCV^f), RV and RBV, are computed using the past (or future) S&P 500 daily logarithmic returns over 30 calendar days. The sample period is from from January 1990 to December 2015 (312 months), and the monthly sample points are on the fourth Wednesday with several exceptions. If SKEW or VIX is not available on the fourth Wednesday, we first use the following Thursday instead, then the preceding Tuesday and then the nearest trading day when both SKEW and VIX are available. The asterisk * indicates that the correlation coefficient is significantly different from zero at the 1% level.

	Mean	Std Dev	Skewness	Kurtosis	Min	Max
RCV	-0.00005	0.0012	-2.6807	72.2281	-0.0133	0.0103
RCV^f	-0.00008	0.0011	-5.8730	77.1740	-0.0138	0.0059
ITC	-0.00576	0.0125	-10.6033	147.8363	-0.1886	-0.0004
RBV	0.02854	0.0507	6.7867	64.7378	0.0025	0.5983
RV	0.03238	0.0586	7.2150	69.0014	0.0026	0.6631
RNV	0.04396	0.0435	4.8124	40.5915	0.0087	0.4851

Correlation	RCV	RCV^f	ITC	RBV	RV	RNV
RCV	1.0000	-0.3093*	0.4766*	-0.0768	-0.2280*	-0.3154*
RCV^f		1.0000	-0.1455*	-0.1997*	-0.0717	0.0353
ITC			1.0000	-0.7682*	-0.8800*	-0.9085*
RBV				1.0000	0.9675*	0.8551*
RV					1.0000	0.9033*
RNV						1.0000

Table 4: Forecasting Jump Risk

We use the implied third cumulant (ITC) and realized bi-power variation (RBV) to forecast future realized cubic variation (RCV^f). The regression is specified by

$$RCV_t^f = \alpha + \beta_1 ITC_t + \beta_2 RBV_t + \varepsilon_t.$$

The realized variance (RV) and risk-neutral variance (RNV) are used as alternatives to RBV for a robustness check, and one-month lagged RCV is used to replace ITC to explore the mean-reverting behaviour of the jump risk. We choose the monthly sample on the fourth Wednesday. If SKEW or VIX is not available on the fourth Wednesday, we first use the following Thursday instead, then the preceding Tuesday and then the nearest trading day when both SKEW and VIX are available. We report the results for the full sample, FS: January 1990 - December 2015, and two equally-divided sub-samples, S1: January 1990 - December 2002 and S2: January 2003 - December 2015. The t-statistics, which are reported in parentheses, are adjusted for heteroscedasticity and serial correlation.

	$\alpha * 10^3$		$\beta_1 * 10^3$		$\beta_2 * 10^3$		Adj R^2
FS ITC	-0.156	(-1.904)	-13.236	(-2.607)			0.018
FS RBV	0.049	(1.209)			-4.494	(-2.189)	0.037
FS RV	-0.035	(-0.702)			-1.397	(-0.946)	0.002
FS RNV	-0.120	(-1.710)			0.926	(0.544)	-0.002
FS RCV	-0.093	(-1.353)	-293.975	(-3.309)			0.093
S1 RCV	-0.018	(-0.443)	-228.400	(-1.851)			0.031
S2 RCV	-0.170	(-1.331)	-302.970	(-3.096)			0.101
FS ITC+RBV	0.026	(0.651)	-66.345	(-4.153)	-17.103	(-3.082)	0.253
FS ITC+RV	-0.006	(-0.123)	-84.125	(-2.440)	-17.255	(-2.101)	0.193
FS ITC+RNV	0.220	(0.734)	-59.083	(-1.557)	-14.565	(-1.101)	0.069
S1 ITC+RBV	-0.068	(-1.468)	2.605	(0.114)	2.425	(0.782)	-0.001
S2 ITC+RBV	0.022	(0.429)	-82.799	(-5.191)	-22.657	(-3.935)	0.386

Table 5: Forecasting Jump Events

We use the implied third cumulant $\ln(|ITC|)$ to forecast future jump events J^f. The regression is specified by

$$P(J_t^f = 1|ITC_t) = \phi_B(\alpha + \beta \ln(|ITC_t|)), \quad \phi_B(x) = 1 - e^{-e^x}.$$

The realized bi-power variation $\ln(RBV)$, the realized variance $\ln(RV)$, the risk-neutral variance $\ln(RNV)$, the lagged cubic variation $\ln(|RCV|)$ and the lagged jump events $\ln(1 + J)$ are are used as alternatives to $\ln(|ITC|)$ for comprison. We choose the monthly sample on the fourth Wednesday. If SKEW or VIX is not available on the fourth Wednesday, we first use the following Thursday instead, then the preceding Tuesday and then the nearest trading day when both SKEW and VIX are available. We report three pseudo R-squareds: Efron, Macfadden and Adjusted Count. The t-statistics are reported in parentheses.

	α		β		Pseudo R^2 Efron	Macfadden	Adj Count
ITC	4.3773	(5.1797)	1.2069	(6.9359)	0.2368	0.2370	0.1795
RBV	3.0572	(5.0273)	1.4051	(7.4863)	0.3046	0.3025	0.2564
RV	5.9005	(6.2087)	2.3952	(7.4359)	0.5318	0.5388	0.5128
RNV	4.2761	(5.5635)	2.0822	(7.3663)	0.2878	0.2942	0.2821
RCV	4.3703	(5.0363)	0.7346	(6.7920)	0.2127	0.2117	0.0513
J	-2.5214	(-11.5513)	2.8954	(6.1972)	0.1448	0.1367	0.0000

Figure 1: Implied Third Cumulant and Skewness

This graph shows the evolutions of the option-implied third cumulant (ITC magnified by 10) and skewness (Sk) from 02 January 1990 to 31 December 2015. The solid line represents ITC multiplied by 10, and the dotted line represents Sk.

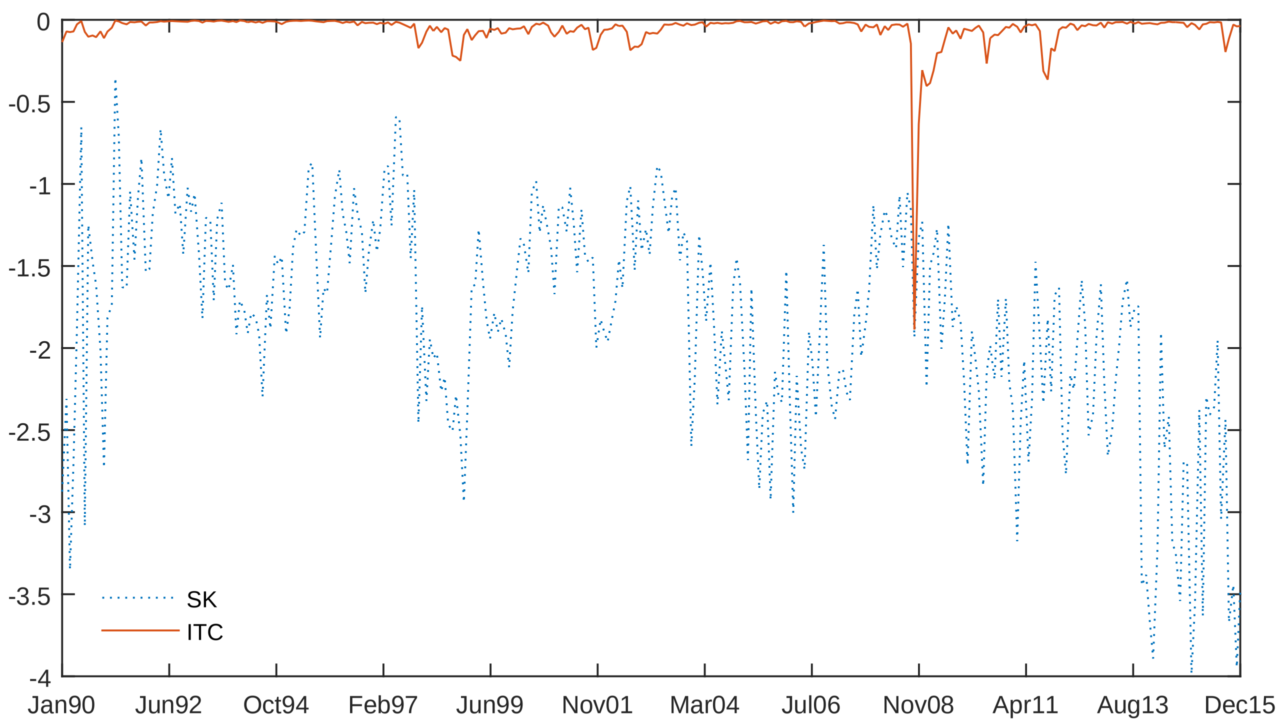

Figure 2: Forecasting Jump Risk

This graph shows the observed and forecasted values from the multivariate regression specified by

$$RCV_t^f = \alpha + \beta_1 ITC_t + \beta_2 RBV_t + \varepsilon_t.$$

The sample period is from 2 January 1990 to 31 December 2015.

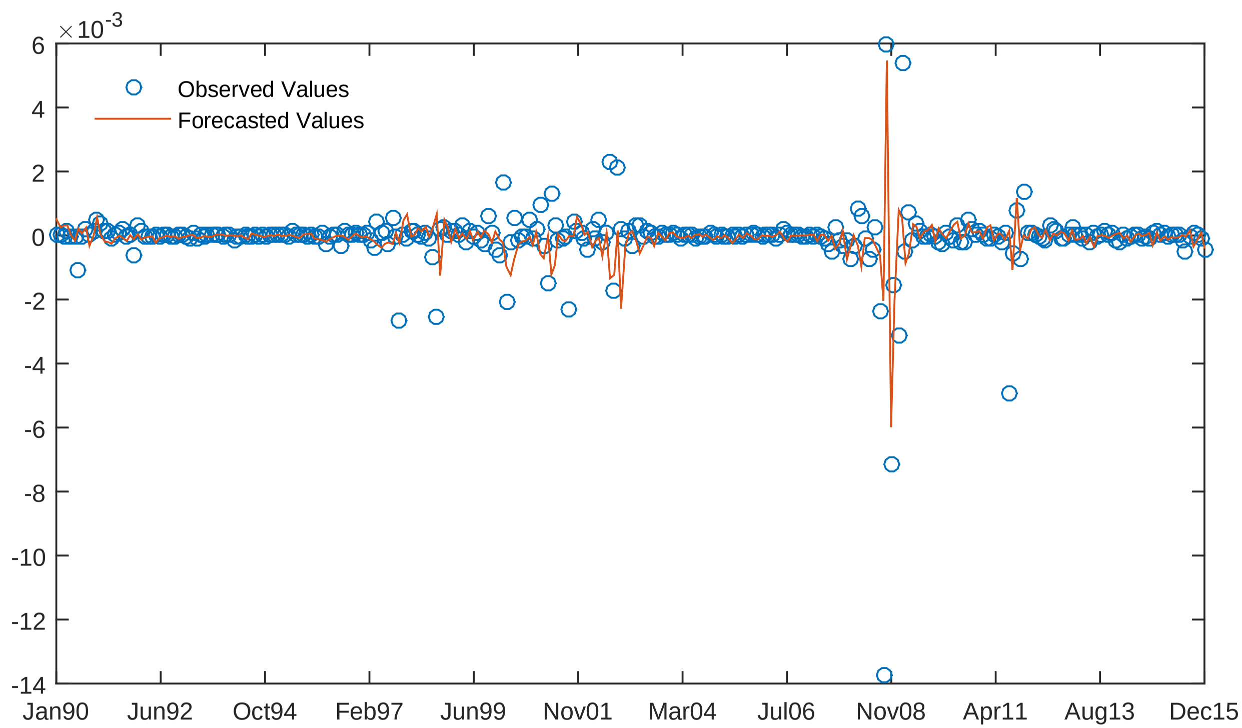

Figure 3: Forecasting Jump Events

This graph shows the observed and forecasted values against the predictor variable $\ln(|ITC|)$ (upper panel) and against the observation date (lower panel) from the binomial regression specified by

$$P(J_t^f = 1 | ITC_t) = \phi_B(\alpha + \beta \ln(|ITC_t|)), \quad \phi_B(x) = 1 - e^{-e^x}.$$

The sample period is from 2 January 1990 to 31 December 2015.