Variance Discount Rates: What Drives Preferences over Variance Risk?

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Abstract

I study time-variation in variance discount rates, defined as the expected returns for investing in variance risk. I show that variance discount rates drive a significant fraction of the variation in prices of S&P 500 variance swaps. This analysis offers important insights into preferences of investors over variance risk. I decompose variation in prices into variation due to variance expectations and variation due to variance discount rates. Variance expectations drive most of the variation in short-term variance swaps, whereas variance discount rates drive most of the variation in long-term variance swaps. I show that prominent asset pricing models, in which variation in the equity premium originates from variation in variance risk, have profoundly different predictions regarding the behavior of variance discount rates. None of the models analyzed are able to match the empirical properties of variance discount rates.

Keywords: Asset pricing, derivatives, variance pricing, variance discount rates.

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1 Introduction

In this paper, I analyze time-series variation in expected returns for investing in stock market variance, which are called variance discount rates. More specifically, I analyze variation in variance discount rates in the S&P 500 variance swap market. I show that variation in variance discount rates is important for the pricing of variance swaps, and I test whether prominent asset pricing models are able to match the observed variation.

The volume in the market for variance has increased significantly over the past decades; nevertheless, important properties of this market, like the time-variation in variance discount rates, have received little attention in the literature. One important reason to invest in stock market variance is that it can provide a hedge against stock market crashes. Moreover, risk premia in this market are substantial,[1] which shows that investors pay a premium to obtain a positive exposure towards stock market variance.

Variation in variance discount rates is either driven by variation in preferences over variance risk or by variation in the quantity of variance risk. Therefore, analyzing variance discount rates offers important insights into preferences of investors over variance risk and into how this risk varies over time. Moreover, in many prominent asset pricing models, variance risk is a central component and the main driver of variation in the equity premium. In the models considered in this paper,[2] variance risk is driven by time-variation in consumption disasters or time-variation in large and sudden movements in the agent's investment opportunity set. However, the literature has not settled on what the appropriate way is to incorporate variance risk in an asset pricing model. Analyzing the variation of variance discount rates allows me to contrast the empirical findings on variance discount rates with the predictions of these asset pricing models. Therefore, this analysis offers important insights into how variance risk should be incorporated in asset pricing models. In sum, my contribution to the literature is twofold.

First, I show that variation in variance discount rates is an important determinant of variation in S&P 500 variance swap prices. Variance swaps are assets that pay their owner the sum of daily squared stock market returns and, thus, give a direct exposure toward stock market variance. Variance discount rates are defined as the expected returns on variance swaps and equal the premium an investor pays to get exposure toward variance risk. I show that variation in variance discount rates drives at least 6.9% of the variation in short-term variance swaps, whereas it drives up to 76.0% of the variation for long-term variance swaps. Therefore, I find strong evidence that the premium associated with hedging variance risk varies over time. Furthermore, my results show that there is a predictable component in variance

[1] In the literature this phenomenon is referred to as the variance risk premium and documented in Bollerslev et al. (2009), Kozhan et al. (2013), Dew-Becker et al. (2017), and Aït-Sahalia et al. (2020).

[2] I study variance risk in the variable rare disaster model by Gabaix (2012), the time-varying disaster model by Wachter (2013), and the long-run risk model by Drechsler and Yaron (2011).

risk, which drives variation in both short-term and long-term variance discount rates.

Second, I show that prominent asset pricing models have profoundly different predictions regarding the pricing of variance risk and, in particular, regarding the variation in prices due to variance discount rates. In the model by Gabaix (2012), all variation in variance swap prices is attributed to variation in variance discount rates, both for short-term and long-term variance swaps. On the other hand, in the model by Wachter (2013) (almost) none of the variation in variance swap prices is attributed to variance discount rates, while the data shows that variance discount rates are the main driver of long-term variance swaps. These results follow from the fact that the model by Wachter (2013) predicts a strong persistence in stock market variance, whereas the model by Gabaix (2012) predicts that, in absence of disasters, stock market variance is constant. Finally, the model by Drechsler and Yaron (2011) matches the empirical result that long-term variance swaps are mostly driven by variance discount rates. However, when I use the calibration of Drechsler and Yaron (2011), the model predicts substantially larger variation in short-term variance discount rates than empirically observed. In sum, these models incorporate variance risk to capture empirical features of the equity premium; however, the models fail to directly match empirically documented features with respect to the pricing of variance risk.

I now explain the methodology to study variance discount rates in more detail. In order to analyze variation in variance discount rates, I derive a log-linear pricing identity for variance swaps in the spirit of the approach by Campbell and Shiller (1988) for equity. More specifically, I show that the variance swap rate with T months to maturity $vs_t^{(T)}$ approximately equals the difference between expected stock market variance $E_{\mathrm{rv},t}^{(T)}$ and variance discount rates $E_{\mathrm{vdr},t}^{(T)}$ up to maturity, as follows:

$$vs_t^{(T)} \approx E_{\mathrm{rv},t}^{(T)} - E_{\mathrm{vdr},t}^{(T)}, \qquad (1)$$

where the full derivation and definitions are given in Section 2.2. Equation (1) offers a key insight into the pricing of variance risk, as it shows that the variance swap rate is high, either due to high stock market variance expectations or low variance discount rates. Moreover, this identity for the variance swap rate is similar to the pricing identity of the price-dividend ratio, which is presented in Campbell and Shiller (1988), where variance expectations take the role of expected cash flows and variance discount rates replace stock market discount rates. In the literature on stock market discount rates, it is standard to model discount rates or dividend growth using a vector autoregression (VAR). I proceed accordingly and model stock market variance with a VAR. Using the VAR, I can decompose variation in variance swap rates into variation due to stock market variance expectations and variation due to variance discount rates. I document the following stylized facts regarding short-term and long-term variance swap rates.

First, regarding short-term variance swaps, I document that variation in stock market variance expectations accounts for, respectively, 123.5% and 87.0% of the total variation in variance swap rates with one and three months to maturity, whereas variation in variance discount rates accounts for 6.9% and 31.3% of the total variation.[3] Additionally, I find that stock market variance expectations and short-term variance discount rates are positively correlated. This finding is at odds with economic intuition, which suggests that the premium to hedge variance risk increases, rather than decreases, during periods of elevated stock market variance. However, the finding that variance discount rates and stock market variance expectations are positively correlated is in line with the result in Cheng (2019), who finds that risk premia in the market for variance tend to decrease during periods of elevated stock market variance.[4] Furthermore, I show that decompositions of short-term variance swap rates using predictive regressions, which are independent of the VAR, yield very similar results.

Second, regarding long-term variance swaps, I document that variation in variance discount rates accounts for, respectively, 60.6% and 76.0% of the total variation in 12-month and 18-month variance swap rates, whereas variation in variance expectations accounts for 48.5% and 28.5% of the total variation. Hence, there is strong evidence for time-variation in variance discount rates, and this variation accounts for a significant part of the total variation in variance swap rates. In order for long-term variance discount rates to vary over time as much as empirically documented, either variance risk should contain a persistent component or preferences over variance risk need to vary over time and in a persistent way. Moreover, the fact that short-term variance swaps mostly move due to variance expectations indicates that stock market variance is highly time-varying and predictable. The result that variance expectations drive only a small fraction of the variation in long-term variance swaps indicates, however, that stock market variance is not very persistent. To show robustness of my results, I show that the decomposition of long-term variance swaps, based on predictive regressions, yields the same results as those based on the VAR.

After having established the stylized facts of the pricing of variance risk, I compare the empirical findings to the predictions of prominent asset pricing models. The first model considered in this paper is the consumption disaster model by Gabaix (2012). The model builds on the framework of Rietz (1988) and Barro (2006) and adds that the size of the disaster varies over time. I show that variation in variance swap rates in the model by Gabaix

[3]The sum of variation due to stock market variance expectations and variance discount rates does not equal one, because these components turn out to be positively correlated.

[4]Lochstoer and Muir (2019) show that a model in which an agent has slow moving beliefs over stock market volatility can reconcile the evidence of the negative correlation of stock market variance and the variance risk premium. Slow moving volatility expectations lead the agent to initially underreact to volatility news followed by a delayed overreaction.

(2012) is solely driven by variation in variance discount rates. This result follows from the fact that, in the absence of disasters, stock market variance is constant. However, during periods of increased disaster size, the investor is willing to pay an even larger premium to hold variance swaps, which drives up the variance swap rate.

The second model considered in this paper is the model by Wachter (2013), in which variance risk is also driven by variation in consumption disaster risk. However, in the model by Wachter (2013) the disaster intensity, rather than the disaster size, varies over time. I show that in this model, (almost) all variation in variance swap rates is driven by variance expectations rather than variance discount rates. In this model, stock market variance is, even in the absence of consumption disasters, time-varying. This feature, which is not present in the model by Gabaix (2012), results from the fact that the variance of the stock market increases in the current level of the disaster intensity. Due to this time-variation in stock market variance and its persistence, the variation in variance expectations is substantially larger than the variation in variance discount rates and, thus, the main driver of variance swap rates.

The third model considered in this paper is the model by Drechsler and Yaron (2011), who extend the long-run risk framework of Bansal and Yaron (2004) by including jumps. In this model, variance risk is driven by jumps in the investment opportunity set, rather than jumps in the consumption process. These jumps appear in the stochastic process governing the long-run mean consumption growth and in the process governing the variance of the model. I show that, in line with the data, short-term variance swap rates are driven mostly by variance expectations and long-term variance swap rates mostly by variance discount rates. However, due to the relatively high frequency of the jumps in the model and the large variation in intensity, short-term variance discount rates drive a much larger part of the variation in short-term variance swaps than empirically observed. Therefore, I conclude from this analysis that jumps in stock market variance are much less predictable over short horizons in the data than in the model by Drechsler and Yaron (2011). Furthermore, stock market variance is more persistent in the model by Drechsler and Yaron (2011) than in the data, and this follows from the fact that the fraction of variation explained by variance expectations decreases less in the maturity than empirically observed.

How do variance discount rates relate to stock market discount rates? To analyze this, I decompose returns on variance swaps into news about future variance and news about future variance discount rates and decompose stock market returns, following Campbell and Vuolteenaho (2004), into news about future dividends and news about future stock market discount rates. I find that the beta of a variance swap is negative, because the correlation between variance swap returns and stock market returns is strongly negative. First, I show that the stock market beta increases from -7.18 for short-term variance swaps to -2.21 for long-term variance swaps. Second, the low stock market beta is, both for short-term and long-term variance swaps, mostly explained by the positive correlation between news about stock market variance and news about stock market discount rates. This result indicates that during periods of increased stock market variance, stock market discount rates are revised upward to compensate for the increased risk of investing in the stock market. Third, I show that news about short-term variance discount rates is positively correlated to news about stock market discount rates, whereas news about long-term variance discount rates is negatively correlated to news about stock market discount rates.

Overall, I contribute to three strands of the literature. First, I contribute to the literature on the variance risk premium. Bollerslev et al. (2009), Kozhan et al. (2013), Dew-Becker et al. (2017), and Aït-Sahalia et al. (2020) show that this risk premium is, on average, sizable. Moreover, Bollerslev et al. (2009) show that the variance risk premium predicts future stock market returns, and Bollerslev and Todorov (2011) show that a substantial fraction of the equity premium is compensation for variance risk. I show that there is sizable time-variation in expected returns to invest in variance risk. Second, I contribute to the literature on how to incorporate variance risk in an asset pricing model. Rietz (1988) and Barro (2006) propose that the equity premium is a compensation for consumption disasters. Gabaix (2012) and Wachter (2013) show how time-variation in consumption disaster risk is able to capture time-variation in the equity premium. On the other hand, Drechsler and Yaron (2011) show that the equity premium and variance risk premium is explained by a long-run risk model that incorporates jumps in the investment opportunity set, rather than in the consumption growth process. The decomposition of variance swap rates in this paper allows me to directly test these mechanisms that drive variance risk in the models. Third I contribute to the literature on discount rates in various financial markets of which the importance is stressed in Cochrane (2011). Shiller (1981), Campbell and Shiller (1988), Campbell and Vuolteenaho (2004), Cochrane (2008), and Campbell et al. (2018) analyze variation in stock market discount rates. Variation in discount rates for bonds and corporate credits is analyzed in Fama and Bliss (1987), Campbell and Ammer (1993), and Nozawa (2017). Johnson (2017) shows that there is predictability in the market for variance swaps, which indicates that there is some time variation in variance discount rates. I show that, indeed, variance discount rates vary over time and drive a significant part of the variation in variance swap rates.

The remainder of this article is organized as follows. Section 2 shows the derivation of the pricing identity for variance swap rates. Section 3 describes the data on variance swaps, reports the stylized facts on variance swap returns, and reports the results of the decomposition. Section 4 explains the predictions of several asset pricing models regarding the decomposition of variance swap rates. Section 5 concludes.

2 Methodology

In this section, I present the methodology to study the time-series variation of variance discount rates in the S&P 500 variance swap market. A variance swap is a derivative security that pays the holder of the contract the realized variance of the underlying up to maturity. Variance swaps are used to manage market variance risk, and the variance discount rates correspond to expected returns for holding a variance swap.

In the next subsection, I formalize the cash flows of a variance swap and will afterward derive a log-linear pricing idenity for the variance swap rate in the spirit of the approach by Campbell and Shiller (1988) for equity.

2.1 Variance swap contract

A variance swap pays its holder the realized variance of the underlying from the inception of the contract up to maturity. At maturity, the holder of the contract pays the variance swap rate in exchange for the realized variance of the underlying from inception up to maturity. The payoff of a variance swap at maturity T-periods from origination time t is defined as follows:

$$\text{payoff}_{t+T} = \sum_{i=1}^{T} RV_{t+i} - VS_t^{(T)}, \tag{2}$$

where RV_{t+i} is the realized variance over period $t+i$ and $VS_t^{(T)}$ is the variance swap rate at time t for a variance swap with maturity T-periods to maturity. Note that a variance swap can last for several periods, and, therefore, the total realized variance at the end of the contract equals the sum of the realized variance over each period. The holder of the variance swap receives the realized variance in exchange for a fixed rate at the end of the contract and, therefore, hedges variance risk until maturity of the contract. Given a risk-neutral pricing measure $\mathbb{Q}$, the variance swap rate with T-periods to maturity at time t is given by the following:

$$VS_t^{(T)} = \sum_{i=1}^{T} \mathbb{E}_t^{\mathbb{Q}}\big(RV_{t+i}\big), \tag{3}$$

where $\mathbb{E}_t^{\mathbb{Q}}$ denotes the expectation under the risk-neutral measure conditional on information available at time t. Therefore, $VS_t^{(T)}$ corresponds to the risk-neutral expectation of the sum of realized variances from period $t+1$ until period $t+T$.

Next, I define the gross return over period t to $t+1$ on a variance swap with T-periods to maturity, as follows:

$$R_{t+1}^{(T)} = \frac{VS_{t+1}^{(T-1)} + RV_{t+1}}{VS_t^{(T)}}. \tag{4}$$

The gross return is defined as the investor holding a variance swap with T-periods to maturity for one period and selling in the period afterward. She buys the variance swap for the current variance swap rate $VS_t^{(T)}$, receives the next period's realized variance RV_{t+1}, and sells the variance swap for the next period's variance swap rate $VS_{t+1}^{(T-1)}$. The definition of the return in equation (4) corresponds to the return on a variance asset, which pays the realized variance at the end of each period rather than at the end of the contract. Under the assumption of no-arbitrage, the price of such an asset equals the variance swap rate discounted with the T-period risk-free rate to time t, and the realized variance payment of such an asset is discounted in a similar way. It is possible to show that the logarithm of the return defined in (4) equals the log-return on this variance asset in excess of the risk-free rate. Moreover, it follows from equation (4) that realized variance for returns on variance swaps is the equivalent to dividend payments for returns on the stock market; that is, realized variance is the cash flow component of a variance swap.

In the next subsection, I show that it is possible to write the variance swap rate as a function of the expectation of future stock market variance and variance discount rates. The derivation is analogous to the derivation of Campbell and Shiller (1988) for equity.

2.2 Pricing identity for the variance swap rate

I show in this subsection how to derive the log-linear pricing identity of equation (1). The goal of this exercise is to write the current variance swap rate as a function of expected variance and variance discount rates only.

The first step in deriving the log-linear pricing identity is to Taylor-expand the log-return on the variance swap and obtain the following log-linear approximation:

$$r_{t+1}^{(T)} \approx k(T) + \rho(T) \cdot vs_{t+1}^{(T-1)} + \big[1 - \rho(T)\big] rv_{t+1} - vs_t^{(T)}, \tag{5}$$

where $r_{t+1}^{(T)} = \log\big(R_{t+1}^{(T)}\big)$, $vs_{t+1}^{(T-1)} = \log\big(VS_{t+1}^{(T-1)}\big)$, $rv_{t+1} = \log\big(RV_{t+1}\big)$, and $vs_t^{(T)} = \log\big(VS_t^{(T)}\big)$. In the following, unless stated otherwise, I refer to the logarithm of these variables as the return, variance swap rate, and realized variance. The interpretation of equation (5) is as follows: The return on a variance swap is approximately equal to a linear function of the next period's variance swap rate $vs_{t+1}^{(T-1)}$, realized variance in the next period rv_{t+1}, and the current variance swap rate $vs_t^{(T)}$. $k(T)$ is the approximation constant and $\rho(T)$ governs the relative importance of the next period's price and the next period's realized variance in the calculation of the return of the variance swap. These constants depend on the maturity T

of the variance swap, because the one-period return on a long-term variance swap is mostly driven by the next period's variance swap rate rather than the one-period realized variance. Intuitively, this indicates that $\rho(T)$ is closer to one for variance swaps with a large maturity. I show in Section 3.3 that $\rho(T)$ can be estimated using a simple regression and, importantly, that equation (5) approximates the return on a variance swap really well.

In order to write the current variance swap rate as a function of future realized variance and future returns only, I substitute the next period's variance swap rate with the following equation:

$$vs_{t+i}^{(T-i)} \approx k(T-i) + \rho(T-i) \cdot vs_{t+i+1}^{(T-i-1)} + \big[1 - \rho(T-i)\big] rv_{t+i+1} - r_{t+i+1}^{(T-i)},$$

where $k(T-i)$ and $\rho(T-i)$ are the log-linearization coefficients of a variance swap with $T-i$-periods to maturity. Next, this equation allows me to substitute the future variance swap rate $vs_{t+i}^{(T-i)}$ in equation (5) and iterate forward up to the point that the current variance swap rate depends on the one-month variance swap rate $vs_{t+T-1}^{(1)}$. The following holds regarding the one-month variance swap rate: $vs_{t+T-1}^{(1)} = rv_{t+T} - r_{t+T}^{(1)}$, which yields $k(1) = \rho(1) = 0$ such that it holds with an equal sign. In the end, the future variance swap rate is substituted out from equation (5) and I obtain the following:

$$vs_t^{(T)} \approx K + \sum_{i=1}^{T} \big[1-\rho(T-i+1)\big] \Big(\prod_{j=1}^{i-1} \rho(T-j+1) \Big) \cdot rv_{t+i} - \sum_{i=1}^{T} \Big(\prod_{j=1}^{i-1} \rho(T-j+1) \Big) \cdot r_{t+i}^{(T-i+1)}, \quad (6)$$

where K is a constant and a function of the constants $k(T-i)$ and $\rho(T-i)$ from the individual log-linearizations. In the remainder of this paper, I will discard the constant K from the pricing identity for simplicity and because the focus of this paper is on time-series variation. The intuition of equation (6) is as follows: A high current variance swap rate $vs_t^{(T)}$ is either due to high future realized variance rv_{t+i} or low future returns on the variance swap $r_{t+i}^{(T-i+1)}$. Equation (6) holds ex post, but the equation also holds in expectation conditional on information at time t, as follows:

$$vs_t^{(T)} \approx E_{\mathrm{rv},t}^{(T)} - E_{\mathrm{vdr},t}^{(T)}, \quad (7)$$

where

$$E_{\mathrm{rv},t}^{(T)} = \mathbb{E}_t \sum_{i=1}^{T} \big[1 - \rho(T-i+1)\big] \Big(\prod_{j=1}^{i-1} \rho(T-j+1) \Big) \cdot rv_{t+i} \quad \text{and} \quad (8)$$

$$E_{\mathrm{vdr},t}^{(T)} = \mathbb{E}_t \sum_{i=1}^{T} \Big(\prod_{j=1}^{i-1} \rho(T-j+1) \Big) \cdot r_{t+i}^{(T-i+1)}. \quad (9)$$

The intuition of equation (7) is similar to before: The current variance swap rate $vs_t^{(T)}$ is high either due to high expected realized variance $E_{\text{rv}}^{(T)}$ or low expected variance discount rates $E_{\text{vdr}}^{(T)}$. Campbell and Shiller (1988) show that the current price-dividend ratio increases in dividend growth expectations and decreases in stock market discount rates. Therefore, identity (7) for the variance swap rate is similar to the pricing identity of the price-dividend ratio, where variance expectations take the role of expected cash flows and variance discount rates replace stock market discount rates. There are two main differences between the identity of equation (7) and the identity in Campbell and Shiller (1988), and these are due to the fact that a variance swap is a finite cash flow, whereas equity is a perpetual cash flow. First, variance discount rates in equation (9) depend on the maturity of the variance swap, which is important because Dew-Becker et al. (2017) show that the term structure variance discount rates is upward sloping. This indicates that expected returns on short-term variance swaps are much lower than expected returns on long-term variance swaps. Second, in the derivation of the pricing identity for equity, Campbell and Shiller (1988) assume the so-called no-bubble condition. This assumption is not needed in case of a variance swap, because it is a finite cash flow rather than a perpetual cash flow in the case of equity.

In Section 3, I take equation (7) to the data and decompose the variance swap rate into variance expectations and variance discount rates for several maturities. This allows me to show how much of the variation in variance swap rates is due to variance expectations and variance discount rates. Furthermore, in the following subsection I derive an equation to decompose realized returns on variance swaps into news about future variance and news about future variance discount rates.

2.3 Variance swap return decomposition

From the log-linear pricing identity of equation (7), it is possible to derive an equation that shows how returns on variance swaps are decomposed into news about future variance and news about future variance discount rates. Similar to what Campbell and Vuolteenaho (2004) do for equity, I calculate changes in expectation of identity (7), which yields the following:

$$\begin{aligned} r_{t+1}^{(T)} - \mathbb{E}_t r_{t+1}^{(T)} \approx \left(\mathbb{E}_{t+1} - \mathbb{E}_t\right) \left[\sum_{i=1}^{T} \left(1 - \rho(T-i+1)\right) \left(\prod_{j=1}^{i-1} \rho(T-j+1) \right) rv_{t+i} \right] \\ - \left(\mathbb{E}_{t+1} - \mathbb{E}_t\right) \left[\sum_{i=2}^{T} \left(\prod_{j=1}^{i-1} \rho(T-j+1) \right) r_{t+i}^{(T-i+1)} \right] = N_{\text{rv},t+1}^{(T)} - N_{\text{vdr},t+1}^{(T)}. \qquad (10) \end{aligned}$$

As previously, equation (10) indicates that unexpected returns on a variance swap with maturity T are associated with news about future variance and news about variance discount rates. An increase in expected variance is associated with a positive return on the variance swap, whereas an increase in variance discount rates is associated with a negative return on the variance swap. The decomposition of returns on variance swaps, rather than variance swap rates, helps to understand how quickly expectations over realized variance of discount rates change over time. Furthermore, using this decomposition I analyze which part of the variance swap return correlates with stock market returns.

In the next section, I first discuss the details of my data and afterward present the empirical results regarding the decomposition of variance swap rates and variance swap returns.

3 Empirical results

In this section, I present the empirical results of the decomposition of variance swap rates and variance swap returns. First, I describe my data on variance swaps and present some sample statistics. Second, I show the results from the decomposition of variance swap rates into variance expectations and variance discount rates. I decompose the variance swap rate in two ways: using predictive regressions—which I call the simple decomposition—and using the VAR. Third, I decompose returns on variance swaps into news about future variance and news about variance discount rates and analyze how variance swap returns are related to stock market returns.

3.1 Data

In this paper, I use data on S&P 500 options from January 1996 until June 2019 from OptionMetrics. Using the methodology described in Kozhan et al. (2013) and discussed more into details in Appendix A.1, I construct variance swaps with maturities ranging from one to 18 months. The maturity of 18 months to maturity is the longest for which I can calculate a variance swap rate every month. I calculate variance swap rates at the end of each month in the sample and interpolate the variance swap rates linearly, such that the maturity equals T months. Note that interpolating variance swap rates linearly is equivalent to taking long posititions in two variance swaps with maturity T_1 and T_2 such that the weighted average of the maturities equals T.

I use the methodology of Kozhan et al. (2013), because these variance swaps are most closely related to the variance swaps that are traded over-the-counter (OTC). In Appendix A.2, I show that my data on variance swaps is highly similar to data from the OTC market, which is analyzed in Dew-Becker et al. (2017).

In addition to the pricing information on variance swaps, I obtain data that is used in the VAR. From the panel of variance swap rates, the first two principal components, $pc_t^{(1)}$

and $pc_t^{(2)}$, are calculated. $pc_t^{(1)}$ captures the level in the term structure of variance swap rates, and $pc_t^{(2)}$ captures the slope of the term structure of variance swap rates. Furthermore, realized variance is defined as in Kozhan et al. (2013) and approximately equals the sum of daily squared returns within a month. Finally, the default spread is obtained from the Federal Reserve Bank of St. Louis and defined as the difference in yield on BAA and AAA credit-rated corporate bonds.

3.2 Returns on variance swaps

In this subsection, I present sample statistics from the panel of variance swap returns. Note that I also present sample statistics on the realized variance of the S&P 500, because realized variance plays the role of the dividend payment in the calculation of the return. Moreover, I calculate simple returns and log-returns in order to quantify its differences in the case of variance swaps. The sample statistics of the realized variance and variance swap returns are shown in Table 1.

Table 1: The table shows sample statistics of realized variance in Panel A, simple variance swap returns in Panel B and log variance swap returns in Panel C. The sample statistics of monthly realized variance in Panel A are scaled to represent the yearly standard deviation. The mean, standard deviation, yearly Sharpe ratio and the 5%, 25%, 50%, 75% and 95% quantiles are presented.

Panel A: Realized variance								
Maturity	Mean	SD	SR	5%	25%	Median	75%	95%
-	0.162	0.095	-	0.068	0.101	0.142	0.188	0.323
Panel B: Simple returns on variance swaps								
18	-0.006	0.187	-0.106	-0.231	-0.123	-0.037	0.071	0.317
12	-0.013	0.227	-0.202	-0.269	-0.151	-0.056	0.081	0.346
6	-0.050	0.316	-0.544	-0.363	-0.232	-0.110	0.058	0.480
3	-0.098	0.447	-0.756	-0.477	-0.353	-0.204	0.002	0.581
1	-0.285	0.676	-1.458	-0.779	-0.630	-0.456	-0.193	0.838
Panel C: Log-returns on variance swaps								
18	-0.021	0.167	-	-0.262	-0.131	-0.037	0.069	0.275
12	-0.034	0.195	-	-0.313	-0.164	-0.057	0.078	0.297
6	-0.090	0.264	-	-0.451	-0.265	-0.117	0.057	0.392
3	-0.181	0.365	-	-0.649	-0.435	-0.228	0.002	0.458
1	-0.572	0.640	-	-1.508	-0.995	-0.608	-0.214	0.609

In Table 1, sample statistics of monthly realized variance on the S&P 500 and returns on variance swaps with different maturities ranging from one to 18 months are presented. Panel A presents sample statistics of realized variance scaled to yearly standard deviation, Panel B presents simple returns on variance swaps, and Panel C presents log-returns on variance swaps. The mean monthly realized variance over the sample is equal to 16.2% p.a., with a standard deviation of 9.5%. Furthermore, the distribution of monthly realized variance is right-skewed, indicated by the quantiles of the distribution.

The first observation from Panel B of Table 1 is that, on average, returns on variance swap returns are negative. This result is in line with a positive variance risk premium for the market portfolio as in Bollerslev et al. (2009) and Drechsler and Yaron (2011). Economically, a negative expected return on a variance swap indicates that if an investor wants to hedge variance risk, she pays a risk premium. The premium the investor pays for holding a variance swap is decreasing in the maturity of the variance swap. A variance swap with maturity longer than one month has exposure toward realized variance in the next month and toward expected variance for the remaining of the contract. Dew-Becker et al. (2017) show that the premium for hedging realized variance the next month is much larger than for hedging expected variance, and, therefore, the premium for holding a variance swap is decreasing in the maturity. Moreover, returns on variance swaps are volatile, as seen in the third column of Panel B in Table 1 and the volatility is decreasing in the maturity of the variance swap. However, the yearly Sharpe ratio is strongly increasing in maturity, and the yearly Sharpe ratio for investing in variance swaps with one month to maturity is very low (≈ -1.46), and similar to what Dew-Becker et al. (2017) find. Furthermore, the distributions of variance swap returns are right-skewed, indicated by the quantiles of the return distribution.

The sample average of the log-returns on variance swaps in Panel C of Table 1 are lower than the sample average for simple returns in Panel B of Table 1. This is driven by the fact that log-return distributions are less right-skewed than the simple return distributions and, therefore, is the sample average lower. The distribution of one-month returns is affected the most by the log-transformation. This result derives from the fact that the approximation of log-returns is equal to simple returns is close if the volatility of the return is low.[5]

3.3 Simple decomposition of variance swap rates

As a starting point, I test the basic intuition from the pricing identity (7): The current variance swap rate is high either due to high future variance or low future returns on the variance swap. Therefore, the current variance swap rate should predict future stock market variance, future returns on the variance swap, or both. In order to do this, I run predictive regressions of future stock market variance and future returns on the variance swap, with the

[5]Note, $\log(1 + x) \approx x$ for x close to zero.

current variance swap rate as the predictor. The present value identity (6) holds ex post and, therefore, the coefficients of the following predictive regressions decompose the variance swap rate into variance expectations and variance discount rates, as follows:

$$y_{\mathrm{rv},t+T} = \sum_{i=1}^{T} \left[1-\rho(T-i+1)\right]\left(\prod_{j=1}^{i-1}\rho(T-j+1)\right)\cdot rv_{t+i} = a_{\mathrm{rv}} + b_{\mathrm{rv}}\cdot vs_t^{(T)} + \epsilon_{t+T}^{\mathrm{rv}} \text{ and } \tag{11}$$

$$y_{\mathrm{vdr},t+T} = \sum_{i=1}^{T}\left(\prod_{j=1}^{i-1}\rho(T-j+1)\right)\cdot r_{t+i}^{(T-i+1)} = a_{\mathrm{vdr}} + b_{\mathrm{vdr}}\cdot vs_t^{(T)} + \epsilon_{t+T}^{\mathrm{vdr}}, \tag{12}$$

where rv_{t+i} and $r_{t+i}^{(T-i+1)}$ are the realized variance and return of a variance swap during period $t+i$. The difference between the regression coefficients of equations (11) and (12) should be approximately one, as follows:

$$1 \approx b_{\mathrm{rv}} - b_{\mathrm{vdr}}.$$

The regression coefficients indicate whether a high current variance swap rate predicts high future stock market variance or low future returns. Therefore, economic intuition suggests that $b_{\mathrm{rv}} > 0$ and $b_{\mathrm{vdr}} < 0$. Moreover, if $b_{\mathrm{rv}} \approx 1$, it indicates that variation in the variance swap rate is mostly driven by stock market variance expectations, whereas if $b_{\mathrm{vdr}} \approx -1$, it indicates that variation in the variance swap rate is mostly driven by variance discount rates. Due to the fact that a variance swap is a finite cash flow, these regressions provide a powerful test if the present value identity holds. This identity holds if I find that, indeed, the difference between the regression coefficients is close to one. In case of a perpetual cash flow like equity, the regressions (11) and 12 do not fully decompose the current price, because the future price of the asset could be important.

In order to compute $y_{\mathrm{rv},t+T}$ and $y_{\mathrm{vdr},t+T}$, I need the log-linear approximation coefficients $\rho(T)$. I estimate $\rho(T)$ using a simple regression, which follows from minimizing the squared error of approximation (5). The results of this exercise are presented in Table 2.

Table 2: This table shows the regression results of $r_{t+1}^{(T)} - rv_{t+1} + vs_t^{(T)} = k + \rho\big(vs_{t+1}^{(T-1)} - rv_{t+1}\big) + \epsilon_{t+1}^{(T)}$, for different maturities T. For each maturity, the log-linearization coefficient $\rho(T)$ is given as well as the R^2 of the regression.

Maturity	1	2	3	4	5	6	7	8	9
$\rho(T)$	0	0.634	0.777	0.841	0.876	0.898	0.914	0.925	0.933
R^2	100%	98.24%	99.28%	99.57%	99.70%	99.78%	99.82%	99.85%	99.88%
Maturity	10	11	12	13	14	15	16	17	18
$\rho(T)$	0.940	0.945	0.950	0.954	0.958	0.961	0.963	0.965	0.967
R^2	99.90%	99.91%	99.92%	99.94%	99.94%	99.95%	99.96%	99.96%	99.96%

As expected, the log-linearization coefficient $\rho(T)$ depends on the maturity of the variance swap. $\rho(T)$ increases in maturity, which indicates that the next period's variance swap rate is relatively more important than the one-period realized variance for long-term variance swaps. The second row in Table 2 shows that the log-linear approximation of the variance swap returns is in fact a good approximation, as indicated by the large R^2's. The R^2's range from 98.24% to 99.96%, with an average of 99.7%.

The next step is to estimate the regression equations (11) and (12) which decompose the variance swap rate into variance expectations and variance discount rates. Table 3 presents the results.

Table 3: This table shows the results of the predictive regressions of equations (11) and (12), in which the variance swap rate is the independent variable. t-statistics are represented in brackets and are computed using Newey-West standard errors with number of lags equal to T.

Dependent variable:	$y_{\text{rv},t+T}$		$y_{\text{vdr},t+T}$	
Maturity	b_{rv} (t-stat.)	R^2	b_{vdr} (t-stat.)	R^2
18	0.245	0.030	-0.728	0.205
	(1.32)		(-3.87)	
12	0.558	0.142	-0.419	0.080
	(3.70)		(-2.65)	
6	0.833	0.315	-0.168	0.017
	(7.00)		(-1.38)	
3	0.957	0.432	-0.040	0.001
	(11.50)		(-0.46)	
1	1.101	0.577	0.101	0.013
	(19.52)		(1.79)	

A positive b_{rv} indicates that a high variance swap rate predicts high future realized variance, whereas a negative b_{vdr} indicates that a high variance swap rates predicts low future returns on the variance swap. The main result from Table 3 is that variation in short-term variance swaps is mostly attributed to variation in realized variance, whereas variation in long-term variance swaps is mostly attributed to variation in variance discount rates. Variation in variance swap rates with maturity less than six months is for at least 83.3% attributed to variation in realized variance and only 16.8% due to variation in variance discount rates. Moreover, these short-term variance swap rates do not predict lower future variance discount rates with a coefficient significantly different from zero. However, in Section 3.4, where the variance swap rate is decomposed using a VAR, I show that short-term variance discount rates vary over time as well. The finding of a postive b_{vdr} indicates that a high one-month variance swap rate predicts a high future return on the variance swap is at odds with economic intuition. However, the finding could be in line with Cheng (2019), who shows that the variance risk premium tends to decline during market turmoil. This finding is explained by falling hedging demand for variance risk during periods when stock market risk rises.

I now turn to the predictive regressions for long-term variance swaps with maturity of 12 or 18 months. Variation in expected variance accounts for, respectively, 55.8% and 24.5% of the total variation in variance swap rates with 12 or 18 months to maturity, whereas variance discount rates account for 41.9% and 72.8% of the total variation. Long-term variance swap rates predict future returns on the variance swaps negatively and with a coefficient significantly different from zero. Furthermore, the 18-month variance swap rate does not predict future realized variance over the lifetime of the contract with a coefficient significantly different from zero. These findings indicate that long-term variance swap rates are mostly driven by variation in variance discount rates.

The fact that $b_{\text{rv}} - b_{\text{vdr}}$ is, indeed, close to one for each maturity indicates that the variation in variance swap rates is effectively attributed to variation in variance expectations and variance discount rates. The difference between the coefficients of the predictive regressions lies between 0.973 and 1.001 for each of the considered maturities.

In the following subsection, I specify the VAR and present the results of the decomposition of variance swap rates.

3.4 Decomposition of variance swap rates using a VAR

In this subsection, I show how the variance swap rate is decomposed using a VAR. This allows me to estimate variance expectations and variance discount rates and analyze its time-series variation.

The results of the simple decomposition in Table 3 show that short-term variance swap rates do no predict future returns on a variance swap with a coefficient significantly different from zero. In the analysis using the VAR, I thoroughly investigate the variation in short-term and long-term variance discount rates. I use the VAR to model variance expectations and obtain variance discount rates as a latent variable from the log-linear pricing identity (7). It is convenient to model stock market variance with a VAR, because this allows me to obtain variance expectations for each period by iterating the VAR forward.

In the benchmark exercise, I focus on the following VAR with four state variables:

$$z_{t+1} = Bz_t + \epsilon_{t+1} \quad \text{and} \tag{13}$$

$$z_t = \begin{pmatrix} rv_t & pc_t^{(1)} & pc_t^{(2)} & DEF_t \end{pmatrix}', \tag{14}$$

where $B \in \mathbb{R}^{4\times 4}$ is a matrix with regressor coefficients and $\epsilon_{t+1} \in \mathbb{R}^{4\times 1}$ is a vector with errors. The vector z_t consists of the following variables: rv_t is the log of realized variance, $pc_t^{(1)}$ is the first principal component of the panel of log variance swap rates with maturity from one to 18 months, $pc_t^{(2)}$ is the second principal component of the panel of log variance swap rates, and DEF_t is the default spread defined as the yield difference between BAA and AAA credits. For simplicity, and standard in the literature, all the variables included in z_t are demeaned such that the intercepts in the VAR are zero.

The first row in the VAR of equation (13) represents the predictive model for stock market variance. In order to decompose the variance swap rates, I calculate variance expectations, and, therefore, the remaining variables in the VAR are included based on its ability to predict stock market variance. $pc_t^{(1)}$ captures the level of the term structure of variance swap rates and predicts future stock market variance well. The level of term strucutre of variance swap rates is highly correlated (≈ 0.93) with the VIX index, which Drechsler and Yaron (2011) show to be a good predictor of stock market variance. $pc_t^{(2)}$ relates to the slope of the term structure of variance swap rates, which is high during episodes of low stock market variance and low during episodes of elevated variance and, therefore, able to predict variance. Finally, the default spread DEF_t is included in the VAR, which Campbell et al. (2018) show to predict variation in long-term variance and a well-known business cycle indicator.

Based on the VAR model, monthly variance expectations for a variance swap with T-months to maturity equals

$$E_{\text{rv},t}^{(T)} = e_1' \left(\big(1-\rho(T)\big)B + \cdots + \big(1-\rho(1)\big)\rho(T) \times \cdots \times \rho(2) B^T \right) z_t, \tag{15}$$

where $e_1 \in \mathbb{R}^{4\times 1}$ is a unit vector with the first element equal to one and the remaining equal to zero. By the pricing identity (7), variance discount rates are a function of variance expectations from equation (15) and the current variance swap rate, as follows:

$$E_{\mathrm{vdr},t}^{(T)} = E_{\mathrm{rv},t}^{(T)} - vs_t^{(T)}. \tag{16}$$

In this way, I obtain an ex ante estimate of variance expectations over the lifetime of the variance swap and an estimate of the variance discount rates that price the variance swap. I use the estimates to decompose variation in the variance swap rate to either variance expectations of equation (15) or variance discount rates of equation (16). Before I show the results of this decomposition, I present the estimation results of the VAR in Table 4.

Table 4: This table shows the estimated coefficients of the VAR of equation (13) with t-values in parentheses in Panel A. All variables are normalized to have mean equal to zero, and $pc_t^{(1)}$ and $pc_t^{(1)}$ are additionally standardized to have standard deviation equal to one. Panel B shows the correlation matrix of the residual vector ϵ_t with the standard deviations on the diagonal. Sample period for the dependent variables is January 1996 – June 2019, with 282 monthly data points.

Panel A: Coefficients VAR model

	rv_t	$pc_t^{(1)}$	$pc_t^{(2)}$	DEF_t	R^2
rv_{t+1}	0.056	0.514	-0.344	0.514	0.589
(t-stat.)	(0.73)	(7.43)	(-7.13)	(3.38)	
$pc_{t+1}^{(1)}$	0.049	0.856	0.003	0.105	0.847
(t-stat.)	(1.02)	(20.00)	(0.09)	(1.11)	
$pc_{t+1}^{(2)}$	-0.022	0.151	0.845	-0.022	0.745
(t-stat.)	(-0.35)	(2.70)	(21.77)	(-0.18)	
DEF_{t+1}	0.013	-0.010	-0.004	0.964	0.936
(t-stat.)	(1.34)	(-1.11)	(-0.66)	(48.81)	

Panel B: Correlation/Std Matrix of Residuals.

corr/std	rv_{t+1}	$pc_{t+1}^{(1)}$	$pc_{t+1}^{(2)}$	DEF_{t+1}
rv_{t+1}	0.627	0.627	-0.484	0.354
$pc_{t+1}^{(1)}$	0.627	0.388	-0.598	0.372
$pc_{t+1}^{(2)}$	-0.484	-0.598	0.505	-0.154
DEF_{t+1}	0.354	0.372	-0.154	0.082

The first row in Panel A of Table 4 presents the model for log realized variance each month. In line with expectation, the level of the term structure of variance swap rates $pc_t^{(1)}$ predicts next month's realized variance positively. This result is expected as variance swap rates rise during episodes of evelvated stock market variance. The slope of the term structure of variance swap rates, $pc_t^{(2)}$, predicts next month's realized variance negatively. This result is in line with expectation due to the fact that the slope of the term structure is high during periods of low stock market variance. Finally, DEF_t predicts future realized variance positively and is in line with Campbell et al. (2018). The R^2 of 58.9% to predict next month's variance indicates that most variation is captured. This is an important validation of the results, given that the model for the realized variance is used to calculate realized variance expectations. Furthermore, the impulse response functions in Appendix A.3 show that $pc_t^{(1)}$ and $pc_t^{(2)}$ mostly capture variation in short- to mid-term variance, whereas DEF_t captures variation in long-term variance.

The remaining rows in Panel A of Table 4 summarize the dynamics of the explanatory variables in the VAR. The level of the term structure of variance swap rates, $pc_t^{(1)}$, is approximately an AR(1) process with an autoregressive coefficient of 0.86. The slope of the term structure of variance swap rates, $pc_t^{(2)}$, has a similar persistence of 0.85 but is also predicted with a positive coefficient by the level of the term structure of variance swap rates. Finally, the default spread, DEF_t, is more persistent with an autoregressive coefficient of 0.96. The persistence of the variables indicates whether the variables capture variation in short- or long-term variance and the implications are similar to the results of the impulse response functions, as shown in Appendix A.3.

The estimates in Panel A of Table 4 are used to calculate variance expectations, using equation (15), and variance discount rates, using equation (16). Variation in the variance swap rate with T-months to maturity is attributed to variation in $E_{\mathrm{rv},t}^{(T)}$, $E_{\mathrm{vdr},t}^{(T)}$ or correlation between $E_{\mathrm{rv},t}^{(T)}$ and $E_{\mathrm{vdr},t}^{(T)}$. This intuition follows from the following equation and is obtained if I calculate the variance of the pricing identity (7) for the variance swap rate, as follows:

$$\mathrm{var}\big(vs_t^{(T)}\big) \approx \mathrm{var}\big(E_{\mathrm{rv},t}^{(T)}\big) + \mathrm{var}\big(E_{\mathrm{vdr},t}^{(T)}\big) - 2 \cdot \mathrm{cov}\big(E_{\mathrm{rv},t}^{(T)}, E_{\mathrm{vdr},t}^{(T)}\big). \tag{17}$$

The results of the variance decomposition of equation (17) using the VAR are presented in Table 5.

Table 5: This table shows the results of the variance decomposition of variance swap rates using equation (17). Note that the (co)variances of the third, fourth, and fifth columns are scaled with the variance of the second column such that the sum of the three (co)variances equals one. Standard errors are computed using the Delta method.

T	$\text{var}(vs)$	$\frac{\text{var}(E_{\text{rv}})}{\text{var}(vs)}$ (s.e.)	$\frac{\text{var}(E_{\text{vdr}})}{\text{var}(vs)}$ (s.e.)	$\frac{-2\cdot\text{cov}(E_{\text{rv}},E_{\text{vdr}})}{\text{var}(vs)}$ (s.e.)
18	0.219	0.285	0.760	-0.045
		(0.204)	(0.294)	(0.330)
12	0.231	0.485	0.606	-0.092
		(0.245)	(0.272)	(0.352)
6	0.280	0.870	0.313	-0.182
		(0.242)	(0.154)	(0.298)
3	0.350	1.128	0.147	-0.275
		(0.189)	(0.070)	(0.219)
1	0.455	1.235	0.069	-0.304
		(0.123)	(0.027)	(0.139)

The second column of Table 5 shows that the variation in variance swap rates decreases in the maturity of the contract. In line with the results of the simple decomposition in Table 3, most of the variation in short-term variance swaps, with at most six months to maturity, is attributed to variance expectations and increases from 87.0% for six-month variance swaps to 123.5% for one-month variance swaps. However, short-term variance discount rates also drive a part of the total variation in short-term variance swap rates, which is significantly different from zero. Variation in variance discount rates explains 6.9% of the total variation in one-month variance swap rates to 31.3% of the variation in six-month variance swap rates. Interestingly, the covariance between expected variance in the next month and the one-month variance discount rate is significantly positive. This result indicates that the variance discount rate increases, rather than decreases, during periods of elevated stock market variance. The positive covariance between expected variance and the variance discount rate is in line with Cheng (2019), who shows that the variance risk premium decreases during periods of elevated stock market variance. The size of the covariance between expected realized variance and variance discount rates decreases in the maturity of the variance swap and is no longer significantly different from zero beyond a horizon of one month.

The variation in long-term variance swap rates with a maturity of 12 or 18 months is mostly attributed to variance discount rates which drives, respectively, 60.6% or 76.0% of the total variation. Moreover, variation in variance expectations only drives 48.5% or 28.5% of the total variation in 12-month or 18-month variance swap rates. Overall, the results from the decomposition using the VAR of Table 5 are remarkably close to the results of the simple decomposition of Table 3. The results from the simple decomposition are model-free and, therefore, the similarity of the results suggests that the VAR is correctly specified. Furthermore, I show in Appendix A.4 that if I do the decomposition using the predictive regressions or the VAR using a quarterly frequency, the results are highly similar.

Up to this point, I have shown that variation in variance discount rates drives a significant portion of the variation in variance swap rates. The next step is to analyze the time-series variation in variance discount rates. In the next figure, I plot the decomposition of the one-month variance swap rate and the decomposition of the 18-month variance swap rate.

Figure 1: The figure plots the decomposition of the variance swap rate (blue line) into variance expectations (dotted line) and variance discount rates (dashed line) over the sample period. The left graph shows the decomposition of the one-month variance swap rate, and the right graph the decomposition of the 18-month variance swap rate. The variables are represented as yearly moving averages. The shaded area corresonds to the NBER recessions.

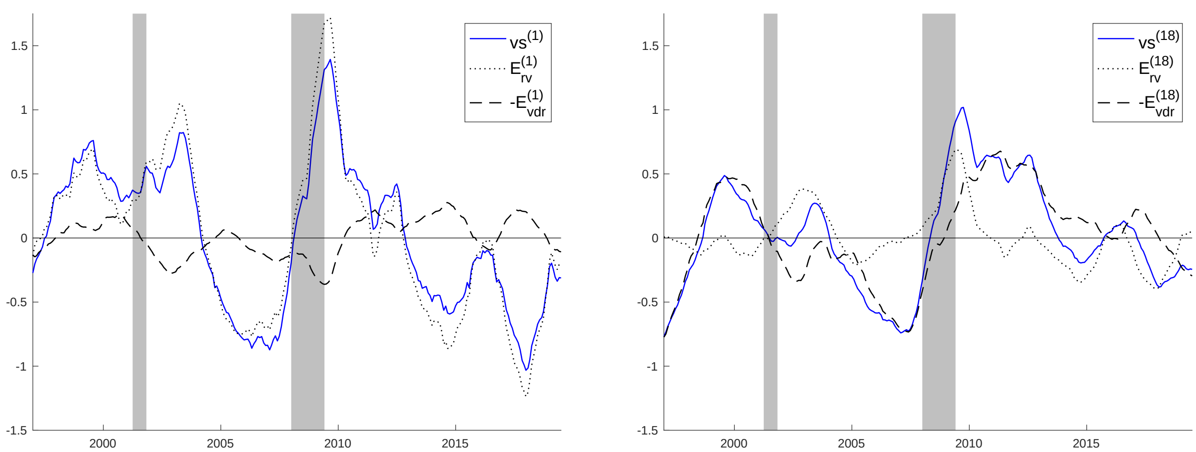

Figure 1 plots the decomposition of the variance swap rate (blue line) into variance expectations (dotted line) and variance discount rates (dashed line). The left graph shows the demeaned variance swap rate with one month to maturity, and it is clearly visible that the monthly variance swap rate closely follows expected stock market variance. Moreover, there is a negative correlation visible between expected variance and the variance discount rate, which indicates that the discount rate decreases when expected variance increases. As mentioned, this is not in line with economic intuition, which would suggest that the cost to hedge variance risk increases rather than decreases when expected variance increases.

The right graph of Figure 1 plots the decomposition of the 18-month variance swap rate (blue line) into variance expectations (dotted line) and variance discount rates (dashed line). Variance discount rates are a more important determinant of the 18-month variance swap rate than they are for the one-month variance swap rate. It follows from the graph that variance discount rates were relatively high during the period following the financial crisis in 2008, whereas variance discount rates were relatively low during the period leading up to the crisis. Moreover, the variation in the short-term variance discount rate (left graph) and the variation in long-term variance discount rates (right graph) is correlated, which indicates that there is a common component in the term structure of variance discount rates. In order to investigate this further, I plot in the following figure the variance discount rates obtained from the variance swap rates analyzed in the benchmark exercise.

Figure 2: The figure plots yearly moving averages of the variance discount rates obtained from the variance swap rate with one month to maturity (solid grey line), three months to maturity (dotted black line), six months to maturity (dash-dotted black line), 12 months to maturity (dashed black line), and 18 months to maturity (solid black line). The grey area corresponds to NBER recessions.

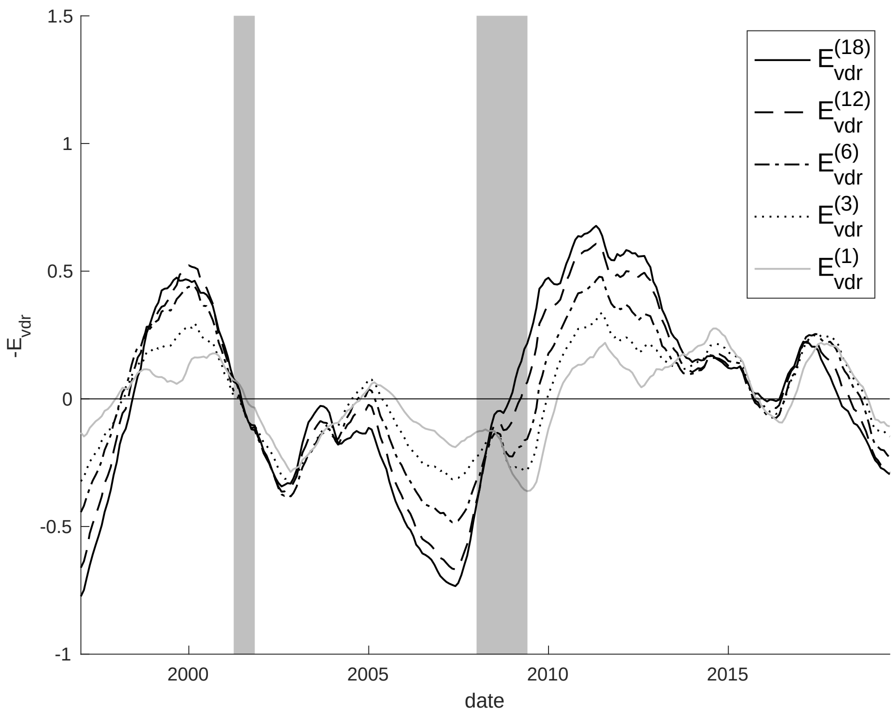

Figure 2 plots the variance discount rates obtained from the variance swap rate one month to maturity (solid grey line), three months to maturity (dotted black line), six months to maturity (dash-dotted black line), 12 months to maturity (dashed black line), and 18 months to maturity (solid black line). The main result from Figure 2 is that the time-variation in term structure of variance discount rates is correlated, such that short-term variance discount rates move in the same direction as long-term variance discount rates do. Overall, the variation in long-term variance discount rates is larger than variation in short-term discount rates. Interestingly, during the financial crisis of 2008, short-term variance discount rates decreased, whereas long-term variance discount rates increased. Therefore, only the price to hedge short-term variance risk decreased during the financial crisis. Finally, in the periods after the dot-com bubble and after 2015, the size of the variation in short-term and long-term variance discount rates is similar in magnitude. In the following, I show that the variation in variance discount rates is large in economic magnitudes.

I show in Figure 2 that variance discount rates are correlated for each of the considered maturities. In the following, I show that this time variation implies economically sizable differences in the expected returns for investing in a variance swap contract. To show this, I run the following regressions:

$$R_{t+1}^{(T)} - 1 = \mu_L + \mu_{H-L} \cdot \mathbb{1}_{\text{vdr},t}^{(T)} + \epsilon_{t+1}, \tag{18}$$

where, $R_{t+1}^{(T)} - 1$ correspond to simple returns on a variance swap with T-months to maturity and $\mathbb{1}_{\text{vdr},t}^{(T)}$ is an indicator function that equals one if $E_{\text{vdr},t}^{(T)}$ exceeds its median level over the sample. Therefore, μ_L equals the average return on a variance swap during periods of low variance discount rates and μ_{H-L} equals by how much the average return increases when variance discount rates are high rather than low. Table 6 shows the results of regression (18).

Table 6: This table presents the estimates of regression equation (18). The second column indicates the average simple return during periods of low variance discount rates, and the third column indicates how much the average return increases during periods of high variance discount rates. t-statistics are represented in parentheses.

Maturity	μ_L (t-stat.)	μ_{H-L} (t-stat.)
18	-0.043	0.074
	(-2.75)	(3.38)
12	-0.062	0.098
	(-3.33)	(3.71)
6	-0.105	0.112
	(-4.01)	(3.01)
3	-0.181	0.167
	(-4.87)	(3.18)
1	-0.367	0.166
	(-6.48)	(2.07)

There are two main takeaways from Table 6. First, during periods of low variance discount rates, the expected return on a one-month variance swap equals -36.7% per month and increases to -4.3% per month for 18-month variance swaps. Therefore, during periods of low variance discount rates—or periods during which hedging variance risk is expensive—expected returns are negative and economically sizable. Second, expected returns on variance swaps increase during periods of high variance discount rates. The expected return on a one-month variance swap increases with 16.7% per month, and the expected return on a 18-month variance swap increases with 7.4% per month. I conclude that the variation in variance discount rates is, both for short-term and long-term variance swaps, economically sizable. Interestingly, the estimates indicate that expected return on a variance swap with a maturity beyond three months is positive during periods of high variance discount rates. In the following, I present evidence that, under the current VAR specification, variance expectations and variance discount rates are obtained effectively.

I run the same predictive regressions of the future variance and future returns on the variance swap as before, only I use variance expectations and variance discount rates as the predictive variables. The regressions are specified as follows:

$$y_{\mathrm{rv},t+T} = \gamma_{0,\mathrm{rv}} + \gamma_{1,\mathrm{rv}} \cdot E^{(T)}_{\mathrm{rv},t} + \gamma_{2,\mathrm{rv}} \cdot \left(- E^{(T)}_{\mathrm{vdr},t}\right) + u^{\mathrm{rv}}_{t+T} \quad \text{and} \tag{19}$$

$$-y_{\mathrm{vdr},t+T} = \gamma_{0,\mathrm{vdr}} + \gamma_{1,\mathrm{vdr}} \cdot E^{(T)}_{\mathrm{rv},t} + \gamma_{2,\mathrm{vdr}} \cdot \left(- E^{(T)}_{\mathrm{vdr},t}\right) + u^{\mathrm{vdr}}_{t+T}. \tag{20}$$

If variance expectations and variance discount rates are obtained effectively, then variance expectations $E^{(T)}_{\mathrm{rv},t}$ should only predict future stock market variance and variance discount rates $E^{(T)}_{\mathrm{vdr},t}$ should only predict future returns on the variance swap. Moreover, the regressions are specified such that I should find the following: $\gamma_{1,\mathrm{rv}} = 1$ and $\gamma_{2,\mathrm{rv}} = 0$ in the case of regression equation (19) and $\gamma_{1,\mathrm{vdr}} = 0$ and $\gamma_{2,\mathrm{vdr}} = 1$ in the case of regression equation (20). Table 7 presents the results.

Table 7: This table presents the results of the predictive regressions of equations (19) and (20) in which the expected realized variance and expected variance discount rates are the independent variables. t-statistics are represented in parentheses and are computed using Newey-West standard errors with number of lags equal to T.

Dependent variable:	$y_{\mathrm{rv},t+T}$			$-y_{\mathrm{vdr},t+T}$		
Maturity	$\gamma_{1,\mathrm{rv}}$ (t-stat.)	$\gamma_{2,\mathrm{rv}}$ (t-stat.)	R^2	$\gamma_{1,\mathrm{vdr}}$ (t-stat.)	$\gamma_{2,\mathrm{vdr}}$ (t-stat.)	R^2
18	0.943	-0.007	0.127	0.095	0.956	0.267
	(3.61)	(-0.03)		(0.34)	(4.06)	
12	1.104	0.124	0.269	-0.089	0.822	0.191
	(4.48)	(0.58)		(-0.34)	(3.68)	
6	1.016	0.178	0.399	-0.005	0.787	0.118
	(7.92)	(0.74)		(-0.03)	(3.13)	
3	0.964	0.143	0.479	0.030	0.839	0.077
	(13.56)	(0.54)		(0.45)	(3.09)	
1	1.043	0.346	0.592	-0.043	0.653	0.045
	(17.82)	(1.29)		(-0.73)	(2.63)	

Table 7 shows the results of two regressions: a regression to predict future stock market variance in the columns labeled $y_{\mathrm{rv},t+T}$ and a regression to predict future variance swap returns in the columns labeled $-y_{\mathrm{vdr},t+T}$. First, from the regression to predict future variance it follows that, indeed, future variance is predicted by variance expectations and not predicted by variance discount rates. Moreover, the regression coefficient of variance expectations $\gamma_{1,\mathrm{rv}}$ to predict future variance is close to, and not significantly different from, one for each maturity, and this shows that the VAR is able to capture stock market variance beyond one month. Finally, $\gamma_{1,\mathrm{rv}}$ is not exactly equal to one for the regression with one month to maturity, because variance discount rates are implied using the current variance swap rates, which contain information that is not included in the VAR.

Second, from the regression to predict future returns on variance swaps, it follows that, indeed, future returns are predicted by variance discount rates and not predicted by variance expectations. The regression coefficient of variance discount rates to predict future returns $\gamma_{2,\mathrm{vdr}}$ are fairly close, and not significantly different from, one, and this indicates that variance discount rates are able to predict future returns on the variance swap for each maturity. In fact, as indicated by the increasing R^2's, returns on long-term variance swaps are predicted more effectively by variance discount rates than returns on short-term variance swaps. Again, this is in line with the result that variation in long-term variance discount rates is larger than variation in short-term variance discount rates.

In the following subsections, I decompose returns on variance swaps. This allows me to study how quickly variance expectations and variance discount rates move over time and analyze their correlation with stock market returns.

3.5 Decomposition of variance swap returns

In this subsection, I show how to decompose returns on variance swaps into news about future expected variance and news about future variance discount rates using identity (10). This analysis is similar to what Campbell and Vuolteenaho (2004) do for realized stock market returns. The decomposition of returns on variance swaps, rather than variance swap rates, helps to understand how quickly expectations over stock market variance and variance discount rates change over time.

I show how to use the VAR of equation (13) to decompose returns on variance swaps using identity (10). The left-hand side of identity (10) corresponds to realizations of variance swap returns. In order to calculate return realizations using the VAR of equation (13), I assume that all variation in variance swap rates is captured by the first two principal components of the panel of variance swap rates. This two-factor assumption implies the following:

$$vs_t^{(T)} = \beta_0(T) + \beta_1(T) \cdot pc_t^{(1)} + \beta_2(T) \cdot pc_t^{(2)}. \tag{21}$$

Simple regressions show that equation (21) captures between 99.1% and 99.8% of the variation, with an average of 99.5% for all variance swap rates with one up to 18 months to maturity. Therefore, the variation in variance swap rates is accurately captured by the first two principal components. Under this assumption, realizations of variance swap returns are defined as follows:

$$\begin{aligned} r_{t+1}^{(T)} &\approx k(T) + \rho(T) \cdot vs_{t+1}^{(T-1)} + \big[1-\rho(T)\big] rv_{t+1} - vs_t^{(T)} \\ \iff \quad r_{t+1}^{(T)} - \mathbb{E}_t r_{t+1}^{(T)} &\approx \Big(\mathbb{E}_{t+1} - \mathbb{E}_t\Big)\Big[\rho(T) vs_{t+1}^{(T-1)} + \big[1-\rho(T)\big] rv_{t+1}\Big] \\ &= e_L(T)' \cdot \epsilon_{t+1}, \end{aligned} \tag{22}$$

where,

$$e_L(T) = \Big(1-\rho(T) \quad \rho(T)\beta_1(T-1) \quad \rho(T)\beta_2(T-1) \quad 0\Big)'.$$

It follows from the definition of $e_L(T)$ that the return realization of a variance swap is a linear combination of the realization toward stock market variance, the realization toward the first principal component and the realization toward the second principal component. Vector $e_L(T)$ depends on the maturity of the variance swap, because the relative importance of each of the individual realizations toward the variables in the VAR depends on the maturity and is captured by the constants $\rho(T)$, $\beta_1(T-1)$, and $\beta_2(T-1)$. In sum, the realizations of variance swap returns follow from the two-factor assumption of equation (21) and from the assumption that log-linear approxmation of the return holds with an equality. I show in Appendix A.5 that both assumptions indeed hold in the data and realizations of variance swap returns are obtained effectively. In the following, I define news about stock market variance and news future variance discount rates.

News about future stock market variance follows directly from the VAR and is defined as follows:

$$N^{(T)}_{\mathrm{rv},t+1} = e_1' \Bigg(\big(1-\rho(T)\big) + \cdots + \big(1-\rho(1)\big)\rho(T) \times \cdots \times \rho(2) B^{T-1} \Bigg) \epsilon_{t+1}. \tag{23}$$

This definition is very similar to definition of $E^{(T)}_{\mathrm{rv},t}$; however, $N^{(T)}_{\mathrm{rv},t}$ depends on the error term rather than the current level of the state variables. The return identity (10) implies that returns on variance swaps are fully explained by news about variance and news about variance discount rates. Therefore, news about variance discount rates is defined in the following way:

$$N^{(T)}_{\mathrm{vdr},t+1} = N^{(T)}_{\mathrm{rv},t+1} - e_L(T)' \cdot \epsilon_{t+1}. \tag{24}$$

I use news about stock market variance of equation (23) and news about variance discount rates of equation (24) to decompose variation in returns on variance swaps. By calculating the variance of identity (10), I obtain the following identity to decompose the variation in returns, as follows:

$$\mathrm{var}\big(\bar{r}^{(T)}_{t+1}\big) \approx \mathrm{var}\big(N^{(T)}_{\mathrm{rv},t+1}\big) + \mathrm{var}\big(N^{(T)}_{\mathrm{vdr},t+1}\big) - 2 \cdot \mathrm{cov}\big(N^{(T)}_{\mathrm{rv},t+1}, N^{(T)}_{\mathrm{vdr},t+1}\big), \tag{25}$$

where $\bar{r}^{(T)}_{t+1} := e_L(T)' \cdot \epsilon_{t+1}$ is the return realization of a variance swap with maturity T at time $t+1$. Equation (25) implies that variation in variance swap returns is attributed to news about variance, news about variance discount rates, or the covariance between the two. Table 8 presents the results of this decomposition.

Table 8: This table shows the results of the variance decomposition of variance swap rates using equation (25). Note that the (co)variances of the third, fourth, and fifth columns are scaled with the variance of the second column such that the sum of the three (co)variances equals one. Standard errors are computed using the Delta method.

T	$\text{var}(\bar{r})$	$\frac{\text{var}(N_{\text{rv}})}{\text{var}(\bar{r})}$ (s.e.)	$\frac{\text{var}(N_{\text{vdr}})}{\text{var}(\bar{r})}$ (s.e.)	$\frac{-2\text{cov}(N_{\text{rv}},N_{\text{vdr}})}{\text{var}(\bar{r})}$ (s.e.)
18	0.023	0.630	0.674	-0.304
		(0.274)	(0.274)	(0.415)
12	0.032	0.966	0.414	-0.380
		(0.325)	(0.192)	(0.420)
6	0.063	1.224	0.123	-0.346
		(0.246)	(0.063)	(0.286)
3	0.115	1.217	0.033	-0.250
		(0.124)	(0.017)	(0.138)
1	0.393	1.000	0.000	0.000
		(-)	(-)	(-)

The first observation of Table 8 is that the variance of variance swap returns is strongly decreasing in the maturity of the variance swap and equals 39.3% for a variance swap with one month to maturity and equal to 2.3% for a variance swap with 18 months to maturity. The return on a one-month variance swap is solely driven by news about stock market variance, because the variance swap matures in the next periods and, therefore, is independent of variance discount rates by definition. Variation in returns on short-term variance swaps is driven by news about stock market variance and drives, respectively, 122.4% and 121.7% of the total variation in variance swap returns with six and three months to maturity. This result indicates that variance expectations change more quickly over time than variance discount rates do. However, even variation in short-term variance swap returns is driven by news about variance discount rates with a coefficient significantly different from zero and accounts for 12.3% of the variation in six-month variance swap returns and 3.3% of the variation in three-month variance swap returns.

Variation in returns of long-term variance swaps is also largely driven by news about variance and drives 96.6% of the total variation in 12-month variance swap returns and 63.0% of the total variation in 18-month variance swap returns. However, variation in long-term variance swap rates is driven by a larger fraction due to news about variance discount rates and drives 67.4% of the total variation in 18-month variance swap returns and 41.4% of the total variation in 12-month variance swap returns. Interestingly, the covariance between news about stock market variance and news about variance discount rates is positive for each maturity and indicates that hedging variance risk becomes less expensive if expected stock market variance increases.

In Appendix A.5, I show that the results of obtaining the realizations to variance swap return using equation (22) or by including the variance swap return $r_{t+1}^{(T)}$ as an additional state variable in the VAR yield very similar results. If $r_{t+1}^{(T)}$ is an additional state variable in the VAR, realizations toward variance swap returns are obtained directly from the VAR. In the following subsection, I analyze the correlation between returns on variance swaps and stock market returns. This allows me to decompose the stock market beta of variance swaps.

3.6 Decomposition of stock market returns

In this subsection, I decompose stock market returns into news about dividends and news about stock market discount rates, following Campbell and Vuolteenaho (2004). A salient feature in the data of stock market returns is the strong negative correlation between stock market variance and stock market returns, which is also known as the leverage effect. Therefore, variance swaps have a negative stock market beta and I analyze the drivers of this negative beta.

In a derivation analogous to Section 2, Campbell and Vuolteenaho (2004) derive an identity to decompose realizations of stock market returns, as follows:

$$\begin{aligned} r_{t+1} - \mathbb{E}_t r_{t+1} &\approx \left(\mathbb{E}_{t+1} - \mathbb{E}_t\right)\Big[\sum_{i=0}^{\infty} \rho^i \Delta d_{t+1+i}\Big] - \left(\mathbb{E}_{t+1} - \mathbb{E}_t\right)\Big[\sum_{i=1}^{\infty} \rho^i r_{t+1+i}\Big] \\ &= N_{\mathrm{cf},t+1} - N_{\mathrm{dr},t+1}, \end{aligned} \tag{26}$$

where r_{t+1} is the stock market return, Δd_{t+1} dividend growth, and ρ the log-linearization coefficient. The stock market is a perpetual cash flow, and, therefore, is ρ is constant. The intuition of identity (26) is as follows: Large stock market returns are driven by news about high dividends or news about low stock market discount rates. In order to decompose the realizations to stock market returns, I follow Campbell and Vuolteenaho (2004) and model discount rates using a VAR.

The VAR is similar to before; only Campbell and Vuolteenaho (2004) estimate the model with an intercept and use the following state variables:

$$z_t = \begin{pmatrix} r_t & TY_t & PE_t & VS_t \end{pmatrix}', \tag{27}$$

where TY_t is the term yield spread, PE_t is the log price earning ratio, and VS_t is the small-stock value spread. The model is estimated on a monthly sample from January 1929 to December 2018. Table 9 presents the estimation results.

Table 9: This table shows the coefficients of the VAR of equation (13) with state variables of equation (27), and a vector with intercepts c. The t-values of the coefficients are denoted in parentheses, sample period for the dependent variables is January 1929 – December 2018, with 1080 monthly data points.

Coefficients VAR model						
	c	r_t	TY_t	PE_t	VS_t	R^2
r_{t+1}	0.064	0.101	0.004	-0.015	-0.012	0.025
(t-stat.)	(3.59)	(3.35)	(2.06)	(-3.26)	(-2.27)	
TY_{t+1}	-0.026	0.067	0.938	-0.004	0.053	0.897
(t-stat.)	(-0.29)	(0.44)	(90.28)	(-0.16)	(2.04)	
PE_{t+1}	0.025	0.511	0.001	0.992	-0.003	0.991
(t-stat.)	(2.10)	(25.07)	(0.76)	(320.99)	(-0.83)	
VS_{t+1}	0.019	0.011	0.000	-0.001	0.989	0.979
(t-stat.)	(1.14)	(0.40)	(0.14)	(-0.12)	(203.71)	

The estimates of the VAR model in Table 9 are very similar to the estimates reported in Campbell and Vuolteenaho (2004). This is expected, because I extend the original sample by 17 years. Using the estimates of Table 9 and $\rho = 0.95^{1/12}$, as in Campbell and Vuolteenaho (2004), news about dividends $N_{\mathrm{cf},t+1}$ and news about stock market discount rates $N_{\mathrm{dr},t+1}$ are defined as follows:

$$N_{\mathrm{cf},t+1} = \Big(e_1' + e_1'\rho B\big(I - \rho B\big)^{-1}\Big)\epsilon_{t+1} \quad \text{and} \tag{28}$$

$$N_{\mathrm{dr},t+1} = e_1'\rho B\big(I - \rho B\big)^{-1}\epsilon_{t+1}. \tag{29}$$

Over the sample period January 1996 until December 2018, I estimate $N^{(T)}_{\mathrm{rv},t+1}$, $N^{(T)}_{\mathrm{vdr},t+1}$, $N_{\mathrm{cf},t+1}$, and $N_{\mathrm{dr},t+1}$. In Figure 3, I plot each standardized news term over the sample period from stock market returns and returns on a 12-month variance swap. Furthermore, Table 10 presents the correlation coefficients of the variables over the sample period.

Figure 3: The figure plots yearly moving averages of $N^{(12)}_{\mathrm{rv},t}$, $N^{(12)}_{\mathrm{vdr},t}$, $N_{\mathrm{cf},t}$ and $N_{\mathrm{dr},t}$ over the sample period from January 1996 – December 2018. Each of the news terms is standardized with its standard deviation. The shaded area corresponds to the NBER recessions.

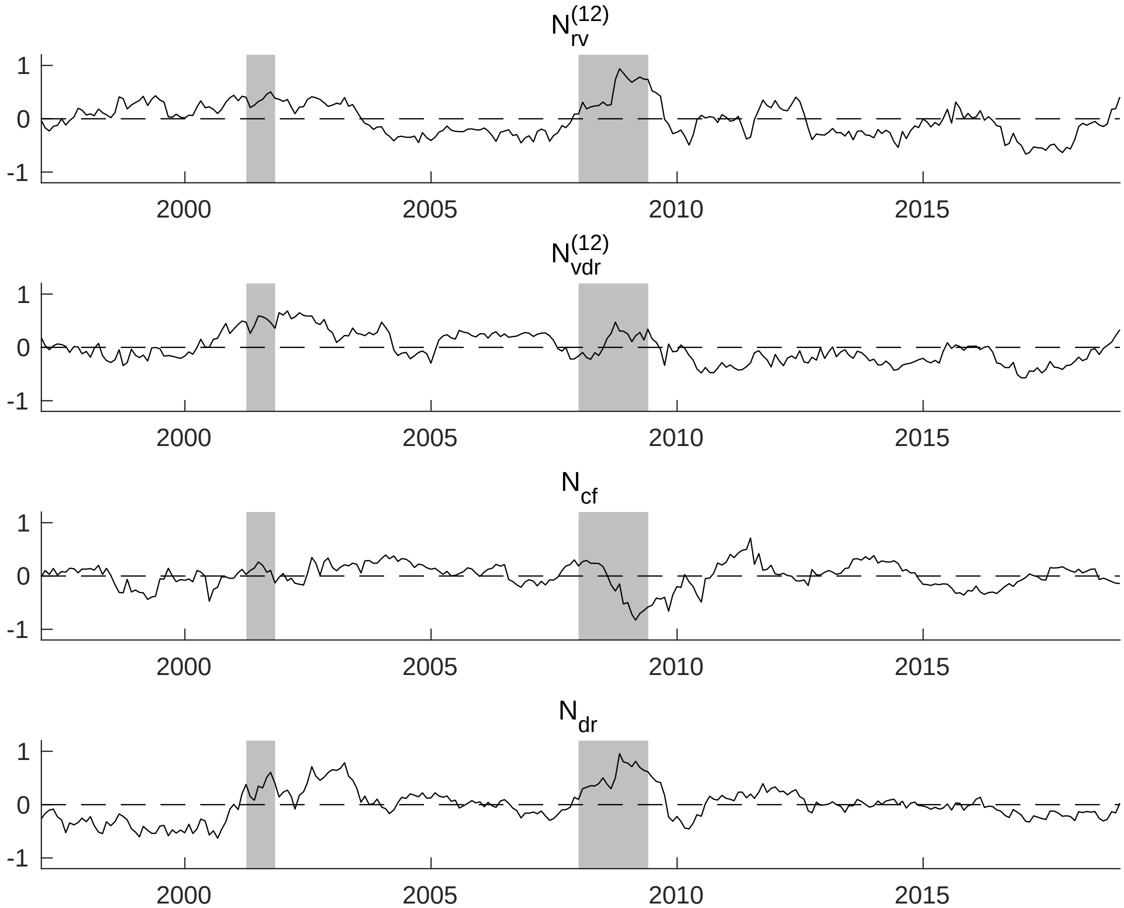

Table 10: This table represent the estimated correlation matrix of $N^{(12)}_{\mathrm{rv},t}$, $N^{(12)}_{\mathrm{vdr},t}$, $N_{\mathrm{cf},t}$, and $N_{\mathrm{dr},t}$ over the sample period from January 1996 – December 2018.

Correlation Matrix				
	$N^{(12)}_{\mathrm{rv},t}$	$N^{(12)}_{\mathrm{vdr},t}$	$N_{\mathrm{cf},t}$	$N_{\mathrm{dr},t}$
$N^{(12)}_{\mathrm{rv},t}$	1.000	0.302	-0.166	0.566
$N^{(12)}_{\mathrm{vdr},t}$	0.302	1.000	0.021	-0.069
$N_{\mathrm{cf},t}$	-0.166	0.021	1.000	0.266
$N_{\mathrm{dr},t}$	0.566	-0.069	0.266	1.000

The first graph of Figure 3 plots news about variance, the second graph plots news about variance discount rates, the third graph plots news about dividends, and the fourth graph plots news about stock market discount rates. The first and fourth graphs show that during episodes of large stock market variance, stock market discount rates tend to increase. This result is supported by the evidence in the correlation matrix of Table 10, which shows that the correlation coefficient between news about variance and news about stock market discount rates is positive.

A salient feature of stock market returns is the negative correlation between realized returns and realized stock market variance, and, therefore, returns on variance swaps and stock market returns are negatively correlated. Therefore, the stock market beta of variance swaps is also negative. Using the decomposition of variance swap returns and stock market returns, I analyze which parts of the variance swap return and stock market return drive the low stock market beta. I define the stock market beta for variance swaps in the following way:

$$N_{\mathrm{rv},t+1} - N_{\mathrm{vdr},t+1} = \alpha + \beta_M \big(N_{\mathrm{cf},t+1} - N_{\mathrm{dr},t+1}\big) + u_{t+1}. \tag{30}$$

The stock market beta β_M of a variance swap is decomposed into four covariances, all divided by $\mathrm{var}\big(N_{\mathrm{cf},t+1} - N_{\mathrm{dr},t+1}\big)$, as follows:

$$\begin{aligned}\beta_M = {} & \frac{\mathrm{cov}\big(N_{\mathrm{rv},t+1}, N_{\mathrm{cf},t+1}\big)}{\mathrm{var}\big(N_{\mathrm{cf},t+1} - N_{\mathrm{dr},t+1}\big)} + \frac{\mathrm{cov}\big(N_{\mathrm{rv},t+1}, -N_{\mathrm{dr},t+1}\big)}{\mathrm{var}\big(N_{\mathrm{cf},t+1} - N_{\mathrm{dr},t+1}\big)} + \frac{\mathrm{cov}\big(-N_{\mathrm{vdr},t+1}, N_{\mathrm{cf},t+1}\big)}{\mathrm{var}\big(N_{\mathrm{cf},t+1} - N_{\mathrm{dr},t+1}\big)} \\ & + \frac{\mathrm{cov}\big(N_{\mathrm{vdr},t+1}, N_{\mathrm{dr},t+1}\big)}{\mathrm{var}\big(N_{\mathrm{cf},t+1} - N_{\mathrm{dr},t+1}\big)} = \beta_{\mathrm{rv,cf}} + \beta_{\mathrm{rv,dr}} + \beta_{\mathrm{vdr,cf}} + \beta_{\mathrm{vdr,dr}}.\end{aligned}$$

Therefore, I decompose β_M into four covariances: the covariance between news about variance and news about dividends, the covariance between news about variance and news about stock market discount rates, the covariance between news about variance discount rates and news about dividends, and the covariance between news about variance discount rates and news about stock market discount rates. Table 11 presents the result of this decomposition.

Table 11: This table shows the results of β_M, estimated using regression equation (30) in the second column for several maturities. The remaining columns show how this β_M is decomposed into covariance between realized variance and dividends $\beta_{\text{rv,cf}}$, realized variance and market discount rates $\beta_{\text{rv,dr}}$, variance discount rates and dividends $\beta_{\text{vdr,cf}}$ and variance discount rates, and stock market discount rates $\beta_{\text{vdr,dr}}$.

T	β_M (t-stat.)	$\beta_{\text{rv,cf}}$ (t-stat.)	$\beta_{\text{rv,dr}}$ (t-stat.)	$\beta_{\text{vdr,cf}}$ (t-stat.)	$\beta_{\text{vdr,dr}}$ (t-stat.)
18	-2.210	-0.230	-1.446	-0.072	-0.462
	(-14.02)	(-2.35)	(-9.01)	(-0.74)	(-1.75)
12	-2.759	-0.352	-2.200	-0.029	-0.178
	(-15.89)	(-2.42)	(-9.79)	(-0.34)	(-0.75)
6	-4.004	-0.562	-3.656	0.017	0.196
	(-16.71)	(-2.36)	(-10.29)	(0.28)	(1.16)
3	-5.237	-0.639	-4.996	0.031	0.366
	(-15.78)	(-2.00)	(-10.13)	(0.72)	(3.47)
1	-7.183	0.097	-7.281	0.000	0.000
	(-9.79)	(0.20)	(-7.76)	(-)	(-)

First, Table 11 shows that β_M increases in the maturity of the variance swap and increases from -7.18 for a variance swap with one month to maturity to -2.21 for a variance swap with 18 months to maturity. Second, this negative β_M is mostly attributed for each maturity to $\beta_{\text{rv,dr}}$ and corresponds to the correlation between news about variance and news about stock market discount rates. This result indicates that during periods of large stock market variance, stock market discount rates are revised upward to compensate for the increased risk of investing in the stock market. Third, the β_M of variance swaps with a maturity beyond one month is driven with a coefficient significantly different from zero by $\beta_{\text{rv,cf}}$ and corresponds to the negative correlation between news about variance and news about dividends. This result shows that during episodes of increased stock market variance, dividends expectations are revised downward. Finally, the sign of the correlation between news about variance discount rates and news about stock market discount rates changes from positive for short-term variance discount rates to negative for long-term variance discount rates.

In the next section, I discuss the implications of several prominent asset pricing models with respect to the pricing of variance risk. In particular, I show the results of the decomposition of variance swap rates in each of the considered models.

4 Variance risk in asset pricing models

In this section, I discuss the predictions of several prominent asset pricing models with respect to the pricing of variance risk. I discuss the implications of the following three models: the variable rare disaster model by Gabaix (2012), the time-varying rare disaster model by Wachter (2013), and the long-run risk model by Drechsler and Yaron (2011). These models are able to match some empirical results on the pricing of variance risk. Dew-Becker et al. (2017) show that the model by Gabaix (2012) matches the empirical results on the term structure of risk premia of variance risk, Seo and Wachter (2019) show that the model by Wachter (2013) matches the implied volatility slope on S&P 500 options, and the model by Drechsler and Yaron (2011) is designed to match the empirical results on the variance risk premium by Bollerslev et al. (2009).

I show that the models considered in this paper have profoundly different predictions regarding the decomposition variance swap rates. The model by Gabaix (2012) predicts that all variation in variance swap rates is attributed to variation in variance discount rates. On the other hand, the model by Wachter (2013) predicts that (almost) all variation in variance swap rates is attributed to variance expectations rather than variance discount rates. This difference follows from the fact that stock market variance is very persistent in the model by Wachter (2013), and this feature is not present in the model by Gabaix (2012). The model by Drechsler and Yaron (2011) predicts that, in line with the data, short-term variance swaps are mostly driven by variance expectations, whereas long-term variance swaps are mostly driven by variance discount rates. However, variation in short-term variance swaps is in the model by Drechsler and Yaron (2011), driven by a larger fraction due to variance discount rates than empirically observed. This results from the fact that the variation in variance risk in the model by Drechsler and Yaron (2011) is large and too predictable.

In the following subsections I briefly discuss the details of each of the models and provide insights into the drivers of variance risk. Furthermore, in Subsection 4.4 the results of the asset pricing models are discussed.

4.1 Variable disaster risk and CRRA preferences

The first model I consider is the variable rare disaster model of Gabaix (2012). I use the following specification from Dew-Becker et al. (2017):

$$\begin{aligned}\Delta c_{t+1} &= \mu_c + \sigma_c \epsilon_{c,t+1} + J_{c,t+1},\\ L_{t+1} &= \big(1-\rho_L\big)\bar{L} + \rho_L L_t + \sigma_L \epsilon_{L,t+1} \quad \text{and}\\ \Delta d_{t+1} &= \eta \sigma_c \epsilon_{c,t+1} - L_t \cdot \mathbb{1}_{J_{c,t}\neq 0},\end{aligned}$$

where $\epsilon_{c,t+1}, \epsilon_{L,t+1} \sim N(0,1)$ and $J_{c,t+1}$ is the jump process (rare disaster). The state variable L_t captures the exposure of the dividend process toward the rare disaster, and this exposure varies over time. During times when L_t is large, the stock market is affected more by consumption disasters than when L_t is low. The rare disaster process is modeled as a compound Poisson process and is defined as follows:

$$J_t = \sum_{i=1}^{N_t} \xi_{i,t}, \text{ where } N_t \sim \text{Poisson}(\lambda_t) \text{ and } \xi_{i,t} \sim N\big(\mu_d, \sigma_d\big). \tag{31}$$

Note that in the model by Gabaix (2012) $\lambda_t = \lambda$; that is the jump intensity does not vary over time. The representative agent in the model has power utility preferences with risk aversion parameter γ, which yields the following stochastic discount factor:

$$M_{t+1} = \delta\Big(\frac{C_{t+1}}{C_t}\Big)^{-\gamma},$$

where δ is the utility discount rate. I use the calibration from Dew-Becker et al. (2017), which is calibrated to match the risk premium on one-month variance swaps, and is given in Table 21 of Appendix B.1.

In order to obtain an equation for the realized stock market variance, I use the following log-linear stock market return approximation:

$$r_{m,t+1} \approx \kappa_0 + \kappa_1 pd_{t+1} - pd_t + \Delta d_{t+1}, \tag{32}$$

where κ_0, κ_1 are log-linearization constants and I use the follow approximation: $pd_t \approx z_0 + z_1 L_t$. The stock market return is then driven by two Gaussian shocks $\big(\epsilon_{c,t+1}$ and $\epsilon_{L,t+1}\big)$ and a jump shock $\big(L_t \cdot \mathbb{1}_{J_{c,t}\neq 0}\big)$. I follow Dew-Becker et al. (2017), who assume that, in the absence of a disaster, the shocks to consumption and the variable disaster have a deterministic variance. In the case of a disaster occurring, Dew-Becker et al. (2017) assume that the largest daily decline in the value of the stock market is $F\%$. Under these assumptions, realized variance over the next period equals the following:

$$RV_{t+1} = \kappa_1^2 z_1^2 \sigma_L^2 + \eta^2 \sigma_c^2 + F \cdot L_t \mathbb{1}_{J_{c,t}\neq 0}, \tag{33}$$

where the first two summands correspond to the variance from the consumption process and variable rare disaster process, and the last summand is the realized variance from the jump process.

Equation (33) offers a first insight into the drivers of variance risk in the model by Gabaix (2012). Realized variance depends on whether the consumption disaster hits the economy.

Given that the consumption disaster is a (very) undersirable outcome of the agent, she is willing to pay a large price to hedge this risk. Furthermore, it follows from equation (33) that the disaster size L_t drives variation in expected stock market variance.

The variance swap rate at time t with T months to maturity is computed as the sum of risk-neutral realized variances of equation (33), as follows:

$$VS_t^{(T)} = \sum_{t=1}^{T} \mathbb{E}_t^{\mathbb{Q}}\big(RV_{t+i}\big) = T \cdot v_0 + v_1 \sum_{i=i}^{T} \mathbb{E}_t\big(L_{t+i-1}\big), \tag{34}$$

where $v_0 = \kappa_1^2 z_1^2 \sigma_L^2 + \eta^2 \sigma_c^2$ is the diffusive variance and $v_1 = F \cdot \mathbb{E}^{\mathbb{Q}}\big(\mathbb{1}_{J_c \neq 0}\big)$. These equations show that the size of the disaster L_t also drives the risk premium embedded in the variance swap rate.

In Subsection 4.4 the stylized facts regarding variance swap rates and the variation of variance swap rates in the model by Gabaix (2012) are discussed. In the following subsection, I briefly discuss the details of the model by Wachter (2013).

4.2 Time-varying disaster risk and Epstein-Zin preferences

In this subsection, I discuss a discrete version of the model by Wachter (2013). Similar to Gabaix (2012), the consumption disaster risk varies over time. However, in the model by Wachter (2013), the disaster intensity, rather than the disaster size, varies over time. Furthermore, the agent in the model has preferences as in Epstein and Zin (1989), rather than CRRA preferences.

Consumption and dividend growth in the model are given by

$$\begin{aligned} \Delta c_{t+1} &= \mu_c + \sigma_c \epsilon_{c,t+1} + J_{t+1} \quad \text{and} \\ \Delta d_{t+1} &= \eta \Delta c_{t+1}, \end{aligned}$$

where $\epsilon_c \sim N(0,1)$ and J_t is a compound-Poisson as in equation (31) of the model by Gabaix (2012). However, in this model the intensity of the consumption disaster is time-varying and follows the following square-root process:

$$\lambda_{t+1} = \phi \lambda_t + (1-\phi)\mu_\lambda + \sigma_\lambda \sqrt{\lambda_t} \epsilon_{\lambda,t+1},$$

where $\epsilon_{\lambda,t} \sim N(0,1)$. The investor has Epstein-Zin utility with elasticity of intertemporal substitution (EIS) equal to one and, therefore, is the log-utility given by

$$v_t = (1-\beta)c_t + \frac{\beta}{1-\alpha} \log \mathbb{E}_t \exp\big(v_{t+1}(1-\alpha)\big),$$

where β is the utility discount rate and $\gamma = 1-\alpha$ is the risk aversion parameter. The calibration of the model is from Dew-Becker et al. (2017) and is given in Table 21 of Appendix B.2.

An equation for the realized variance in the model by Wachter (2013) follows from the log-linear market return which is given by

$$r_{m,t+1} \approx \kappa_0 + \kappa_1 pd_{t+1} - pd_t + \Delta d_{t+1},$$

where κ_0, κ_1 are log-linearization constants for the log-market return and pd_t is the log price-dividend ratio and is approximately linear in the state variable: $pd_t \approx z_0 + z_1 \lambda_t$. Under these assumptions, realized variance in this model given by

$$RV_{t+1} = \eta^2 \sigma_c^2 + \kappa_1^2 z_1^2 \sigma_\lambda^2 \lambda_t - F\eta J_{t+1}, \tag{35}$$

where the first two summands correspond the variances of the diffusive shocks $\epsilon_{c,t+1}$ and $\epsilon_{\lambda,t+1}$ and the last summand corresponds to the realized variance from the consumption disaster.

Equation 35 offers a first insight into the pricing of variance risk in the model by Wachter (2013). The first summand of equation (33) is constant over time; however, the second summand scales with the level of the intensity of the consumption disaster. This results from the fact that the disaster intensity follows a square-root process, which indicates that future variance of the disaster intensity scales with the current level of the disaster intensity. Therefore, even in the absence of consumption disasters, stock market variance is time-varying in this model. This result is different from the model by Gabaix (2012) in which the variance of the stock market only varies if a disaster hits the economy. The third summand of equation (35) corresponds to the stock market variance that follows from the disaster process.

Variance swap rates are computed as the risk-neutral expectation of the sum of realized variances of equation (35) of period $t+1$ until $t+T$, as follows:

$$VS_t^{(T)} = \sum_{t=1}^{T} \mathbb{E}_t^{\mathbb{Q}}\big(RV_{t+i}\big) = T \cdot v_0 + v_1 \sum_{i=1}^{T} \mathbb{E}^{\mathbb{Q}}(\lambda_{t+i-1}), \tag{36}$$

where $v_0 = \eta^2 \sigma_c^2$ and $v_1 = \kappa_1^2 z_1^2 \sigma_\lambda^2 - F\eta \exp\big(-\alpha\mu_d + \frac{1}{2}\alpha\sigma_d^2\big)(\mu_d - \alpha\sigma_d^2)$. Due to the Epstein-Zin preferences of the agent, the risk-neutral dynamics of the disaster intensity are different from the real-world dynamics in the sense that states with low lifetime utility, which correspond to states with high disaster intensity, receive a larger risk-neutral probability. This yields the agent a premium for instruments that offer protection against states in which disaster intensity is high, and this feature is not present in a model with CRRA preferences.

In Subsection 4.4 the stylized facts regarding the pricing of variance risk of the model by Wachter (2013) are discussed. In the following subsection, I briefly discuss the details of the model by Drechsler and Yaron (2011).

4.3 Long-run risk

In this subsection, I discuss the long-run risk model by Drechsler and Yaron (2011). This model is a generalization of the long-run risk model by Bansal and Yaron (2004) in order to incorporate stylized facts regarding the variance risk premium. The model is generalized in the sense that the long-run mean consumption growth and the stochastic volatility incorporate jump shocks. Moreover, the long-run mean of the stochastic volatility process varies over time. The agent in the model has Epstein-Zin preferences, as is standard in long-run risk models. An important difference between the long-run risk model and the previously discussed consumption disaster models is that there are no consumption disasters in the long-run risk model. However, the state variables, which govern the future consumption growth rate and future consumption volatility, are exposed to jump risk.

Drechsler and Yaron (2011) specify the state vector of the economy as a VAR with Gaussian and jump shocks, as follows:

$$Y_{t+1} = \begin{pmatrix} \Delta c_{t+1} \\ x_{t+1} \\ \bar{\sigma}^2_{t+1} \\ \sigma^2_{t+1} \\ \Delta d_{t+1} \end{pmatrix} = \mu + FY_t + G_t z_{t+1} + J_{t+1}, \tag{37}$$

where, μ is a vector with the means of each state variable, F is specified as follows:

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \rho_x & 0 & 0 & 0 \\ 0 & 0 & \rho_{\bar{\sigma}} & 0 & 0 \\ 0 & 0 & (1-\tilde{\rho}_\sigma) & \rho_\sigma & 0 \\ 0 & \phi & 0 & 0 & 0 \end{pmatrix}, \tag{38}$$

$G_t G_t'$ is the variance-covariance matrix, $z_{t+1} \sim N(0, I)$ is a vector of Gaussian shocks, and J_{t+1} is a vector of jump shocks. Jumps are compound-Poisson as in equation (31) with intensity λ_t, which can vary over time, similar to the model by Wachter (2013). Drechsler and Yaron (2011) consider a specification with jumps in x_t and σ^2_t, where $J_{x,t}$ is compound normal distributed and $J_{\sigma,t}$ is compound gamma distributed.

The first and last element of Y_t are the consumption and dividend growth, respectively. These processes have a time-varying mean, which is driven by the persistent process x_t, the second element of Y_t. The third element of Y_t is the long-run mean $\bar{\sigma}_t^2$ of the stochastic volatility process σ_t^2, the fourth element of Y_t.

The variance-covariance matrix, $G_t G_t'$, which governs the stochastic volatility of the model, and the jump intensity, λ_t, are affine in the state variable σ_t^2:

$$G_t G_t' = h + H_\sigma \sigma_t^2 \quad \text{and}$$
$$\lambda_t = l_0 + l_1 \sigma_t^2,$$

and, therefore, all variation in either the jump intensity or stochastic volatility is driven by σ_t^2.

The representative agent in the model has Epstein-Zin utility for which the stochastic discount factor is given by

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1},$$

where $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$, δ is the utility discount rate, γ is the risk aversion, ψ is the EIS, and $r_{c,t+1}$, the return on wealth. Drechsler and Yaron (2011) solve a log-linear version of the model and use $pd_{t+1} \approx A_{0,m} + A_m' Y_{t+1}$, which says that the log price-dividend ratio is linear in the state variables. Under these conditions is the log-linearized market return, written as follows:

$$r_{m,t+1} = r_0 + \left(B_r' F - A_m'\right) + B_r' G_t z_{t+1} + B_r' J_{t+1}, \tag{39}$$

where $A_{0,m}$, A_m', r_0 and B_r' are given in equations (8) and (9) of Drechsler and Yaron (2011). Realized variance during period $t+1$ is equal to

$$RV_{t+1} = B_r' h B_r + B_r' H_\sigma \sigma_t^2 B_r + B_r' J_{t+1} J_{t+1}' B_r. \tag{40}$$

The assumption underlying this realized variance equation is that the Gaussian shocks z_{t+1} occur diffusively during period $t+1$, while jumps happen on a single day.

Equation (40) offers a first insight into the pricing of variance risk in the model by Drechsler and Yaron (2011). The first summand corresponds to the constant variance coming from the Gaussian shocks in the model. The second summand corresponds to the stochastic variance coming from the Gaussian shocks for which the variance is governed by the state variable σ_t^2. Finally, the third summand corresponds to the realized variance coming from the jump realizations in the state variables x_t and σ_t^2. Similar to the model by Wachter (2013) is time-variation in the realized variance coming from stochastic variance of Gaussian shocks and from the jump shocks.

The variance swap rate at time t with maturity T is computed as the risk-neutral expectation of the realized variance of equation (40), as follows:

$$VS_t^{(T)} = \sum_{t=1}^{T} \mathbb{E}_t^{\mathbb{Q}}\big(RV_{t+i}\big) = T \cdot v_0 + v_1 \sum_{t=1}^{T} \mathbb{E}_t^{\mathbb{Q}}\big(\sigma_{t+i-1}^2\big),$$

where,

$$v_0 = B_r' h B_r \text{ and } v_1 = B_r' H_\sigma B_r + l_1 \cdot B_r' \Psi^{\mathbb{Q}} B_r.$$

In the last equation, $\Psi^{\mathbb{Q}}$ is a matrix that has the risk-neutral variance of the disaster realization on the diagonal and corresponds to equation (21) of Drechsler and Yaron (2011). In order to derive these equations, I used $l_{0,x} = l_{0,\sigma} = 0$ and $l_{1,x} = l_{1,\sigma}$ from the calibration of Drechsler and Yaron (2011). The full calibration of the model is from Drechsler and Yaron (2011) and presented in Table 5 of their paper.

In the following subsection, I present the predictions of the previously discussed asset pricing models with respect to the pricing of variance risk in combination with what I found in data.

4.4 Results from asset pricing models

In this subsection, I compare the results regarding the pricing of variance risk in the data to the predictions of the previously discussed asset pricing models. In each of the models the following stylized facts are calculated: the expected returns on variance swaps, the standard deviation of variance swap returns, variance of the variance swap rates, and how this variation is attributed to realized variance expectations and variance discount rates. These results from the models are obtained from a simulation study.[6] Furthermore, in Appendices B.1—B.3 I show that the decomposition of variance swap rates is stable across the simulation sets.

In the following, I show the results from the decomposition of variance swap rates in each of the considered models. I focus on the results from the simple decomposition in Subsection 3.3 and run the analogous predictive regressions in each of the models. Figure 4 plots the results.

[6]For each model, 1,000 independent simulation sets of a time-series with 1,000 data points are obtained. On the basis of these simulation sets, each of the considered statistics is calculated.

Figure 4: The left (right) figure plots how much of the variation in variance swap rates is driven by variance expectations (variance discount rates) in the data (solid line), the model by Gabaix (2012) (dashed line), the model by Wachter (2013) (dash-dotted line), and the model by Drechsler and Yaron (2011) (dotted line). The grey area corresponds to a 95% confidence interval. The results are plotted for variance swap rates with 1, 3, 6, 12, and 18 months to maturity. The y-axis corresponds to how much of the variation is attributed to variance expectations or variance discount rates in percentages.

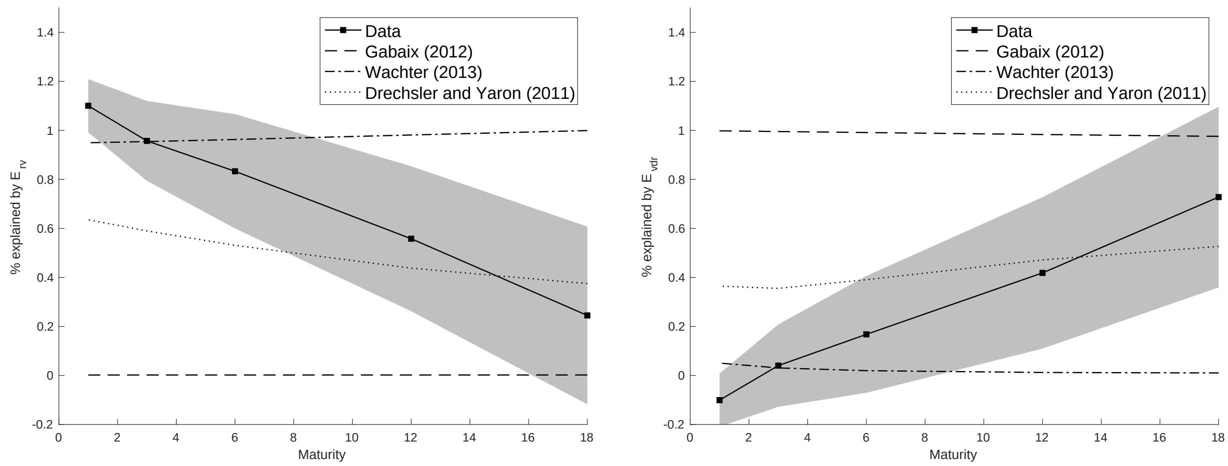

Figure 4 plots how much of the variation in variance swap rates is attributed to variance expectations (variance discount rates) in the left (right) graph. I show the results from the data (solid line) as well as the predictions of the models. In the model by Gabaix (2012) (dashed line), (almost) none of the variation in variance swap rates is driven by variation in realized variance expectations. Instead, variance discount rates drive all the variation in variance swap rates, as indicated by the right graph of Figure 4. The time-variation in the disaster size only affects the realized variance, conditional on a disaster hitting the economy. Given that this probability is low (1% p.a.), expected stock market variance only increases marginally when the disaster size increases. However, given that this consumption disaster is highly undesirable for the investor, variance discount rates adjust accordingly when the disaster size increases. In order to match the observed variation in short-term variance swap rates, the model has to incorporate some form of stochastic volatility.

The model by Wachter (2013) (dash-dotted line) predicts the exact opposite: (Almost) all of the variation in variance swap rates is driven by variation in realized variance expectations. This result is driven by the heteroskedastic nature of the disaster intensity process; that is high levels of the disaster intensity scale future variance of the disaster intensity upward. Therefore, stock market variance varies, even in the absence of disasters, over time. At the same time, high levels of the disaster intensity correspond to low variance discount rates; however, variation in variance discount rates drives a much smaller fraction in the total variation in variance swap rates, as seen in the right graph of Figure 4. Even if realizations in which the consumption disaster hits the economy are excluded, the variation in 18-month variance swap rates due to variance discount rates only increases to 6%. Therefore, even in the absence of consumption disasters, the model by Wachter (2013) is not able to match the data. Overall, the results from the decomposition in the model by Wachter (2013) show that stock market variance is too persistent in this model, which is driven by the persistence of the disaster intensity.

In the model by Drechsler and Yaron (2011) (dotted line), and in line with the data, most of the variation in short-term variance swap rates is driven by realized variance expectations, whereas long-term variance swap rates are driven mostly by variance discount rates. However, the right graph of Figure 4 shows that the fraction explained by variance discount rates is considerably larger in the model by Drechsler and Yaron (2011) than observed in the data. In the data the variation due to variance discount rates of the one-month variance swap rate is close to zero (even negative), whereas in the model by Drechsler and Yaron (2011), variance discount rate variation accounts for 36% of the variation in one-month variance swap rates. The variation in short-term variance discount rates is considerably larger in the model by Drechsler and Yaron (2011), because variation in variance risk is sizable and it varies in a predictable way. At the same time, the model is not able to capture the empirical result that the attribution of the variation in variance swap rates due to variance expectations strongly decreases in maturity. This result indicates that stock market variance is more persistent in the model than empirically observed, which is driven by the persistence of the state variables that govern the variance in the model ($\bar{\sigma}_t$ and σ_t).

In the following, I show the results of the total variation of variance swap rates in the data and in the considered asset pricing models. Figure 5 plots the results.

Figure 5: The figure plots the term structure of the variance of variance swap rates in the data (solid line), the model by Gabaix (2012) (dashed line), the model by Wachter (2013) (dash-dotted line), and the model by Drechsler and Yaron (2011) (dotted line). The grey area corresponds to a 95% confidence interval. The variance of variance swap rates with 1, 3, 6, 12, and 18 months to maturity. The y-axis corresponds to monthly volatility.

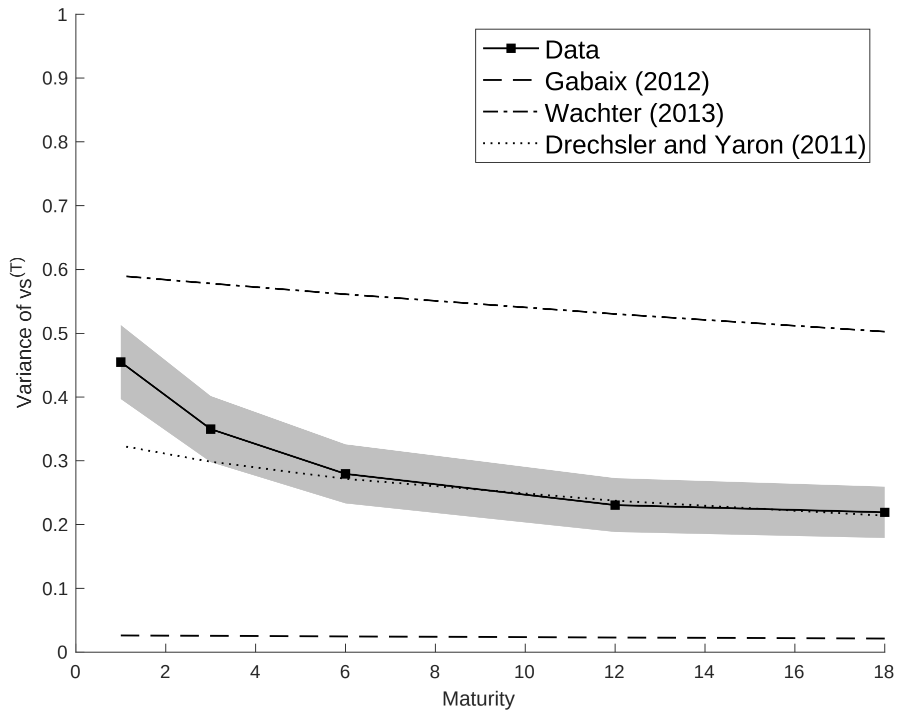

Figure 5 plots the term structure of the variance of variance swap rates in the data (solid line) and in each of the asset pricing models. The model by Gabaix (2012) (dashed line) predicts a flat term structure of the variance of variance swap rates, and its level is much lower than empirically observed. All variation in the model by Gabaix (2012) is driven by variation in the disaster size, and therefore, the variance of the stock market is only affected conditional on the disaster hitting the economy. Given that this probability is low (1% p.a.), variance swap rates only move marginally due to variation in stock market variance expectations. Due to the CRRA preferences in the model, the agent invests myopically such that she does not price shocks that affect the investment opportunity set, and, therefore, the term structure of the variance of variance swaps rates is flat. Furthermore, in the model by Gabaix (2012) all variation in variance swap rates is driven by variance discount rates, and, therefore, it follows that the size of the variation due to variance discount rates is relatively small.

The model by Wachter (2013) (dash-dotted line) overstates the empirically observed variation in variance swap rates. One of the reasons is that the disaster intensity process is very persistent, and, therefore, the unconditional variance of the disaster intensity is large, which drives most of the variation in variance swap rates. Another reason is that the process of the disaster intensity is heteroskedastic such that it scales with the level of the disaster intensity.

This effect yields that, if the current disaster intensity is high, future variance of the disaster intensity is high. Together, the large persistence and heteroskedastic nature of the disaster intensity make variance swap rates too volatile compared to the data.

The model by Drechsler and Yaron (2011) (dotted line) is best able to match the data on the term structure of variance of variance swap rates. Especially for long-term variance swap rates, the model by Drechsler and Yaron (2011) matches the data surprisingly well. However, the model generates slightly less variation in short-term variance swap rates than observed empirically. This result indicates that the variation in stock market variance is larger in the data than in the model by Drechsler and Yaron (2011).

In the following, I analyze the predictions of the models with respect to the expected return on variance swaps and volatility of variance swap returns. Figure 6 plots the results.

Figure 6: The left (right) graph plots the term structure of expected returns (return volatility) on variance swaps in the data (solid line), the model by Gabaix (2012) (dashed line), the model by Wachter (2013) (dash-dotted line), and the model by Drechsler and Yaron (2011) (dotted line). The grey area corresponds to a 95% confidence interval. The results are shown for variance swaps with 1, 3, 6, 12, and 18 months to maturity. The y-axis corresponds to monthly returns.

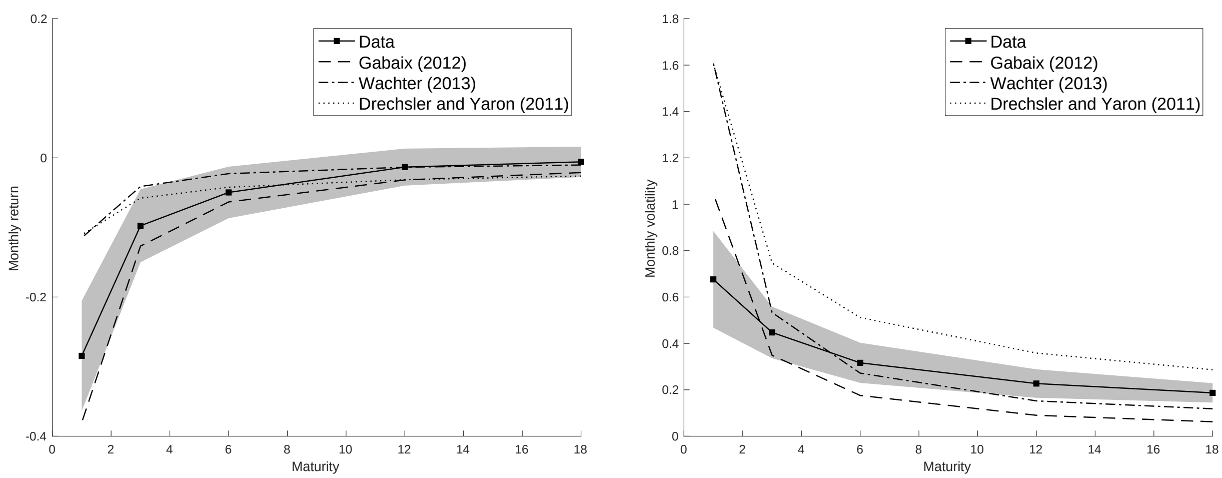

Figure 6 shows the term structure of expected returns (return volatility) on variance swaps in the left (right) graph. I show the results from the data (solid line) as well as the predictions of the models. In the left graph, I show that the model by Gabaix (2012) (dashed line) is able to match the strongly upward sloping term structure of expected returns on variance swaps found in the data. Overall, the expected returns in the model are only slightly lower compared to the data, and this could be resolved by adjusting the calibration, for example, decrease the average disaster size. The fact that the model by Gabaix (2012) is able to capture the term structure risk premia on the pricing of variance risk is in line with the findings of Dew-Becker et al. (2017). In the right graph, I show that the model by Gabaix (2012) (dashed line) predicts a stronger downward sloping term structure of return volatility on variance swaps than empirically observed. The model predicts a larger return volatility on variance swaps with one month to maturity and a lower return volatility for variance swaps with maturity beyond one month. A potential reason for this pattern is the fact that, conditional on a disaster not hitting the economy, the model by Gabaix (2012) does not have time-varying stock market volatility. Furthermore, an explanation for the large one-month return volatility is the severity of the consumption disaster (-30%) in the current calibration, which makes the return on the one-month variance swap, and thus the return volatility, (extremely) large, if a disaster hits the economy.

In the left graph of Figure 6, I show that the model by Wachter (2013) (dash-dotted line) is not able to capture the (extreme) low expected returns on short-term variance swaps, and, therefore, it cannot match the empirical term structure of expected returns. The main reason is that the mean consumption disaster in the calibration by Wachter (2013) (-15%) is much smaller than in the calibration by Gabaix (2012) (-30%). In the calibration by Wachter (2013), the mean disaster intensity is increased; however the left graph of Figure 6 shows that increasing the mean disaster size has a much larger effect on the expected returns of variance swaps. Interestingly, the expected returns for long-term variance swaps are quite similar to the data in the model by Wachter (2013). In the right graph, I show that the model by Wachter (2013) (dash-dotted line) predicts a larger return volatility on one-month variance swaps than empirically observed. Again, this result indicates that the mean of the consumption disaster is too extreme (-15%) or the frequency too large (3.55% p.a.). However, from the left graph it follows that this model cannot match the expected return on one-month variance swaps. A potential solution is to increase the size of the consumption disaster. This will, indeed, attenuate the mispricing in terms of expected returns on short-term variance swaps; however, at the same time the return volatility on short-term variance swaps will increase and thus exacerbate the discrepancy between the data and the model. Beyond a maturity of one month, the model by Wachter (2013) predicts only slightly lower return volatility compared to the data.

I show in the left graph of Figure 6, that the model by Drechsler and Yaron (2011) (dotted line) is not able to capture the strong upward sloping term structure of expected returns, and, in particular, the model does not predict the (extreme) low expected returns for short-term variance swaps. Moreover, the model has the opposite prediction for long-term variance swaps, as the expected returns are lower than empirically observed. Therefore, simply increasing the agent's risk aversion will alleviate the mispricing at short horizons, as the expected return will decrease; however, it will exacerbate the mispricing at long horizons. The reason for this mispricing at the short horizon is that jumps in the long-run risk are not sufficiently severe to capture the empirically observed risk premium. In the right graph of Figure 6, I show that the model by Drechsler and Yaron (2011) (dotted line) predicts a larger return volatility than empirically observed, both for short-term and long-term variance swaps. This results from the fact that in order for this model to generate a variance risk premium, the jumps in the volatility process σ_t^2 and long-run consumption mean x_t have to be sufficiently frequent and large. If the jumps appear in either of these state variables, volatility of the stock market spikes and results in a large return volatility for short-term variance swaps. The return volatility for long-term variance swaps is also large and this is driven by the persistence and volatility of the long-term volatility $\bar{\sigma}_t$ and volatillity σ_t processes.

5 Conclusion

I show that variance discount rates vary over time, which indicates that during some periods investors worry more about variance risk than during others. Moreover, the variation in variance discount rates drives a significant part of the variation in prices in the market for variance and, in particular, for longer horizons. Short-term variance swap rates are driven by variation in variance expectations, whereas long-term variance swap rates are mostly driven by variation in variance discount rates. Interestingly, prominent asset pricing models in which variance risk drives variation in the equity premium have profoundly different predictions regarding the decomposition of variance swap rates. The disaster model by Gabaix (2012) predicts that all variation in variance swap rates is attributed to variation in variance discount rates. On the other hand, the disaster model by Wachter (2013) predicts that all variation is attributed to variance expectations, and this is driven by the fact that this model incorporates a strong persistence in stock market variance. This feature is not present in the model by Gabaix (2012). The long-run risk model by Drechsler and Yaron (2011) predicts, in line with the data, that most of the variation in short-term variance swaps is driven by variance expectations, whereas most of the variation in long-term variance swaps is driven by variance discount rates. However, due to the large variation in short-term disaster risk, short-term variance discount rates move more of the variation in short-term variance swaps than empirically observed.

In sum, this paper presents new key stylized facts about the market for variance risk. I show that these stylized facts pose a challenge for state-of-the-art asset pricing models, and augmenting the asset pricing models to better describe the pricing of variance risk is thus an interesting avenue for future research.

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A Appendix: Additional empirical results

A.1 Variance swaps in Kozhan et al. (2013).

The realized variance of a variance swap entered at time t with maturity T is caluclated in the following way:

$$RV_t^{(T)} = \sum_{j=1}^{T} \Big[2\big(e^{r_{t+j}} - 1 - r_{t+j}\big) \Big], \tag{41}$$

where r_{t+j} is daily log return realized on day $t+j$. Note that equation (41) is similar to the sum of squared daily returns as $r^2 \approx 2\big(e^r - 1 - r\big)$. The variance swap rate is defined as the the risk-neutral expectation of the realized variance specified in equation (41). Kozhan et al. (2013) show how to calculate the variance swap rate with maturity T at time t from option prices, as follows:

$$VS_t^{(T)} = \frac{2}{B_t^{(T)}} \left[\int_0^{F_t^{(T)}} \frac{P_t^{(T)}(K)}{K^2} dK + \int_{F_t^{(T)}}^{\infty} \frac{C_t^{(T)}(K)}{K^2} dK \right], \tag{42}$$

where $B_t^{(T)}$ is the risk-free bond price at time t with maturity T, $F_t^{(T)}$ is the forward price at time t with maturity T and $P_t^{(T)}(K)$ and $C_t^{(T)}(K)$ are prices of European put and call options at time t with maturity T and strike price K.

Kozhan et al. (2013) show how to approximate equation (42) using a finite number of available put and call options. Given the set of available option prices $P_t^{(T)}(K_i)$ and $C_t^{(T)}(K_i)$ for $0 \leq i \leq N$ where prices are mid points from bid and ask quotes, Kozhan et al. (2013) compute variance swap rates as follows. Define the following function:

$$\Delta I(K_i) = \begin{cases} \frac{K_{i+1} - K_{i-1}}{2}, & \text{for } 0 \leq i \leq N \text{ (with } K_{-1} := 2K_0 - K_1,\ K_{N+1} := 2K_N - K_{N-1}) \\ 0, & \text{otherwise.} \end{cases}$$

Then the variance swap rate is computed as follows:

$$VS_t^{(T)} \approx 2 \sum_{K_i \leq F_t^{(T)}} \frac{P_t^{(T)}(K_i)}{B_t^{(T)} K_i^2} \Delta I(K_i) + 2 \sum_{K_i > F_t^{(T)}} \frac{C_t^{(T)}(K_i)}{B_t^{(T)} K_i^2} \Delta I(K_i). \tag{43}$$

Options on the S&P 500 expire every month on the third Friday. Using linear interpolation, I calculate variance swap rates that expire on the last trading day of each month. The linear interpolation works in the following way: Variance swap rates with maturity $T_1 < T$ and $T_2 > T$ are calculated by equation (43); then the variance swap rate with maturity T is constructed as follows:

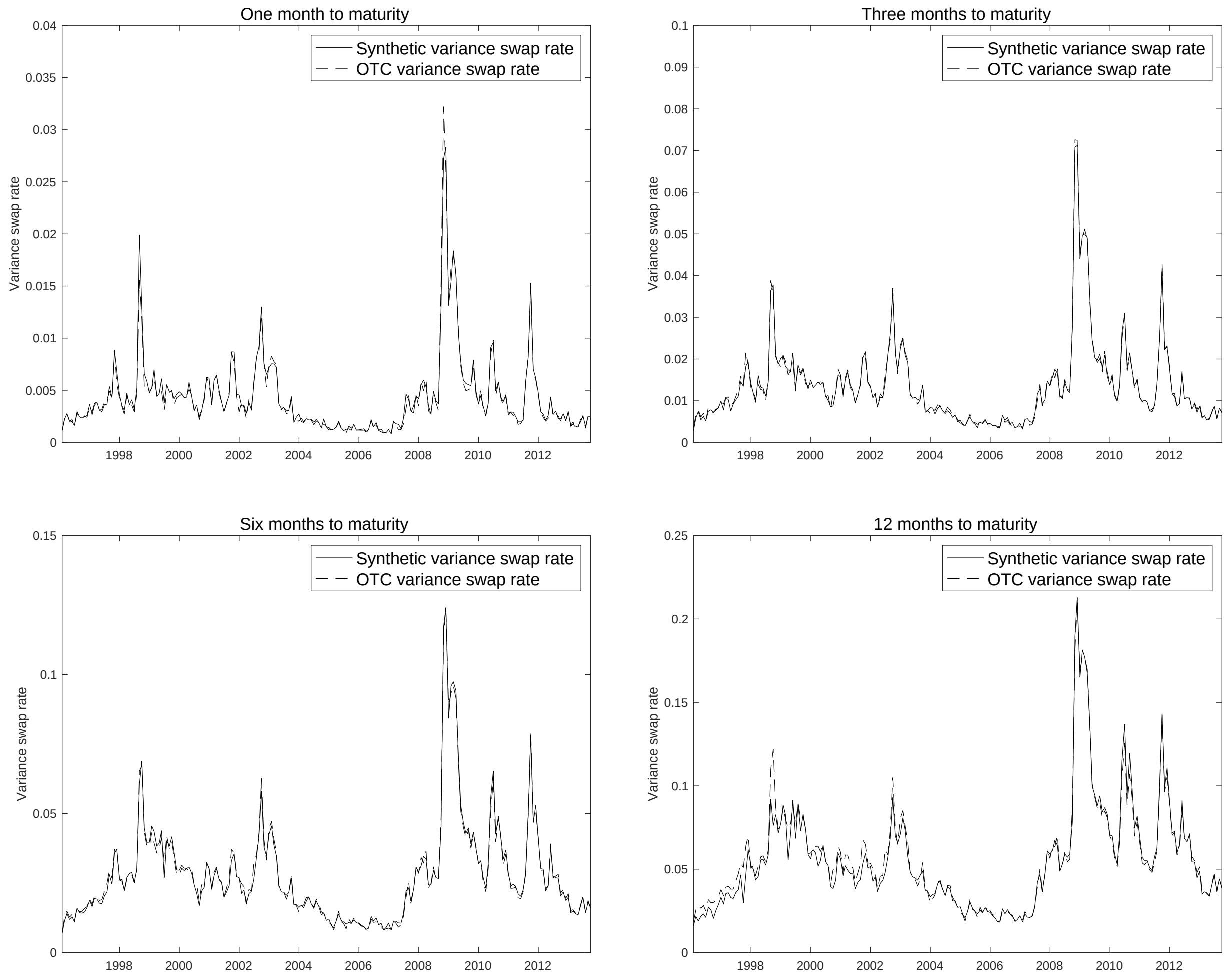
$$VS_t^{(T)} = \alpha VS_t^{(T_1)} + (1-\alpha)VS_t^{(T_2)},$$

where $T = \alpha T_1 + (1-\alpha)T_2$. With the data from OptionMetrics I calculate a panel of variance swap rates with one month up to 18 months to maturity.

A.2 Compare synthetic variance swaps to OTC variance swaps

In this section, I compare the data on synthetic variance swaps that are obtained from option pricing to the data on variance swaps from the OTC market. The data on variance swaps from the OTC market is from Dew-Becker et al. (2017). Their sample covers the period from December 1995 to September 2013 and variance swap rates up to a maturity of 12 months. During the period from January 1996 to September 2013, I observe a synthetic variance swap rate obtained using my methodology and a variance swap rate from the actual OTC data. I plot these rates in following graphs for one, three, six, and 12 months to maturity, which are the maturities of my benchmark analysis.

Figure 7: This figure plots the synthetic variance swap rate and OTC swap rate from Dew-Becker et al. (2017) for four maturities. The top-left graph plots the one-month swap rate, the top-right graph plots the three-month swap rate, the bottom-left graph plots the six-month swap rate, and the bottom-right graph plots the 12-month swap rate.



Overall, Figure 7 provides strong evidence that the synthetic variance swap rate is very similar to the swap rate in the OTC market. This indicates that the option market and the variance swap market are integrated markets and contain the same information regarding the pricing of variance risk. Notable differences include the difference in the one-month swap rate during the financial crisis and the difference in the 12-month swap rates during the first years of my sample. Furthermore, the average correlations between synthetic and OTC swap rate of the four maturities equals 0.991.

A.3 Impulse response functions of the VAR

In this subsection, I show the impulse response function of stock market variance in response to a change of each of the other variables in the VAR of equation (13). The impulse responses are presented in Figure 8.

Figure 8: This figure plots the monthly impulse response functions of stock market variance in response to a change of the variables in the VAR. The scale of the y-axis is in standard deviation of stock market variance, where each of the variables increases by one standard deviation at time 0. The top-left graph plots the responses to a change in rv, the top-right graph plots the responses to a change in $pc^{(1)}$, the bottom-left graph plots the responses to a change in $pc^{(2)}$, and the bottom-right graph plots the responses to a change in DEF.

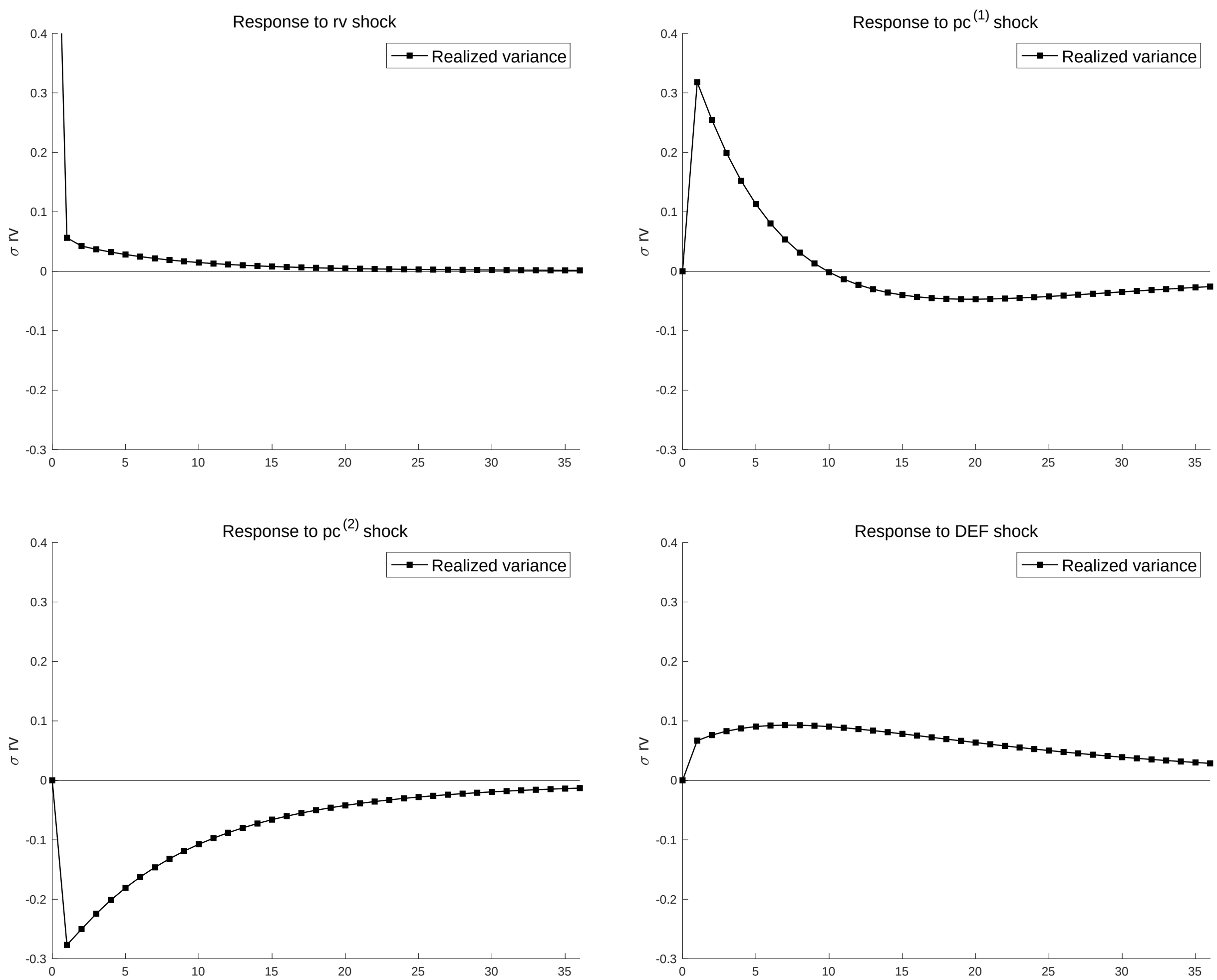


Figure 8 plots the impulse response functions of stock market variance for a horizon up to 36 months. The top-left graph shows that there is very little persistence in stock market variance, if the current level increases. The top-right graph shows that an increase in $pc^{(1)}$ increases future stock market variance up to 10 months forward. The bottom-left graph shows that an increase in $pc^{(2)}$ decreases future stock market variance and the shock is more persistent than a shock in $pc^{(1)}$. Finally, the bottom-right graph shows that shocks towards DEF are the most persistent and, therefore, affect long-term stock market variance.

A.4 VAR using quarterly data

In this subsection, I estimate the VAR based on data with a quarterly rather than monthly frequency. I do this to alleviate concerns that the estimates based on monthly frequency over-state the persistence of the variables and, therefore, create a bias in the variance expectations. The VAR is used to calculate quarterly stock market variance expectations and for this reason the variance swap rate with three months to maturity is the shortest maturity considered in this exercise. Table 12 presents the estimation results.

Table 12: This table shows the estimated coefficients of the VAR of equation (13) with t-values in parentheses. All variables are normalized to have mean equal to zero, and $pc_t^{(1)}$ and $pc_t^{(1)}$ are additionally standardized to have standard deviation equal to one. The sample period for the dependent variables is March 1996 to June 2019, with 94 quarterly data points.

Coefficients VAR model					
	rv_t	$pc_t^{(1)}$	$pc_t^{(2)}$	DEF_t	R^2
rv_{t+1}	-0.031	0.468	-0.267	0.448	0.459
(t-stat.)	(-0.17)	(3.07)	(-3.28)	(1.66)	
$pc_{t+1}^{(1)}$	-0.013	0.750	0.003	0.104	0.585
(t-stat.)	(-0.07)	(5.02)	(0.04)	(0.40)	
$pc_{t+1}^{(2)}$	-0.108	0.294	0.696	0.140	0.619
(t-stat.)	(-0.60)	(1.99)	(8.82)	(0.54)	
DEF_{t+1}	0.024	-0.025	-0.007	0.847	0.709
(t-stat.)	(0.46)	(-0.59)	(-0.33)	(11.50)	

Overall, the estimation results based on quarterly frequency are very similar to the results based on monthly frequency. Variance expectations and variance discount rates are calculated using these estimates and by adjusting equations (15) and (16) accordingly. Table 13 presents the results.

Table 13: This table shows the results of the variance decomposition of variance swap rates using equation (17), based on the VAR estimated on quarterly data. Note that the (co)variances of the third, fourth, and fifth columns are scaled with the variance of the second column such that the sum of the three (co)variances equals one.

T	$\mathrm{var}(vs)$	$\frac{\mathrm{var}(E_{\mathrm{rv}})}{\mathrm{var}(vs)}$	$\frac{\mathrm{var}(E_{\mathrm{vdr}})}{\mathrm{var}(vs)}$	$\frac{-2\cdot\mathrm{cov}(E_{\mathrm{rv}},E_{\mathrm{vdr}})}{\mathrm{var}(vs)}$
18	0.220	0.218	0.665	0.117
12	0.227	0.434	0.505	0.060
6	0.274	0.824	0.206	-0.030
3	0.348	1.021	0.085	-0.107

The results of the decomposition of variance swap rates in Table 12 are remarkably close to the results of Table 5. Therefore, my results are robust whether the frequency of the VAR is monthly or quarterly. Finally, I also decompose the variance swap rate using the predictive regressions of equations (11) and (12). Table 14 presents the results.

Table 14: This table shows the results of the predictive regressions of equations (11) and (12) in which the variance swap rate is the independent variable. The frequency of the data is quarterly. t-statistics are represented in parentheses and are computed using Newey-West standard errors with number of lags equal to $\frac{1}{3} \cdot T$.

Dependent variable:	$y_{\text{rv},t+T}$		$y_{\text{vdr},t+T}$	
Maturity	b_{rv} (t-stat.)	R^2	b_{vdr} (t-stat.)	R^2
18	0.240	0.029	-0.741	0.218
	(1.21)		(-3.76)	
12	0.540	0.129	-0.452	0.092
	(3.32)		(-2.71)	
6	0.836	0.303	-0.162	0.016
	(5.48)		(-1.04)	
3	0.964	0.417	-0.036	0.001
	(8.05)		(-0.30)	

The similarity between the results of Table 3 and Table 14 indicate that the results of predictive regressions are robust to decreasing the frequency to the quarterly level.

A.5 VAR with five variables

In this section, I show that adding the variance swap return to the VAR yields very similar results, as in Section 3.5. The VAR of equation (13) is estimated using the following state variables:

$$z_t = \left(r_t^{(T)} \quad rv_t \quad pc_t^{(1)} \quad pc_t^{(2)} \quad DEF_t \right)',$$

where $r_t^{(T)}$ are the returns on a variance swap with T-periods to maturity. Therefore, to decompose the returns of different maturities using identity (10), the VAR has to be re-estimated for each maturity. In this appendix, I show the estimation results of the VAR with returns on 12-month variance swaps $r_t^{(12)}$ and the results of the decomposition. The estimation results of the VAR are in Table 15.

Table 15: This table shows the estimated coefficients of the VAR of equation (13) with t-values in parentheses. All variables are normalized to have the mean equal to zero, and $pc_t^{(1)}$ and $pc_t^{(1)}$ are additionally standardized to have the standard deviation equal to one. The sample period for the dependent variables is January 1996 to June 2019, with 282 monthly data points.

Coefficients VAR model						
	$r_t^{(12)}$	rv_t	$pc_t^{(1)}$	$pc_t^{(2)}$	DEF_t	R^2
$r_{t+1}^{(12)}$	0.018	0.017	-0.043	-0.047	0.064	0.085
	(0.26)	(0.71)	(-2.02)	(-3.13)	(1.42)	
rv_{t+1}	0.357	0.029	0.511	-0.308	0.523	0.594
	(1.53)	(0.38)	(7.24)	(-6.15)	(3.49)	
$pc_{t+1}^{(1)}$	-0.039	0.037	0.863	-0.002	0.106	0.848
	(-0.27)	(0.76)	(19.76)	(-0.07)	(1.13)	
$pc_{t+1}^{(2)}$	0.260	-0.016	0.133	0.869	-0.017	0.746
	(1.39)	(-0.25)	(2.34)	(21.64)	(-0.14)	
DEF_{t+1}	0.092	0.008	-0.012	0.002	0.967	0.938
	(3.06)	(0.78)	(-1.28)	(0.36)	(46.63)	

The inclusion of $r_t^{(12)}$ into the VAR does not alter the models of the four other variables much. Only $r_t^{(12)}$ positively predicts the default spread in the next period. Low returns on 12-month variance swaps are predicted by a large level of the term structure of variance swap rates $pc_t^{(1)}$ and a large slope of the term structure of variance swap rates $pc_t^{(2)}$.
Using the estimates of this VAR, $\bar{N}_{\mathrm{rv},t}^{(T)}$ is obtained, as follows:

$$\bar{N}_{\mathrm{rv},t}^{(T)} = e_2' \Bigg(\big(1-\rho(T)\big) + \cdots + \big(1-\rho(1)\big)\rho(T) \times \cdots \times \rho(2) B^{T-1} \Bigg) \epsilon_t,$$

where e_2 is vector of zeros and the second element a one (as rv_t is the second element in z_t). Furthermore, $\bar{N}_{\mathrm{vdr},t}^{(T)}$ is using this VAR obtained in the following way:

$$\bar{N}_{\mathrm{vdr},t}^{(T)} = \bar{N}_{\mathrm{rv},t}^{(T)} - e_1' \cdot \epsilon_t.$$

The results of the decomposition of variance swap returns are shown in Table 16.

Table 16: This table shows the results of the variance decomposition of variance swap rates using equation (25). Note that the (co)variances of the third, fourth, and fifth columns are scaled with the variance of the second column such that the sum of the three (co)variances equals one. Standard errors are computed using the Delta method.

Maturity	$\text{var}\left(e_1'\epsilon_t\right)$	$\text{var}\left(\bar{N}_{\text{rv},t}^{(T)}\right)$	$\text{var}\left(\bar{N}_{\text{vdr},t}^{(T)}\right)$	-2cov$\left(\bar{N}_{\text{rv},t}^{(T)}, \bar{N}_{\text{vdr},t}^{(T)}\right)$
18	0.026	0.593	0.542	-0.134
		(0.265)	(0.224)	(0.348)
12	0.035	0.893	0.369	-0.262
		(0.307)	(0.165)	(0.379)
6	0.064	1.172	0.152	-0.324
		(0.236)	(0.068)	(0.276)
3	0.123	1.151	0.037	-0.188
		(0.147)	(0.014)	(0.156)
1	0.378	1.000	0.000	0.000

Note that the decomposition of the variance swap returns with 12 months to maturity is obtained using the estimates of the VAR of Table 15. To decompose the returns on variance swaps with T months to maturity, the VAR is re-estimated with $r_t^{(T)}$ as a state variable. Overall, the results of Table 16 are very similar to the results of the decomposition using only four state variables represented in Table 8.

Furthermore, each of the decomposition objects in identity 10 are directly compared using the methodology with five variables in the VAR and the method with only four variables. The results of the comparison are represented in the next table.

Table 17: This table shows the results of the comparison between the method in which the variance swap return is modeled directly in the VAR and the method in which the realization is obtain using equation (22). $e_1'\epsilon_t$, $\bar{N}_{\text{rv},t}^{(T)}$, and $\bar{N}_{\text{vdr},t}^{(T)}$ correspond to the variables obtained using the VAR with five variables. $e_L(T)'\epsilon_t$, $N_{\text{rv},t}^{(T)}$, and $N_{\text{vdr},t}^{(T)}$ correspond to the variables obtained using the VAR with four variables.

Maturity	$\text{corr}\left(e_1'\epsilon_t, e_L(T)'\epsilon_t\right)$	$\text{corr}\left(\bar{N}_{\text{rv},t}^{(T)}, N_{\text{rv},t}^{(T)}\right)$	$\text{corr}\left(\bar{N}_{\text{vdr},t}^{(T)}, N_{\text{vdr},t}^{(T)}\right)$
18	0.963	0.998	0.940
12	0.981	0.998	0.952
6	0.986	0.999	0.917
3	0.990	0.999	0.767
1	0.987	0.987	-

The correlations in Table 17 are all very high except for the correlation of the revised discount rate expectations of variance swap returns with a maturity of three months. However, the variance of this object is very small, and, therefore, a tiny deviation yields large changes in correlation as indicated by the large correlations in the other columns.

B Appendix asset pricing models

In the following subsections, I discuss more results from the models considered in this paper. In Section 4, I discuss the implications of the models based on 1,000 independent simulation sets with a time-series of 1,000 data points. In the following, I analyze the variation in the variance decomposition of variance swap rates across the independent simulation sets to assess the stability of the results from the model.

B.1 Variable disaster risk and CRRA preferences

The calibration of the model by Gabaix (2012) is from Dew-Becker et al. (2017) and given in the following table.

Table 18: Calibration of the model by Gabaix (2012).

Parameter	Value	Parameter	Value
μ_c	$0.01/12$	σ_c	$0.02/\sqrt{12}$
μ_d	-0.3	σ_d	0.15
$\bar{L}$	$-\log(0.5)$	σ_L	0.04
ρ_L	$0.87^{1/12}$	η	5
β	$0.96^{1/12}$	γ	7
λ	$\frac{0.01}{12}$		

Note that in the calibration of Dew-Becker et al. (2017) the risk-aversion is raised to 7 in order to match the Sharpe ratio on one-month variance swaps.

In the following, I present more details of the results from the simulation study for the model by Gabaix (2012). First, I present sample statistics of realized variance and variance discount rates in the model. The results of this simulation study are represented in the following table.

Table 19: This table presents sample statistics of the realized variance and variance discount rates in the model by Gabaix (2012). The mean, standard deviation and Sharpe Ratio of the 18-, 12-, 6-, 3-, and 1-month simple variance discount rates are presented. The second column consists of the empirical result, and the third, fourth, and fifth columns represent the 5%, 50%, and 95% quantile of the simulation study, respectively.

Statistic	Data	Model		
	Est.	5%	50%	95%
Realized variance				
$\mathbb{E}(RV)$	0.162	0.115	0.116	0.117
$\sigma(RV)$	0.095	0.000	0.021	0.043
Variance discount rates				
$\mathbb{E}(r^{(18)})$	-0.006	-0.024	-0.021	-0.017
$\sigma(r^{(18)})$	0.187	0.021	0.056	0.109
$SR(r^{(18)})$	-0.106	-4.091	-1.278	-0.572
$\mathbb{E}(r^{(12)})$	-0.013	-0.037	-0.032	-0.025
$\sigma(r^{(12)})$	0.227	0.021	0.080	0.161
$SR(r^{(12)})$	-0.202	-5.976	-1.326	-0.578
$\mathbb{E}(r^{(6)})$	-0.050	-0.075	-0.063	-0.050
$\sigma(r^{(6)})$	0.316	0.023	0.155	0.319
$SR(r^{(6)})$	-0.544	-11.028	-1.362	-0.584
$\mathbb{E}(r^{(3)})$	-0.098	-0.150	-0.127	-0.099
$\sigma(r^{(3)})$	0.447	0.028	0.308	0.636
$SR(r^{(3)})$	-0.756	-17.644	-1.370	-0.585
$\mathbb{E}(r^{(1)})$	-0.285	-0.451	-0.380	-0.296
$\sigma(r^{(1)})$	0.676	0.068	0.926	1.905
$SR(r^{(1)})$	-1.458	-22.305	-1.366	-0.585

Table 19 confirms the finding of Figure 6 that the model by Gabaix (2012) is able to capture the strongly increasing term structure of expected variance swap returns documented in the data. Moreover, Table 19 shows that the volatility of variance swap returns varies a lot across simulation sets, and this results from the fact that the probability of a disaster is small (1% p.a.). If no disasters occur in a simulation set, the volatility of variance swap returns is very low. Finally, I conclude from Table 19 that the model by Gabaix (2012) is not able to capture the dynamics of empirical stock market volatility.

In the following, I decompose variance swap rates in the model by Gabaix (2012) for each simulation set seperately. Table 20 presents the results.

Table 20: This table presents the results of the simple variance decomposition of variance swap rates in the data and in the model by Gabaix (2012). The results of the data are from Table 3, with standard errors in parentheses. The regression coefficients of the model are estimated for each simulation set, and the mean and standard deviation of the regression coefficients are represented in the table.

Maturity	Data		Model	
	b_{rv}	b_{vdr}	b_{rv}	b_{vdr}
18	0.245	-0.728	0.002	-0.985
	(0.185)	(0.188)	(0.028)	(0.125)
12	0.558	-0.419	0.002	-0.989
	(0.151)	(0.158)	(0.028)	(0.101)
6	0.833	-0.168	0.002	-0.994
	(0.119)	(0.122)	(0.028)	(0.069)
3	0.957	-0.040	0.002	-0.997
	(0.083)	(0.086)	(0.027)	(0.047)
1	1.101	0.101	0.002	-0.998
	(0.056)	(0.056)	(0.027)	(0.027)

Table 20 shows that the result of Figure 4 is stable across the simulation sets. In particular, short-term variance swap rates are solely driven by variance discount rates, and this number is very similar across simulations, and, therefore, it is strong evidence that the model is not in line with the data.

In the following subsection, I discuss the results for the model by Wachter (2013).

B.2 Time-varying disaster risk and Epstein-Zin preferences

In this subsection, the calibration of the model is given as well as some details from the simulation study. The calibration of the model is given in Table 21.

Table 21: This table shows the calibration of the model by Wachter (2013).

Parameter	Value	Parameter	Value
μ_c	$0.0252/12$	σ_c	$0.02/\sqrt{12}$
μ_d	-0.15	σ_d	0.10
μ_λ	$0.0355/12$	σ_λ	$0.067/12$
ϕ	$\exp(-0.08/12)$	β	$\exp(-0.012/12)$
η	2.6	γ	$4.9 = 1 - \alpha$

Note that in the calibration of Dew-Becker et al. (2017) the risk-aversion is raised to 4.9 in order to match the Sharpe ratio on one-month variance swaps as closely as possible.

In the following, I present more details of the results from the simulation study for the model by Wachter (2013). First, I present sample statistics of realized variance and variance discount rates in the model. The results of this simulation study are represented in the following table.

Table 22: This table presents sample statistics of the realized variance and variance discount rates in the model by Wachter (2013). The mean, standard deviation, and Sharpe ratio of the 18-, 12-, 6-, 3-, and 1-month simple variance discount rates are presented. The second column consists of the empirical result, and the third, fourth, and fifth columns represent the 5%, 50%, and 95% quantile of the simulation study, respectively.

Statistic	Data	Model		
	Est.	5%	50%	95%
Realized variance				
$\mathbb{E}(RV)$	0.162	0.094	0.123	0.168
$\sigma(RV)$	0.095	0.028	0.048	0.073
Variance discount rates				
$\mathbb{E}(r^{(18)})$	-0.006	-0.015	-0.010	-0.005
$\sigma(r^{(18)})$	0.187	0.080	0.101	0.179
$SR(r^{(18)})$	-0.106	-0.556	-0.357	-0.097
$\mathbb{E}(r^{(12)})$	-0.013	-0.019	-0.014	-0.005
$\sigma(r^{(12)})$	0.227	0.084	0.118	0.256
$SR(r^{(12)})$	-0.202	-0.721	-0.412	-0.075
$\mathbb{E}(r^{(6)})$	-0.050	-0.032	-0.024	-0.007
$\sigma(r^{(6)})$	0.316	0.086	0.183	0.502
$SR(r^{(6)})$	-0.544	-1.284	-0.464	-0.048
$\mathbb{E}(r^{(3)})$	-0.098	-0.060	-0.045	-0.009
$\sigma(r^{(3)})$	0.447	0.074	0.338	1.008
$SR(r^{(3)})$	-0.756	-2.674	-0.459	-0.031
$\mathbb{E}(r^{(1)})$	-0.285	-0.173	-0.127	-0.018
$\sigma(r^{(1)})$	0.676	0.045	0.993	3.058
$SR(r^{(1)})$	-1.458	-12.294	-0.442	-0.019

Table 22 confirms the finding of Figure 6 that the model by Wachter (2013) is not able to capture the strongly increasing term structure of expected variance swap returns documented in the data. Moreover, Table 22 shows that also in the model by Wachter (2013) the volatility of variance swap returns varies a lot across simulation sets, and this results from the fact that the probability of a disaster is, on average, small (3.55% p.a.). If no disasters occur in a simulation set, the volatility of variance swap returns is very low. Finally, I conclude from Table 19 that the model by Wachter (2013) does a better job than the model by Gabaix (2012) of capturing the empirical dynamics of stock market volatility.

In the following, I decompose variance swap rates in the model by Wachter (2013) for each simulation set separately. Table 23 presents the results.

Table 23: This table presents the results of the simple variance decomposition of variance swap rates in the data and in the model by Wachter (2013). The results of the data are from Table 3, with standard errors in parentheses. The regression coefficients of the model are estimated for each simulation set, and the mean of the regression coefficients is represented in the table with the standard deviation in parentheses.

Maturity	Data		Model	
	b_{rv}	b_{vdr}	b_{rv}	b_{vdr}
18	0.245	-0.728	0.973	-0.037
	(0.185)	(0.188)	(0.040)	(0.040)
12	0.558	-0.419	0.963	-0.033
	(0.151)	(0.158)	(0.029)	(0.032)
6	0.833	-0.168	0.954	-0.033
	(0.119)	(0.122)	(0.021)	(0.024)
3	0.957	-0.040	0.950	-0.039
	(0.083)	(0.086)	(0.019)	(0.020)
1	1.101	0.101	0.948	-0.052
	(0.056)	(0.056)	(0.018)	(0.018)

Table 23 confirms the finding of Figure 4 that variance swap rates are driven by variance expectations in the model by Wachter (2013). Moreover, this result is very stable across the simulation sets, as indicated by the low standard deviation of b_{rv}. Therefore, this is strong evidence that the model is not in line with the data because my analysis shows that long-term variance swaps are mostly driven by variance discount rates.

In the following subsection, I discuss the results for the model by Drechsler and Yaron (2011).

B.3 Long-run risk

The calibration is from Table 5 of the paper by Drechsler and Yaron (2011), and I use the calibration in which jump shocks in the x_t process follow a compound-Poisson in combination with a normal distribution.

In the following, I present more details of the results from the simulation study for the model by Drechsler and Yaron (2011).[7] First, I present sample statistics of realized variance

[7] I thank Friedrich Lorenz for sharing the codes to solve the model.

and variance discount rates in the model. The results of this simulation study are represented in the following table.

Table 24: This table presents sample statistics of the realized variance and variance discount rates in the model by Drechsler and Yaron (2011). The mean, standard deviation, and Sharpe ratio of the 18-, 12-, 6-, 3-, and 1-month simple variance swap returns are presented. The second column consists of the empirical result, and the third, fourth, and fifth columns represent the 5%, 50%, and 95% quantile of the simulation study, respectively.

Statistic	Data	Model		
	Est.	5%	50%	95%
Realized variance				
$\mathbb{E}(RV)$	0.162	0.157	0.169	0.187
$\sigma(RV)$	0.095	0.051	0.087	0.134
Variance discount rates				
$\mathbb{E}(r^{(18)})$	-0.006	-0.036	-0.026	-0.014
$\sigma(r^{(18)})$	0.187	0.191	0.276	0.387
$SR(r^{(18)})$	-0.106	-0.654	-0.333	-0.128
$\mathbb{E}(r^{(12)})$	-0.013	-0.045	-0.032	-0.015
$\sigma(r^{(12)})$	0.227	0.232	0.343	0.488
$SR(r^{(12)})$	-0.202	-0.659	-0.326	-0.109
$\mathbb{E}(r^{(6)})$	-0.050	-0.063	-0.043	-0.017
$\sigma(r^{(6)})$	0.316	0.304	0.477	0.734
$SR(r^{(6)})$	-0.544	-0.686	-0.314	-0.084
$\mathbb{E}(r^{(3)})$	-0.098	-0.089	-0.060	-0.020
$\sigma(r^{(3)})$	0.447	0.394	0.671	1.144
$SR(r^{(3)})$	-0.756	-0.736	-0.309	-0.064
$\mathbb{E}(r^{(1)})$	-0.285	-0.176	-0.116	-0.027
$\sigma(r^{(1)})$	0.676	0.708	1.352	2.697
$SR(r^{(1)})$	-1.458	-0.820	-0.292	-0.036

Table 24 confirms the finding of Figure 6 that the model by Drechsler and Yaron (2011) is not able to capture the strongly increasing term structure of expected variance swap returns documented in the data. Moreover, it shows that the model predicts, for each maturity, a volatility of variance swap returns, which is larger than observed empirically. Finally, I conclude from Table 19 that the model by Drechsler and Yaron (2011) does a good job of capturing the empirical dynamics of stock market volatility.

In the following, I decompose variance swap rates in the model by Drechsler and Yaron (2011) for each simulation set separately. Table 25 presents the results.

Table 25: This table presents the results of the simple variance decomposition of variance swap rates in the data and in the model by Drechsler and Yaron (2011). The results of the data are from Table 3, with standard errors in parentheses. The regression coefficients of the model are estimated for each simulation set and the mean of the regression coefficients are represented in the table with the standard deviation in parentheses.

Maturity	Data		Model	
	b_{rv}	b_{vdr}	b_{rv}	b_{vdr}
18	0.245	-0.728	0.349	-0.560
	(0.185)	(0.188)	(0.105)	(0.108)
12	0.558	-0.419	0.412	-0.502
	(0.151)	(0.158)	(0.105)	(0.107)
6	0.833	-0.168	0.506	-0.418
	(0.119)	(0.122)	(0.097)	(0.098)
3	0.957	-0.040	0.567	-0.379
	(0.083)	(0.086)	(0.087)	(0.086)
1	1.101	0.101	0.615	-0.385
	(0.056)	(0.056)	(0.075)	(0.075)

Table 25 confirms the finding of Figure 4 that short-term variance swap rates are driven by variance expectations and long-term variance swap rates by variance discount rates. Moreover, this result is stable across the simulation sets, as indicated by the low standard deviations of b_{rv} and b_{vdr}. Therefore, this is strong evidence that the model predicts a variation in short-term variance discount rates, which is substantially larger than observed empirically.