No Good Deals - No Bad Models

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Abstract

Faced with the problem of pricing complex contingent claims, an investor seeks to make her valuations robust to model uncertainty. We construct a notion of a model-uncertainty-induced preference functional and extend the "No Good Deals" methodology of Cochrane and Saá-Requejo [2000] to compute lower and upper good deal bounds in the presence of model uncertainty. Illustrating the methodology with numerical examples, we show how model uncertainty reduces the benefit of dynamic hedging relative to static hedging and increases the early exercise premia for American options.

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1 Introduction

The long, continuing saga of derivatives losses by financial institutions, in the face of the best efforts of regulators to prevent them, highlights the need for economists to develop better techniques to deal with model uncertainty. We address this issue by proposing a methodology to compute lower and upper bounds on prices of complex (potentially non-traded) securities that is robust to misspecifications of the model of the underlying assets and cash-flows. More specifically, we incorporate a concern for robustness to model uncertainty into the "No Good Deals" methodology of Cochrane and Saá-Requejo [2000]. Their methodology refines the lower and upper arbitrage bounds on securities prices by imposing a maximal Sharpe ratio. We extend the intuition of Cochrane and Saá-Requejo [2000] and argue that investors should incorporate a concern for model uncertainty in computing the maximal Sharpe ratio. In our analysis, besides the maximal Sharpe ratio, one additional parameter is needed to quantify the degree of aversion to model uncertainty. While an investor might have an estimate of the distribution of future payoffs to financial assets, she recognizes that her estimate may not be the true data-generating process. Thus, she considers a set of alternative models, with her preference for robustness forcing her to choose the "worst-case" model in computing the good deal bounds. In particular, we assume that her preferences are based on those of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [2011].

Model uncertainty impacts the good deal bounds in three ways. First, the uncertainty averse investor assigns higher marginal utility to states with lower payoffs (or higher losses). Second, the good deal bounds are wider in the presence of model uncertainty than in its absence. Intuitively, the lower (respectively, upper) bound on the price of the security is the bid (ask) that an investor buying (selling) the security is willing to submit; in the presence of model uncertainty, an uncertainty averse investor fears that the security is less (more) valuable. Third, while the right to dynamically hedge is always a valuable one, we show how model uncertainty reduces the benefit of dynamic hedging relative to static hedging.[1]

[1]See Carr, Ellis, and Gupta [1998] for a discussion of static hedging strategies for complex options.

Building on Černý [2003], we solve for the good deal bounds in the presence of model uncertainty. We show how the degree of aversion to model uncertainty may be estimated. Illustrating with numerical examples, we show that greater model uncertainty can lead to larger early exercise premia for American options, since, intuitively, exercising the option removes model uncertainty. We also demonstrate how model uncertainty leads to economically different outcomes than choosing a higher maximal Sharpe ratio.

The seminal contribution of Cochrane and Saá-Requejo [2000] is to refine the arbitrage bounds by additionally requiring that the second moment of the pricing kernel is bounded which, in view of Hansen and Jagannathan [1991], is the same as imposing a maximal Sharpe ratio.[2] The resulting good deal bounds both rule out arbitrage and rule out the possibility of forming a portfolio of the complex security (termed the focus asset) and of a set of hedging assets (termed basis assets) which has more than some given Sharpe ratio. Just as arbitrages are ruled out for giving investors a free lunch, so very high Sharpe ratios are ruled out on the grounds that, if allowed, a very high Sharpe ratio would represent such a good deal that (Ross [1976]) it should not exist in equilibrium.[3]

Although the No Good Deals methodology provides a compelling and economically-motivated way for investors to consider the impact of unhedgeable market risks on the prices of complex securities, the recent history of financial institutions suggest that these

[2] Bernardo and Ledoit [2000] simultaneously developed an alternative framework for constructing good deal bounds which uses gain-loss ratios rather than Sharpe ratios. Černý [2003] and Černý and Hodges [2001] explain how the use of gain-loss ratios puts their framework into a rather different category compared to the Sharpe ratio based framework and, for this reason, we don't consider it further. Carr, Geman, and Madan [2001] and Cherny and Madan [2009] also consider the problem of pricing and hedging derivatives in incomplete markets (in the absence of model uncertainty), using metrics which could, broadly speaking, be interpreted as a reward-for-risk, but, for similar reasons, their frameworks also fall outside the scope of our paper. Hodges [2009] is a comprehensive review of follow-up papers. He points out that most of them are highly mathematical and the economic intuition of Cochrane and Saá-Requejo [2000] and its potential use as a practical tool for practitioners and regulators alike has been obscured.

[3] Hence, the No Good Deals methodology incorporates a partial equilibrium consideration into the pricing of complex securities but, crucially, without having to specify the details of the equilibrium-formation mechanism or to give the precise specification of investors' utility functions. The latter is especially pertinent to the pricing and trading of complex securities where agents (typically employees of investment banks or hedge funds) are rarely acting on their own account and hence personalised measures of preferences such as a utility function (even if estimable) may be inappropriate.

considerations are insufficient to incentivize market participants to be sufficiently conservative in their valuations. Even before the global financial crisis of 2007–2009, there had been a series of financial institutions (such as Bank of Tokyo/Mitsubishi in 1997, Nat-West in 1997, Bankers Trust in 1998, Amaranth Advisors LLC in 2006) reporting large losses on their (supposedly, hedged) positions in over-the-counter (OTC) derivatives. The losses have continued after the crisis. For example, in April 2011, Reuters reported that Mitsubishi UFJ Morgan Stanley Securities (a joint venture between a Japanese bank and Morgan Stanley) had incurred losses of more than 1.75 billion dollars on its positions in complex derivatives. Whilst the exact details of how these losses arose is still unclear, the Reuters report suggests that the positions were incorrectly marked-to-market or suffered from gradual hedge-slippage as the prices of hedging assets diverged from their model predictions.

The aftermath of the global financial crisis has intensified political and regulatory scrutiny of OTC derivatives. Indeed the Basel Committee on Banking Supervision notes that: "Failure to capture major on– and off–balance sheet risks, as well as derivative related exposures, was a key destabilising factor during the crisis".

We describe an extension of good deal bounds to incorporate a concern for model uncertainty. These could be used as fair reservation prices at which complex derivatives or illiquid securities are marked-to-market - using the lower (respectively, upper) good deal bound for long (short) positions. If an investor (or, more generally, a financial institution) would be prepared to enter into any trade that is either an arbitrage or that is expected to deliver more than a specified Sharpe ratio, then she will, in all likelihood, be prepared to trade a complex security priced by the same criterion or by one expected to deliver an even larger Sharpe ratio. Further, Sharpe ratios are simple and widely used so it is likely that there will be other investors (or financial institutions) who would be prepared to trade on the same terms and who would therefore be prepared to take the other side of the trade should the first investor decide to liquidate her position. Hence the use of Sharpe ratios gives a market (as opposed to an individualistic) perspective. This makes good deal bounds attractive candidates as fair reservation prices. Incorporating a concern for model uncertainty adds further economic realism and, in particular, recognizes the difficulty of evaluating the probability of infrequent large adverse moves.

Literature Review

A rapidly growing literature studies the behavior of asset prices in the presence of model uncertainty. The original insight of Gilboa and Schmeidler [1989] – that model uncertainty can explain the Ellsberg [1961] paradox – has been extended by Hansen and Sargent [2001, 2008] and by Cerreia-Vioglio et al. [2011]. They, together with Hansen, Sargent, and Tallarini [1999], Cagetti, Hansen, Sargent, and Williams [2002], Barillas, Hansen, and Sargent [2009], Maenhout [2004], Garlappi, Uppal, and Wang [2007] and Uppal and Wang [2003] show that investors have a fundamentally different aversion to model uncertainty than to market risk.

For example, Barillas et al. [2009] and Maenhout [2004] show that the classical equity premium puzzle of Mehra and Prescott [1985] can be resolved by allowing investors to have robust preferences over alternative models; Uppal and Wang [2003] show that model uncertainty can also be used to explain the home-bias puzzle of Cooper and Kaplanis [1994] and Huberman [2001][4]. Furthermore, recent studies have demonstrated the importance of model uncertainty in modeling the prices of complex securities. Liu, Pan, and Wang [2005] find that aversion to model uncertainty plays an important role in explaining the skew in implied volatilities among options on the S&P 500 stock index. Drechsler [2013] finds that concerns for model misspecification explain the large premia in index options. Finally, Boyarchenko [2012] argues that model uncertainty can explain the behavior of credit swap spreads (CDS) on financial institutions during the recent financial crisis.

[4]Additional studies examining the implications of model uncertainty for asset pricing include Anderson, Ghysels, and Juergens [2009], Bossaerts, Ghirardato, Guarnaschelli, and Zame [2010], Leippold, Trojani, and Vanini [2008], Cao, Wang, and Zhang [2005], Boyle, Garlappi, Uppal, and Wang [2012], Cvitanic, Lazrak, Martellini, and Zapatero [2011] and Trojani, Wiehenkamp, and Wrampelmeyer [2013].

While the implications for asset pricing and portfolio choice under model uncertainty have been extensively considered, incorporating model uncertainty into the pricing of contingent claims has been less well-developed. Boyle, Feng, Tian, and Wang [2008] argue that, in incomplete markets, the multiplicity of available pricing kernels naturally leads to a concern for model misspecification. Instead of following the literature on robust preferences, however, they choose the optimal pricing kernel to limit the variation in the price of a contingent claim when the underlying asset's payoff is slightly perturbed. The main advantage of this approach is to construct perturbations that are based on the volatility of the basis asset, which is the pertinent quantity in pricing options on the asset. This natural link comes at a cost, however, with the Black and Scholes [1973] model used as the benchmark model. Furthermore, they focus on standard European options and their continuous-time setup may make extensions to path-dependent or American (early exercise) options non-trivial.

In contrast, our approach uses a discrete-time, discrete space (lattice) formulation. The lattice approach simplifies the pricing of finite horizon options while emphasizing their illiquid nature. Further, any continuous space stochastic process can be well approximated by a discrete lattice, provided that the lattice has sufficient states. Finally, continuous trading is not always possible in real markets; our discrete-time formulation naturally incorporates these breaks in trading activity.

2 Good deal bounds without model uncertainty

We briefly review the No Good Deals methodology of Cochrane and Saá-Requejo [2000] (in the absence of model uncertainty) and its subsequent extensions (Hodges [1998], Černý and Hodges [2001] and Černý [2003]) to alternative utility function settings. This methodology assumes that the distribution of future payoffs to financial assets or outcomes of economic variables (such as interest-rates) is known; that is, the methodology specifies a reference (real-world, data-generating) probability measure $\mathbb{P}$ over the possible states of the world.

While the focus of this paper is to allow for uncertainty about the probability measure $\mathbb{P}$, in this section we ignore model uncertainty concerns and take $\mathbb{P}$ as given. We outline the relationship between the No Good Deals methodology and expected utility maximization in a way that will allow us to, more naturally, introduce model uncertainty in Section 3.

Consider pricing a complex security or contingent claim (or a portfolio of these), and assume that this focus asset (in the terminology of Cochrane and Saá-Requejo [2000]) pays c at time 1. In addition, assume that there are N_b basis assets, traded in an active, liquid market. The basis assets pay $\mathbf{x}$ at time 1, with time 0 market prices given by $\mathbf{p}$, where $\mathbf{p}$ and $\mathbf{x}$ are N_b - dimensional vectors. Any pricing kernel m used to value the focus asset must exactly price the basis assets, so that $\mathbb{E}^{\mathbb{P}}[m\mathbf{x}] = \mathbf{p}$, and imply the absence of arbitrage opportunities, so that $m \geq 0$. Since we do not assume that markets are complete, there can be multiple candidate pricing kernels that satisfy these conditions.

The innovation of Cochrane and Saá-Requejo [2000] is to restrict the set of candidate pricing kernels by requiring that the second moment of a candidate pricing kernel is bounded, which implies (see Hansen and Jagannathan [1991]) a maximal Sharpe ratio. The time 0 lower $\underline{C}^{\text{NoMU}}$ and upper $\overline{C}^{\text{NoMU}}$ good deal bounds on the focus asset satisfy

$$\underline{C}^{\text{NoMU}} = \inf_{m} \left\{ \mathbb{E}^{\mathbb{P}}[mc] \text{ such that } \mathbb{E}^{\mathbb{P}}[m\mathbf{x}] = \mathbf{p}, m \geq 0, \mathbb{E}^{\mathbb{P}}[m^2] \leq \Gamma \right\}, \quad (1)$$

with the infimum replaced by a supremum for $\overline{C}^{\text{NoMU}}$. Rearranging the solutions of Cochrane and Saá-Requejo [2000], the good deal bounds can be expressed:

$$\underline{C}^{\text{NoMU}} = \inf_{C \in \mathcal{NA}} \left\{ C \text{ such that } \max_{\mu > 0, \mathbf{v}} \left\{ 2\mathbf{v}'\mathbf{p} - 2\mu C + \mathbb{E}^{\mathbb{P}}[\tilde{U}^{\text{TQ}}\left(\mu c - \mathbf{v}'\mathbf{x}\right)] \right\} \leq \Gamma \right\}, \quad (2)$$

$$\overline{C}^{\text{NoMU}} = \sup_{C \in \mathcal{NA}} \left\{ C \text{ such that } \max_{\mu < 0, \mathbf{v}} \left\{ 2\mathbf{v}'\mathbf{p} - 2\mu C + \mathbb{E}^{\mathbb{P}}[\tilde{U}^{\text{TQ}}\left(\mu c - \mathbf{v}'\mathbf{x}\right)] \right\} \leq \Gamma \right\}, \quad (3)$$

where $\mathcal{NA}$ denotes the set of C consistent with the absence of arbitrage. Further, $\tilde{U}^{\text{TQ}}(V) \equiv -\left(\max(-V, 0)\right)^2$, μ is a scalar and $\mathbf{v}$ is a N_b - dimensional vector. The maximizations are made over choices of μ and $\mathbf{v}$ which play the role of positions taken in the focus asset and in the basis assets. The restriction $\mu > 0$ in (2) (respectively, $\mu < 0$ in (3)) corresponds to taking a long position (a short position of size $|\mu|$), at time 0, at a price of $\underline{C}^{\text{NoMU}}$ (respectively, $\overline{C}^{\text{NoMU}}$) to solve for the lower (upper) good deal bound. Since $\tilde{U}^{\text{TQ}}(V)$ is non-decreasing and concave, so is $2\mathbf{v}'\mathbf{p} - 2\mu C + \tilde{U}^{\text{TQ}}\left(\mu c - \mathbf{v}'\mathbf{x}\right)$. Hence, it has the usual properties of a utility function. More generally, $2\mathbf{v}'\mathbf{p} - 2\mu C + \mathbb{E}^{\mathbb{P}}[\tilde{U}^{\text{TQ}}\left(\mu c - \mathbf{v}'\mathbf{x}\right)]$ can be interpreted as a reward-for-risk which has the convenient normalization that redundant assets (including one having zero payoff) have zero reward-for-risk. The terms $\max_{\mu,\mathbf{v}}$ in (2) and (3) say that the investor chooses μ and $\mathbf{v}$ so as to maximize her reward-for-risk. The good deal bounds solve for the target level of reward-for-risk Γ, if it can be achieved; if it can't, the good deal bounds are the arbitrage bounds. Further, $\mathbf{v}$ has the interpretation of the optimal hedges in the basis assets for μ units of the focus asset in seeking to achieve the target level of reward-for-risk Γ. If the investor trades the focus asset at a price strictly inside the range $\left(\underline{C}^{\text{NoMU}}, \overline{C}^{\text{NoMU}}\right)$, she will not have a good deal.

An alternative link with utility functions comes from Černý [2003] who considers an investor, without a concern for model uncertainty, endowed with wealth V_0 who maximizes her expected utility $U(V)$ over future wealth V. For[5] truncated quadratic utility $U(V) = -\left(\max\left(\overline{V} - V, 0\right)\right)^2$, Černý [2003] shows that the bound on the second moment of the pricing kernel implies a bound on the maximum achievable certainty equivalent CE and *vice versa*:

$$\text{CE} \leq \Lambda \Longleftrightarrow \mathbb{E}^{\mathbb{P}}[m^2] \leq \Gamma, \qquad \text{where} \qquad \Gamma \equiv \left(\frac{1}{1 - \Lambda/\left(\overline{V} - V_0\right)}\right)^2, \tag{4}$$

or $\Lambda \equiv \left(\overline{V} - V_0\right)\left(1 - 1/\sqrt{\Gamma}\right)$. This gives us a dual interpretation: No Good Deals can be seen either as ruling out Sharpe ratios which are too high or as ruling out too high levels of certainty equivalent (or expected utility) relative to V_0 (or $U(V_0)$). The second interpretation will allow us to connect good deal bounds with model uncertainty.

[5] Černý [2003] also derives restrictions on the pricing kernel corresponding to exponential and CRRA utility and shows that they are equivalent to bounds on maximal "generalized" Sharpe ratios.

We conclude this section by summarizing the basic assumptions that we maintain throughout the paper. We assume that there are S possible time 1 states of the world. We denote the probability, under $\mathbb{P}$, of attaining state s by π_s, for each $s = 1, \ldots, S$. We assume that

Assumption 1 *(1) S, the number of possible states of the world at time 1, is finite,*

(2) $\pi_s > 0$, $\forall s = 1, \ldots, S$, so that zero probability states have been pruned,

(3) the time 1 payoffs are finite in each state,

(4) there are no arbitrage opportunities amongst the basis assets, and

(5) redundant basis assets have been pruned.

3 Introducing model uncertainty

In this section, we introduce the notion of a model-uncertainty-induced preference functional and, building on Černý [2003], derive the restrictions on pricing kernels equivalent to bounds on certainty equivalents. For the rest of the paper, we assume

Assumption 2 *(1) Utility $U(V)$ over wealth V is bounded above by a finite positive constant[6] which, without loss of generality, can be taken to be zero. Hence, $U(V)$ is non-positive i.e. $U(V) \leq 0$, for all V. Further, for $-\infty < V < \infty$, $\quad -\infty < U(V) \leq 0$.*

(2) $U(V)$ is non-decreasing, concave, continuous and differentiable. Furthermore, $\lim_{V \to -\infty} V/U(V) = 0$ and $\lim_{V \to \infty} U'(V) = 0$.

For future use, we define $\text{CARA}(U(V)) = -U''(V)/U'(V)$ to be the coefficient of absolute risk aversion of $U(V)$.

[6] This is satisfied by the three utility functions that we will be interested in: Quadratic, exponential and CRRA: $U(V) = \overline{\beta}\frac{V^{1-\gamma}}{1-\gamma}$, for (the empirically relevant) $\gamma > 1$.

3.1 The model-uncertainty-induced preference functional

We begin by describing the model uncertainty structure considered in this paper. As in the previous section, denote by $\mathbb{P}$ the reference probability measure over the possible states of the world. While the investor knows the model $\mathbb{P}$ to be the best estimate of the data-generating measure given the information at her disposal, she recognizes that the model is estimated from a finite data-set. Thus, she worries that the true model may be in a set of alternative models that are difficult for her to reject empirically. The investor guards against model uncertainty by considering portfolio allocations of basis assets that are robust across the set of alternative models.

None of the existing approaches to modeling model uncertainty suit our purpose well. Besides satisfying standard properties of ambiguity aversion, we wish to satisfy three additional criteria. First, we require preferences for which a theory of No Good Deals is tractable. Second, like Maenhout [2004], we prefer a multiplicative adjustment to the utility function to an additive form. Finally, we require a free parameter to be available controling the degree of aversion to model uncertainty. After experimenting with a variety of formulations, our approach is to introduce distortions ξ and a function $F(\xi)$ and assume that the uncertainty averse investor evaluates any prospect over future wealth V as:

$$J(\mathbf{V}) \equiv \inf_{\xi \in \Xi} \left\{ \mathbb{E}^{\mathbb{P}}[F(\xi) U(V)] \right\}, \qquad \text{where}$$
$$F(\xi) \equiv \xi (1 - \Omega \log \xi) \qquad \text{and} \qquad \Xi = \{\xi : \mathbb{E}^{\mathbb{P}}[\xi] = 1, \xi > 0, F(\xi) \geq 0\}. \tag{5}$$

Here, Ξ denotes the set of feasible distortions ξ. Similarly to Maenhout [2004], $U(V)$ is multiplicatively scaled by $F(\xi)$. The constraint $F(\xi) \geq 0$ ensures that $F(\xi) U(V)$ is non-decreasing in V. The constant Ω satisfies $1 \leq \Omega < \infty$ and captures the degree of aversion to model uncertainty. As Ω increases, the degree of aversion to model uncertainty increases, and the investor considers a (weakly) larger set of alternative models. In the limit $\Omega \to 1$, the investor only considers the reference model $\mathbb{P}$, corresponding to the case of no model uncertainty. In Section 5, we show how Ω may be estimated from historical time-series data. Note that Ω captures both the presence of and the investor's aversion to model uncertainty - the latter being measured by what we term the Model Confidence Level (see Section 5 for more details). A higher Model Confidence Level corresponds to the investor having a greater aversion to model uncertainty which leads to a higher value of Ω.

We can re-express (5) as $J(\mathbf{V}) = \inf_{\xi\in\Xi}\{\mathbb{E}^{\mathbb{P}}[\xi U(V) + \Omega\xi\log\xi(-U(V))]\}$, in which the first term is expected utility (under a different measure) and the second has the form of relative entropy scaled by $U(V)$. Despite some similarities between this and the formulations of Hansen and Sargent [2008] and Maenhout [2004], our formulation falls outside theirs.

The next proposition shows, however, that our formulation possesses all the main desirable characteristics of uncertainty averse preferences:

Proposition 3 *The model-uncertainty-induced preference functional*[7] $J(\boldsymbol{V})$ *defined by* (5) *is a well-defined uncertainty averse preference functional which provides a weak ordering of prospects with the following properties: (1) Monotonicity; (2) Convexity; (3) Risk Independence; (4) Continuity; (5) Uniform Continuity; (6) Homotheticity; where all the properties are defined as in Cerreia-Vioglio et al. [2011].*

The next proposition considers further properties of $J(\mathbf{V})$, focusing on the effects of the degree of aversion to model uncertainty Ω.

Proposition 4 *(1)* $J(\boldsymbol{V}) \equiv \inf_{\xi\in\Xi}\left\{\mathbb{E}^{\mathbb{P}}[F(\xi)U(V)]\right\} \leq \mathbb{E}^{\mathbb{P}}[U(V)]$.

(2) In the special case of $\Omega = 1$, $\xi_s \equiv 1$ *in all states* $s = 1,\ldots,S$ *and the inequality in the first part holds with equality:* $J(\boldsymbol{V}) = \mathbb{E}^{\mathbb{P}}[U(V)]$.

(3) As Ω *increases above one,* $J(\boldsymbol{V})$ *is non-increasing (and strictly decreasing except in the degenerate case of* $U(V)$ *being constant across all states* s*).*

[7] We describe it as a functional rather than as a function, and use the argument $\mathbf{V}$, since $J(\mathbf{V})$ depends on the entire distribution of V and cannot be represented as the expectation of a new function of a scalar V.

(4) As Ω increases, (a) the investor is more uncertainty averse in the sense described by Cerreia-Vioglio et al. [2011] (their Proposition 6), and (b) the investor considers a (weakly) larger set of alternative models.

(5) $J(\boldsymbol{V})$ is non-positive, non-decreasing, concave and continuous.

Thus, intuitively, the uncertainty averse investor cannot do better than her expected utility in the absence of model uncertainty. The second, third and fourth parts of the proposition show that the case $\Omega = 1$ corresponds to no aversion to model uncertainty, while the limit $\Omega \to \infty$ corresponds to total aversion to model uncertainty.

The next proposition solves for the infimum in (5). We use a subscript s to denote the state s. For an arbitrary function f_s, let $\min_s \{f_s\}$ and $\max_s \{f_s\}$ denote the minimum and maximum values of f_s across the S possible states. Introduce

$$\Psi \equiv \frac{1}{\Omega} - 1 \quad (\text{note } \Psi \leq 0), \quad \xi^{\max} \equiv \exp(1/\Omega) \quad \text{and} \quad G(\xi) \equiv 1 + \Omega(\xi - \log \xi - 1).$$

Proposition 5 *Define β_s to be the Lagrange multipliers on the constraints $F(\xi_s) \geq 0$. In the degenerate case that $\min_s \{-U(V_s)\} = \max_s \{-U(V_s)\} \equiv -U_c$, say, then $\xi_s \equiv 1$, set $\beta_s \equiv 0$ and $J(\boldsymbol{V}) \equiv \inf_{\xi \in \Xi} \left\{\mathbb{E}^{\mathbb{P}}[F(\xi) U(V)]\right\} = U_c$. Otherwise,*

$$\beta_s = \begin{cases} 0, & \text{if } U(V_s) \neq 0 \text{ and } \left(1 + \frac{\eta}{U(V_s)}\right)\Psi \leq \log \xi^{\max}; \\ \frac{\eta \Psi}{(\Psi - \log \xi^{\max})} + U(V_s), & \text{otherwise}; \end{cases} \tag{6}$$

$$\xi_s = \exp\left(\left(1 + \frac{\eta}{(U(V_s) - \beta_s)}\right)\Psi\right), \qquad \text{and} \tag{7}$$

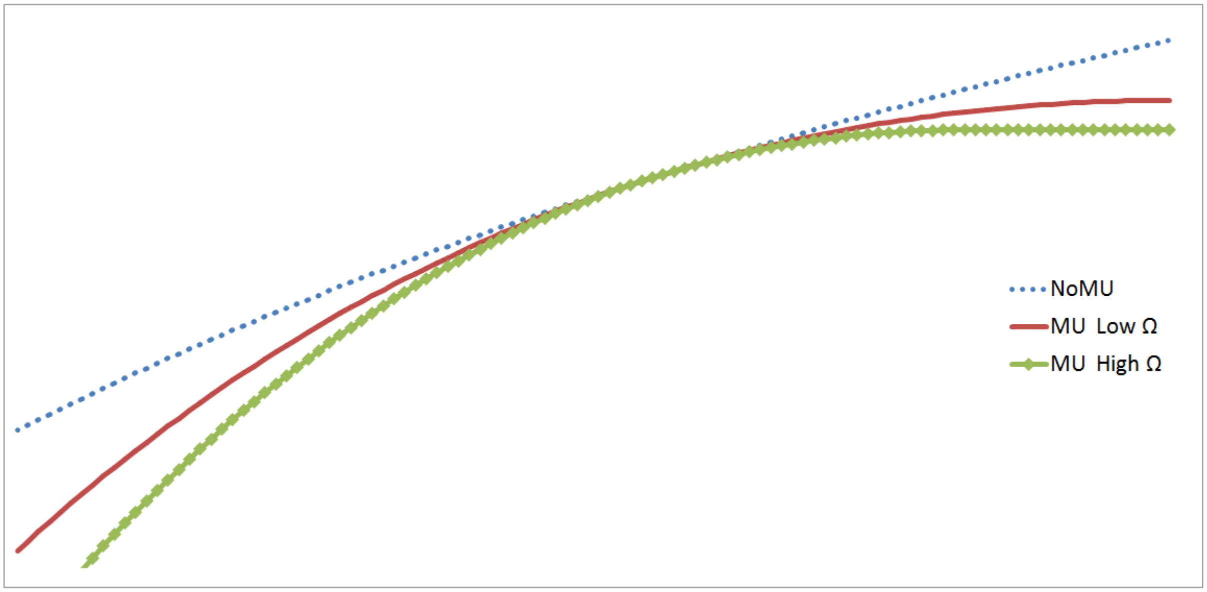
$$J(\boldsymbol{V}) \equiv \inf_{\xi \in \Xi} \left\{\mathbb{E}^{\mathbb{P}}[F(\xi) U(V)]\right\} = \max_{\eta} \left\{\sum_{s=1}^{S} \pi_s \left[G(\xi_s)(U(V_s) - \beta_s)\right]\right\}. \tag{8}$$

Furthermore:

$$0 \leq \min_s \{-U(V_s)\} \leq \eta \leq \max_s \{-U(V_s)\}, \tag{9}$$

and for all $s = 1, \ldots, S$: $$\xi_s \in [\exp(\Psi), \xi^{\max}]. \tag{10}$$

Figure 1: $U(V)$ and $U^{\mathrm{MU}}(V) \equiv G(\xi)(U(V)-\beta)$ as a function of outcomes of wealth V.



NOTES: $U(V)$ (labeled NoMU) and $U^{\mathrm{MU}}(V) \equiv G(\xi)(U(V)-\beta)$ (labeled MU) (vertical axis) for a range of outcomes of wealth V (plotted along the horizontal axis). $U^{\mathrm{MU}}(V)$ is shown for two different (low and high) values of Ω (>1), labeled MU Low Ω and MU High Ω respectively.

We refer to ξ given by (7), evaluated at the maximizing value $\hat{\eta}$ of η in (8), as the worst-case distortion. Since it depends on the reference measure $\mathbb{P}$, the distribution of V and $U(V)$, the model-uncertainty-induced preference functional $J(\mathbf{V})$ is setting-specific.

In Figure 1, we plot, for a range of outcomes of wealth V, $U(V)$ and $U^{\mathrm{MU}}(V) \equiv G(\xi)(U(V)-\beta)$ (the term inside the square brackets in (8) - note that β, ξ and $G(\xi)$ depend on V and $U(V)$ through (6) and (7)) for two different (low and high) values of Ω (>1). Intuitively,[8] the uncertainty averse investor assigns higher marginal utility to lower utility outcomes. We also see that $U^{\mathrm{MU}}(V)$ has greater curvature than $U(V)$. Both these statements are strengthened as Ω increases (cf Proposition 4).

We examine further economic properties of our model uncertainty structure in Appendix D and focus here on the implications for No Good Deals.

[8] By convexity, $G(\xi_s) \geq 1$ for any ξ_s. Hence, since $\beta_s \geq 0$, $U^{\mathrm{MU}}(V_s)$ is less than or equal to $U(V_s)$ for all states s. For outcomes of V_s near the the middle of its distribution, $U(V_s) \approx -\hat{\eta}$, $\xi_s \approx 1$ and $U^{\mathrm{MU}}(V_s) \approx U(V_s)$. The amount by which $U^{\mathrm{MU}}(V_s)$ is less than $U(V_s)$ increases if ξ_s is further away from 1. Higher outcomes of V_s result in $U(V_s)$ being larger (i.e. less negative), ξ_s being larger, $F(\xi_s)$ being smaller and marginal utility being smaller. Conversely, lower outcomes of V_s result in marginal utility being greater - captured by the steeper gradient of $U^{\mathrm{MU}}(V_s)$ for lower V_s. In the special case that $\Omega = 1$, $U^{\mathrm{MU}}(V) \equiv U(V)$.

3.2 Restriction on the pricing kernel under model uncertainty

We now use $J(\mathbf{V})$ and the results of the previous sub-section to derive the restriction on the pricing kernel in the presence of model uncertainty analogous to equation (4) in its absence.

We assume that the uncertainty averse investor, endowed with wealth V_0 at time 0, maximizes her preference functional $J(\mathbf{V})$, at time 1, subject to the time 0 budget constraint

$$\begin{aligned}\sup_V \left\{ \inf_{\xi\in\Xi} \left\{ \mathbb{E}^{\mathbb{P}}[F(\xi)\,U(V)] \right\} \text{ such that } \mathbb{E}^{\mathbb{P}}[mV] = V_0 \right\} &= \inf_{\xi\in\Xi} \left\{ \mathbb{E}^{\mathbb{P}}[F(\xi)\,U(V_0+\mathrm{CE})] \right\}, \quad \text{or} \\ \sup_V \left\{ [J(\mathbf{V})] \text{ such that } \mathbb{E}^{\mathbb{P}}[mV] = V_0 \right\} &= U(V_0+\mathrm{CE}), \end{aligned} \tag{11}$$

where CE denotes the certainty equivalent and where we have used the results of Proposition 5 in the second line. We solve for the certainty equivalent CE using the approach of Cox and Huang [1989] and Černý [2003].

Proposition 6 *(1) The first-order condition from (11) implies that the pricing kernel m is proportional to $F(\xi)\,U'(V)$, with ξ the worst-case distortion.*

(2) For the case of truncated quadratic utility, $U(V) = -\overline{\beta}\left(\max\left(\overline{V}-V,0\right)\right)^2$, where $\overline{\beta}>0$ is the investor's subjective discount factor and $\overline{V}$ is the bliss point (assumed to satisfy $\overline{V} > V_0+\mathrm{CE}$), the certainty equivalent CE is

$$\mathrm{CE} = \frac{1}{\mathrm{CARA}\,(U(V_0))}\Big(1 - \frac{1}{\sqrt{\inf_{\xi\in\Xi}\left\{\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right]\right\}}}\Big), \qquad \textit{and furthermore} \tag{12}$$

$$\mathrm{CE} \le \Lambda \Longleftrightarrow \inf_{\xi\in\Xi}\left\{\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right]\right\} \le \left(\frac{1}{1-\mathrm{CARA}\,(U(V_0))\;\Lambda}\right)^2. \tag{13}$$

Equation (13) is the analog to (4) and, hence, also to the Cochrane and Saá-Requejo [2000] bound and, clearly, our bound collapses to theirs if $\xi \equiv 1$. Equation (13) focuses on truncated quadratic utility - the cases of exponential and CRRA utility are in Appendix B.

4 No Good Deals - No Bad Models

In this section, we define and solve for good deal bounds in the presence of model uncertainty, which we term "No Good Deals - No Bad Models". Their definition is a natural extension of good deal bounds in the absence of model uncertainty.

Definition 7 *In the presence of model uncertainty, the time* 0 *lower* $\underline{C}$ *and upper* $\overline{C}$ *good deal bounds on the focus asset, paying* c *at time* 1*, solve*

$$\underline{C} = \inf_{C \in \mathcal{NA}} \left\{ C \text{ such that } \max_{\mu>0, \boldsymbol{v}} \left\{ 2\boldsymbol{v}'\boldsymbol{p} - 2\mu C + \inf_{\xi \in \Xi} \left\{ \mathbb{E}^{\mathbb{P}}[F(\xi)\, \tilde{U}^{\mathrm{TQ}}\left(\mu c - \boldsymbol{v}'\boldsymbol{x}\right)] \right\} \right\} \leq \Gamma \right\},$$

$$\overline{C} = \sup_{C \in \mathcal{NA}} \left\{ C \text{ such that } \max_{\mu<0, \boldsymbol{v}} \left\{ 2\boldsymbol{v}'\boldsymbol{p} - 2\mu C + \inf_{\xi \in \Xi} \left\{ \mathbb{E}^{\mathbb{P}}[F(\xi)\, \tilde{U}^{\mathrm{TQ}}\left(\mu c - \boldsymbol{v}'\boldsymbol{x}\right)] \right\} \right\} \leq \Gamma \right\}.$$

Definition 7 replaces $\tilde{U}^{\mathrm{TQ}}$, in equations (2) and (3), with $F(\xi)\,\tilde{U}^{\mathrm{TQ}}$ in the definition of the reward-for-risk and takes the infimum over the set of alternative models. In the presence of model uncertainty, the good deal bounds both rule out arbitrage and rule out the possibility of forming a portfolio of the focus asset and the basis assets which has a greater reward-for-risk, computed under the worst-case model, than Γ. The following proposition provides alternative characterizations (or equivalent definitions) of the good deal bounds.

Proposition 8 *Equivalently to Definition 7,* $\underline{C}$ *and* $\overline{C}$ *solve*

$$\underline{C} = \inf_{m} \left\{ \mathbb{E}^{\mathbb{P}}[mc] \text{ such that } \mathbb{E}^{\mathbb{P}}[m\boldsymbol{x}] = \boldsymbol{p}, m \geq 0, \inf_{\xi \in \Xi} \left\{ \mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \right\} \leq \Gamma \right\} \tag{14}$$

$$= \inf_{\xi \in \Xi} \left\{ \inf_{m} \left\{ \mathbb{E}^{\mathbb{P}}[mc] \text{ such that } \mathbb{E}^{\mathbb{P}}[m\boldsymbol{x}] = \boldsymbol{p}, m \geq 0, \mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \leq \Gamma \right\} \right\}, \quad \textit{and} \tag{15}$$

$$\overline{C} = \sup_{m} \left\{ \mathbb{E}^{\mathbb{P}}[mc] \text{ such that } \mathbb{E}^{\mathbb{P}}[m\boldsymbol{x}] = \boldsymbol{p}, m \geq 0, \inf_{\xi \in \Xi} \left\{ \mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \right\} \leq \Gamma \right\} \tag{16}$$

$$= \sup_{\xi \in \Xi} \left\{ \sup_{m} \left\{ \mathbb{E}^{\mathbb{P}}[mc] \text{ such that } \mathbb{E}^{\mathbb{P}}[m\boldsymbol{x}] = \boldsymbol{p}, m \geq 0, \mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \leq \Gamma \right\} \right\}. \tag{17}$$

The first two constraints in (14) to (17) are the same as for good deal bounds in the absence of model uncertainty and enforce, respectively, that the candidate pricing kernel m prices the basis assets and absence of arbitrage opportunities. The third constraint in (14) and (16) replaces the bound on the second moment of the pricing kernel in (4) by the bound in (13). Equations (15) and (17) show that good deal bounds in the presence of model uncertainty correspond to those of Cochrane and Saá-Requejo [2000] in its absence but computed under the worst-case model. The latter equations are more convenient for computing numerical solutions and simplify the analysis of good deal bounds in a multi-period setting (Section 6).

For (14) to (17) to have a solution, the target level of reward-for-risk Γ has to be large enough to reprice the basis assets. This requires the assumption that $\Gamma \geq \Gamma^{\star}$, where

$$\Gamma^{\star} \quad = \quad \inf_{\xi \in \Xi} \left\{ \inf_{m} \left\{ \mathbb{E}^{\mathbb{P}} \left[\frac{m^2}{F(\xi)} \right] \quad \text{such that} \;\; \mathbb{E}^{\mathbb{P}}[m\mathbf{x}] = \mathbf{p}, m \geq 0 \right\} \right\}. \tag{18}$$

Similarly to Cochrane and Saá-Requejo [2000], we consider two different combinations of slack and binding constraints in (15) and (17):

In case (1), the constraint $\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \leq \Gamma$ binds. The solution is in Proposition 9.

In case (2), the constraint $\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \leq \Gamma$ is slack. The solution is in Proposition 10.

To simplify the solutions, we introduce a binary variable $\mathbf{1}_{L/U}$ which takes the value 1 (respectively, -1) if we are computing the lower (upper) good deal bound $\underline{C}$ (respectively, $\overline{C}$). Let δ and φ be scalars and let $\mathbf{w}$ be a N_b - dimensional vector. We define the loss function $Z_s\left(\mathbf{1}_{L/U}, \mathbf{w}, \varphi, \delta\right)$ in state s as

$$Y_s\left(\mathbf{1}_{L/U}, \mathbf{w}, \varphi, \delta\right) \;\; \equiv \;\; \max[-\mathbf{1}_{L/U} \frac{\left(\varphi c_s - \mathbf{w}^{'}\mathbf{x}_s\right)}{\delta}, 0], \tag{19}$$

$$Z_s\left(\mathbf{1}_{L/U}, \mathbf{w}, \varphi, \delta\right) \;\; \equiv \;\; \left(Y_s\left(\mathbf{1}_{L/U}, \mathbf{w}, \varphi, \delta\right)\right)^2 = -\tilde{U}^{\mathrm{TQ}}\left(\mathbf{1}_{L/U} \frac{\left(\varphi c_s - \mathbf{w}^{'}\mathbf{x}_s\right)}{\delta}\right). \tag{20}$$

4.1 Case (1): The constraint $\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \leq \Gamma$ binds

We solve for the good deal bounds $\underline{C}$ and $\overline{C}$ by forming the Lagrangian of the constrained optimization problem in equations (15) and (17).

Proposition 9 *When* $\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \leq \Gamma$ *binds, the pricing kernel in state* s *is* $m_s = F(\xi_s) Y_s\left(\mathbf{1}_{L/U}, \boldsymbol{w}, 1, \delta\right)$, *where the worst-case distortion* ξ *and the Lagrange multipliers* β_s *on* $F(\xi_s) \geq 0$ *are given in Proposition 5, with* $U(V_s)$ *replaced by* $-Z_s\left(\mathbf{1}_{L/U}, \boldsymbol{w}, 1, \delta\right)$. *The lower* $\underline{C}$ *and upper* $\overline{C}$ *good deal bounds solve*

$$\begin{aligned} \underline{C} &= \max_{\delta>0,\boldsymbol{w}}\left\{\boldsymbol{w}'\boldsymbol{p} - \frac{1}{2}\delta\Gamma + \right. \\ &\qquad \left. \max_{\eta}\left\{\sum_{s=1}^{S}\pi_s\left[-\frac{1}{2}\delta G(\xi_s)\left(Z_s\left(\mathbf{1}_{L/U}, \boldsymbol{w}, 1, \delta\right) + \beta_s\right)\right]\right\}\right\}, \qquad (21) \\ \overline{C} &= \min_{\delta>0,\boldsymbol{w}}\left\{\boldsymbol{w}'\boldsymbol{p} + \frac{1}{2}\delta\Gamma + \right. \\ &\qquad \left. \min_{\eta}\left\{\sum_{s=1}^{S}\pi_s\left[\frac{1}{2}\delta G(\xi_s)\left(Z_s\left(\mathbf{1}_{L/U}, \boldsymbol{w}, 1, \delta\right) + \beta_s\right)\right]\right\}\right\}. \qquad (22) \end{aligned}$$

Equivalently, with $\tilde{U}^{\mathrm{TQ}}(V) \equiv -\left(\max(-V, 0)\right)^2$, $\mu = \mathbf{1}_{L/U}/\delta$ *and* $\boldsymbol{v} = \mathbf{1}_{L/U}\,\boldsymbol{w}/\delta$,

$$\underline{C} \text{ solves}: \quad \max_{\mu>0,\boldsymbol{v}}\left\{2\boldsymbol{v}'\boldsymbol{p} - 2\mu\underline{C} + \inf_{\xi\in\Xi}\left\{\mathbb{E}^{\mathbb{P}}\left[F(\xi)\,\tilde{U}^{\mathrm{TQ}}\left(\mu c - \boldsymbol{v}'\boldsymbol{x}\right)\right]\right\}\right\} = \Gamma, \qquad (23)$$

$$\overline{C} \text{ solves}: \quad \max_{\mu<0,\boldsymbol{v}}\left\{2\boldsymbol{v}'\boldsymbol{p} - 2\mu\overline{C} + \inf_{\xi\in\Xi}\left\{\mathbb{E}^{\mathbb{P}}\left[F(\xi)\,\tilde{U}^{\mathrm{TQ}}\left(\mu c - \boldsymbol{v}'\boldsymbol{x}\right)\right]\right\}\right\} = \Gamma. \qquad (24)$$

Equations (21), (22) are solved numerically by choice of η and then by choice of $\delta > 0$, $\mathbf{w}$.

Consider first the pricing kernel m. The quantity $Y_s\left(\mathbf{1}_{L/U}, \mathbf{w}, 1, \delta\right)$ is the Cochrane and Saá-Requejo [2000] pricing kernel (in the absence of model uncertainty). In the special case of $\Omega = 1$ (which corresponds to no aversion to model uncertainty), $\xi_s = 1$ in each state, the pricing kernel m reduces to the pricing kernel of Cochrane and Saá-Requejo [2000] and the lower and upper good deal bounds (21) and (22) coincide with those of Cochrane and Saá-Requejo [2000]. Similarly, when the loss function $Z_s\left(\mathbf{1}_{L/U}, \mathbf{w}, 1, \delta\right)$ is independent of the state s, $\xi_s = 1$ in each state and the solution again reduces to that of Cochrane and Saá-Requejo [2000].

In the more general case of Ω strictly greater than one (so that the investor exhibits aversion to model uncertainty) and $\min_s \left\{ Z_s \left(\mathbf{1}_{L/U}, \mathbf{w}, 1, \delta \right) \right\} \neq \max_s \left\{ Z_s \left(\mathbf{1}_{L/U}, \mathbf{w}, 1, \delta \right) \right\}$ (so that the investor is not indifferent amongst the different states of the world), ξ_s decreases when $Z_s \left(\mathbf{1}_{L/U}, \mathbf{w}, 1, \delta \right)$ increases. Since $F(\xi_s)$ is decreasing in ξ_s (in the range $[\exp(\Psi), \xi^{\max}]$), the pricing kernel m_s in state s increases as $Z_s \left(\mathbf{1}_{L/U}, \mathbf{w}, 1, \delta \right)$ increases. Thus, the uncertainty averse investor assigns higher marginal utility (i.e. higher than that in the absence of model uncertainty) to states with larger losses. The maximization (respectively, minimization) over $\delta > 0$, $\mathbf{w}$ for the lower good deal bound $\underline{C}$ (upper good deal bound $\overline{C}$) then has the effect of minimizing the average weighted losses.

The Lagrange multipliers $\mathbf{w}$ on the constraints $\mathbb{E}^{\mathbb{P}}[m\mathbf{x}] = \mathbf{p}$ have the interpretation of optimal hedges for the focus asset using the basis assets. This property allows us to interpret $Z_s \left(\mathbf{1}_{L/U}, \mathbf{w}, 1, \delta \right)$ as a (post-hedge) *loss function*.

4.2 Case (2): The constraint $\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \leq \Gamma$ is slack

When $\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \leq \Gamma$ is slack, the good deal bounds reduce to the arbitrage bounds (i.e. those enforceable by sub- or super-replication). In particular, since the infimum (respectively, supremum) over ξ for the lower (upper) good deal bound becomes irrelevant, we solve

$$\begin{aligned} \underline{C} &= C_{\text{Arb}}(1) \equiv \inf_m \left\{ \mathbb{E}^{\mathbb{P}}[mc] \;\text{ such that }\; \mathbb{E}^{\mathbb{P}}[m\mathbf{x}] = \mathbf{p}, m \geq 0 \right\}, \quad \text{and} \\ \overline{C} &= C_{\text{Arb}}(-1) \equiv \sup_m \left\{ \mathbb{E}^{\mathbb{P}}[mc] \;\text{ such that }\; \mathbb{E}^{\mathbb{P}}[m\mathbf{x}] = \mathbf{p}, m \geq 0 \right\}, \end{aligned} \tag{25}$$

the solution of which can always be obtained numerically (see Cochrane and Saá-Requejo [2000]), since it is a linear program. The respective arbitrage bound, denoted $C_{\text{Arb}}\left(\mathbf{1}_{L/U}\right)$, is the good deal bound if the constraint $\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \leq \Gamma$ is slack. To check, we solve

$$\inf_{\xi \in \Xi} \left\{ \inf_m \left\{ \mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \quad \text{such that } C_{\text{Arb}}\left(\mathbf{1}_{L/U}\right) = \mathbb{E}^{\mathbb{P}}[mc], \; \mathbb{E}^{\mathbb{P}}[m\mathbf{x}] = \mathbf{p}, m \geq 0 \right\} \right\}. \tag{26}$$

If the solution to (26) is less than Γ, then $C_{\mathrm{Arb}}\left(\mathbf{1}_{L/U}\right)$ is the good deal bound. Otherwise, the constraint binds and case (1) is the relevant one. The solution is summarized below, with μ a scalar and $\mathbf{v}$ a N_b - dimensional vector.

Proposition 10 *The pricing kernel in state s is $m_s = F\left(\xi_s\right) Y_s\left(1, \boldsymbol{v}, \mu, 1\right)$, where the worst-case distortion ξ and the Lagrange multipliers β_s on $F\left(\xi_s\right) \geq 0$ are given in Proposition 5, with $U\left(V_s\right)$ replaced by $-Z_s\left(1, \boldsymbol{v}, \mu, 1\right)$. The solution to (26) is*

$$\max_{\boldsymbol{v},\mu}\left\{2\boldsymbol{v}^{'}\boldsymbol{p} - 2\mu C_{\mathrm{Arb}}\left(\boldsymbol{1}_{L/U}\right) + \max_{\eta}\left\{\sum_{s=1}^{S}\pi_s\left[-G\left(\xi_s\right)\left(Z_s\left(1, \boldsymbol{v}, \mu, 1\right) + \beta_s\right)\right]\right\}\right\}. \tag{27}$$

Equivalently, $\underline{C}$ (respectively, $\overline{C}$) equals the lower (upper) arbitrage bound $C_{\mathrm{Arb}}\left(\boldsymbol{1}_{L/U}\right)$, if

$$\max_{\mu,\boldsymbol{v}}\left\{2\boldsymbol{v}^{'}\boldsymbol{p} - 2\mu C_{\mathrm{Arb}}\left(\boldsymbol{1}_{L/U}\right) + \inf_{\xi\in\Xi}\left\{\mathbb{E}^{\mathbb{P}}\left[F\left(\xi\right)\tilde{U}^{\mathrm{TQ}}\left(\mu c - \boldsymbol{v}^{'}\boldsymbol{x}\right)\right]\right\}\right\} < \Gamma. \tag{28}$$

Problem (27) is solved numerically by choice of η and then by choice of $\mathbf{v}$, μ.

Unlike in Proposition 9, the Lagrange multipliers $\mathbf{v}$ on the constraints $\mathbb{E}^{\mathbb{P}}\left[m\mathbf{x}\right] = \mathbf{p}$ do not have the interpretation of optimal hedges for the focus asset. Instead, the optimal hedges $\mathbf{w}$ enforce the arbitrage bounds in (25) and can be computed explicitly by solving the dual to the linear program. Specifically, $\mathbf{w}$ solves

$$\max_{\mathbf{w}}\mathbf{w}^{'}\mathbf{p} \quad \text{such that} \quad \mathbf{w}^{'}\mathbf{x}_s \leq c_s, \text{ for each state } s,\ s = 1, \ldots, S, \tag{29}$$

for the lower bound (for the upper bound, replace max by min and $\leq$ by $\geq$).

To conclude this sub-section, we emphasize the intuition of the No Good Deals methodology. Interpreting Γ as the target level of reward-for-risk (a Sharpe ratio or certainty equivalent), the No Good Deals - No Bad Models methodology computes lower and upper good deal bounds which either: Case (1), achieve the target level of reward-for-risk under the worst-case model, or; Case (2), are the arbitrage bounds $C_{\mathrm{Arb}}\left(\mathbf{1}_{L/U}\right)$ when the target level cannot be achieved. Analogously to (26) and (28), the minimum level $\Gamma^\star$ (defined in equation (18)) of the restriction on the pricing kernel has the dual representation

$$\Gamma^\star = \max_{\mathbf{v}} \left\{ 2\mathbf{v}^{'}\mathbf{p} + \inf_{\xi\in\Xi} \left\{ \mathbb{E}^{\mathbb{P}} \left[F\left(\xi\right) \tilde{U}^{\mathrm{TQ}} \left(-\mathbf{v}^{'}\mathbf{x} \right) \right] \right\} \right\}, \tag{30}$$

$$\Gamma^\star = \max_{\mathbf{v}} \left\{ 2\mathbf{v}^{'}\mathbf{p} + \max_{\eta} \left\{ \sum_{s=1}^{S} \pi_s \left[-G\left(\xi_s\right) \left(Z_s\left(1, \mathbf{v}, 0, 1\right) + \beta_s \right) \right] \right\} \right\}, \tag{31}$$

where β_s and ξ_s are defined as in (27) but replacing $Z_s\left(1, \mathbf{v}, \mu, 1\right)$ with $Z_s\left(1, \mathbf{v}, 0, 1\right)$. Thus, $\Gamma^\star$ is the maximum reward-for-risk available under the worst-case model from trading in the basis assets. The restriction $\Gamma \geq \Gamma^\star$ is then interpretable as the requirement that the target level of reward-for-risk weakly exceeds that available from trading in the basis assets alone.

4.3 Model uncertainty versus reward-for-risk

In this sub-section, we show that model uncertainty leads to economically different outcomes than higher target levels of reward-for-risk. We begin with the following proposition.

Proposition 11 *(a) Good deal bounds in the presence of model uncertainty are never narrower than those in its absence.*

(b) As either Γ increases or Ω increases, the lower good deal bound $\underline{C}$ is non-increasing and the upper good deal bound $\overline{C}$ is non-decreasing.

Property (a) is intuitive since $\xi_s \equiv 1$, for each s, is feasible in (14) to (17). Property (b) shows that good deal bounds in the presence of model uncertainty weakly widen as the target level of reward-for-risk Γ increases, similarly to the good deal bounds of Cochrane and Saá-Requejo [2000]. We discuss further similarities in Appendix B.

Here, we focus upon the differences. We show that the the distance between good deal bounds in the presence of model uncertainty and those in its absence weakly increases with the dispersion and negative skewness of the loss function Z. Define $H(\hat{Z}) \equiv G\left(\xi\right)\hat{Z}$, where

$\hat{Z} \equiv Z + \beta$. Note that Z can refer to either $Z_s\left(\mathbf{1}_{L/U}, \mathbf{w}, 1, \delta\right)$ or $Z_s\left(1, \mathbf{v}, \mu, 1\right)$ and that $G\left(\xi\right)$ depends on $\hat{Z}$ through ξ. Applying a Taylor expansion around $\hat{Z} - \mathbb{E}^{\mathbb{P}}[\hat{Z}]$ and using the result $\mathbb{E}^{\mathbb{P}}[\hat{Z}] \geq \mathbb{E}^{\mathbb{P}}[Z]$ implies

$$\mathbb{E}^{\mathbb{P}}\left[H\left(\hat{Z}\right)\right] - \mathbb{E}^{\mathbb{P}}[Z] \;\geq\; \frac{1}{2}\sum_{s=1}^{S}\pi_s\left[\left(\hat{Z} - \mathbb{E}^{\mathbb{P}}[\hat{Z}]\right)^2 H''\left(\hat{Z}^{\star}\right)\right], \quad \text{or} \tag{32}$$

$$\mathbb{E}^{\mathbb{P}}\left[H\left(\hat{Z}\right)\right] - \mathbb{E}^{\mathbb{P}}[Z] \;\geq\; \frac{1}{2}\mathbb{V}ar^{\mathbb{P}}[\hat{Z}]H''\left(\mathbb{E}^{\mathbb{P}}[\hat{Z}]\right) + \frac{1}{6}\sum_{s=1}^{S}\pi_s\left[\left(\hat{Z} - \mathbb{E}^{\mathbb{P}}[\hat{Z}]\right)^3 H'''\left(\hat{Z}^{\star\star}\right)\right],$$

for some $\hat{Z}^{\star}$, $\hat{Z}^{\star\star}$. It is straightforward to verify that $H''\left(\hat{Z}\right) \geq 0$ and $H'''\left(\hat{Z}\right) \leq 0$. Now comparing with (21), (22) and (27), we see the sense in which a wide dispersion in Z increases the distance between good deal bounds in the presence of model uncertainty and those in its absence. In the latter case, good deal bounds depend only upon $\mathbb{E}^{\mathbb{P}}[Z\left(\mathbf{1}_{L/U}, \mathbf{w}, 1, \delta\right)]$ (or $\mathbb{E}^{\mathbb{P}}[Z\left(1, \mathbf{v}, \mu, 1\right)]$ in determining if $\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \leq \Gamma$ is slack), but in the former case, they depend upon $\mathbb{E}^{\mathbb{P}}\left[H\left(\hat{Z}\right)\right]$. *Ceteris paribus*, when Z has greater variance and/or more negative skewness, $\mathbb{E}^{\mathbb{P}}\left[H\left(\hat{Z}\right)\right] - \mathbb{E}^{\mathbb{P}}[Z]$ is weakly greater and hence so is the distance.

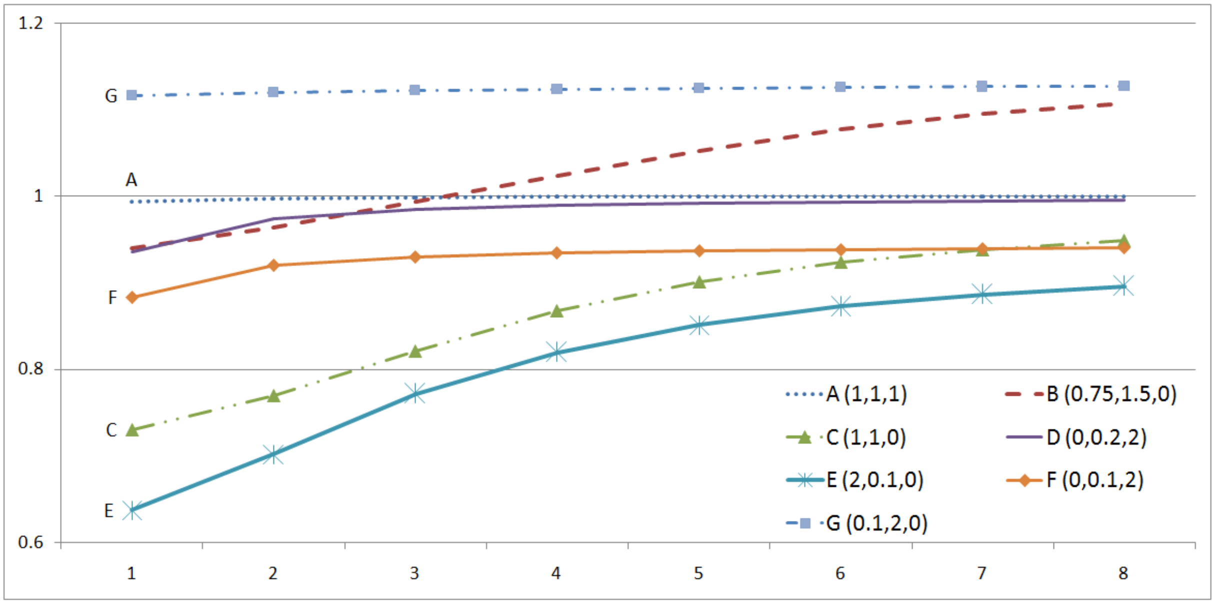
Although good deal bounds in the presence of model uncertainty weakly widen with increases in both Γ and Ω (Proposition 11), we now show that increases in Ω lead to fundamentally different outcomes than increases in Γ.

Suppose there are three states with $\mathbb{P}$ probabilities $(1/8, 1/2, 3/8)$. There is a single basis asset with time 0 price equal to zero which pays $(1, -0.8, 1)$ in the three states, at time 1. We consider seven different focus assets, labeled A to G, with payoffs: A $(1, 1, 1)$; B $(0.75, 1.5, 0)$; C $(1, 1, 0)$; D $(0, 0.2, 2)$; E $(2, 0.1, 0)$; F $(0, 0.1, 2)$; G $(0.1, 2, 0)$.

For these simple payoffs, it is straightforward to show that the upper arbitrage bound is infinite and the constraint $\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \leq \Gamma$ always binds. Hence, using equation (24), the upper good deal bound, denoted $\overline{C}\left(\mathcal{J}, \Gamma, \Omega\right)$, for each $\mathcal{J} = A, \ldots, G$, can be expressed[9] as

[9]Equation (24) simplifies to $\max_{\delta>0,v}\left\{2\overline{C} - \delta\Gamma + \inf_{\xi\in\Xi}\left\{\mathbb{E}^{\mathbb{P}}\left[F\left(\xi\right)\tilde{U}^{\mathrm{TQ}}\left(-c+wx\right)\right]\right\}/\delta\right\} = 0$, after setting $\mu = -1/\delta$, $v = -w/\delta$ and $p = 0$. The first-order condition for δ implies $-\inf_{\xi\in\Xi}\left\{\mathbb{E}^{\mathbb{P}}\left[F\left(\xi\right)\tilde{U}^{\mathrm{TQ}}\left(-c+wx\right)\right]\right\} = \delta^2\Gamma$. Now solve for δ, substitute and simplify.

Figure 2: Upper good deal bounds $\overline{C}(\mathcal{J},1,\Omega)$ for the seven focus assets $\mathcal{J}=A,\ldots,G$.



NOTES: Upper good deal bounds $\overline{C}(\mathcal{J},1,\Omega)$ as a function of the degree of aversion to model uncertainty Ω (with the target level of reward-for-risk $\Gamma=1$). $\mathbb{P}$ probabilities of the three states are $(1/8,1/2,3/8)$.

$\overline{C}(\mathcal{J},\Gamma,\Omega)=\sqrt{\Gamma}\ \ \min_w\{\sqrt{-\inf_{\xi\in\Xi}\{\mathbb{E}^{\mathbb{P}}[F(\xi)\,\tilde{U}^{\mathrm{TQ}}(-c+wx)]\}}\ \}$, after simplifying. Hence, $\overline{C}(\mathcal{J},\Gamma,\Omega)$ is linear in $\sqrt{\Gamma}$ but, in sharp contrast, no such simple relationship holds for Ω.

In Figure 2, we plot $\overline{C}(\mathcal{J},\Gamma,\Omega)$, setting $\Gamma=1$, for each focus asset $\mathcal{J}$ as a function of Ω. As Ω increases, the upper good deal bounds increase (in line with Proposition 11), but their sensitivity to changes in Ω varies significantly across the seven different focus assets. Compare, for example, E, F and G. Although they have the same combinations of payoffs, they receive them in different states. The upper good deal bound for E (which receives the largest payoff in the least probable state) has much greater sensitivity to Ω than that for F, which, in turn, has greater sensitivity than that for G (which receives the largest payoff in the most probable state). Overall, A is least sensitive to Ω. These results are in line with the theoretical analysis above (in (32)) since E (respectively, A) has the payoff, and losses after hedging with the basis asset, with the most (respectively, least) variance and skew.

We now show that higher levels of Γ do not achieve the same economic outcomes as higher levels of Ω. We begin by finding, for focus asset F, the value of Γ that matches, in the absence of model uncertainty, the upper good deal bound for $\Gamma = 1$ and $\Omega = 4$. This requires $\Gamma = 1.1193$, since $\overline{C}(F, 1.1193, 1) = \overline{C}(F, 1, 4) = 0.9342$. Recomputing the upper good deal bounds on E and G with these values of Γ and Ω, we find $\overline{C}(E, 1.1193, 1) = 0.6740$ while $\overline{C}(E, 1, 4) = 0.8192$ and $\overline{C}(G, 1.1193, 1) = 1.1804$ while $\overline{C}(G, 1, 4) = 1.1233$. Thus, although we can match the upper good deal bound on one focus asset by increasing Γ, the upper good deal bounds for other focus assets still deviate from those in the presence of model uncertainty. Further, if we consider focus asset A, we see that, even for a single focus asset, there is not a one-to-one mapping between good deal bounds computed for different Γ and those computed for different Ω. For example, from Figure 2, $\overline{C}(A, 1, 1) = 0.9939$. Now increase Γ slightly to $\Gamma = 1.0133$: The upper good deal bound in the absence of model uncertainty becomes $\overline{C}(A, 1.0133, 1) = \sqrt{1.0133}\ 0.9939 = 1.0005$. Can one find a value of Ω such that $\overline{C}(A, 1, \Omega) = 1.0005$? No - because no such Ω exists since, as is apparent from Figure 2, $\overline{C}(A, 1, \Omega)$ asymptotes at 1 and never reaches 1 for any finite Ω. In contrast, by changing Γ, any finite, positive value of $\overline{C}(A, \Gamma, 1)$ whatsoever can be attained. Hence, model uncertainty leads to fundamentally different economic outcomes.

5 Estimating Ω and the choice of Γ

In this section, we describe a procedure for estimating the degree of aversion to model uncertainty Ω and discuss the choice of the target level of reward-for-risk Γ.

We choose Ω so that the maximum reward-for-risk achievable under the worst-case model, from trading in the basis assets, is difficult to distinguish from the maximum reward-for-risk achievable under the reference model $\mathbb{P}$. Note that, while our methodology for the estimation of Ω is similar in spirit to the detection error probability methodology of Hansen and Sargent [2008], Anderson, Hansen, and Sargent [2003] and Maenhout [2004], their methodology requires that the whole probability measure under the worst-case distortion is difficult to distinguish from that under $\mathbb{P}$. In contrast, our methodology requires that the worst-case model and the reference model $\mathbb{P}$ are difficult to distinguish only in terms of the maximum reward-for-risk available. The latter is more appropriate in our setting since the good deal bounds are defined (see Definition 7) in terms of reward-for-risk. We set the level of distinguishability by the Model Confidence Level. While acknowledging that some investors (or, more generally, financial institutions) may exhibit more uncertainty aversion than others, we choose Ω based on a Model Confidence Level[10] in the region of 80%. We compute Ω from a historical data-set of asset prices, making the parameter Ω context-specific, with different data-sets or different basis assets leading to different estimates of Ω.

Similarly to $\Gamma^{\star}$ in equations (18), (30) and (31), we define $\Gamma^{\star\text{NoMU}}$ to be the maximum reward-for-risk available from trading in the basis assets under the reference measure $\mathbb{P}$:

$$\begin{aligned} \Gamma^{\star\text{NoMU}} &\equiv \max_{\mathbf{v}} \left\{ 2\mathbf{v}^{'}\mathbf{p} + \sum_{s=1}^{S} \pi_s \left[- \left(Z_s \left(1, \mathbf{v}, 0, 1\right)\right)\right] \right\} \\ &\equiv 2\hat{\mathbf{v}}^{\text{NoMU}\,'}\mathbf{p} + \sum_{s=1}^{S} \pi_s \left[- \left(Z_s \left(1, \hat{\mathbf{v}}^{\text{NoMU}}, 0, 1\right)\right)\right], \end{aligned} \tag{33}$$

where $\hat{\mathbf{v}}^{\text{NoMU}}$ is the maximizing value of $\mathbf{v}$ in the first line. We compute $\hat{\mathbf{v}}^{\text{NoMU}}$ by solving problem (33) using historical data[11], setting the number of states S equal to the number of available observations J and assigning equal probabilities to each state $\pi_s = 1/J$. We bootstrap[12] the historical data, K times, by repeatedly sampling from the data with replacement. Sampling with replacement means that sometimes we sample a given historical data point

[10] Hansen and Sargent [2008], Anderson et al. [2003] and Maenhout [2004] recommend detection error probabilities of around 20% which, given the "opposite parameterization", provides some support for our choice of Model Confidence Level of around $100 - 20\% = 80\%$.

[11] Note that when solving (33), (34) and (35), we use returns (i.e. we divide each historically observed price by its price at the preceding time point). Hence, $Z_s\left(1, \hat{\mathbf{v}}^{\text{NoMU}}, 0, 1\right)$ now reflects "payoffs in return form". As an alternative, we could work with excess (i.e. over and above the risk-free rate) returns $\mathbf{R}$ which may be better conditioned. The analogs of problems (33), (34) and (35) are easily obtained by comparison with (18) and (31). In particular, the constraints $\mathbb{E}^{\mathbb{P}}[m\mathbf{x}] = \mathbf{p}$ are replaced by $\mathbb{E}^{\mathbb{P}}[m\mathbf{R}] = \mathbf{0}$ and $\mathbb{E}^{\mathbb{P}}[m] = 1/\left(1 + R^f\right)$, where R^f is the average one period risk-free rate. We need the extra condition on $\mathbb{E}^{\mathbb{P}}[m]$ to normalise m.

[12] If the returns are not iid or cannot satisfactorily be approximated as such, then, alternatively, we could simulate returns using the reference model $\mathbb{P}$.

more than once and sometimes not at all. This is equivalent to using different probabilities $\pi_s^{(k)}$, for each $k = 1, \ldots, K$. For each bootstrapped sample, we compute

$$\Gamma^{\star(k)} \equiv 2\hat{\mathbf{v}}^{\text{NoMU}\,\prime}\mathbf{p} + \sum_{s=1}^{S} \pi_s^{(k)} \left[-\left(Z_s\left(1, \hat{\mathbf{v}}^{\text{NoMU}}, 0, 1\right)\right)\right]. \tag{34}$$

We sort the K values $\Gamma^{\star(k)}$ into order and select the percentile, $\Gamma^{\star MCL}$ say, corresponding to the chosen Model Confidence Level. We then set Ω to be the value which solves

$$2\hat{\mathbf{v}}^{\text{NoMU}\,\prime}\mathbf{p} + \max_{\eta} \left\{ \sum_{s=1}^{S} \pi_s \left[-G\left(\xi_s\right) \left(Z_s\left(1, \hat{\mathbf{v}}^{\text{NoMU}}, 0, 1\right) + \beta_s \right)\right] \right\} = \Gamma^{\star MCL}. \tag{35}$$

Note that the left-hand side of (35) uses the probabilities from the original (non-bootstrapped) historical sample and can be interpreted as the maximum reward-for-risk available under the worst-case model from taking positions in the basis assets given by $\hat{\mathbf{v}}^{\text{NoMU}}$ (as opposed to the optimal values in (31)). In words, Ω is chosen so that, conditional on maintaining the same[13] positions $\hat{\mathbf{v}}^{\text{NoMU}}$ in the basis assets, the maximum reward-for-risk achievable under the worst-case model is difficult to distinguish from the maximum reward-for-risk under the reference model $\mathbb{P}$. Thus, our methodology for estimating Ω amounts to using an economic definition (maximum reward-for-risk) rather than a statistical definition (for example, detection error probability) of "difficult to distinguish".

We now turn our attention to choosing the target level of reward-for-risk Γ in the constraint $\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \leq \Gamma$: There are different alternatives: (1) Choose Γ to be some margin over (or some multiple ≥ 1 of) either (a) $\Gamma^{\star\text{NoMU}}$ or (b) $\Gamma^{\star}$ (with the former being more conservative); (2) Choose Γ through the choice of an annualized Sharpe ratio h_{Ann}.

[13] Keeping the same positions $\hat{\mathbf{v}}^{\text{NoMU}}$ in the basis assets is consistent with the premise (Section 3) that the investor guards against model uncertainty by considering positions in the basis assets that are robust across the set of alternative models. In this section, we are essentially solving the "inverse" problem, namely, does the maximum reward-for-risk change significantly across the K bootstrapped samples? If so, then there is a significant amount of model uncertainty, leading to a higher estimate of Ω; if not, there is less model uncertainty, leading to a lower estimate of Ω.

In our numerical examples in Section 7, we use the second alternative. Černý [2003] shows that the certainty equivalent CE of any utility function $U(V)$ and its coefficient of absolute risk aversion $\text{CARA}(U(V_0))$ are linked, under Assumptions 1 and 2, to investment opportunities, over a time period Δt years, with a small per period Sharpe ratio h_{PerP}, by:

$$\text{CARA}(U(V_0))\ \text{CE} \approx \frac{1}{2}h^2_{\text{PerP}} = \frac{1}{2}h^2_{\text{Ann}}\Delta t. \tag{36}$$

Hence, a bound $\text{CE} \leq \Lambda$ in equation (13) is approximately the same as a bound

$$\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \leq \left(\frac{1}{1-(\text{CARA}(U(V_0))\Lambda)}\right)^2 \approx \left(\frac{1}{1-\frac{1}{2}h^2_{\text{PerP}}}\right)^2 \approx 1 + h^2_{\text{PerP}} = 1 + h^2_{\text{Ann}}\Delta t.$$

In our numerical examples, we (following Cochrane and Saá-Requejo [2000]) set $\Gamma = \left(1 + h^2_{\text{Ann}}\Delta t\right) / \left(1 + R^f\right)^2$, where R^f is the one period risk-free rate. Cochrane and Saá-Requejo [2000] use a value of h_{Ann} equal to one, arguing that a value twice that of the annualised Sharpe ratio on the S&P 500 (based on stylized data of an 8% excess return and 16% volatility) is appropriate. However, we show in Appendix D.1 that, with a Model Confidence Level of 80%, model uncertainty can account for half, possibly a little more, of the excess return on equity markets. Hansen and Sargent [2008], Anderson et al. [2003] and Maenhout [2004] come to broadly the same conclusion, within their model uncertainty frameworks, using a detection error probability of 20%. This suggests that smaller values of h_{Ann} may be appropriate in the presence of model uncertainty - we choose various values of h_{Ann} (typically) around 0.5. There is some anecdotal support[14] for this choice from senior management within the banking industry, based on target returns on equity.

[14] In 2000, the then chairman of a leading global bank made a public statement saying that he wanted its investment banking arm to achieve a 20% target return on equity (when risk-free rates were around 5% and the volatility of its shares was around 30%) - which could be interpreted as implying he wanted it to achieve an ex-post Sharpe ratio of $(20-5)/30 = 0.5$. Choosing $h_{\text{Ann}} = 0.5$ is then equivalent to seeking to achieve his target, even under the worst-case model. Following the public statement, other leading banks also adopted similar target returns on equity.

6 Multiple time periods

The previous analysis considered a one period problem. In this section, we extend the results to a multi-period setting, focusing on the two period case. There are three dates, indexed by $t = 0, 1, 2$. At time $t = 0$, the investor takes a position in the focus asset, which pays c at time $t = 2$. Time $t = 1$ is an intermediate rebalancing time, when the investor can adjust her hedging portfolio in the basis assets and update her valuation of the focus asset. Denote the time $t = 0, 1$ conditional expectation under the reference measure $\mathbb{P}$ by $\mathbb{E}_t^{\mathbb{P}}$ and the lower and upper good deal bounds by $\underline{C}_t$ and $\overline{C}_t$. Denote the time $t = 1, 2$ pricing kernel by m_t and distortion by ξ_t. The two period problem (for the lower good deal bound) is

$$\underline{C}_0 = \inf_{\xi_1 \in \Xi_1, \xi_2 \in \Xi_2} \left\{ \inf_{m_1, m_2} \left\{ \mathbb{E}_0^{\mathbb{P}} \left[m_1 m_2 c \right] \text{ such that } \mathbb{E}_0^{\mathbb{P}} \left[m_1 \mathbf{p}_1 \right] = \mathbf{p}_0, m_1 \geq 0, \mathbb{E}_0^{\mathbb{P}} \left[\frac{m_1^2}{F(\xi_1)} \right] \leq \Gamma_0, \right. \right.$$
$$\left. \left. \mathbb{E}_1^{\mathbb{P}} \left[m_2 \mathbf{x} \right] = \mathbf{p}_1 \quad \forall \Im_1, m_2 \geq 0, \mathbb{E}_1^{\mathbb{P}} \left[\frac{m_2^2}{F(\xi_2)} \right] \leq \Gamma_1 \quad \forall \Im_1 \right\} \right\}, \tag{37}$$

where $\Xi_t = \{\xi_t : \mathbb{E}_{t-1}^{\mathbb{P}}[\xi_t] = 1, \xi_t > 0, F(\xi_t) \geq 0\}$ and $\Im_1$ is the information set at time $t = 1$. Thus, the time 0 good deal bound imposes sequential constraints on the pricing kernels at each date. Applying the Law of Iterated Expectations (see the detailed proof in Cochrane and Saá-Requejo [2000]), we can rewrite (37) as two (sequential) one period problems

$$\underline{C}_{1,s} = \inf_{\xi_2 \in \Xi_2} \left\{ \inf_{m_2} \left\{ \mathbb{E}_1^{\mathbb{P}} \left[m_2 c \right] \text{ such that } \mathbb{E}_1^{\mathbb{P}} \left[m_2 \mathbf{x} \right] = \mathbf{p}_{1,s}, m_2 \geq 0, \mathbb{E}_1^{\mathbb{P}} \left[\frac{m_2^2}{F(\xi_2)} \right] \leq \Gamma_1 \right\} \right\},$$

$$\underline{C}_0 = \inf_{\xi_1 \in \Xi_1} \left\{ \inf_{m_1} \left\{ \mathbb{E}_0^{\mathbb{P}} \left[m_1 \underline{C}_1 \right] \text{ such that } \mathbb{E}_0^{\mathbb{P}} \left[m_1 \mathbf{p}_1 \right] = \mathbf{p}_0, m_1 \geq 0, \mathbb{E}_0^{\mathbb{P}} \left[\frac{m_1^2}{F(\xi_1)} \right] \leq \Gamma_0 \right\} \right\}, \tag{38}$$

where (38) takes the solution to the time 1 problem as given. For $\overline{C}_0$, we solve (37) or (38) with the infimum replaced by supremum. The separation of the non-sequential problem (37) into two sequential problems is possible because the constraints on the pricing kernels and distortions are applied sequentially and the former preclude arbitrage opportunities. Finally, notice that we can easily extend our analysis to accommodate more than two periods.

Consider now solving for the optimal hedges, denoted $\mathbf{w}_t$, at each time $t = 0, 1$ and assume that the constraints $\mathbb{E}_0^{\mathbb{P}}\left[\frac{m_1^2}{F(\xi_1)}\right] \leq \Gamma_0$ and $\mathbb{E}_1^{\mathbb{P}}\left[\frac{m_2^2}{F(\xi_2)}\right] \leq \Gamma_1$ bind. After substituting in (38) from first-order conditions and simplifying, we find

$$\begin{aligned}
\underline{C}_{1,s} &= \max_{\mathbf{w}_{1,s}} \left\{ \mathbf{w}_{1,s}^{'} \mathbf{p}_{1,s} - \hat{\delta}_{1,s}\Gamma_1 \right\}, \qquad \text{and defining} \\
\hat{R}_{1,s} &\equiv \mathbf{w}_0^{'}\mathbf{p}_{1,s} - \underline{C}_{1,s} = -\min_{\mathbf{w}_{1,s}} \left\{ \left(\mathbf{w}_{1,s}^{'} - \mathbf{w}_0^{'} \right) \mathbf{p}_{1,s} - \hat{\delta}_{1,s}\Gamma_1 \right\},
\end{aligned}$$

$$\underline{C}_0 = \max_{\delta_0 > 0, \mathbf{w}_0} \left\{ \mathbf{w}_0^{'}\mathbf{p}_0 - \frac{1}{2}\delta\Gamma_0 + \max_{\eta_0} \left\{ \sum_{s=1}^{S} \pi_s \left[-\frac{1}{2}\delta_0 G\left(\xi_{1,s}\right) \left(\frac{\max[\hat{R}_{1,s}, 0]^2}{\delta_0^2} + \beta_{1,s} \right) \right] \right\} \right\}.$$

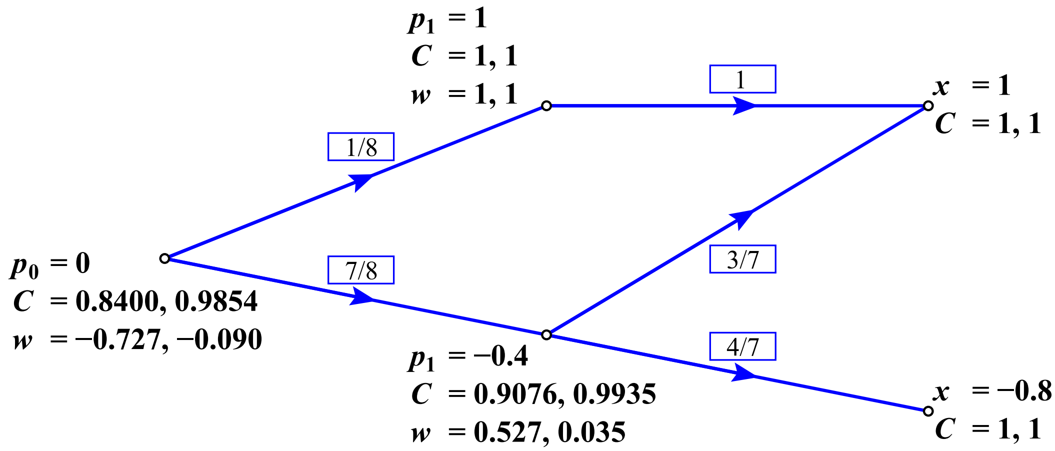
Using (32), model uncertainty has a larger impact on the time 0 good deal bounds if there is more dispersion in $\max[\hat{R}_{1,s}, 0]^2$. Consider the situation where $\hat{\delta}_{1,s}\Gamma_1$ is relatively insensitive to the realization of the state s; then greater dispersion in $\max[\hat{R}_{1,s}, 0]^2$ corresponds to greater dispersion in $\left(\mathbf{w}_{1,s}^{'} - \mathbf{w}_0^{'}\right)\mathbf{p}_{1,s}$. Conversely, the impact of model uncertainty is reduced when $\left(\mathbf{w}_{1,s}^{'} - \mathbf{w}_0^{'}\right)\mathbf{p}_{1,s}$ becomes less sensitive to the realization of the state s. In particular, $\left(\mathbf{w}_{1,s}^{'} - \mathbf{w}_0^{'}\right)\mathbf{p}_{1,s}$ is constant and independent of s when $\mathbf{w}_{1,s} = \mathbf{w}_0$ for all states s. The latter corresponds to a static hedging strategy,[15] which keeps the hedging portfolio constant across time and state realizations.

More generally, compare two strategies for hedging the focus asset with the basis assets: (1) A dynamic hedging strategy, with hedge rebalancing allowed at the intermediate time $t = 1$; (2) A static hedging strategy, with the hedges chosen at time $t = 0$ maintained at time $t = 1$. Clearly, an investor following the dynamic strategy can always choose to maintain the position in the hedging portfolio at time $t = 1$, so dynamic hedging is always weakly better than static hedging. However, model uncertainty may decrease the distance (and, hence, the benefit to dynamic hedging) between the good deal bounds under dynamic and static hedging. We now give a simple illustration of this.

Suppose there is a single basis asset whose price at time 0 is $p_0 = 0$ and which can move up, with probability 1/8, to a price of $p_1 = 1$ at time 1 at which price it stays with

[15] Carr et al. [1998] is a comprehensive reference on static hedging strategies, with a number of examples.

Figure 3: Dynamic hedging strategy for $\Omega = 1$ and $\Omega = 8$.



NOTES: The transition probabilities, under $\mathbb{P}$, are in boxes. At each node, the price of the basis asset is shown above the upper good deal bounds (labeled C, for $\Omega = 1$, then $\Omega = 8$) which are above the optimal hedges in the basis asset (labeled w, for $\Omega = 1$, then $\Omega = 8$). As an example of how to read the figure, 0.9935 means that, at time 1, in the node when the price of the basis asset is -0.4, for $\Omega = 8$, the upper good deal bound is 0.9935 and the optimal hedge in the basis asset is 0.035.

probability 1 at time 2 or it can move down, with probability 7/8, to a price of $p_1 = -0.4$ at time 1. In the latter case, it can move up to 1 with probability 3/7 or down to -0.8 with probability 4/7 at time 2. Hence, the basis asset pays $x = 1$ or $x = -0.8$, at time 2, with (two step) probabilities 1/2 and 1/2 respectively. The focus asset pays 1 in all states at time 2. We consider a static hedging strategy and a dynamic hedging strategy which rebalances the hedge at time 1. We compute the upper good deal bounds with $\Gamma = 1$, for $\Omega = 1$ and $\Omega = 8$. The static hedging strategy has the same solution as the one period problem in Section 4.3 for focus asset A. For $\Omega = 1$ ($\Omega = 8$), the time 0 upper good deal bound is 0.9939 (respectively, 0.9995) with optimal hedge 0.122 (respectively, 0.009). For the dynamic hedging strategy, for $\Omega = 1$ ($\Omega = 8$), the time 0 upper good deal bound is 0.8400 (respectively, 0.9854). The dynamic hedging strategy is shown in Figure 3.

We acknowledge that this is a very simple example. Nonetheless, we see clearly that the distance between the upper good deal bounds under dynamic and static hedging is significantly decreased in the presence of model uncertainty. This is true even though, conditional on $p_1 = 1$, the focus asset is a redundant asset, the basis asset is risk-free and, hence, in this state there is no model uncertainty. Furthermore, the choice of the possible values for p_1 is essentially arbitrary in the sense that the same qualitative effect would be observed for any other possible values for p_1 (consistent with the absence of arbitrage).

7 Numerical examples

In this section, we consider numerical examples which illustrate the No Good Deals methodology and its wide applicability. The first example values options on a non-traded asset. The second values a type of option traded in the fx markets. We have relegated a third example to Appendix C for brevity. Finally, we also examine the time-series properties of Ω.

7.1 Options on non-traded asset

We now consider options written on a non-traded asset. This example is based on one in Cochrane and Saá-Requejo [2000], which we modify to consider the impact of model uncertainty. There is a traded asset, labeled 1, whose price is denoted by S, and a non-traded asset, labeled 2, whose value is denoted by V. The option (focus asset) payoff is $\max(V-K,0)$ for a call (or, $\max(K-V,0)$ for a put), where K is a fixed strike (for simplicity, we assume that the payoff is paid in cash). We consider a European call with $K=65$, a European call with $K=55$ and a European put with $K=65$. The option maturity is 2 years. The traded asset 1 is correlated (but not perfectly) with the non-traded asset 2 and can be used to partially hedge the option.

We model the underlying dynamics as double trinomial (a pyramid rather than a triangle) with steps one month apart (24 steps over 2 years). At each time-step t ($t=0,1,\ldots,23$) of the trinomial tree, when the price of the traded asset 1 is S_t, the price can stay the same, go up to $S_t\exp\left(\lambda_1\sigma_1\sqrt{\Delta t}\right)$ or down to $S_t\exp\left(-\lambda_1\sigma_1\sqrt{\Delta t}\right)$, where $\lambda_1=\sqrt{(3/2)}$, $\sigma_1=0.25$ and $\Delta t=2/24$ (one month) is the time period corresponding to each step. When the value of the non-traded asset 2 is V_t, the value can change to: $V_t \exp\left(\lambda_2 \sigma_2 \sqrt{\Delta t}\left(\sqrt{1-\rho^2} Z_2 + \rho Z_1\right)\right)$ for $Z_1 = -1, 0, 1$ and $Z_2 = -1, 0, 1$ and where $\lambda_2 = \sqrt{(3/2)}$, $\sigma_2 = 0.28$. The correlation between log-changes in S and V is ρ. In the limit of small time-steps, this tree construction will approximate S and V being jointly log-normal (but we will not be interested in this small time-step limit - we regard these discrete-time dynamics as specifying the actual dynamics, rather than of being an approximation to some continuous-time dynamics). We denote the (continuously-compounded) excess return on asset 1 by μ_1 and that on the non-traded asset 2 by μ_2. The probabilities in the double trinomial tree are computed by requiring that they sum to one and match each of the first two moments, under $\mathbb{P}$, of S and V.

The (continuously-compounded) risk-free rate is 0.03. The traded asset 1 has an initial price of asset 20 and pays a dividend yield of 2% (expressed as a continuous yield proportional to its price) or 0.02. The initial value of the non-traded asset 2 is 60.

The investor can trade in the basis assets (traded asset 1 and a one period risk-free bond) and rebalance her hedges every three months at the start of steps 0, 3, 6, 9, 12, 15, 18, 21 (not every step). There are two sources of incompleteness: The discrete-time hedging and the fact that she can only hedge with a partially correlated traded asset.

In order to estimate the degree of aversion to model uncertainty Ω, we simulate a data-set with 128 data points, setting $\mu_1 = 0.03$. Each data point represents the return on the traded asset 1 over a period of three months consistent with the (one month) probabilities computed in the lattice (as well as the (constant) return on a risk-free bond). Given the simulated historical data-set, we use the bootstrap procedure of Section 5 to estimate Ω. We bootstrap the data-set 65000 times, and, for a Model Confidence Level of 80%, we find $\Omega = 2.81$. Note that 128 data points (each representing quarterly returns) would, in practice, require 32 years of data. In Table 1, we report also the estimate of Ω for simulated data-sets with 8, 32 and 512 data points (the latter would require (an unrealistic) 128 years of data), keeping the same lattice probabilities and the same Model Confidence Level of 80%. We see that as the length of the simulated data-set decreases, the estimate of Ω increases, implying greater model uncertainty. In Table 2, we report the estimate of Ω with 128 data points but with different Model Confidence Levels. As the Model Confidence Level is increased, implying greater aversion to model uncertainty, the estimate of Ω increases.

In the second panel of Table 2, we also report, for comparison, estimates of Ω setting $\mu_1 = 0.08$. We see that increasing μ_1 to 0.08 (from 0.03) dramatically lowers the estimate of Ω. More color is provided in the third panel, where, fixing the Model Confidence Level at 80%, we consider a range of values of μ_1 - negative as well as positive. We see that when the absolute value of μ_1 increases, Ω decreases. Further, the estimate of Ω is almost symmetric around the point $\mu_1 = 0$. This is intuitive, because the bootstrap procedure works by considering the uncertainty in the reward-for-risk achievable by trading in the basis assets. In this simple example, the latter depends upon the magnitude of μ_1. The sign is largely irrelevant in the sense that, if μ_1 were to be multiplied by minus one, $\hat{\mathbf{v}}^{\text{NoMU}}$, defined in (33), would also be approximately multiplied by minus one (switching a long position to short and *vice versa*), leaving the reward-for-risk largely unchanged. The small asymmetry around $\mu_1 = 0$ is due to the small asymmetry in the return distribution of asset 1. Finally, in the fourth panel, we consider different values of both μ_1 and σ_1. We see that if μ_1/σ_1 (which is approximately the Sharpe ratio on asset 1) is kept constant, Ω hardly changes. However, if σ_1 is increased, holding μ_1 constant, Ω increases. Hence, the degree of aversion to model uncertainty Ω increases, if there is more model uncertainty (a shorter data-set or a lower Sharpe ratio) or if the investor has greater aversion to it (a higher Model Confidence Level).

We turn now to computing the good deal bounds on the options on the non-traded asset 2. For illustration, we consider different values of Ω (16, 8, 4, 2, 1.75, 1.5, 1.25 and 1) which span the values estimated in Tables 1 and 2. We set the excess return on asset 1 to be $\mu_1 = 0.03$ and that on the non-traded asset 2 to be $\mu_2 = 0.04$. We set the correlation ρ to be 0.80. We set the Sharpe ratio bound h_{Ann} to 0.5. The results[16] are in Table 3.

[16] We also report the risk-neutral price (meaning with the excess returns set to zero but computed with the same placements of tree nodes) and the Black and Scholes [1973] price just for illustration. Neither of these prices has a real financial meaning here since one cannot even trade the non-traded asset 2 (but the

We see that as Ω is increased, the good deal bounds widen - in fact, changing Ω has quite a large impact. For the call options, this may be partly due to the fact that the lower and upper arbitrage bounds are very wide.[17] Following Cochrane and Saá-Requejo [2000] (p85), we see the width of the good deal bounds as being as much an advantage as a disadvantage. It warns that hedging is poor and that valuation is very sensitive to assumptions about the model and about the market prices of risk - things which standard risk-neutral valuation techniques, and the financial insitutions which use them, usually ignore.

We now extend our analysis by considering American options which additionally give the holder[18] of the option the right to exercise at the same frequency as the hedge rebalancing (i.e. every three months). We assume that the American option is exercised when the immediate exercise value exceeds the continuation value of the lower good deal bound. The results are in Table 4. Since the non-traded asset 2 does not pay dividends, the risk-neutral prices of the American call options are the same as their European counterparts. Defining the early exercise premium to be the value of the relevant American option minus that of its European counterpart, the early exercise premia of the lower good deal bounds are substantial - even for call options - and typically increase as Ω increases. For values of $\Omega \in \{4, 8, 16\}$, it is actually optimal to immediately exercise the options (call with $K = 55$ and put with $K = 65$) which are initially in-the-money and, for these options, the early exercise premium increases very significantly as Ω increases.

Intuitively, greater model uncertainty can lead to larger early exercise premia, since exercising the option removes model uncertainty, providing the uncertainty averse investor with an additional incentive to exercise early. Thus, model uncertainty gains extra significance in the context of real options, since real options are often American in nature.

difference between the former and the latter would give an idea of the discretization error if there were to be an interest in approximating joint geometric Brownian motion by the double trinomial tree).

[17]The lower arbitrage bound is zero while the upper is governed by the maximum value of V in the lattice.

[18]For simplicity, we consider only the lower good deal bounds $\underline{C}$ (the upper good deal bounds $\overline{C}$ (corresponding to reservation prices for short positions) depend upon the exercise strategy of the option buyer).

7.1.1 Sensitivity of the good deal bounds to ρ, h_{Ann}, μ_1 and μ_2.

For further comparison, we now consider the sensitivity of the good deal bounds to ρ, h_{Ann}, μ_1 and μ_2. We focus on the good deal bounds on the European call with strike $K = 65$. We now set the correlation ρ between log-changes in S and V to be 0.95. The results are in Table 5. We see (comparing the first two lines of Tables 3 and 5) increasing ρ from 0.80 to 0.95 significantly tightens the good deal bounds. This is especially true for large values of Ω. Keeping ρ equal to 0.95, we also compute the good deal bounds for smaller values of the Sharpe ratio bound h_{Ann}, namely 0.12 (for the case $\mu_1 = 0.03$, $\mu_2 = 0.04$) and 0.3156 (where we set $\mu_1 = 0.08$, $\mu_2 = 0.09$). Note that, in each case, the chosen value of h_{Ann} is such that Γ is only slightly greater than $\Gamma^{\star\text{NoMU}}$ defined in (33). We see that lowering h_{Ann} tightens the good deal bounds (in line with Proposition 11). Again, this is especially true for large values of Ω. Increasing μ_1 and μ_2 tightens the good deal bounds in the absence of model uncertainty. However, Table 5 shows that, for large values of Ω, the story is more complicated. For $\Omega = 16, 8, 4$, the lower good deal bounds are rather lower for $\mu_1 = 0.08$, $\mu_2 = 0.09$, $h_{\text{Ann}} = 0.5$ than those for $\mu_1 = 0.03$, $\mu_2 = 0.04$, $h_{\text{Ann}} = 0.5$. On the other hand, from Table 2, the estimate of Ω, for a given Model Confidence Level, is much smaller for $\mu_1 = 0.08$. Taking this into account implies that the good deal bounds are dramatically narrower for $\mu_1 = 0.08$, $\mu_2 = 0.09$, $h_{\text{Ann}} = 0.5$ than for $\mu_1 = 0.03$, $\mu_2 = 0.04$, $h_{\text{Ann}} = 0.5$.

7.2 Fx options in the Heston [1993] stochastic volatility model

In this sub-section, we value a binary cash-or-nothing (BCON) option, a type of exotic option written by many investment banks and popular in the foreign exchange (fx) markets. We denote the spot fx rate, at time t, by S_t (quoted as the number of units of domestic currency per unit of foreign currency). The option pays one unit of domestic currency at maturity T if the spot fx rate S_T is greater than or equal to the strike K and zero otherwise. In our examples, the domestic currency is USD, the foreign is STG and the maturity T is two months. We assume that the dynamics of S_t, under $\mathbb{P}$, are those of Heston [1993]:

$$dV_t = \kappa\,(\theta - V_t)\,dt + \zeta\sqrt{V_t}dW_t^1, \quad dS_t/S_t = (r_d - r_f + \Upsilon\sqrt{V_t})dt + \sqrt{V_t}\,dW_t^2, \tag{39}$$

where r_d and r_f are the domestic and foreign interest-rates, $\kappa > 0$, $\theta > 0$, $\zeta \geq 0$ are constants and W_t^1 and W_t^2 are independent[19] $\mathbb{P}$ standard Brownian motions. The parameter Υ is a constant and $\Upsilon\sqrt{V_t}$ is the excess return on the fx rate. Similarly to the online appendix of Cochrane and Saá-Requejo [2000], we include the scaling by $\sqrt{V_t}$ in order to have a constant instantaneous Sharpe ratio. We use historical time-series data from Bloomberg for STGUSD spanning the period from 2nd October 2003 until 12th September 2011 (2000 working days) and estimate the parameters in (39) using the Maximum Likelihood method of Ait-Sahalia and Kimmel [2007]. The results are: $\Upsilon = 0.3439$, $\kappa = 2.821$, $\theta = 0.01315$ and $\zeta = 0.164$.

We build a multinomial lattice, with 45 time-steps (equivalent to one time-step per working day for the two month option), which approximates the Heston [1993] dynamics in (39). The details of the lattice construction are relegated to Appendix E.

We consider two different strategies for hedging the BCON option. In the first, we dynamically hedge using two basis assets, namely, a one period domestic risk-free bond and the spot fx rate, rebalancing every working day (i.e. on every step of the lattice). In the second, we use a static hedging strategy, with hedges computed only at time 0, and we use three basis assets, namely a domestic risk-free bond and vanilla call options of two different strikes, denoted $\{K_1, K_2\}$, all with two month maturity. In each case, the domestic (USD) risk-free bond is constructed synthetically from USD Libor quotes (obtained from Bloomberg). Our aim is to compare the two different hedging[20] strategies. Note that our

[19] We also considered allowing for a non-zero correlation ρ between the Brownian motions but found that, for STGUSD, ρ was very close to zero and the maximum likelihood was not significantly improved.

[20] The rationale for the static hedging strategy is that the payoff $\mathbf{1}_{S_T \geq K}$ of the BCON option can be (Carr et al. [1998]) approximately replicated by a long position $1/(2\epsilon)$ in a call option with strike $K - \epsilon$ and a short position $1/(2\epsilon)$ in a call option with strike $K + \epsilon$, for $\epsilon > 0$. In the limit $\epsilon \to 0$, the replication is exact, but

analysis does not fall within the scope of the theoretical analysis in Section 6, because the two different hedging strategies use different basis assets.

Corresponding to the two different hedging strategies, we consider two estimates of Ω. The first uses, as basis assets, a domestic risk-free bond and the spot foreign exchange rate with returns computed over a time period of one working day. The second uses, as basis assets, a domestic risk-free bond and a synthetic variance swap, with two month maturity, constructed from the market prices of options of different strikes (Britten-Jones and Neuberger [2000], Carr and Wu [2009]), with non-overlapping returns over a period of two months. Note that the second estimate of Ω does not use exactly the same basis assets as the ones we use for static hedging but, in a Heston [1993] model, they capture broadly the same risks and represent a pragmatic choice. We estimate Ω, with a Model Confidence Level of 80%, using the bootstrap procedure described in Section 5 with the same historical time-series data used to estimate the parameters in (39) and report the results in Table 6.

We compute the good deal bounds on the BCON option using market data from 12th September 2011, the last day of the period used for the estimates of Ω. For ease of comparison, we scale the spot fx rate to 100. For the static hedging strategy, we consider three different combinations of call option strikes: $\{K_1 = 99.5, K_2 = 100.5\}$, $\{99, 101\}$ and $\{98, 102\}$. The market prices of these call options are determined by parabolically interpolating the implied Black and Scholes [1973] volatilities of options trading in the market with strikes closest to K_1 or K_2 (obtained from Bloomberg) and then substituting the interpolated volatility into the Black and Scholes [1973] formula. The strike K of the BCON option is 100. We use the estimated values of Ω in Table 6, as well as $\Omega \in \{1, 2, 16\}$ for comparison, setting the Sharpe ratio bound h_{Ann} to 0.5, and display the results in Table 7. We see that hedging with call options of strikes $\{99.5, 100.5\}$, the static hedging strategy is much more resilient to model uncertainty in that, as Ω is increased, the good deal bounds widen out only slowly compared to how much they widen for dynamic hedging. On the other hand,

this is a limit only available in theory - in practice, options with strikes infinitesimally close together will not trade in the market.

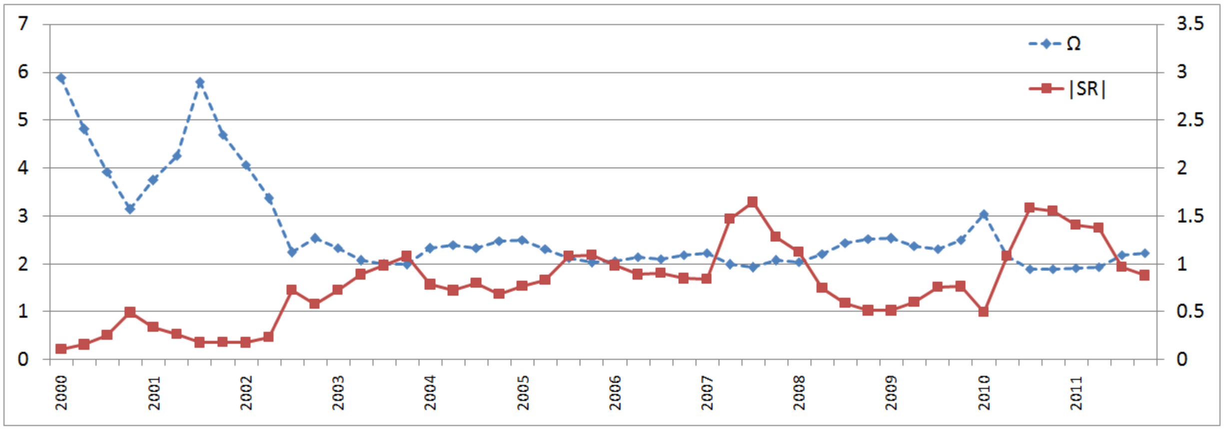
for strikes $\{98, 102\}$, the picture is reversed. This shows that static hedging is more resilient to the presence of model uncertainty but only if the static hedges are actually good hedges in the first place. If the static hedges are poor hedges, resulting in wide good deal bounds even in the absence of model uncertainty, then they may widen significantly further in its presence. For the values of Ω estimated in Table 6, dynamic hedging is better, in the sense of giving tighter good deal bounds, than static hedging with strikes $\{99, 101\}$ or $\{98, 102\}$.

We asked traders at leading investment banks what bid-ask spread they would quote on a BCON option, whose strike is around the current forward fx rate, on a major currency pair if they were asked for a two-way price and the consensus was that it would be around 3 cents. We see from Table 7 that the differences between the lower and upper good deal bounds are around 3 cents for the dynamic hedging strategy and for the static hedging strategy with strikes $\{99.5, 100.5\}$ for the values of Ω estimated in Table 6. This gives us confidence that our methodology is capable of generating valuations which are not only realistic from the viewpoint of an economist but also from that of a trader.

7.3 Time-series properties of model uncertainty

In this sub-section, we investigate the time-series behavior of the degree of aversion to model uncertainty Ω. We consider two asset markets: The spot market for gold and the S&P 500. For each, we construct a quarterly time-series of real time estimates of Ω using observations for the preceding 600 trading days with a Model Confidence Level of 80%. For gold, Ω is estimated using a one day risk-free bond and a one day forward (using the convenience yield implied from market quotes for one month forward contracts) as the basis assets. We estimate two different versions of Ω for the S&P 500. The first version also uses a one day risk-free bond and a one day forward as the basis assets. The second version adds the daily return on a synthetic variance swap (approximated using VIX data) as a third basis asset.

Figure 4: Time-series of estimates of Ω for gold.

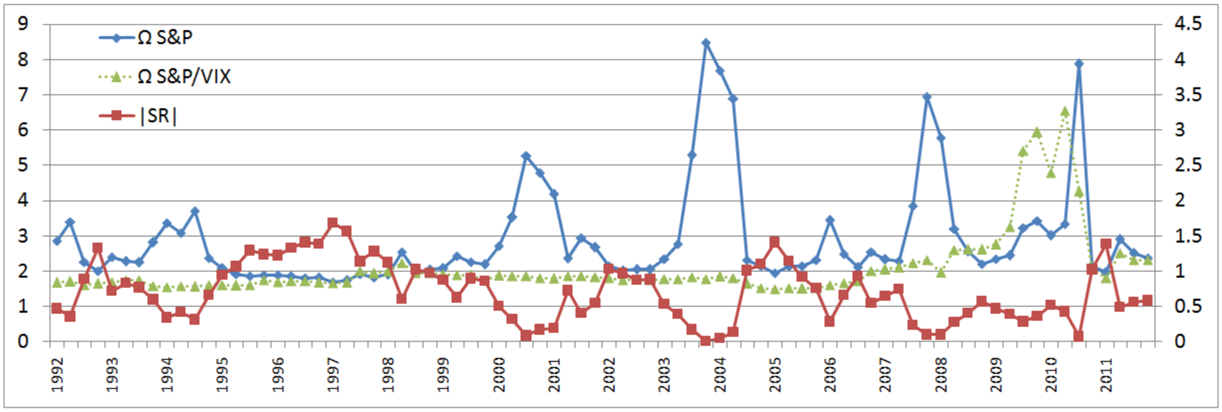


NOTES: Rolling 600 day estimates of the degree of aversion to model uncertainty Ω (left axis) and the absolute value of the Sharpe ratio $|SR|$ (right axis) for gold. The estimates of Ω span the period from 2000 to 2012. All data are from Bloomberg.

We plot the time-series of Ω estimates for gold and the S&P 500 in Figures 4 and 5, respectively, together with the absolute value of the corresponding Sharpe ratio $|SR|$ (computed using the same 600 day rolling windows). While for gold, $|SR|$ appears to be almost perfectly negatively correlated with Ω, the relationship between $|SR|$ and Ω for the S&P 500 is slightly less strong. In Table 8, we report the pairwise contemporaneous correlations between our estimates of Ω and $|SR|$ and the spot price for the market in question, as well as the correlations between Ω and some other indicators of market conditions. We find that, while Ω for both gold and the S&P 500 exhibits significant contemporaneous correlation with $|SR|$, the realized excess return, the spot price, the risk-free rate and the Libor rate, the sign of the correlations differ. For example, Ω for gold is negatively correlated with the spot price of gold, while that for the S&P 500 is positively correlated with the level of the S&P 500.

In Table 9, we regress changes in Ω on changes in indicators of market conditions. For both gold and the S&P 500, increases in Ω coincide with decreases in $|SR|$. For the S&P 500, increases in Ω coincide with decreases in realized volatility, the Libor rate, and the Baa-Aaa credit spread, and increases in the Chicago Fed Financial Conditions Index. The latter implies that increases in the degree of aversion to model uncertainty Ω coincide with worsening conditions for the overall financial sector.

Figure 5: Time-series of estimates of Ω for S&P 500.



NOTES: Rolling 600 day estimates of the degree of aversion to model uncertainty Ω (left axis) and the absolute value of the Sharpe ratio $|SR|$ (right axis) for the S&P 500. "Ω S&P" refers to Ω estimated using a one day risk-free bond and a one day forward as the basis assets; "Ω S&P/VIX" refers to Ω estimated using a one day risk-free bond, a one day forward, and the daily return to a synthetic variance swap as the basis assets. The estimates of Ω span the period from 1992 to 2012. All data are from Bloomberg.

8 Summary and conclusions

We describe a new and practical approach for incorporating a concern for robustness to model uncertainty into the No Good Deals methodology of Cochrane and Saá-Requejo [2000]. We introduce the notion of a model-uncertainty-induced preference functional and show how bounds on the certainty equivalent translate into bounds on the pricing kernel. The uncertainty averse investor assigns higher probability to lower utility states. We demonstrate how model uncertainty leads to economically different outcomes than higher target levels of reward-for-risk (for example, higher Sharpe ratio bounds). We show how model uncertainty reduces the benefit of dynamic hedging relative to static hedging (meaning, for example, hedging an option by taking a static ("buy-and-hold") position in other options (as opposed to dynamic hedging with the underlying asset)). Illustrating our methodology with American options, we show how greater model uncertainty can lead to larger early exercise premia, since exercising the option removes model uncertainty, providing the uncertainty averse investor with an additional incentive to exercise early.

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Table 1: Degree of aversion to model uncertainty Ω

Length of data-set	8	32	128	512
Value of Ω	7.67	4.08	2.81	2.11

Notes: Degree of aversion to model uncertainty Ω as a function of the number of simulated historical quarterly observations (labeled Length of data-set). Note that, for example, 32 quarterly observations are equivalent to 8 years of data. Ω is estimated using the procedure in Section 5, with 65000 bootstrapped samples, setting the Model Confidence Level to 80%. The basis assets are the traded asset 1 and a risk-free bond, with hedge rebalancing possible every quarter. The excess return on the traded asset 1 is $\mu_1 = 0.03$ and its volatility is $\sigma_1 = 0.25$.

Table 2: Degree of aversion to model uncertainty Ω

$\mu_1 = 0.03$					
Model Confidence Level	75	80	85	90	95
Value of Ω	2.55	2.81	3.12	3.49	4.06

$\mu_1 = 0.08$					
Model Confidence Level	75	80	85	90	95
Value of Ω	1.72	1.85	1.99	2.19	2.47

μ_1	-0.09	-0.07	-0.05	-0.03	-0.01	0.01	0.03	0.05	0.07	0.09	0.11
Ω	1.90	2.07	2.35	2.94	5.60	5.50	2.81	2.23	1.95	1.78	1.69

μ_1	0.018	0.030	0.042	0.048	0.080	0.112	0.080	0.080	0.080
σ_1	0.15	0.25	0.35	0.15	0.25	0.35	0.15	0.25	0.35
$\frac{\mu_1}{\sigma_1}$	0.120	0.120	0.120	0.320	0.320	0.320	0.533	0.320	0.229
Ω	2.84	2.81	2.80	1.86	1.85	1.84	1.63	1.85	2.07

Notes: Degree of aversion to model uncertainty Ω as a function of the Model Confidence Level and of the excess return μ_1 on the traded asset 1 and its volatility σ_1. Ω is estimated using the procedure in Section 5, with 65000 bootstrapped samples, with the number of historical observations set to 128 (which is equivalent to 32 years of data). The basis assets are the traded asset 1 and a risk-free bond, with hedge rebalancing possible every quarter. In the first panel, $\mu_1 = 0.03$, $\sigma_1 = 0.25$; in the second panel, $\mu_1 = 0.08$, $\sigma_1 = 0.25$. In the third and fourth panels, the Model Confidence Level is set to 80%. In the third panel, different values of μ_1 are considered (keeping $\sigma_1 = 0.25$). In the fourth panel, different values of both μ_1 and σ_1 are considered and the ratio μ_1/σ_1 is also shown.

Table 3: Good deal bounds on options on the non-traded asset 2

Ω	16	8	4	2	1.75	1.5	1.25	1	NoMU
		Call, strike $K = 65$							
$\underline{C}$	0.000	0.058	0.738	2.601	3.051	3.560	4.092	4.445	4.445
$\overline{C}$	69.974	47.643	31.606	21.580	20.275	19.035	17.983	17.471	17.471
Risk-neutral price (same tree)					8.937		BS	8.917	
		Call, strike $K = 55$							
$\underline{C}$	0.006	0.360	2.052	5.212	5.880	6.610	7.336	7.782	7.782
$\overline{C}$	79.143	56.299	39.356	28.307	26.830	25.417	24.211	23.623	23.623
Risk-neutral price (same tree)					13.483		BS	13.463	
		Put, strike $K = 65$							
$\underline{C}$	0.001	0.188	1.422	3.738	4.196	4.674	5.117	5.350	5.350
$\overline{C}$	33.503	27.507	21.406	16.466	15.753	15.068	14.495	14.229	14.229
Risk-neutral price (same tree)					10.151		BS	10.132	

Notes: The lower $\underline{C}$ and upper $\overline{C}$ good deal bounds on 2 year European options on the non-traded asset 2 as a function of the degree of aversion to model uncertainty Ω. The correlation ρ is set to 0.80. The excess return on asset 1 is $\mu_1 = 0.03$ and that on the non-traded asset 2 is $\mu_2 = 0.04$. The Sharpe ratio bound is $h_{\text{Ann}} = 0.5$. The basis assets are the traded asset 1 and a one period risk-free bond, with hedge rebalancing possible every quarter. "BS" refers to the Black and Scholes [1973] price of the same option. The case of no model uncertainty is labeled NoMU and is computed by setting $\xi \equiv 1$.

Table 4: Good deal bounds on American options on the non-traded asset 2

Ω	16	8	4	2	1.75	1.5	1.25	1	NoMU
		Call, strike $K = 65$							
$\underline{C}$	0.000	0.234	1.295	3.153	3.555	4.006	4.481	4.809	4.809
Risk-neutral price (same tree)					8.937				
		Call, strike $K = 55$							
$\underline{C}$	5.000	5.000	5.000	6.927	7.360	7.844	8.355	8.695	8.695
Risk-neutral price (same tree)					13.483				
		Put, strike $K = 65$							
$\underline{C}$	5.000	5.000	5.000	6.162	6.475	6.803	7.114	7.284	7.284
Risk-neutral price (same tree)					10.671				

Notes: The lower good deal bound $\underline{C}$ on 2 year American options on the non-traded asset 2 as a function of the degree of aversion to model uncertainty Ω. The correlation ρ is set to 0.80. The excess return on asset 1 is $\mu_1 = 0.03$ and that on the non-traded asset 2 is $\mu_2 = 0.04$. The Sharpe ratio bound is $h_{\text{Ann}} = 0.5$. The basis assets are the traded asset 1 and a one period risk-free bond, with hedge rebalancing possible every quarter.

Table 5: Good deal bounds on a European call option as a function of the excess returns μ_1 and μ_2 and the Sharpe ratio bound h_{Ann}.

Ω	16	8	4	2	1.75	1.5	1.25	1	NoMU
$\mu_1 = 0.03$, $\mu_2 = 0.04$, $h_{\text{Ann}} = 0.5$									
$\underline{C}$	0.126	0.902	2.559	4.646	5.023	5.423	5.814	6.066	6.066
$\overline{C}$	33.266	25.639	19.602	15.356	14.759	14.179	13.676	13.430	13.430
$\mu_1 = 0.03$, $\mu_2 = 0.04$, $h_{\text{Ann}} = 0.12$									
$\underline{C}$	5.115	6.454	7.572	8.542	8.722	8.930	9.168	9.372	9.372
$\overline{C}$	14.763	12.866	11.492	10.420	10.234	10.027	9.795	9.596	9.596
$\mu_1 = 0.08$, $\mu_2 = 0.09$, $h_{\text{Ann}} = 0.5$									
$\underline{C}$	0.111	0.848	2.472	4.611	5.032	5.506	6.017	6.377	6.377
$\overline{C}$	32.900	25.255	19.138	14.630	13.957	13.277	12.666	12.350	12.350
$\mu_1 = 0.08$, $\mu_2 = 0.09$, $h_{\text{Ann}} = 0.3156$									
$\underline{C}$	1.007	2.563	4.493	6.603	7.059	7.626	8.354	9.167	9.167
$\overline{C}$	24.502	18.988	14.905	11.786	11.253	10.658	9.983	9.215	9.215

NOTES: The lower $\underline{C}$ and upper $\overline{C}$ good deal bounds on a 2 year European call option, with strike $K = 65$, on the non-traded asset 2 for different values of the degree of aversion to model uncertainty Ω. The correlation ρ is set to 0.95. The basis assets are the traded asset 1 and a one period risk-free bond, with hedge rebalancing possible every quarter. We consider different values of the excess return μ_1 on asset 1, the excess return μ_2 on the non-traded asset 2 and of the Sharpe ratio bound h_{Ann}. The case of no model uncertainty is labeled NoMU and is computed by setting $\xi \equiv 1$.

Table 6: Estimates of Ω for STGUSD

Ω	1.16	1.96

NOTES: The left-hand side estimate of Ω (1.16) uses as basis assets a USD risk-free bond and the spot foreign exchange rate with returns computed over a time period of one working day. The right-hand side estimate of Ω (1.96) uses as basis assets a USD risk-free bond and a synthetic variance swap constructed from the market prices of options on STGUSD of different strikes with returns computed over a time period of two months. We use historical time-series data for STGUSD spanning 2000 working days until 12th September 2011. The Model Confidence Level is set to 80% and 65000 bootstrapped samples are used. The USD risk-free bond is constructed synthetically from USD Libor quotes. All data are from Bloomberg.

Table 7: Good deal bounds on a binary cash-or-nothing (BCON) option

	Ω	1	1.16 / 1.96	2	16
Dynamic	$\underline{C}$	0.4761	0.4656	0.4546	0.3766
	$\overline{C}$	0.4899	0.5004	0.5115	0.5911
Static					
$K_1 = 99.5,\ K_2 = 100.5$	$\underline{C}$	0.4840	0.4793	0.4792	0.4614
	$\overline{C}$	0.4895	0.4944	0.4945	0.5114
$K_1 = 99,\ K_2 = 101$	$\underline{C}$	0.4780	0.4617	0.4614	0.4070
	$\overline{C}$	0.4947	0.5114	0.5117	0.5609
$K_1 = 98,\ K_2 = 102$	$\underline{C}$	0.4671	0.4261	0.4254	0.3063
	$\overline{C}$	0.5039	0.5459	0.5466	0.6484

NOTES: The lower $\underline{C}$ and upper $\overline{C}$ good deal bounds on a 2 month BCON option, as a function of the degree of aversion to model uncertainty Ω, computed using market data (from Bloomberg) for STGUSD as of 12th September 2011. The Sharpe ratio bound is $h_{\text{Ann}} = 0.5$. For the dynamic hedging strategy, the basis assets are a USD risk-free bond and the spot foreign exchange rate, with rebalancing every working day. For the static hedging strategy, the basis assets are a USD risk-free bond and and vanilla call options of two different strikes, denoted K_1 and K_2. We consider different values of Ω. The column labeled 1.16/1.96 means that the dynamic hedging strategy uses the value of Ω (1.16) estimated in Table 6 for the same basis assets and the same frequency of hedge rebalancing whiles the static hedging strategies use the value of Ω (1.96) estimated in Table 6 using a USD risk-free bond and a synthetic variance swap. The USD risk-free bond is constructed synthetically from USD Libor quotes (from Bloomberg).

Table 8: Contemporaneous pairwise correlations between Ω and risk indicators

Indicator	S&P 500		Gold
	S&P only	S&P/VIX	
|SR|	-0.74 ***	-0.32 ***	-0.80 ***
Realized Excess Return	-0.23**	-0.33 ***	-0.80***
Realized Volatility	0.03	0.43 ***	-0.19
VIX	-0.05	0.26**	0.14
Spot Price	0.20 *	0.19 *	-0.49 ***
Risk-free rate	-0.27 **	-0.50 ***	0.34 **
Libor	-0.28 **	-0.51 ***	0.27 *
Baa-Aaa	0.04	0.24 **	-0.06
TED	0.08	-0.01	-0.19
Fin. Cond. Index	0.10	0.26 **	-0.05

NOTES: *** significant at 1%, ** significant at 5%, *significant at 10%. Pairwise correlations between rolling 600 day estimates of the degree of aversion to model uncertainty Ω and risk indicators. Ω is estimated using the procedure of Section 5 with a Model Confidence Level of 80%. For S&P 500, "S&P only" refers to Ω estimated using a one day risk-free bond and a one day forward on the S&P 500 as the basis assets; "S&P/VIX" refers to Ω estimated using additionally the daily return on a synthetic variance swap as a third basis asset. The absolute value of the Sharpe ratio |SR| and the spot price are market-specific. Fin. Cond. Index is the Federal Reserve Bank of Chicago Financial Conditions Index. All data are from Bloomberg.

Table 9: Model uncertainty and risk indicators

Indicator	S&P 500		Gold
	S&P only	S&P/VIX	
|SR|	-3.31	-0.12	-3.15
	[0.51]***	[0.26]	[1.16]**
Realized Excess Return	0.04	-0.05	0.12
	[0.02]*	[0.04]	[0.05]**
Realized Volatility	-0.11	-0.04	-0.12
	[0.03]***	[0.03]	[0.07]*
VIX	0.01	0.01	-0.04
	[0.04]	[0.02]	[0.04]
Libor	-0.94	0.06	0.49
	[0.17]***	[0.23]	[0.53]
Baa-Aaa	-1.34	-0.73	-0.39
	[0.48]***	[0.40]*	[0.34]
TED	-0.89	0.20	-1.08
	[0.83]	[0.47]	[0.63]*
Fin. Cond. Index	2.14	-0.24	1.24
	[0.94]**	[0.46]	[1.02]
Risk-free rate	6.38	-1.61	-4.68
	[1.87]***	[2.27]	[6.41]
Const.	-0.06	0.07	-0.16
	[0.13]	[0.15]	[0.13]
Adj. R^2	0.765	0.232	0.451
N. obs	76	76	44

NOTES: *** significant at 1%, ** significant at 5%, *significant at 10%. Newey-West standard errors with 4-quarter truncation shown in brackets. Dependent and independent variables in 4-quarter changes. Time-series dependence of rolling 600 day estimates of the degree of aversion to model uncertainty Ω on risk indicators. Ω is estimated using the procedure of Section 5 with a Model Confidence Level of 80%. For the spot market for gold, Ω is estimated using a one day risk-free bond and a one day forward as the basis assets. For S&P 500, "S&P only" refers to Ω estimated using a one day risk-free bond and a one day forward on the S&P 500 as the basis assets; "S&P/VIX" refers to Ω estimated using additionally the daily return on a synthetic variance swap as a third basis asset. The absolute value of the Sharpe ratio |SR| and the spot price are market-specific. TED is the spread between the interest-rate on interbank loans and the risk-free rate. Fin. Cond. Index is the Federal Reserve Bank of Chicago Financial Conditions Index. All data are from Bloomberg.

A Proofs of propositions

Proofs of Propositions 3 and 4. See after the proof of Proposition 5.

Proof of Proposition 5.
Introducing Lagrange multipliers α, $\beta_s \geq 0$ and $\varpi_s \geq 0$, for each state s, on the set of constraints Ξ, the minimization in problem (5) becomes:

$$J(\mathbf{V}) = \max_{\varpi_s \geq 0, \beta_s \geq 0, \alpha} \{\min_{\xi_s} \{\sum_{s=1}^{S} \pi_s [F(\xi_s) U(V_s) + \alpha(\xi_s - 1) - \beta_s F(\xi_s) - \varpi_s \xi_s]\}\}. \tag{A.1}$$

The first-order condition from taking partial derivatives with respect to ξ_s in each state s implies:

$$\alpha - \varpi_s \quad = \quad (\beta_s - U(V_s)) \, \Omega (\Psi - \log \xi_s). \tag{A.2}$$

From (*A.2*), we can solve for ξ_s. If $U(V_s)$ is the same in every state s and/or $\Psi = 0$, then the only solution is $\varpi_s = \beta_s = 0$, $\xi_s = 1$. Consider now opposing cases. We conjecture that the constraint $\xi_s > 0$ is not binding which means one could set $\varpi_s = 0 \ \forall s$ (we check this later). We need $\beta_s \geq 0$ and $F(\xi_s) \geq 0$ for all s. If $F(\xi_s) > 0$, then $\beta_s = 0$ and we can solve for ξ_s from (*A.2*), whereas if $F(\xi_s) = 0$ (which, since $\xi_s > 0$, means $\xi_s = \xi^{\max}$), then $\beta_s > 0$ but this implies that β_s must be such that (*A.2*) holds. This enables us to solve for ξ_s and β_s as in the statement of the proposition, in terms of which $\eta \equiv \alpha / \Omega\Psi = \alpha / (1 - \Omega)$ for $\Psi < 0$. (If $\Psi = 0$, then $\Omega = 1$ and the maximization over η is irrelevant). With these values of ξ_s, the constraint $\xi_s > 0$ is automatically satisfied and hence $\varpi_s \equiv 0$ as conjectured.

Substituting from (*A.2*) implies (*A.1*) can be re-written in the form of (8). Equation (8), with ξ_s and β_s substituted, can be solved numerically by choice of η. To see what values of η are possible, note that since $\Psi \leq 0$, for $\xi_s \geq 1$, $(\Psi - \log \xi_s)$ cannot be strictly positive (and must be strictly negative if $\Psi < 0$). Since $\mathbb{E}^{\mathbb{P}}[\xi] = 1$, ξ_s must be greater than or equal to 1 in at least one of the S possible states. Hence, α must be less than or equal to zero (this is implied by the expression for ξ_s if $1 \leq \xi_s < \xi^{\max}$ and by the expression for β_s if $\xi_s = \xi^{\max}$). Hence, $\eta \geq 0$. Actually, we can strengthen this result. Since ξ_s must be less than or equal to one in at least one state, $1 + \eta / U(V_s)$ must be greater than or equal to zero in at least one state. Hence, η must be less than or equal to $-U(V_s)$ for at least one state s - giving an upper bound. A similar argument gives a lower bound and hence (10). The requirement that $\alpha \leq 0$, $\eta \geq 0$ implies (from (*A.2*)) $\xi_s \geq \exp(\Psi)$ which, since $F(\xi_s) \geq 0$, implies that $\xi_s \leq \xi^{\max}$.

For future reference, note that $F(\xi_s)$ is decreasing in ξ_s in the range $[\exp(\Psi), \xi^{\max}]$, with a minimum value of $F(\xi^{\max}) = 0$ and a maximum value of $F(\exp(\Psi)) = \Omega \exp(\Psi)$.

Proof of Proposition 3.
The proof is a direct consequence of Theorem 26 of Cerreia-Vioglio et al. [2011]. In order to invoke this theorem, from which all the properties in the proposition would follow, we need to show that the following property holds:

We can express $J(\mathbf{V})$ in the form $J(\mathbf{V}) = \inf_{\omega \in \mathcal{N}} \mathbb{E}^{\mathbb{P}}[\omega U(V)] / c_2(\omega)$ where $U(V) \leq 0$, ω is a non-negative change of measure (so $\mathbb{E}^{\mathbb{P}}[\omega] = 1$) belonging to a set $\mathcal{N}$ and $c_2(\omega)$ is convex and lower semicontinuous with $\min_{\omega \in \mathcal{N}} c_2(\omega) = 1$.

Our formulation is parameterized by ξ and takes the form:

$$J(\mathbf{V}) \equiv \inf_{\xi \in \Xi} \left\{ \mathbb{E}^{\mathbb{P}}[F(\xi) U(V)] \right\} = \inf_{\xi \in \Xi} \left\{ \mathbb{E}^{\mathbb{P}} \left[\left(F(\xi) / \mathbb{E}^{\mathbb{P}}[F(\xi)] \right) U(V) \right] \mathbb{E}^{\mathbb{P}}[F(\xi)] \right\}.$$

Thus, we need to show that we can parameterize our formulation in terms of $\omega \equiv F(\xi)/\mathbb{E}^{\mathbb{P}}[F(\xi)]$ and that $c_2(\omega) = 1/\mathbb{E}^{\mathbb{P}}[F(\omega)]$ with the requisite properties.

The definition of $F(\xi) \equiv \xi(1 - \Omega \log \xi)$ implies that, for $\xi \in [\exp(\Psi), \xi^{\max}]$, it is non-negative, concave and monotonic decreasing in ξ and that (by Jensen's inequality) $\mathbb{E}^{\mathbb{P}}[F(\xi)] \leq 1$. The inverse function $\xi(F)$ is well-defined (for $F \in [0, \Omega \exp(\Psi)]$), positive, concave and monotonic decreasing in F. Feasible $\omega \in \mathcal{N}$ are characterized: $\omega = F/\mathbb{E}^{\mathbb{P}}[F]$ subject to the restriction on F that $\mathbb{E}^{\mathbb{P}}[\xi(F)] = 1$. Consider now the convexity of $c_2(\omega) = 1/\mathbb{E}^{\mathbb{P}}[F(\omega)]$. Notice first that there is a one-to-one mapping between ω and ξ. For any given ω, F must be proportional to ω. The constraint $\mathbb{E}^{\mathbb{P}}[\xi(F)] = 1$ together with the monotonicity of $\xi(F)$ ensure that there is a unique constant of proportionality, and $\xi(F)$ itself then completes the mapping. Given $\omega_i = F_i/\mathbb{E}^{\mathbb{P}}[F_i]$ (with $\mathbb{E}^{\mathbb{P}}[\xi(F_i)] = 1$), for $i = 1, 2$, for any $\theta \in (0, 1)$, we form:

$$\omega_\theta \equiv \theta\omega_1 + (1-\theta)\,\omega_2 = \frac{\theta\mathbb{E}^{\mathbb{P}}[F_2]F_1 + (1-\theta)\,\mathbb{E}^{\mathbb{P}}[F_1]F_2}{\mathbb{E}^{\mathbb{P}}[F_1]\mathbb{E}^{\mathbb{P}}[F_2]}.$$

Suppose we form the weighted average G_θ:

$$\begin{aligned} G_\theta &= \frac{\theta\mathbb{E}^{\mathbb{P}}[F_2]F_1 + (1-\theta)\,\mathbb{E}^{\mathbb{P}}[F_1]F_2}{\theta\mathbb{E}^{\mathbb{P}}[F_2] + (1-\theta)\,\mathbb{E}^{\mathbb{P}}[F_1]} \\ &= \phi F_1 + (1-\phi)\,F_2, \quad \text{where} \quad 0 < \phi \equiv \frac{\theta\mathbb{E}^{\mathbb{P}}[F_2]}{\theta\mathbb{E}^{\mathbb{P}}[F_2] + (1-\theta)\,\mathbb{E}^{\mathbb{P}}[F_1]} < 1. \end{aligned}$$

We can write ω_θ as $\omega_\theta = F_\theta/\mathbb{E}^{\mathbb{P}}[F_\theta]$ (with $\mathbb{E}^{\mathbb{P}}[\xi(F_\theta)] = 1$) and $F_\theta = \lambda G_\theta$ for some $\lambda > 0$. Then

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[\xi(G_\theta)] &= \mathbb{E}^{\mathbb{P}}[\xi(\phi F_1 + (1-\phi)\,F_2)] \\ &\geq \phi\mathbb{E}^{\mathbb{P}}[\xi(F_1)] + (1-\phi)\,\mathbb{E}^{\mathbb{P}}[\xi(F_2)] \qquad \text{(since } \xi(F) \text{ is concave)} \\ &= \phi + (1-\phi) = 1. \end{aligned}$$

In order to satisfy $\mathbb{E}^{\mathbb{P}}[\xi(F_\theta)] = 1$ we must therefore have $\lambda \geq 1$ (since $\xi(F)$ is monotonic decreasing). Finally, we calculate:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[F_\theta] = \mathbb{E}^{\mathbb{P}}[\lambda G_\theta] &\geq \mathbb{E}^{\mathbb{P}}[G_\theta] = \phi\mathbb{E}^{\mathbb{P}}[F_1] + (1-\phi)\,\mathbb{E}^{\mathbb{P}}[F_2] \\ &= \frac{\mathbb{E}^{\mathbb{P}}[F_1]\mathbb{E}^{\mathbb{P}}[F_2]}{\theta\mathbb{E}^{\mathbb{P}}[F_2] + (1-\theta)\,\mathbb{E}^{\mathbb{P}}[F_1]}, \quad \text{which rearranges as:} \\ \frac{1}{\mathbb{E}^{\mathbb{P}}[F_\theta]} &\leq \frac{\theta}{\mathbb{E}^{\mathbb{P}}[F_1]} + \frac{1-\theta}{\mathbb{E}^{\mathbb{P}}[F_2]}. \end{aligned}$$

Thus, $c_2(\omega) = 1/\mathbb{E}^{\mathbb{P}}[F(\omega)]$ is convex in ω. It is finite and continuous (the latter by Rockafellar [1970], Corollary 10.1.1), hence, *a fortiori*, also lower semicontinuous. Finally, $\min_{\omega \in \mathcal{N}} c_2(\omega) = 1$, since $\mathbb{E}^{\mathbb{P}}[F(\omega)] \leq 1$, with equality when $\omega = 1$.

Proof[21] of Proposition 4.
1: This is immediate from the definition of infimum, since $\xi_s \equiv 1$ is always feasible and $F(1) = 1$.
2: This follows since $\Omega = 1$ implies $\xi_s \equiv 1$ in all states s by Proposition 5 and $F(1) = 1$.
3: The proof of the degenerate case is clear from Proposition 5. Examining (8), it is sufficient to show that the expression in square brackets is decreasing in Ω for any fixed η. That expression is, after some algebra, $U_1^{\text{MU}}(\Omega) \equiv G(\xi)\,U(V)$ for $\Omega < \Omega^{\text{crit}}$ (corresponding to $\beta = 0$) and

[21] We are greatly indebted to Volker Bosserhoff for assistance in proving this proposition.

$U_2^{\mathrm{MU}}(\Omega) \equiv \eta(\exp(1/\Omega)-1)(1-\Omega)$ for $\Omega \geq \Omega^{\mathrm{crit}}$ (corresponding to $\beta > 0$) where Ω^{crit} solves $\exp\left(1/\Omega^{\mathrm{crit}}\right) = \exp\left((1+(\eta/U))\left(\left(1/\Omega^{\mathrm{crit}}\right)-1\right)\right)$ i.e. $\Omega^{\mathrm{crit}} = 1/(1+U/\eta)$. One can easily confirm that $U_1^{\mathrm{MU}}\left(\Omega^{\mathrm{crit}}\right) = U_2^{\mathrm{MU}}\left(\Omega^{\mathrm{crit}}\right)$ i.e. that the expression in square brackets in (8) is continuous across $\Omega = \Omega^{\mathrm{crit}}$. So it is sufficient to show that (A) $U_1^{\mathrm{MU}}(\Omega)$ is decreasing for $1 < \Omega < \Omega^{\mathrm{crit}}$ and (B) $U_2^{\mathrm{MU}}(\Omega)$ is decreasing for $\Omega \geq \Omega^{\mathrm{crit}}$.

For case (A), we use the following lemma: For any b_1 and b_2 satisfying $b_2 < b_1 < 0$ and any a, $\exp(ab_2) - ab_2 - 1 \geq \exp(ab_1) - ab_1 - 1$, with equality only if $a = 0$. Consider $\Omega_b > \Omega_s > 1$ and define $\Psi_b \equiv (1/\Omega_b) - 1$, $\Psi_s \equiv (1/\Omega_s) - 1$. The lemma implies $\exp(a\Psi_b) - a\Psi_b - 1 \geq \exp(a\Psi_s) - a\Psi_s - 1 \Rightarrow (1 + \Omega_b(\exp(a\Psi_b) - a\Psi_b - 1))U \leq (1 + \Omega_s(\exp(a\Psi_s) - a\Psi_s - 1))U$ with equality only if $a = 0$. Now set $a = 1 + \eta/U$ and case A is proven.

Case (B) is proven by taking derivatives of $U_2^{\mathrm{MU}}(\Omega)$.

4: (a): From Proposition 6 of Cerreia-Vioglio et al. [2011]. (b): In contrast to Hansen and Sargent [2008], for example, the set of alternative models is governed, not by ξ, but by $\omega \in \mathcal{N}$ defined in the proof of Proposition 3. In the degenerate case of $U(V)$ being constant across all states s or if $\Omega = 1$, we have $\xi \equiv 1$, $F(\xi) \equiv 1$, $\omega \equiv F(\xi)/\mathbb{E}^{\mathbb{P}}[F(\xi)] \equiv 1$ and only the reference model $\mathbb{P}$ is considered. Consider now opposing cases. The range of $F(\xi)$ is $[0, \Omega\exp(\Psi)]$. The investor considers a larger set of alternative models, as Ω increases, if the range of ω increases. This is true if $\Omega\exp(\Psi)/\mathbb{E}^{\mathbb{P}}[F(\xi)] = \Omega\exp(\Psi)/\left(1 - \Omega\mathbb{E}^{\mathbb{P}}[\xi\log\xi]\right)$ increases, with increasing Ω. But the derivative of this expression is $\Omega\exp(\Psi)\left(\Omega - \mathbb{E}^{\mathbb{P}}[F(\xi)]\right)/\left(\Omega\mathbb{E}^{\mathbb{P}}[F(\xi)]^2\right)$ which is always positive since $\mathbb{E}^{\mathbb{P}}[F(\xi)] = 1 - \Omega\mathbb{E}^{\mathbb{P}}[\xi\log\xi] \leq 1$.

5: For any feasible $\xi \in \Xi$, $\mathbb{E}^{\mathbb{P}}[F(\xi)U(V)]$ is finite, non-positive, non-decreasing, concave and continuous (as the non-negative weighted sum of the same). Since this is true for any feasible $\xi \in \Xi$, it is also true for the infimum and hence for $J(\mathbf{V})$ (where we have also used Rockafellar [1970], Corollary 10.1.1, for the proof of continuity).

Proof of Proposition 6.

For simplicity, within this proof, we interpret V_0 as forward initial wealth and normalize $\mathbb{E}^{\mathbb{P}}[m] = 1$ - but this is, essentially, without loss of generality. We can write the left-hand side of equation (11) as $\sup_V \left\{\inf_{\xi\in\Xi}\left\{\mathbb{E}^{\mathbb{P}}[F(\xi)U(V)]\right\} \text{ such that } \mathbb{E}^{\mathbb{P}}[mV] = V_0\right\} =$ $\inf_{\xi\in\Xi}\left\{\sup_V\left\{\mathbb{E}^{\mathbb{P}}[F(\xi)U(V)] \text{ such that } \mathbb{E}^{\mathbb{P}}[mV] = V_0\right\}\right\}$, because the objective function is concave in V, convex in ξ, continuous, and only takes finite values, both the domains of maximization and minimization are closed and convex and the domain of minimization is bounded. So the conditions of Rockafellar [1970], Corollary 37.3.2, are fulfilled which justifies the exchange of inf and sup. Consider the inner sup, over V, for any feasible $\xi \in \Xi$. We introduce a Lagrange multiplier λ and re-express it in the form: $\sup_V\{\mathbb{E}^{\mathbb{P}}[F(\xi)U(V) - \lambda(mV - V_0)]\}$. Differentiating with respect to V gives the first order condition: $F(\xi)U'(V) = \lambda m$, or $m \propto F(\xi)U'(V)$.

For the special case of $U(V) = -\overline{\beta}\left(\max\left(\overline{V} - V, 0\right)\right)^2$, we have $2F(\xi)\overline{\beta}\left(\overline{V} - V\right) = \lambda m$. The constraint $\mathbb{E}^{\mathbb{P}}[mV] = V_0$ then implies: $V_0 = \mathbb{E}^{\mathbb{P}}[m]\overline{V} - \frac{\lambda}{2\overline{\beta}}\mathbb{E}^{\mathbb{P}}[\frac{m^2}{F(\xi)}]$. Hence, solving for λ, we find $\lambda = 2\overline{\beta}\left(\overline{V} - V_0\right)/\mathbb{E}^{\mathbb{P}}[\frac{m^2}{F(\xi)}]$. Substituting for V, $\sup_V\{\mathbb{E}^{\mathbb{P}}[F(\xi)U(V)]\}$, subject to the constraint $\mathbb{E}^{\mathbb{P}}[mV] = V_0$, is

$$\sup_V\{\mathbb{E}^{\mathbb{P}}[F(\xi)U(V)\} = \mathbb{E}^{\mathbb{P}}[-F(\xi)\overline{\beta}\frac{1}{4}\frac{\lambda^2}{\overline{\beta}^2}\frac{m^2}{(F(\xi))^2}] = -\overline{\beta}\frac{\left(\overline{V} - V_0\right)^2}{\mathbb{E}^{\mathbb{P}}[\frac{m^2}{F(\xi)}]}, \quad \text{whence}$$

$$\inf_{\xi\in\Xi}\left\{\sup_V\{\mathbb{E}^{\mathbb{P}}[F(\xi)U(V)\}\right\} = \inf_{\xi\in\Xi}\left\{-\overline{\beta}\frac{\left(\overline{V} - V_0\right)^2}{\mathbb{E}^{\mathbb{P}}[\frac{m^2}{F(\xi)}]}\right\} = -\overline{\beta}\frac{\left(\overline{V} - V_0\right)^2}{\inf_{\xi\in\Xi}\left\{\mathbb{E}^{\mathbb{P}}[\frac{m^2}{F(\xi)}]\right\}}. \tag{A.3}$$

Setting ($A.3$) equal to $-\overline{\beta}\left(\overline{V}-\left(V_0+\text{CE}\right)\right)^2$, we solve for CE as in (12).

As a final observation, we note that the expression $m \propto F(\xi)\,U^{'}(V)$ implies that we should read $\frac{m^2}{F(\xi)}$, in (A.3) and in equations (12) to (18), with the formal convention $1/0=\infty$ and $0/0=0$.

Proof of Proposition 8. See after the proof of Proposition 10.

Proof of Proposition 9.
We focus on $\underline{C}$ ($\overline{C}$ is similar and omitted). Introducing Lagrange multipliers $\delta>0$, $\mathbf{w}$ (a N_b - dimensional vector), α, $\beta_s\geq 0$ and $\varpi_s\geq 0$, for each state s, we can re-express (15) as:

$$\begin{aligned}\underline{C} &= \max_{\varpi_s\geq 0,\beta_s\geq 0,\alpha}\{\min_{\xi_s}\{\max_{\delta>0,\mathbf{w}}\{\min_{m_s\geq 0}\{\sum_{s=1}^{S}\pi_s[m_s c_s-\mathbf{w}^{'}\left(m_s\mathbf{x}_s-\mathbf{p}\right)+\frac{1}{2}\delta\left(\frac{m_s^2}{F\left(\xi_s\right)}-\Gamma\right)]\}\} \\ &\quad +\sum_{s=1}^{S}\pi_s[\frac{1}{2}\alpha\left(\xi_s-1\right)-\frac{1}{2}\delta\beta_s F\left(\xi_s\right)-\frac{1}{2}\varpi_s\xi_s]\}\}. \end{aligned} \tag{A.4}$$

The first-order condition obtained by taking partial derivatives with respect to m_s in each state s and, if necessary, enforcing the condition $m_s\geq 0$ (we say if necessary because this constraint may or may not bind) implies the expression for m_s (we have anticipated the corresponding form for the upper good deal bound and introduced $\mathbf{1}_{L/U}$ accordingly). Substituting for m_s and interchanging the orders of max and min (justified by Rockafellar [1970], Corollary 37.3.2), we get:

$$\begin{aligned}\underline{C} &= \max_{\delta>0,\mathbf{w}}\{\max_{\varpi_s\geq 0,\beta_s\geq 0,\alpha}\{\min_{\xi_s}\{\mathbf{w}^{'}\mathbf{p}-\frac{1}{2}\delta\Gamma+ \\ &\quad \sum_{s=1}^{S}\pi_s[-\frac{1}{2}\delta F\left(\xi_s\right)Z_s\left(\mathbf{1}_{L/U},\mathbf{w},1,\delta\right)+\frac{1}{2}\alpha\left(\xi_s-1\right)-\frac{1}{2}\delta\beta_s F\left(\xi_s\right)-\frac{1}{2}\varpi_s\xi_s]\}\}\}. \end{aligned} \tag{A.5}$$

The first-order condition from taking partial derivatives with respect to ξ_s in each state s implies:

$$\alpha-\varpi_s = \delta\left(Z_s\left(\mathbf{1}_{L/U},\mathbf{w},1,\delta\right)+\beta_s\right)\Omega\left(\Psi-\log\xi_s\right). \tag{A.6}$$

The rest of the proof is similar to that of Proposition 5 - in fact, the solution can almost be read off (identifying $Z_s\left(\mathbf{1}_{L/U},\mathbf{w},1,\delta\right)$ here with $-U(V)$ there). As in Proposition 5, $\varpi_s=0$ and we identify $\eta\equiv\alpha/\left(\delta\Omega\Psi\right)$.

Equation (23) (similarly, (24)) is derived by taking a step back to ($A.5$), removing the Lagrange multipliers α, β_s and ϖ_s and then rearranging.

Proof of Proposition 10.
Introducing Lagrange multipliers $\mathbf{v}$ (a N_b - dimensional vector), μ, α, $\beta_s\geq 0$ and $\varpi_s\geq 0$, for each state s, problem (26) is equivalent to:

$$\begin{aligned}&\max_{\varpi_s\geq 0,\beta_s\geq 0,\alpha}\{\min_{\xi_s}\{\max_{\mu,\mathbf{v}}\{\min_{m_s\geq 0}\{\sum_{s=1}^{S}\pi_s[\frac{m_s^2}{F\left(\xi_s\right)}+2\mu\left(m_s c_s-C_{\text{Arb}}\left(\mathbf{1}_{L/U}\right)\right) \\ &\quad -2\mathbf{v}^{'}\left(m_s\mathbf{x}_s-\mathbf{p}\right)]\}\}+\sum_{s=1}^{S}\pi_s[\alpha\left(\xi_s-1\right)-\beta_s F\left(\xi_s\right)-\varpi_s\xi_s]\}\}. \end{aligned} \tag{A.7}$$

The rest of the proof is similar to that of Proposition 5 (identifying $Z_s\left(1,\mathbf{v},\mu,1\right)$ here with $-U(V)$ there) and Proposition 9. Again, $\varpi_s=0$ and we identify $\eta\equiv\alpha/\left(\Omega\Psi\right)$.

Proof of Proposition 8.
Equations (14) and (15) are completely equivalent to each other (likewise (16) and (17)) since their Lagrangian duals are identical. Hence, to prove the proposition, we show that Definition 7 implies equations (15) and (17) and that (15) and (17) imply the equations for $\underline{C}$ and $\overline{C}$ in Definition 7.

Proof that Definition 7 implies equations (15) and (17):
We focus on $\underline{C}$ ($\overline{C}$ is similar and omitted) and consider two cases: Case (1) when the constraint $\max_{\mu>0,\mathbf{v}}\left\{2\mathbf{v}'\mathbf{p}-2\mu C+\inf_{\xi\in\Xi}\left\{\mathbb{E}^{\mathbb{P}}[F(\xi)\tilde{U}^{\text{TQ}}\left(\mu c-\mathbf{v}'\mathbf{x}\right)]\right\}\right\}\leq\Gamma$ binds and case (2) when the constraint is slack. In case (1), we conjecture that the constraint $C\in\mathcal{NA}$ is not binding (we check this later by showing that the pricing kernel is non-negative), then Definition 7 implies equations (23) and (24). The first-order conditions from differentiating with respect to $\mathbf{v}$ and μ imply $\mathbf{p}=\mathbb{E}^{\mathbb{P}}[F(\xi)\max\left(-\left(\mu c-\mathbf{v}'\mathbf{x}\right),0\right)\mathbf{x}]$ and $\underline{C}=\mathbb{E}^{\mathbb{P}}[F(\xi)\max\left(-\left(\mu c-\mathbf{v}'\mathbf{x}\right),0\right)c]$ respectively. This means we can write the pricing kernel as $m=F(\xi)\max\left(-\left(\mu c-\mathbf{v}'\mathbf{x}\right),0\right)$. This is the same as that in Proposition 9 and is clearly non-negative and therefore (by Cochrane [2005], chapter 4) implies the absence of arbitrage. This confirms that the constraint $C\in\mathcal{NA}$ is automatically satisfied. It also shows that the constraints $\mathbb{E}^{\mathbb{P}}[m\mathbf{x}]=\mathbf{p}$ and $m\geq 0$ in equation (15) are satisfied. Reversing the logic of Proposition 9 shows equation (15) is satisfied. In case (2), $\underline{C}$ is the minimum lower bound in the set $\mathcal{NA}$ i.e. $\underline{C}$ is the minimum lower bound consistent with the absence of arbitrage and is therefore given by (25). Putting the two cases together implies (15) (and (17)).

Proof that equations (15) and (17) imply the equations for $\underline{C}$ and $\overline{C}$ in Definition 7:
By Propositions 9 and 10, equations (15) and (17) imply equations (23), (24) and (28). Note that the maximization over μ in equation (28) can be replaced by one over $\mu\mathbf{1}_{L/U}>0$ because for the lower bound for which $\mathbf{1}_{L/U}=1$ (respectively, upper bound for which $\mathbf{1}_{L/U}=-1$), the constraint $C_{\text{Arb}}\left(\mathbf{1}_{L/U}\right)=\mathbb{E}^{\mathbb{P}}[mc]$ could be weakened to $C_{\text{Arb}}(1)\geq\mathbb{E}^{\mathbb{P}}[mc]$ implying $\mu>0$ (respectively, weakened to $C_{\text{Arb}}(-1)\leq\mathbb{E}^{\mathbb{P}}[mc]$ implying $\mu<0$), without loss of generality. Now rearranging equations (23), (24) and (28) implies the equations for $\underline{C}$ and $\overline{C}$ in Definition 7.

Proof of Proposition 11.
Part (a): Note that $\xi_s\equiv 1$, for each state s, (corresponding to that which would obtain in the absence of model uncertainty) is feasible (i.e. an element of the set Ξ). In view of (15) and (17), the result now follows from the definition of infimum and supremum. Part (b): The statement concerning Γ is obvious from (15) and (17) (since increasing Γ weakens the constraint on $\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right]$) while that concerning Ω is proven in the same way as part (3) of Proposition 4.

Internet appendix

B Exponential and CRRA utility functions

In this appendix, we detail the analysis of good deal bounds for the cases of exponential and CRRA utility functions (as opposed to the (truncated) quadratic utility function case).

The extension of Proposition 6 is stated in the following proposition whose proof follows Proposition 6 and section 2 of Černý [2003].

Proposition B.1 *(a) For the case of exponential utility, $U(V) = -\overline{\beta}\exp(-BV)$, with constant absolute risk aversion $B > 0$, the certainty equivalent* CE *is*

$$\mathrm{CE} = \frac{1}{\mathrm{CARA}\left(U\left(V_0\right)\right)} \inf_{\xi \in \Xi} \left\{ \mathbb{E}^{\mathbb{P}} \left[m \log\left(\frac{m}{F(\xi)} \right) \right] \right\}, \qquad \textit{and furthermore}$$

$$\mathrm{CE} \leq \Lambda \Longleftrightarrow \inf_{\xi \in \Xi} \left\{ \mathbb{E}^{\mathbb{P}} \left[m \log\left(\frac{m}{F(\xi)} \right) \right] \right\} \leq \mathrm{CARA}\left(U\left(V_0\right)\right) \Lambda. \tag{B.1}$$

(b) For the case of constant relative risk aversion (CRRA) utility with CRRA coefficient $\gamma > 1$, $U(V) = \overline{\beta}\frac{V^{1-\gamma}}{1-\gamma}$, the certainty equivalent CE *is*

$$\mathrm{CE} = V_0 \left(\inf_{\xi \in \Xi} \left\{ \mathbb{E}^{\mathbb{P}} \left[m \left(\frac{m}{F(\xi)} \right)^{\frac{-1}{\gamma}} \right] \right\} \right)^{\frac{\gamma}{1-\gamma}} - V_0, \qquad \textit{and furthermore}$$

$$\mathrm{CE} \leq \Lambda \Longleftrightarrow \inf_{\xi \in \Xi} \left\{ \mathbb{E}^{\mathbb{P}} \left[m \left(\frac{m}{F(\xi)} \right)^{\frac{-1}{\gamma}} \right] \right\} \leq \left(1 + \mathrm{CARA}(U\left(V_0\right)) \frac{\Lambda}{\gamma} \right)^{\frac{1}{\gamma} - 1}. \tag{B.2}$$

The constraint $\inf_{\xi \in \Xi} \left\{ \mathbb{E}^{\mathbb{P}} \left[\frac{m^2}{F(\xi)} \right] \right\} \leq \Gamma$ in equations (14) and (16) is replaced by $\inf_{\xi \in \Xi} \left\{ \mathbb{E}^{\mathbb{P}} \left[m \log\left(\frac{m}{F(\xi)} \right) \right] \right\} \leq \Gamma$ or by $\inf_{\xi \in \Xi} \left\{ \mathbb{E}^{\mathbb{P}} \left[m \left(\frac{m}{F(\xi)} \right)^{\frac{-1}{\gamma}} \right] \right\} \leq \Gamma$.

Similarly to the dual utility function $\tilde{U}^{\mathrm{TQ}}(V)$, define the dual utility functions $\tilde{U}^{E}(V) \equiv -\exp(-V-1)$ and $\tilde{U}^{\mathrm{CRRA}}(V) \equiv -(\max(-V,0))^{1-\gamma}/(\gamma(1-1/\gamma)^{1-\gamma})$ and similarly to (19) and (20), define $Y_s^E\left(\mathbf{1}_{L/U}, \mathbf{w}, \varphi, \delta\right)$, $Z_s^E\left(\mathbf{1}_{L/U}, \mathbf{w}, \varphi, \delta\right)$ and $Y_s^{\mathrm{CRRA}}\left(\mathbf{1}_{L/U}, \mathbf{w}, \varphi, \delta\right)$, $Z_s^{\mathrm{CRRA}}\left(\mathbf{1}_{L/U}, \mathbf{w}, \varphi, \delta\right)$ for exponential and CRRA utility, respectively, as

$$\begin{aligned}
Y_s^E\left(\mathbf{1}_{L/U}, \mathbf{w}, \varphi, \delta\right) &\equiv \exp\left(\left(-\mathbf{1}_{L/U}\frac{(\varphi c_s - \mathbf{w}'\mathbf{x}_s)}{\delta}\right) - 1\right), \\
Z_s^E\left(\mathbf{1}_{L/U}, \mathbf{w}, \varphi, \delta\right) &\equiv Y_s^E\left(\mathbf{1}_{L/U}, \mathbf{w}, \varphi, \delta\right) = -\tilde{U}^E\left(\mathbf{1}_{L/U}\frac{(\varphi c_s - \mathbf{w}'\mathbf{x}_s)}{\delta}\right), \\
Y_s^{\mathrm{CRRA}}\left(\mathbf{1}_{L/U}, \mathbf{w}, \varphi, \delta\right) &\equiv \frac{1}{(1-1/\gamma)^{-\gamma}}\left(\max\left[-\mathbf{1}_{L/U}\frac{(\varphi c_s - \mathbf{w}'\mathbf{x}_s)}{\delta}, 0\right]\right)^{-\gamma}, \\
Z_s^{\mathrm{CRRA}}\left(\mathbf{1}_{L/U}, \mathbf{w}, \varphi, \delta\right) &\equiv -\tilde{U}^{\mathrm{CRRA}}(\mathbf{1}_{L/U}\frac{(\varphi c_s - \mathbf{w}'\mathbf{x}_s)}{\delta}).
\end{aligned}$$

Notice that $\tilde{U}^{\text{TQ}}$, $\tilde{U}^{E}$ and $\tilde{U}^{\text{CRRA}}$ and the loss functions Z, Z^{E} and Z^{CRRA} inherit the structure of the utility function used to derive the restriction on the pricing kernel while Y, Y^{E} and Y^{CRRA} take the form of the first derivative of the respective dual utility function.

Definition 7, Propositions 9 and 10, and more specifically equations (21) to (35), hold in their entirety provided: Y, Z and $\tilde{U}^{\text{TQ}}$ are replaced by Y^{E}, Z^{E} and $\tilde{U}^{E}$ or by Y^{CRRA}, Z^{CRRA} and $\tilde{U}^{\text{CRRA}}$, respectively and the factors 1/2 and 2 are replaced by 1.

So as examples, for exponential utility, the analogs of (21) and (33), respectively, read

$$\underline{C} \;=\; \max_{\delta>0,\mathbf{w}} \left\{ \mathbf{w}^{'}\mathbf{p} - \delta\Gamma + \max_{\eta} \left\{ \sum_{s=1}^{S} \pi_s \left[-\delta G\left(\xi_s\right) \left(Z_s^{E}\left(\mathbf{1}_{L/U}, \mathbf{w}, 1, \delta\right) + \beta_s \right) \right] \right\} \right\},$$

$$\Gamma^{\star\text{NoMU}} \equiv \max_{\mathbf{v}} \left\{ \mathbf{v}^{'}\mathbf{p} + \sum_{s=1}^{S} \pi_s \left[- \left(Z_s^{E}\left(1, \mathbf{v}, 0, 1\right)\right)\right] \right\}.$$

The latter means that the estimate of Ω and the choice of Γ also depend upon the choice of utility function. In choosing Γ, the same alternatives are available but there is a particularly simple way for the exponential utility case via the choice of an annualized exponential (Hodges [1998]) Sharpe ratio h^{E}_{Ann} because then (36) holds exactly, for an arbitrary time period Δt years - indeed Hodges [1998] takes that to be the definition of h^{E}_{Ann}. Hence, we can set $\Gamma = \frac{1}{2} h^{E\,2}_{\text{Ann}} \Delta t$. Furthermore, for normally distributed returns and in the limit of small time periods, Hodges [1998] and Černý [2003] show that the exponential Sharpe ratio h^{E}_{Ann} and the (standard) Sharpe ratio h_{Ann} coincide.

We now state some general properties of the good deal bounds, all of which are equally applicable to good deal bounds constructed from restrictions on the pricing kernel in (13), (*B*.1) or (*B*.2).

For the following, let $\mathbf{u}$ denote an arbitrary N_b - dimensional vector.

Proposition B.2 *Let λ be a non-negative constant. If the lower and upper good deal bounds for a focus payoff c are $\underline{C}$ and $\overline{C}$, respectively, with corresponding optimal hedges $\underline{\mathbf{w}}$ and $\overline{\mathbf{w}}$, then the lower and upper good deal bounds for a focus payoff $\lambda c + \mathbf{u}^{'}\mathbf{x}$ are $\lambda\underline{C} + \mathbf{u}^{'}\mathbf{p}$ and $\lambda\overline{C} + \mathbf{u}^{'}\mathbf{p}$, respectively, with corresponding optimal hedges $\lambda\underline{\mathbf{w}} + \mathbf{u}$ and $\lambda\overline{\mathbf{w}} + \mathbf{u}$.*

Corollary B.3 *Consider a focus payoff of the form $\boldsymbol{u}^{'}\boldsymbol{x}$. Then the lower and upper good deal bounds coincide and are both equal to $\boldsymbol{u}^{'}\boldsymbol{p}$, and the optimal hedges are $\boldsymbol{u}$.*

Thus, the good deal bounds satisfy linearity with respect to adding portfolios of basis assets and homogeneity with respect to positive multiples of the focus asset payoff. Furthermore, as shown in Corollary B.3, redundant assets are priced exactly at their replication cost. Proposition *B*.4 shows that the good deal bounds also satisfy dominance and portfolio diversification properties.

Proposition B.4 *Consider two focus assets, A and B, with payoffs c^{A} and c^{B}, with lower and upper good deal bounds $\underline{C}^{A}$, $\overline{C}^{A}$, $\underline{C}^{B}$ and $\overline{C}^{B}$ (computed using the same basis assets).*

(1) If $c^{A}_{s} \leq c^{B}_{s}$ for each and every state s, then $\underline{C}^{A} \leq \underline{C}^{B}$ and $\overline{C}^{A} \leq \overline{C}^{B}$.

(2) Let λ^{A} and λ^{B} be non-negative constants. The lower $\underline{C}^{P}$ and upper $\overline{C}^{P}$ good deal bounds on the payoff $P \equiv \lambda^{A} c^{A} + \lambda^{B} c^{B}$ satisfy $\lambda^{A}\underline{C}^{A} + \lambda^{B}\underline{C}^{B} \leq \underline{C}^{P} \leq \overline{C}^{P} \leq \lambda^{A}\overline{C}^{A} + \lambda^{B}\overline{C}^{B}$.

These propositions are easily proven by substituting into (21), (22) and (27).

Table C.1: Degree of aversion to model uncertainty

Length of data-set	20	40	100	200	500	1000	2000	4000	8000
Value of Ω	3.65	2.81	2.31	2.02	1.77	1.61	1.49	1.40	1.32

NOTES: Degree of aversion to model uncertainty Ω as a function of the number of simulated historical observations. Ω is estimated using the procedure in Section 5, with 65000 bootstrapped samples, setting the Model Confidence Level to 80%. The basis assets are a risk-free and a defaultable bond.

Table C.2: Good deal bounds on Arrow-Debreu securities

				F_{good}				
Ω	1	1.32	1.49	1.77	2.31	3.65	4	8
$\underline{C}$	0.7364	0.7334	0.7308	0.7259	0.7162	0.6937	0.6882	0.6667*
$\overline{C}$	0.7893	0.7917	0.7940	0.7986	0.8086	0.8331	0.8333*	0.8333*
				F_{poor}				
Ω	1	1.32	1.49	1.77	2.31	3.65	4	8
$\underline{C}$	0.0882	0.0832	0.0787	0.0694	0.0495	0.0000*	0.0000*	0.0000*
$\overline{C}$	0.1939	0.1998	0.2050	0.2148	0.2343	0.2793	0.2903	0.3333*
				F_{disaster}				
Ω	1	1.32	1.49	1.77	2.31	3.65	4	8
$\underline{C}$	0.0697	0.0668	0.0642	0.0593	0.0495	0.0270	0.0215	0.0000*
$\overline{C}$	0.1226	0.1251	0.1273	0.1319	0.1419	0.1664	0.1667*	0.1667*

NOTES: The lower $\underline{C}$ and upper $\overline{C}$ good deal bounds as a function of Ω. * means the good deal bound equals the relevant arbitrage bound.

C Defaultable bond and Arrow-Debreu securities

In this appendix, we provide a further numerical example. Consider an economy with two basis assets, a defaultable bond and a risk-free bond. At time 1, there are three possible states of the world, labeled "good", "poor" and "disaster". The defaultable bond has a time 0 price of 1 and a payoff at time 1 equal to 1.2, 0.6 and 0 in states "good", "poor" and "disaster" respectively. The "poor" and "disaster" states are states in which the issuer of the defaultable bond defaults and the holder of the bond receives either 60% (partial recovery) or 0% (zero recovery) of the time 0 price. The risk-free bond has a time 0 price of 1 and a payoff at time 1 equal to 1 in all three states.

To illustrate our theoretical analysis, we simulate a data-set with 2000 data points. The defaultable bond pays 1.2, 0.6 and 0 on 1700, 200 and 100 dates in the sample, respectively. Hence, the probability, under $\mathbb{P}$, of states "good", "poor" and "disaster" occurring are set at 0.85, 0.1 and 0.05 respectively. Given the simulated historical data-set, we use the bootstrap procedure (with 65000 samples) of Section 5 to estimate Ω and, for a Model Confidence Level of 80%, we find $\Omega = 1.49$. In Table C.1, we report also the estimate of Ω for simulated data-sets with different numbers of data points, keeping the same probabilities of the three states. We see that as the length of the simulated data-set decreases, the estimate of Ω increases, implying greater model uncertainty.

We consider three Arrow-Debreu securities as the focus assets. In particular, Arrow-Debreu security F_j, for $j \in \{\text{good}, \text{poor}, \text{disaster}\}$ pays one dollar if state j is realized. Notice that F_{disaster} is a catastrophe insurance contract. It is straightforward to verify (or by solving (25)) that the lower and upper arbitrage bounds for the three focus assets are $2/3, 5/6$ (F_{good}), $0, 1/3$ (F_{poor}) and

Table C.3: Optimal hedging portfolios for F_{disaster}

Ω	1	1.25	1.5	1.75	2	4	8
$\underline{C}$	0.0697	0.0678	0.0640	0.0596	0.0551	0.0215	0.0000*
w_1	-1.1055	-1.0646	-0.9958	-0.9371	-0.8949	-0.8010	0
w_2	1.8624	1.7992	1.6972	1.6168	1.5668	1.6011	0
$\overline{C}$	0.1226	0.1242	0.1275	0.1316	0.1361	0.1667*	0.1667*
w_1	-0.0483	-0.0730	-0.1070	-0.1302	-0.1454	-0.8333	-0.8333
w_2	-0.5163	-0.4804	-0.4377	-0.4191	-0.4175	1	1

Ω	5	5.4	5.45	5.46	5.46125	5.4625	5.46375	5.465
$\underline{C}$	0.00655	0.00088	0.00019	0.00005	0.00003	0.00001	0.00000*	0.00000*
w_1	-0.7875	-0.7832	-0.7827	-0.7826	-0.7826	-0.7826	0.0000	0.0000
w_2	1.6799	1.7133	1.7175	1.7184	1.7185	1.7186	0.0000	0.0000

NOTES: The optimal hedging position in the risk-free (w_1) and the defaultable (w_2) bond for the lower good deal bound $\underline{C}$ as a function of Ω. * means the good deal bound equals the relevant arbitrage bound.

$0, 1/6$ (F_{disaster}). We set the Sharpe ratio bound h_{Ann} equal to $1/3$. In Table C.2, we report the lower $\underline{C}$ and upper $\overline{C}$ good deal bounds for different values of Ω. Note that $\Omega = 1$ is the no model uncertainty case. As Ω increases, the good deal bounds widen and the values of $\underline{C}$ and $\overline{C}$ get closer to the lower and upper arbitrage bounds and, for $\Omega = 8$, the good deal bounds equal the arbitrage bounds.

Consider now the catastrophe insurance contract, F_{disaster}, in detail. In Table C.3 (upper panel), we report the lower $\underline{C}$ and upper $\overline{C}$ good deal bounds for different values of Ω, as well as the optimal hedging positions (denoted w_1 and w_2 for the defaultable bond and risk-free bond respectively). The constraint $\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \leq \Gamma$ is slack when $\Omega = 8$ for the lower bound and when $\Omega \in \{4, 8\}$ for the upper bound. For these values of Ω, the good deal bounds correspond to the arbitrage bounds. The positions in the defaultable bond and risk-free bond respectively (computed from (29)) which enforce the lower and upper arbitrage bounds are $0, 0$ and $-5/6, 1$.[22] We see that, as the value of Ω changes, the optimal hedges w_1 and w_2 change significantly.

In Table C.3 (lower panel), we focus just on $\underline{C}$ and consider a wider range of values of Ω in order to see where the boundary lies between when the constraint $\mathbb{E}^{\mathbb{P}}\left[\frac{m^2}{F(\xi)}\right] \leq \Gamma$ binds and when it is slack. We see that the boundary lies at a value of $\Omega \approx 5.463$. Notice that, since the nature of the good deal bounds is discontinuous, the optimal hedges are discontinuous across the critical value of $\Omega \approx 5.463$.

D Asset pricing puzzles

As remarked in Section 3, our principle motivation for developing a new model uncertainty framework is in order to develop a theory of No Good Deals which is analytically tractable. This entails

[22] That is, the upper arbitrage bound $1/6$ can be enforced by sellng $5/6$ units of the defaultable bond and buying one unit of the risk-free bond which costs, at time 0, a price of $1 - 5/6 = 1/6$ dollar while the lower arbitrage bound is enforced by zero position in each basis asset.

that our framework differs from, for example, those of Hansen and Sargent [2008] and Maenhout [2004] in important ways. First, while entropy plays an important role in their frameworks, it has no role in ours. Instead, our framework derives its economic intuition from Propositions 3, 4 and 5 and from Cerreia-Vioglio et al. [2011]. Second, the set of alternative models is governed, not by ξ, as in Hansen and Sargent [2008] and Maenhout [2004], but by $\omega \in \mathcal{N}$ defined in the proof of Proposition 3. Besides allowing for an analytically tractable theory of No Good Deals, our framework also has the possible advantage of not suffering from a so-called breakdown point (Hansen and Sargent [2008], p32). Given the differences, a pertinent question is whether our framework, similarly to those of Hansen and Sargent [2008] and Maenhout [2004], can explain well-known asset pricing puzzles. In the remainder of this appendix, we answer this question in the affirmative. In summary, our model uncertainty framework seems to lose little or nothing in comparison to those of Hansen and Sargent [2008] and Maenhout [2004] whilst allowing for a theory of No Good Deals, in the presence of model uncertainty, which is tractable and economically intuitive.

D.1 Equity premium puzzle

In this appendix, we show that our model uncertainty framework, whilst differing from those of Hansen and Sargent [2008], Barillas et al. [2009] and Maenhout [2004], is like theirs, potentially able to explain the equity premium puzzle of Mehra and Prescott [1985].

We consider two one period economies - in the first economy, the representative agent has no model uncertainty (quantities are denoted with a superscript $^{\text{NoMU}}$) and has CRRA utility $U(c) = \overline{\beta}\frac{c^{1-\gamma}}{1-\gamma}$, $\gamma > 1$, $\overline{\beta} > 0$, and, in the second economy, the representative agent seeks robustness against model uncertainty but is otherwise identical. Mean consumption growth is denoted by μ_c and the volatility of consumption growth is denoted by σ_c. Consumption, at time t, is denoted c_t, for $t = 0, 1$. We make three possible distributional assumptions concerning consumption growth.

LN: In the first (labeled LN), we assume consumption has a log-normal distribution so that consumption c_1 at time 1 has the form $c_1 = c_0 \exp\left(\left(\mu_c - \frac{1}{2}\sigma_c^2\right)\Delta t + \sigma_c\sqrt{\Delta t}\,\epsilon\right)$ for ϵ distributed $N(0,1)$. We then simulate from this distribution.

3S: In the second (labeled 3S), we assume consumption c_1 at time 1 can take on one of three values: $c_0 \exp\left(\mu_c\Delta t + \lambda\sigma_c\sqrt{\Delta t}\right)$, $c_0 \exp\left(\mu_c\Delta t\right)$ or $c_0 \exp\left(\mu_c\Delta t - \lambda\sigma_c\sqrt{\Delta t}\right)$, where $\lambda = \sqrt{(3/2)}$. We compute the probabilities of the three states occuring by requiring that they sum to 1 and match the mean μ_c and volatility σ_c.

Hist: In the third (labeled Hist), we use the annual U.S. consumption data for the period 1891-1998 (a time period spanning 108 years) from Campbell [2003]. We assume that log-changes in consumption from time 0 to time 1 take one of the 108 values actually observed, with probability, under $\mathbb{P}$, of 1/108. The advantage of this approach is that the (negative) skewness and fat tails in the Campbell [2003] data-set are automatically captured.

We set $\Delta t = 1$ (i.e. consider a time period of one year). The one period risk-free returns are denoted R_f^{NoMU} and R_f. There is a stock, with volatility σ_S, whose price is assumed (counterfactually - but similarly to Maenhout [2004]) to be perfectly correlated with consumption. Its time 0 price is normalised to 1 and at time 1 it has a price equal to $(c_1/c_0)^{(\sigma_S/\sigma_c)}$.

Standard results (Cochrane [2005]) imply that, for the first economy

$$ER^{\text{NoMU}} \equiv \mathbb{E}^{\mathbb{P}}[R^{\text{NoMU}}] - R_f^{\text{NoMU}} = -R_f^{\text{NoMU}}\text{Cov}\,(m, R) \quad \text{and} \quad R_f^{\text{NoMU}} = \frac{1}{\mathbb{E}^{\mathbb{P}}[m]}, \tag{D.1}$$

Table D.1: Excess returns and risk-free rates for different values of γ and Ω.

LN	γ	Ω	$\sigma(m)$	$\sigma(F(\xi)\,m)$	r_f^{NoMU}	r_f	r_e^{NoMU}	r_e
	2	2	6.22	7.77	3.52	3.44	1.33	1.66
	2	8	6.22	25.23	3.52	2.57	1.33	5.26
	2	16	6.22	49.04	3.52	1.40	1.33	9.79
	5	2	15.00	20.86	7.64	7.03	3.26	4.52
	5	8	15.00	76.77	7.64	1.19	3.26	14.32
3S	γ	Ω	$\sigma(m)$	$\sigma(F(\xi)\,m)$	r_f^{NoMU}	r_f	r_e^{NoMU}	r_e
	2	2	6.20	7.75	3.52	3.44	1.32	1.65
	2	8	6.20	25.19	3.52	2.57	1.32	5.23
	2	16	6.20	49.79	3.52	1.36	1.32	9.97
	5	2	14.81	20.65	7.65	7.04	3.25	4.49
	5	8	14.81	81.71	7.65	0.63	3.25	15.74
Hist	γ	Ω	$\sigma(m)$	$\sigma(F(\xi)\,m)$	r_f^{NoMU}	r_f	r_e^{NoMU}	r_e
	2	2	6.33	7.91	3.52	3.44	1.30	1.62
	2	8	6.33	25.55	3.52	2.57	1.30	5.10
	2	16	6.33	49.56	3.52	1.40	1.30	9.50
	5	2	15.79	21.70	7.60	6.99	3.25	4.47
	5	8	15.79	77.95	7.60	1.23	3.25	13.85

NOTES: $\overline{\beta}$ is set to 0.9975. The first panel is for LN (log-normal) (using 250,000 simulations), the second for 3S and the third is for Hist (historical).

where $m \equiv \overline{\beta}\frac{U'(c_1)}{U'(c_0)} = \overline{\beta}\left(\frac{c_1}{c_0}\right)^{-\gamma}$ denotes the pricing kernel. For the second economy,

$$ER \equiv \mathbb{E}^{\mathbb{P}}[R] - R_f = -R_f \text{Cov}\left(F(\xi)\,m, R\right) \quad \text{and} \quad R_f = \frac{1}{\mathbb{E}^{\mathbb{P}}[F(\xi)\,m]}, \tag{D.2}$$

where we have used results in Sections 3 and 4 and where ξ is the worst-case distortion given by (7) so that $F(\xi)\,m$ is the pricing kernel under model uncertainty. We define r_f^{NoMU}, r_f, r_e^{NoMU} and r_e to be the continuously-compounded returns equivalent to R_f^{NoMU}, R_f, ER^{NoMU} and ER respectively: For example, $r_f = \log(R_f)$, $r_e = \log(1 + ER)$.

We use the following U.S. consumption data for 1891-1998 from Campbell [2003]:

$$\mu_c = 1.789\%, \qquad \sigma_c = 3.218\%, \qquad \sigma_S = 18.599\%.$$

Using (D.1) and (D.2), we compute the standard deviations of m and $F(\xi)\,m$, denoted $\sigma(m)$ and $\sigma(F(\xi)\,m)$, as well as r_f^{NoMU}, r_f, r_e^{NoMU} and r_e, all expressed as percentages, for different values of γ and Ω and display the results in Table D.1, where we set $\overline{\beta} = 0.9975$.

We see that the presence of model uncertainty lowers the risk-free rate, increases the excess return on the stock and increases the standard deviation of the pricing kernel (thus raising the Hansen and Jagannathan [1991] bound) relative to its absence. Further, these conclusions are very robust to the three different distributional assumptions. Note also that, while γ and Ω both, loosely speaking, quantify the representative agent's aversion to risk or uncertainty, their effect on the risk-free rate is very different: Raising γ significantly increases the risk-free rate while raising Ω significantly decreases it.

Table D.2: Estimates of Ω for different Model Confidence Levels (MCL).

MCL	80	80	80	93	93	93	94	94	94
Distribution	LN	3S	Hist	LN	3S	Hist	LN	3S	Hist
Ω	6.23	6.38	6.26	9.63	9.58	9.70	10.05	10.14	10.13

NOTES: Estimates of the degree of aversion to model uncertainty Ω for three different Model Confidence Levels (labeled MCL) - namely 80%, 93% and 94% - for the three different distributional assumptions (LN, 3S and Hist) on annual U.S. consumption growth for the period 1891-1998. The results for LN (log-normal) use 250,000 simulations.

For comparison with Table D.1, Campbell [2003] gives historical values for the U.S. for the period 1891-1998 of $r_f = 2.020\%$ and $r_e = 7.169\%$ - the latter being the excess return on the U.S. equity market. We compute numerically the values of γ and Ω which match these historical values (since the values in Table D.1 are quite close, we focus on the 3S case for simplicity) - now setting $\overline{\beta} = 1$ - and obtain

$$\gamma = 2.15, \quad \text{and} \quad \Omega = 9.85. \tag{D.3}$$

We observe that the value of the risk aversion parameter γ is not implausibly high - indeed it is well within the range of 1 to 5 which is usually (Barillas et al. [2009], Maenhout [2004] or chapter 21, Cochrane [2005]) considered reasonable. But what about the value of Ω? To estimate Ω, we simulate consumption data consistent with the parameters in equation (D.3) for a time period spanning 108 years, for each of the three distributional assumptions (LN, 3S and Hist). We estimate Ω for three different Model Confidence Levels (labeled MCL) - namely 80%, 93% and 94% - using the bootstrap procedure[23] described in Section 5. The results are in Table D.2.

We see that the value of $\Omega = 9.85$ in equation (D.3) estimated to match the historical values of $r_f = 2.020\%$ and $r_e = 7.169\%$ is consistent with Model Confidence Levels in the region of 93% or 94%. These Model Confidence Levels are high but not implausible and are, in fact, very close to those reported in Barillas et al. [2009] (who find detection error probabilities of around 5%).

We recompute the quantities in Table D.1 with the values of Ω (6.23 for LN, 6.38 for 3S, 6.26 for Hist) corresponding to Model Confidence Levels of 80% as well as the value of Ω in equation (D.3), setting $\overline{\beta} = 1$ and $\gamma = 2.15$, and report the results in Table D.3.

We see that Model Confidence Levels of 80% allow us to approximately triple the Hansen and Jagannathan [1991] bound $\sigma(m)/\mathbb{E}^{\mathbb{P}}[m]$ applicable in the absence of model uncertainty and takes $\sigma(F(\xi)\, m)/\mathbb{E}^{\mathbb{P}}[F(\xi)\, m]$ to around two-thirds of the value required to match the historical excess return of 7.169%. Hence, we find that, similarly to Barillas et al. [2009] and Maenhout [2004], model uncertainty can potentially explain the equity premium puzzle of Mehra and Prescott [1985] - at least in part and possibly in whole - without increasing the risk-free rate and without implausibly high values of risk aversion γ.

The mechanism through which $\sigma(F(\xi)\, m)$ is increased is displayed in Figure D.1. We use the Hist distributional assumption and plot the consumption values (labeled c) after one year, scaled

[23] More precisely, there are no basis assets in this problem so the procedure can be simplified. Using similar notation to (34) and (35), the procedure is as follows: For each bootstrapped sample, $k = 1, \ldots, K$, we compute $\Gamma^{\star(k)} \equiv \sum_{s=1}^{S} \pi_s^{(k)} [U(c_{1,s})]$. We sort the K values $\Gamma^{\star(k)}$ into order and select the percentile, $\Gamma^{\star MCL}$ say, corresponding to the chosen Model Confidence Level. We then set Ω to be the value which solves $\max_\eta \left\{ \sum_{s=1}^{S} \pi_s [G(\xi_s)(U(c_{1,s}) - \beta_s)] \right\} = \Gamma^{\star MCL}$.

Table D.3: Excess returns and risk-free rates for different values of Ω.

LN	γ	Ω	$\sigma(m)$	$\sigma(F(\xi)\,m)$	r_f^{NoMU}	r_f	r_e^{NoMU}	r_e
	2.15	6.23	6.70	22.42	3.50	2.68	1.43	4.69
	2.15	9.85	6.70	35.09	3.50	2.02	1.43	7.20
3S	γ	Ω	$\sigma(m)$	$\sigma(F(\xi)\,m)$	r_f^{NoMU}	r_f	r_e^{NoMU}	r_e
	2.15	6.38	6.68	22.88	3.50	2.65	1.43	4.76
	2.15	9.85	6.68	35.07	3.50	2.02	1.43	7.17
Hist	γ	Ω	$\sigma(m)$	$\sigma(F(\xi)\,m)$	r_f^{NoMU}	r_f	r_e^{NoMU}	r_e
	2.15	6.26	6.83	22.79	3.50	2.67	1.40	4.57
	2.15	9.85	6.83	35.53	3.50	2.02	1.40	6.99

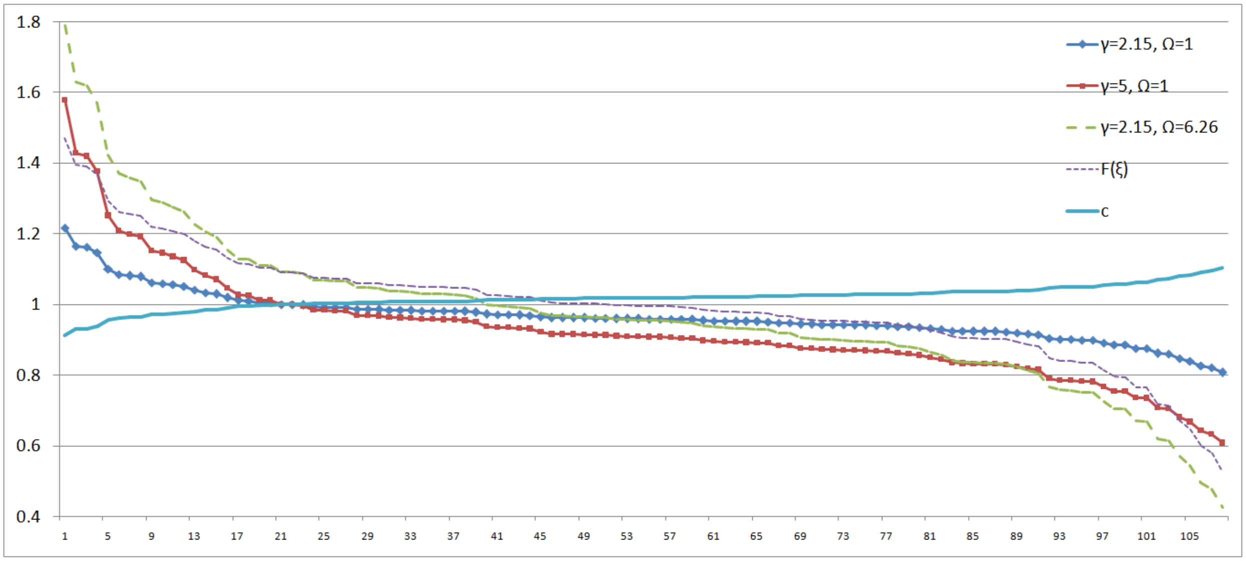
NOTES: $\overline{\beta}$ is set to 1. The first panel is for LN (log-normal) (using 250,000 simulations), the second for 3S and the third is for Hist (historical).

to start at one, in increasing order. Thus, the left-hand most value corresponds to the more than 9% fall in U.S. consumption in 1932. We also display the pricing kernel for each of the 108 states for the cases of $\{\gamma = 2.15, \Omega = 1\}$, $\{\gamma = 5, \Omega = 1\}$ and $\{\gamma = 2.15, \Omega = 6.26\}$ and in addition, for the final case, the value of $F(\xi)$, corresponding to the worst-case distortion. We see that, for $\gamma = 2.15$, introducing model uncertainty by increasing Ω from 1 to 6.26, significantly increases the pricing kernel in the state corresponding to 1932. In fact, the price of an Arrow-Debreu security which pays 100 dollars in that state (which could be interpreted as a catastrophe insurance contract or as, roughly speaking, a credit derivative paying a fixed amount in the event of widespread defaults) is increased from $100 \times (1/108) \times 1.2171 = 1.127$ to $100 \times (1/108) \times 1.7894 = 1.657$ dollars - an increase of nearly 50%. The increase is much more than that which occurs by raising γ from 2.15 to 5, with $\Omega = 1$. The large increase in the price of the Arrow-Debreu security or catastrophe insurance contract, under model uncertainty, brings sharply into focus the losses suffered during the 2008 financial crisis by, for example, AIG - who, in effect, were selling such contracts.

D.2 Variance and skewness risk premia

In this appendix, we build upon the analysis in Appendix D.1 and show that our model uncertainty framework is potentially able to explain variance and skewness risk premia in equity index options. Drechsler [2013] shows that model uncertainty can potentially explain variance risk premia but doesn't consider skewness risk premia. Kozhan, Neuberger, and Schneider [2011] show empirically that variance risk premia and skew risk premia derive from exposure to the same underlying risk factor in the sense that strategies designed to exploit one of the risk premia but to hedge out the other make zero excess returns. Both Drechsler [2013] (using a model uncertainty framework very similar to that of Hansen and Sargent [2008] - see also Liu et al. [2005]) and Kozhan et al. [2011] (who do not consider model uncertainty) develop equilibrium models, with Epstein and Zin [1989] preferences, which result in equity index or stock prices having dynamics which incorporate both jumps and stochastic volatility correlated with stock prices. In this appendix, we show that our model uncertainty framework is also potentially able to explain variance and skewness risk premia. However, our analysis differs significantly from that of Drechsler [2013] and Kozhan et al. [2011] in that (a) we do not need jumps or stochastic volatility in stock price dynamics, (b) we use CRRA (power) utility and (c) we work in a discrete-time one period economy.

Figure D.1: Pricing kernels under the Hist distributional assumption for different γ and Ω.



NOTES: Pricing kernels under the Hist distributional assumption for different values of γ and Ω. For the case $\{\gamma = 2.15, \Omega = 6.26\}$, $F(\xi)$, corresponding to the worst-case distortion, is also shown. The quantity labeled c is U.S. consumption after one year, scaled to start at one, in increasing order for the period 1891-1998.

Our economic framework is identical to that in Appendix D.1. We consider two distibutional assumptions, namely LN and Hist, for consumption growth. We use the value of γ estimated in (D.3) and values of Ω corresponding to no model uncertainty, an 80% Model Confidence Level (6.23 for LN and 6.26 for Hist) and that in (D.3) (matching the Campbell [2003] data-set). We set $\overline{\beta} = 1$.

We price stock options with a wide range of strikes and use these prices to compute the implied log-contract variance and the implied entropy-contract variance using the methodology of Carr and Wu [2009], Kozhan et al. [2011] and Britten-Jones and Neuberger [2000]. Kozhan et al. [2011] show that the implied entropy-contract variance minus the implied log-contract variance is, essentially, a model-independent measure of the slope of the implied volatility skew and hence a measure of the implied risk-neutral skew.

The difference between implied log-contract variance IV and realised variance RV is the variance risk premium and Carr and Wu [2009], Kozhan et al. [2011] and Neuberger [2012] show empirically that it is negative for equity index options. We consider two different measures of the variance risk premium. The first is $\sqrt{IV} - \sqrt{RV}$ while the second is LVRP $\equiv \log(IV/RV)$ and expresses, in continuously-compounded form, the excess return to holding a variance swap and receiving realised variance and paying implied. We use the annual data (since the Campbell [2003] data-set is annual), for S&P 500 options spanning the time period from December 1997 to September 2009, in Table 4 of Neuberger [2012] for comparison. The square root of the sample mean realised variance reported in his Table 4 is $\sqrt{0.0475} \approx 21.79\%$. We cannot hope to match that since the stock volatility in the Campbell [2003] data-set is $\sigma_S = 18.599\%$ i.e. rather lower. So we use $\sqrt{RV} = 21.79\%$ for the Neuberger [2012] data and $\sqrt{RV} = 18.599\%$ for our variance risk premia calculations. Finally, we compute the skewness risk premium (labeled Skew rp) using the methodology of Kozhan et al. [2011] and Neuberger [2012]. Our results are in Table D.4 where we also report the Neuberger [2012] data for comparison.

Table D.4: Comparisons of the variance risk premium, the skewness risk premium and the slope of the implied volatility skew (labeled Slope) with Neuberger [2012].

			LN				Hist		
	Neuberger	Ω	1	6.23	9.85	Ω	1	6.26	9.85
$\sqrt{IV} - \sqrt{RV}$ %	-1.66		-0.86	-1.13	-0.48		-0.49	-1.44	-1.37
LVRP %	-14.66		-9.02	-11.78	-5.08		-5.24	-14.91	-14.26
Slope %	-1.09		-0.23	-0.41	-0.47		-0.40	-0.70	-0.83
Skew rp %	0.34		0.21	0.39	0.45		0.33	0.63	0.76

NOTES: $\gamma = 2.15$, $\overline{\beta} = 1$. The results for LN (log-normal) use 250,000 simulations.

We see that, compared with the case $\Omega = 1$, increasing Ω above 1 increases the variance risk premium $\sqrt{IV} - \sqrt{RV}$, the log variance risk premium LVRP, the skewness risk premium and the implied risk-neutral skew. In the absence of model uncertainty, the variance risk premium $\sqrt{IV} - \sqrt{RV}$ is around a quarter to a half that observed historically while the slope of the implied volatility skew is around a fifth to a third of that observed historically. A value of Ω corresponding to an 80% Model Confidence Level, increases $\sqrt{IV} - \sqrt{RV}$ and LVRP to almost the levels observed historically in Neuberger [2012] and approximately doubles the slope of the implied volatility skew. The Hist distributional assumption is closer to the Neuberger [2012] historical values than LN. Both distributional assumptions result in a skewness risk premium that is of the correct sign and order of magnitude.

Thus, we see that our model uncertainty framework is potentially able to explain variance and skewness risk premia in equity index options. We acknowledge, however, that Carr and Wu [2009], Kozhan et al. [2011] and Neuberger [2012] show that there is an appreciable amount of time-series variation in these risk premia which, of course, we are not able to match with our very simple consumption dynamics. Nevertheless, using the same parameters which explain approximately two-thirds of the equity premium puzzle, we can also explain approximately two-thirds of the slope of the implied volatility skew and nearly all of the variance and skewness risk premia.

D.3 Limited participation in equity markets

In this appendix, we show that our model uncertainty framework, whilst differing from that of Maenhout [2004], is like his (see also Cao et al. [2005]), potentially able to explain limited participation in equity markets i.e. why investors, in practice, hold smaller portfolio fractions in risky assets than classical (Merton [1971]) portfolio theory would suggest is optimal.

We consider a one period economy with a single risky asset and a one period risk-free bond. The investor solves problem (11) in order to choose what fraction of her initial wealth 1 she should invest in the risky asset. Matching the Campbell [2003] data-set used in Appendix D.1, we assume that the risky asset has expected excess return $r_e = 7.169\%$, volatility $\sigma_S = 18.599\%$ and that the risk-free rate $r_f = 2.020\%$.

For illustration, we make two assumptions about the investor's utility function $U(V)$: (I) exponential with absolute risk aversion B for $B \in \{1, 2, 5\}$ and (II) CRRA with CRRA coefficient $\gamma \in \{2, 2.15, 5\}$ (the middle value 2.15 being that estimated in equation (D.3)).

In order to establish appropriate values of Ω, we simulate risky asset returns consistent with $r_e = 7.169\%$ and $\sigma_S = 18.599\%$ for a time period spanning 108 years. Similarly to Appendix D.1,

Table D.5: Exponential utility: Fraction of investor's initial wealth 1 invested in the risky asset as a function of Ω and B.

	LN				3S				
Ω	1	1.86	5	9	1	2.08	5	9	$r_e/B\sigma_S^2$
B									
1	2.01	1.39	0.44	0.23	1.67	1.16	0.44	0.23	2.07
2	1.00	0.70	0.22	0.11	0.84	0.58	0.22	0.11	1.04
5	0.40	0.28	0.09	0.05	0.33	0.23	0.09	0.05	0.41

NOTES: Exponential utility $U(V) = -\overline{\beta}\exp(-BV)$.

Table D.6: CRRA utility: Fraction of investor's initial wealth 1 invested in the risky asset as a function of Ω and γ.

Ω	1	1.50	1.75	2.00	2.50	5	9
γ							
2	0.85	0.79	0.75	0.70	0.61	0.35	0.20
2.15	0.79	0.73	0.69	0.64	0.56	0.31	0.18
5	0.34	0.31	0.28	0.26	0.21	0.11	0.06

NOTES: CRRA utility $U(V) = \overline{\beta}\frac{V^{1-\gamma}}{1-\gamma}$.

we make two possible distributional assumptions about the price of the risky asset. In the first (labeled LN), we assume it has a log-normal distribution, and in the second (labeled 3S), we assume it can, starting at S_0, take on one of the three values $S_0\exp(\lambda\sigma_S\sqrt{\Delta t})$, S_0 or $S_0\exp(-\lambda\sigma_S\sqrt{\Delta t})$, where $\lambda = \sqrt{(3/2)}$ and $\Delta t = 1$, with probabilities which match r_e and σ_S and sum to 1. We then use the bootstrap procedure described in Section 5, with a Model Confidence Level of 80%, to estimate Ω. We find, for the exponential case, values of Ω of 1.86 for LN and 2.08 for 3S. This leads us to consider values of $\Omega \in \{1, 1.86, 2.08, 5, 9\}$ for the exponential case. For the CRRA utility case, for brevity, we focus on the 3S distributional assumption and simply choose a range of values of Ω. The results are reported in Tables D.5 and D.6. In Table D.5, we also, for comparison, report the fraction of initial wealth given by $r_e/B\sigma_S^2$ which is applicable (p154, Cochrane [2005]) for exponential utility under the assumption of normally distributed risky asset returns.

Fractions of initial wealth greater than 1 indicate that the investor takes a leveraged position in the risky asset financed by selling short the one period risk-free bond. We see that increasing values of Ω leads to a smaller position in the risky asset and hence a correspondingly larger (or less negative) position in the risk-free bond. These results are in line with those in Maenhout [2004] and Uppal and Wang [2003] and are, of course, consistent with model uncertainty increasing the investor's effective risk aversion. For the exponential case, we see that a Model Confidence Level of 80% leads to estimates for Ω (1.86 and 2.08) which reduce the fraction of initial wealth invested in the risky asset to about 70% of that in the absence of model uncertainty (corresponding to $\Omega = 1$).

E The Heston [1993] lattice construction

In this appendix, we detail the construction of the Heston [1993] lattice. The lattice is a pyramid with one dimension capturing the evolution of the instantaneous variance V_t and the other capturing the evolution of the fx rate S_t. It is essentially septanomial in each dimension (except on the edges of the pyramid in the V_t dimension) which gives more degrees of freedom to approximate a continuous-time stochastic volatility process compared with, say, trinomial branching. We denote by V_0 the time 0 instantaneous variance and by Δt the time in years corresponding to one working day.

At each time-step t ($t = 0, 1, \ldots, 44$) of the lattice, at node j, when the instantaneous variance is $V_{t,j}$, the instantaneous variance can essentially change to $V_{t,j} + \lambda_1 \zeta \sqrt{V_0 \Delta t} Z^1_{t,j}$, for $Z^1_{t,j}$ an integer satisfying $-3 \leq Z^1_{t,j} \leq 3$ and $\lambda_1 = \sqrt{(3/2)}/2$. However, the lattice is also pruned in the V_t dimension, which means that the lattice is not grown further if V_t would either go below zero or above $2V_0$. Hence, $Z^1_{t,j}$ is restricted to the range $[Z^1_{t,\min}, Z^1_{t,\max}]$ where $-3 \leq Z^1_{t,\min} \leq 0$, $0 \leq Z^1_{t,\max} \leq 3$. The probabilities, under $\mathbb{P}$, of transitions in the V_t dimension solve the linear program:

$$\min \pi\left(Z^1_{t,k} = 0\right) \quad \text{such that} \quad \Bigg\{ \text{for each } k = Z^1_{t,\min}, \ldots, Z^1_{t,\max}, \;\; \pi\left(Z^1_{t,k}\right) \geq 0.005 \text{ and}$$

$$\sum_{k=Z^1_{t,\min}}^{Z^1_{t,\max}} \pi\left(Z^1_{t,k}\right) = 1, \quad \sum_{k=Z^1_{t,\min}}^{Z^1_{t,\max}} \pi(Z^1_{t,k}) \left(V_{t,j} + \lambda_1 \zeta \sqrt{V_0 \Delta t} Z^1_{t,k}\right) = \theta + \left(V_{t,j} - \theta\right) e^{-\kappa \Delta t},$$

$$\sum_{k=Z^1_{t,\min}}^{Z^1_{t,\max}} \pi\left(Z^1_{t,k}\right) \left(V_{t,j} + \lambda_1 \zeta \sqrt{V_0 \Delta t} Z^1_{t,k} - \left(\theta + (V_{t,j} - \theta) e^{-\kappa \Delta t}\right)\right)^2 = \zeta^2 V^2_{t,j} \Delta t \quad \Bigg\}.$$

At each time-step t, at node ℓ in the spot dimension, when the the spot fx rate is $S_{t,\ell}$, it can change to $S_{t,\ell} \exp(\lambda_2 \sqrt{V_0 \Delta t} Z^2_{t,k})$, for $Z^2_{t,k}$ an integer (indexing the 7 branching possibilities) taking the values $-3, \ldots, 3$ and $\lambda_2 = \sqrt{(3/2)}/2$. The probabilities, under $\mathbb{P}$, of transitions in the S_t dimension solve the linear program:

$$\min \pi\left(Z^2_{t,k} = 0\right) \quad \text{such that} \quad \Bigg\{ \text{for each } k = -3, \ldots, 3, \;\; \pi\left(Z^2_{t,k}\right) \geq 0.005 \text{ and}$$

$$\sum_{k=-3}^{3} \pi\left(Z^2_{t,k}\right) = 1, \quad \sum_{k=-3}^{3} \pi\left(Z^2_{t,k}\right) S_{t,\ell} \exp\left(\lambda_2 \sqrt{V_0 \Delta t} Z^2_{t,k}\right) = S_{t,\ell}\; e^{(r_d - r_f + \Upsilon \sqrt{V_{t,j}}) \Delta t},$$

$$\sum_{k=-3}^{3} \pi\left(Z^2_{t,k}\right) \left(\lambda_2 \sqrt{V_0 \Delta t} Z^2_{t,k}\right)^2 = V^2_{t,j} \Delta t + \left((r_d - r_f + \Upsilon \sqrt{V_{t,j}} - \frac{1}{2} V_{t,j}) \Delta t\right)^2 \quad \Bigg\}.$$

The probabilities sum to one and match the conditional expectation and variance of $V_{t,j}$ or the conditional expectation of $S_{t,\ell}$ and second moment of $\log S_{t,\ell}$. The objectives, $\min \pi\left(Z^1_{t,k} = 0\right)$ and $\min \pi\left(Z^2_{t,k} = 0\right)$, are somewhat arbitrary choices and say that the linear programs minimise the "stay-the-same" probabilities. The conditions $\pi\left(Z^1_{t,k}\right) \geq 0.005$, $\pi\left(Z^2_{t,k}\right) \geq 0.005$ force all the probabilities to be strictly positive but the choice of 0.005 is, again, essentially arbitrary.

To see that this lattice can accurately approximate a Heston [1993] process, we compute the risk-neutral price of the BCON option (meaning with Υ set to zero and the same values of κ, θ and ζ and the same placements of lattice nodes) and get a price of 0.48296. The price of the same option computed using the Fourier method of Heston [1993] is 0.48298 - a negligible difference.