Slow Capital Movement and Asset Pricing Puzzles

Abstract

We consider an economy where investors trade the stock market to which they can allocate only with delays. We find a linear equilibrium in this economy and calibrate the slow capital movement by using a typical portfolio turnover observed empirically. Then we show that the risk premium and volatility of the stock market returns are many times bigger than in the economy with no delays in capital allocations. Meantime, the difference between the volatilities of aggregate consumption in the two economies is many times smaller than the corresponding difference between the volatilities of the stock market returns. It follows that the presence of delays in capital allocation can help to solve the risk premium puzzle and the excess volatility puzzle. Furthermore, in agreement with empirical literature, our model predicts the stock price reversals at short horizons.

Keywords: General Equilibrium; Asset Pricing; Slow Capital Movement

JEL Classification: G11, G12

1 Introduction

It is known that most of the models aiming to explain asset pricing puzzles assume a representative agent framework. A shortcoming of this framework is that it does not capture trading between investors in the stock market. In particular, this framework cannot explain a typical pattern of trading observed in the market when a lot of investors sell when the market is falling and buy when it is on the rise. Intuitively, these strategies should generate high volatility of the stock returns which in turn should result in a significant risk premium for a reasonable choice of coefficients of risk aversions of investors. Moreover, models which allow aggregation to the representative agent suggest that a marginal investor takes an advantage of substitution effect and buys (sells) securities which prices are falling (rising). Consequently, volatilities of securities are very small and the risk premium is significant only if the risk aversion of investors is unrealistically high. In this paper we suggest a model in which a marginal investor with a small risk aversion trades in the direction of the market movement resulting in a very high volatility of the market returns and a very high risk premium.

Marginal investor in our economy trades in the direction of the market due to the delays in capital allocations. It is known that allocation of a capital to an investment opportunity can only be made with a delay which may last from a few seconds to a few weeks or even months. The reasons for delays in allocations could be various and include but not limited to the costs of search for trading counterparties, time to raise capital by intermediaries, opportunity costs from time delays caused by valuable alternative activities, difficulties in borrowing, and behavioral. See Duffie (2010) for more examples and causes of slow moving capital in a few sectors of financial markets. Furthermore, Duffie and Strulovichi (2012) provide a model of the capital mobility and study the impact on the capital mobility of search costs, discounting, asset volatility, and other parameters. The related studies are also concerned with understanding of the nature of slow capital movement.[1] We take the presence of slow capital movement for granted and study how the capital inertia affects the risk premium and volatility of the stock market. Our conclusions could be used to explain the risk premium puzzle and the excess volatility puzzle. See the reviews by LeRoy (2006) and Mehra (2008) for the detailed lists of studies towards resolving the excess volatility puzzle and the risk premium puzzle, respectively. To the best of our knowledge, none of these studies explain these puzzles based on a slow capital movement.

We make capital move slowly by using adjustment costs that an investor faces for changing her allocations to risky securities. The adjustment costs are modeled by convex trading costs. These costs assume that the bigger a trade the higher cost per share will be. Moreover, because we treat the economy in continuous time, the convexity in the number of shares is converted into the convexity in the rate of share trading. Consequently, an investor endogenously chooses to trade a finite volume per time interval and is not able to adjust

[1] See, for example, Mitchell, Pedersen, and Pulvino (2007), Acharya, Shin, and Yorulmazer (2009), Malliaris and Yan (2012), and Fuchs, Green, and Papanikolaou (2013).

her allocations quickly enough in response to new information. The approach of modeling trading delays in continuous time by means of convex trading costs is used by Almgren (2003), Isaenko (2010, 2015), Rogers and Singh (2010), Vayanos and Woolley (2013), and by practitioners [see Grinold and Kahn (2000)]. Convex trading costs are also used in Heaton and Lucas (1996) and Garleanu and Pedersen (2012) in the discrete time settings.

We consider competitive investors with CARA preferences who receive non–hedgeable endowments and trade in the stock market while facing delays in their allocations. Investors can also borrow and lend at a fixed rate by using an exogenous production technology. We assume that investors could be heterogeneous in their coefficients of risk aversions, patience, endowments, and accesses to the capital mobility of the stock market. Endowments are essential for creating trading incentives for investors and could be representing labor income, or income from business and real estate projects. We find a linear equilibrium in the stock market which is defined up to the coefficients that solve the system of algebraic equations. These equations are reduced to one nonlinear equation for one unknown coefficient. Then we assume a simple basic calibration where investors experiencing endowment shocks trade with investors with no endowment. The former trade to diversify these shocks while the later trade to take advantage of investment opportunities. All investors under our calibration have the same risk aversion and the same access to mobility of the stock market.

Asset pricing puzzles are usually tackled within models in which investors have CRRA preferences. Such models offer a convenience in calibration. We assume investors who have CARA preferences rendering the calibration to be less straightforward. We choose a calibration of the exogenous dividend and endowment processes such that the Sharpe ratio in the economy with no capital inertia is close to this ratio predicted by the model with investors having CRRA preferences and facing no capital inertia. Moreover, we choose the coefficient in trading costs such that the portfolio turnover in the economy with trading delays is equal to its historical average value observed in the US stock market.

In the economy with capital inertia investors with endowments are willing to give up some of their returns to be able to hedge endowment shocks. Consequently, investors with no endowment become marginal. Let us assume that investors are receiving bad news leading to a decrease of prices. The marginal investor can buy securities whose Sharpe ratio is increasing and enjoy a substitution effect or she can sell securities and reduce her losses on the current positions. She ignores the benefits from the substitution effect and sells securities. The marginal investor sells securities with decreasing price and buy those with increasing price causing the volatility of returns to increase dramatically in the market with a substantial capital inertia. Moreover, trading costs and an excessive volatility of stock returns are compensated by a dramatic rise of the risk premium of the stock market returns. In particular, we find that making capital move slowly results in more than a thirtyfold increase of the risk premium and the volatility of the stock market returns.

Receiving the income from trading in the stock market with a much higher volatility of returns makes the consumption of investors more volatile. Interestingly, the comparison of the markets with and without capital inertia shows that the volatility of the aggregate consumption increases many times less than the volatility of the stock market returns. The aggregate consumption is much more smoother than the stock price due to investors' hedging of their consumption with the respect to the endowment shocks affecting their portfolios.

The presence of delays in allocation to the stock market can be related to the illiquidity of this market. It was argued by Amihud and Mendelson (1986) and and Longstaff (2009). that investors who are less patient will favor liquid securities more than those who are more patient. This is so–called "clientele effect". We find that in the setting of a stationary economy, where investors have very long investment horizon, less patient investors do not hold more liquid securities than more patient investors do. While impatience of an investor does affect her utility function, it does not matter for her trading strategy and, therefore, for the stock prices.

The risk premium and excess volatility puzzles come together with the risk–free rate puzzle according to which the theoretical risk–free rate turns out to be much bigger than that observed historically. While we assume that the risk–free rate is exogenous, it is easy to predict the trends in this rate should the risk–free rate be endogenous. If the capital mobility gets poor, the stock market becomes less attractive for investors resulting in a decrease of its price and an increase of its risk premium. In the meantime, the demand for the locally risk–free security rises causing the risk–free rate to become smaller. We expect that the risk–free rate would be very small given a dramatic increase of the risk premiums in the markets with slow moving capital.

The presence of capital delays makes investors consider future trading opportunities explicitly. For example, marginal investors will start buying securities even if bad news continue to arrive given that they expect that good news are around the corner. Consequently, the presence of capital inertia makes price changes precede the expected news. That is, if there is a positive probability of a news arrival in the future, investors correct prices accordingly in advance. The latter trait is often observed in practice. We call this trait "news lagging" and measure it by introducing a predictable shock into the economy. News lagging is absent in the economy with no capital inertia in which investors adjust their portfolios instantly and prices move with news simultaneously even if the news were expected. Besides news lagging, delays in trading imply that the stock prices do not mirror good or bad news like they do in the economy with no capital inertia, instead they overreact to these news. It is common to relate the price overreaction to the negative sign of the autocorrelation coefficient of the stock returns. See, for example, Lehmann (1990) and Jegadeesh (1990). We find this relation inaccurate and call a price overreaction the situation when the price amplifies news and then adjusts to the level corresponding to this news. It is measured by introducing a sudden shock into the economy and detecting the stock price response to this shock. Finally we show that, while the price overreaction and news lagging are absent, the autocorrelation correlation between the stock returns is negative in the economy with no capital delays. Meantime, in the economy with capital delays price overreaction and news lagging add to the negativity of the autocorrelation coefficient but almost exclusively only in the short run. The last result is consistent with empirical literature.[2]

We carry a sensitivity analysis of our results with respect to changes in the capital inertia while assuming that it is the same for both agents, then we do it with respect to the heterogeneity in the capital inertia across the two agents and with respect to changes in the volatility of endowment. We find that adding heterogeneity in access to the capital mobility across investors has a very strong impact on the risk premium and volatility and allows more flexibility in adjusting these outcomes. Moreover, assuming a better access to the capital mobility of investors with no endowment leads to even more dramatic increase of the risk premium and volatility of the stock market returns. We also find that the main conclusions of our study remain intact for the whole range of values of the endowment volatility for which the linear equilibrium exists. Furthermore, we find that the heterogeneity in the access to the capital mobility translates into a very strong impact of this mobility on the predictability of the stock returns. Both the strength and the time decay of this impact can be controlled by changing the capital mobility of each investor. In particular, if an investor without endowment has a better capital mobility than the other investor then the half–life of the autocorrelation coefficient is always smaller than two and a half weeks.

Our study is related to the paper by Lo, Mamaysky, and Wang (2004) who find a general equilibrium in the stock market where investors pay fixed transaction costs. The trading needs of investors emerges from endowment shocks. The authors show that the fixed transaction costs may result in a significant liquidity premium. Our paper extends this result by showing that a combination of endowment shocks and slow moving capital leads to a dramatic increase in volatility of the stock market making investors demand much higher risk premium than in the market without capital delays.

Finally, our article contributes to the list of papers studying general equilibrium in the presence of trading costs. Besides the papers quoted above, this list includes but is not limited to studies by Vayanos (1998) and Buss and Dumas (2013, 2015). These papers consider trading of risky securities in the presence of proportional transaction costs. They report that transaction costs have a noticeable impact on trading strategies but only minor effects on prices.

The rest of the article is organized as follows. Section 2 describes the model. Section 3 presents the solutions and discussions of the equilibria in economies with and without delays in capital allocations. Section 4 concludes. Appendix A describes a solution of an equilibrium with a capital inertia and proves the results of section 2. The solution of the equilibrium with no capital inertia is found in Appendix B.

[2]See, for example, Lehmann (1990), Jegadeesh (1990) for the early documented results and Collin-Dufresne and Daniel (2015) for the latest related findings.

2 Economic Setting

We consider a markovian economy with an infinite horizon. We assume a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$, with uncertainty in the model being generated by a standard 3–dimensional Brownian motion (W_0, W_Y, W_D) which is also adapted.

2.1 Financial Securities

There is a stock market that follows the processes

$$dS_t + D_t dt = \mu_{St} dt + \sigma_{S_D t}\, dW_{Dt} + \sigma_{S_Y t}\, dW_{Yt}, \tag{1}$$

where W_Y, and W_D have a constant correlation coefficient ρ_{DY}. Stock market pays a dividend flow D that follows an exogenous process

$$dD_t = \kappa_D(\bar{D} - D_t)dt + \sigma_D dW_{Dt}, \quad D(0) = D_0, \tag{2}$$

where κ_D, $\bar{D}$, σ_D are positive constants. Investors can borrow and lend at a constant positive interest rate r.

2.2 Economic Agents

We assume two types of competitive investors who are different in their risk aversions, patience, endowments, and access to capital mobility. Investors of different types trade with each other the stock market and face trading costs. In particular, trading ΔN_n shares of the stock by investor n $(n = 1, 2)$ within a small time interval Δt imposes trading losses on her equal to $\alpha_n(u_n)^2\Delta t$ dollars, where α_n is constant and u_n is an average rate of trading by investor n within this time interval. For simplicity of exposition, we require that coefficient α_n be the same for purchasing and selling the stock. The trading costs do not allow the rate of trading to be infinite due to their convex shape. Naturally, share holding of the stock by investor n, becomes absolutely continuous and can be written as

$$dN_{nt} = u_{nt} dt, \quad n = 1, 2. \tag{3}$$

We conclude that investors change their allocations slowly (at a finite rate). The latter implies that the rate of trading (or trading volume) becomes a control variable of an investor while an allocation becomes her state variable. The convex trading costs are appealing since they endogenize a slow movement of capital. In this regard, they are similar to the adjustment costs that are used in the literature on production economy to prevent firms from changing their capital stock too quickly.

Following Lo, Mamaysky, and Wang (2004), we assume that investor n receives a non-tradable cumulative endowment $I_t = \int_0^t Y_{ns} dW_{0s}$ where $Y_n = \beta_n Y$ and Y follows the process

$$dY_t = \kappa_Y(\bar{Y} - Y_t)dt + \sigma_Y dW_{Yt}, \quad Y(0) = Y_0,$$

in which β_n, κ_Y, $\bar{Y}$, and σ_Y are constant. We require that κ_Y and σ_Y be positive. The correlation coefficients of W_0 with W_Y and W_D, are constant and given by ρ_{0Y} and ρ_{0D}, respectively. Clearly, an endowment can be negative due, for example, a negative shock in business enterprizes. The presence of stochastic endowment is essential to make investors trade dynamically in the long run. Finally, we will call process Y as an endowment volatility.

2.3 Portfolio Choice by an Investor

Let us consider an optimization problem faced by an investor of type $n \in \{1, 2\}$. She maximizes her expected CARA utility function that supports intermediate consumption, subject to the dynamic budget constraint:

$$\max_{c_n \in R^+, u_n \in R} E_0 \int_0^\infty \left(-\frac{1}{\gamma_n} \exp(-\tau_n t - \gamma_n c_{nt}) \right) dt \tag{4}$$

$$\begin{aligned} dX_{nt} &= \left[rX_{nt} + N_{nt}(\mu_{St} - rS_t) - c_{nt} - \alpha_n (u_{nt})^2 \right] dt + Y_{nt} dW_{0t} \\ &+ N_{nt}(\sigma_{S_D t} dW_{Dt} + \sigma_{S_Y t} dW_{Yt}), \end{aligned} \tag{5}$$

where $\gamma_n > 0$ is a coefficient of absolute risk aversion, τ_n is a time discount rate, u_n and X_n, stand for an investor's rate of trading and wealth.

The problem faced by an investor can be solved only in the dynamic programming approach. For tractability, we assume that each investor of a given type has the same initial allocation to the stock. It implies that the rates of trading of the stock are identical across all investors of a given type who also have identical allocations to this security at any time. It follows from equation (5) that the state variables for an investor's problem should include X_n, D, Y, and N_n. We formulate the dynamic programming problem in Appendix A . Solving this problem will provide us with the indirect utility function of an investor $V^n(t, X_n, N_n, Y)$. We show in Appendix A that $V^n(t, X_n, N_n, Y) = -\frac{1}{\gamma_n r} \exp[-\tau_n t - \gamma_n r X_n + g^n(N_n, Y)]$ and the trading rate of an investor is given by

$$u_n(N_n, Y) = -\frac{g^n_{N_n}(N_n, Y)}{2\alpha_n r \gamma_n}. \tag{6}$$

We describe function g^n in details in the section below. The presence of transaction costs forces an investor to be in a state in which her expected utility is not maximal for a given value of Y. Therefore, an investor would like to change her allocations even in the absence of information shocks in the economy.

3 Equilibria

In this section we define and find Radner equilibrium in the economy with no capital inertia and in the economy with such inertia.[3] We find the equilibria in the linear form and then

[3]The choice of Radner equilibria imposes severe restrictions on the predictions of the model. In particular, it results in the efficient prices which in real markets prevail only approximately in their best. This implication

use a calibration below to determine their characteristics.

3.1 Calibration

We choose the following calibration of the model unless pointed otherwise: $\gamma_1 = 5$, $\gamma_2 = 5$, $\beta_1 = 2.0$, $\beta_2 = 0.0$, $r = 0.03$, $\tau_1 = \tau_2 = 0.05$, $\alpha_1 = \alpha_2 = 3.0$, $\kappa_Y = 0.5$, $\bar{Y} = 0$, $\sigma_Y = 0.75$, $\kappa_D = 0.5$, $\bar{D} = 1$, $\sigma_D = 1.25$, $\rho_{DY} = 0$, $\rho_{0D} = 0$, $\rho_{0Y} = 0$. Coefficients α_1 and α_2 will be denoted as α since they are the same. This calibration is applied after a general solution of an equilibrium is found.

Traditionally, the asset pricing puzzles are tackled in the assumption that investors have CRRA utility functions. These utility functions allow a straightforward calibration of the risk aversion and dividend returns. We choose CARA utility function for the purpose of computational convenience. Unfortunately this choice renders the returns to be measured in dollars which complicates calibration. Therefore, we apply the following approach in choosing our parameters. It is known that in the pure exchange economy with a CRRA representative agent the risk premium, the volatility of the stock return, and the Sharpe ratio are given by $\gamma_R\sigma_C^2$, σ_C, and $\gamma_R\sigma_C$, respectively, where γ_R is the coefficient of relative risk aversion of the representative agent and σ_C is the volatility of the aggregate consumption rate. Since the historical values for σ_C is of the order of a few percent, the risk premium and the volatility of stock returns are too small at a reasonable choice of γ_R, while the Sharpe ratio is of the same order of magnitude as its historical value. Hence, we choose the coefficients of absolute risk aversions, betas, the volatilities of dividends, and an endowment such that the Sharpe ratio in the economy with no capital inertia is very close to that found in the model with a CRRA representative agent. Furthermore, in the agreement with the predictions of the latter model we assume that the resulting risk premium and the volatility of stock returns are very low in comparison to their historical counterparts.

Parameter α defining the delays in capital allocations is determined by matching the average portfolio turnover predicted by our model with a typical portfolio turnover observed historically. Moreover, our choice for the values of correlation coefficients ρ_{DY}, ρ_{0D}, and ρ_{0Y} is justified by their low historical values. Finally, we will mitigate the restrictions imposed by a chosen calibration by carrying a sensitivity analysis with respect to the following key parameters for the stock market returns: α, σ_Y, and the difference between α_1 and α_2.

3.2 Equilibrium with no Allocations Delays

As a benchmark case, we first consider an equilibrium in economy with no trading costs so that investors can allocate their capital instantly. In this economy an investor controls her exposure to the risky security, N_n, rather than the trading volume, u_n.

is particularly important in the context of trading incentives of investors who often trade due to price inefficiently in the real markets. The latter shortcoming could be partially mitigated by assuming that the endowment received by investors results from taking advantages in inefficient markets.

Definition 1 *An equilibrium is a price system* $(\mu_S, \sigma_{S_D}, \sigma_{S_Y})$ *and a set of trading strategies* (N_1, N_2) *of investors of type 1 and 2, respectively, such that (i) individual agents choose their optimal portfolio strategies and (ii) the stock market clears, that is* $\forall t \in [0, \infty)$

$$N_{1t} + N_{2t} = 1. \tag{7}$$

Appendix B proves the following theorem and corollary 2 that describe the equilibrium.

Theorem 1 *Assume that coefficient* H_2 *defined below is positive and coefficient* ρ_{DY} *is nonnegative. Let also equation (B-26) has a solution given by* C^{1*}*. Moreover, assume that matrix* $\boldsymbol{\omega}$ *defined by equations (B-32) and (B-33) is not singular. Then a linear equilibrium exists with the utility functions of investors, trading strategies and prices given by*

$$g^n = A^n + B^n Y + C^n Y^2, \tag{8}$$

$$N_n = G_1^n + G_2^n Y, \tag{9}$$

$$c_n = -\frac{1}{\gamma_n}[\ln(r) + A^n + B^n Y + C^n Y^2] + rX_n, \tag{10}$$

$$S = \frac{D}{r + \kappa_D} + H_0 + H_2 Y, \tag{11}$$

$$\mu_S - rS = R_0 + R_2 Y, \tag{12}$$

$$\sigma_{S_D} = \frac{\sigma_D}{r + \kappa_D}, \tag{13}$$

$$\sigma_{S_Y} = \sigma_Y H_2, \tag{14}$$

where the coefficients are given by $C^1 = C^{1*}$ *and by equations (B-8), (B-9),(B-14),(B-21),(B-23), (B-27)–(B-30).*[4][5]

Despite that investors are forward looking, their spot allocations depend only on the current level of the endowment volatility due to the ability of investors to react instantly to trading opportunities. Moreover, the stock returns follow 2–factor model with the factors being the dividend rate and the endowment volatility. Using the calibration introduced above we find the equilibrium coefficients: $A^1 = -2.9107$, $A^2 = 25.2143$, $B^1 = B^2 = 0.3311$, $C^1 = -0.2789$, $C^2 = 1.2211$, $G_1^1 = G_1^2 = 0.5$, $G_2^1 = -G_2^2 = -1.5028$, $H_0 = 17.5408$, $H_2 = 4.4140$, $R_0 = 0.4172$, $R_2 = -2.3394$, implying that $\sigma_{S_D} = 2.3585$ and $\sigma_{S_Y} = 3.3105$.

In the economy with no allocation delays investor 1 trades to hedge shocks in the endowment volatility and to share the risk while investor 2 seeks the risk sharing only. Let us suppose that Y is negative and undergoes a negative shock. It follows from the values of the found coefficients R_0 and R_1 that the risk premium of the stock market is positive and increases. Here investor 1 has a choice of buying the stock since it is getting cheaper or

[4] These equations should be applied to find the coefficients in the following order: C^1, C^2, H_2, G_2^1, G_2^2, B^1, B^2, G_1^1, G_1^2, R_0, H_0, R_2, A^1, A^2.

[5] There exists two equivalent solutions to the equilibrium problem. The solutions are different by the signs of coefficients in front of process Y. We choose the shown solution to make the stock price and endowment process procyclical.

selling the stock to avoid losses on her current holdings. Investor 1 prefers to get advantage of substitution effect and increases her holdings. That is, the higher the risk premium of the stock, the longer the position of investor 1 in this security. On the other end of the trade, investor 2 cares more about the value of her current holdings and sells shares when their Sharpe ratio increases. The competition between the two investors makes the price move with low volatility.

We define volatility of the aggregate consumption rate, $C = c_1 + c_2$, as

$$\sigma_C = \sqrt{\sigma_{C0}^2 + \sigma_{CD}^2 + \sigma_{CY}^2 + 2\sigma_{CD}\sigma_{C0}\rho_{0D} + 2\sigma_{CY}\sigma_{C0}\rho_{0Y} + 2\sigma_{CY}\sigma_{CD}\rho_{DY}}, \tag{15}$$

where

$$dC = \mu_C dt + \sigma_{C0} dW_0 + \sigma_{CD} dW_D + \sigma_{CY} dW_Y. \tag{16}$$

Theorem 1 implies

$$\sigma_{C0} = r(\beta_1 + \beta_2)Y, \tag{17}$$

$$\sigma_{CD} = \frac{r\sigma_D}{r + \kappa_D}, \tag{18}$$

$$\sigma_{CY} = [rH_2 - \frac{1}{\gamma_1}(B^1 + 2C^1 Y) - \frac{1}{\gamma_2}(B^2 + 2C^2 Y)]\sigma_Y. \tag{19}$$

It follows from equations (10) and (19) that investors' consumption includes a fixed proportion of their current wealth and that they hedge their exposure to the endowment shocks rising from their position in the stock market.

Based on Theorem 1 and equations (16)–(19) we derive

Corollary 1 *The long–term risk premium, volatility, and the Sharpe ratio of the stock market as well as the long–term volatility of the aggregate consumption rate are*

$$\lim_{t\to\infty} E_0(\mu_{St} - rS_t) = R_0 + R_2\bar{Y} = 0.42, \tag{20}$$

$$\sigma_S = \sqrt{\frac{\sigma_D^2}{(r+\kappa_D)^2} + (H_2)^2\sigma_Y^2} = 4.06, \tag{21}$$

$$\lim_{t\to\infty}(S_r) = \frac{R_0 + R_2\bar{Y}}{\sigma_S} = 0.10, \tag{22}$$

$$\lim_{t\to\infty} E_0(\sigma_{Ct}) = 0.195. \tag{23}$$

Notice that result (23) is found by using Monte–Carlo simulations.

For the purpose of comparison, we also derive the autocorrelation coefficient between the stock price change for time interval $[t, t+dt]$ and the stock price change for time interval $[t', t'+dt']$, where $t' \geq t + dt$ and dt, dt' are small:

Corollary 2 *The autocorrelation coefficient between return at time t and return at time t' is*

$$\frac{Cov_t(dS_t, dS_{t'})}{\sigma_S^2 dt dt'} = -\left[\frac{\kappa_D \sigma_D \exp[-\kappa_D(t'-t)]}{r+\kappa_D}\left(\frac{\sigma_D}{r+\kappa_D} + \rho_{DY} H_2 \sigma_Y\right)\right. \tag{24}$$

$$+ \left. H_2 \kappa_Y \sigma_Y \exp[-\kappa_Y(t'-t)]\left(\frac{\sigma_D \rho_{DY}}{r+\kappa_D} + H_2 \sigma_Y\right)\right]\left[\frac{\sigma_D^2}{(r+\kappa_D)^2} + (H_2)^2 \sigma_Y^2\right]^{-1}.$$

The last corollary implies that under our calibration

$$\frac{Cov_t(dS_t, dS_{t'})}{\sigma_S^2 dt dt'} = -0.50 e^{-0.5(t'-t)}.$$

It follows that the stock changes are negatively autocorrelated due to a time variation of the risk premium. Moreover, the autocorrelation is long–term since it subsides rather slowly over time. We emphasize that the stock changes are negatively autocorrelated due to autocorrelations of the dividend flow and the endowment volatility. Despite a negative autocorrelation, the price adjusts to an expected event myopically when it arrives.

The values of the risk premium, the volatilities of the stock returns and the aggregate consumption rate, the Sharpe ratio, and the autocorrelation coefficient provide benchmarks for the economy with a capital inertia considered below. Note that the main shortcomings of the equilibrium with no allocation delays are an infinitely large portfolio turnover for every investor as well as a relatively low risk premium and stock return volatility.[6] These shortcomings will be rectified in the economy in which capital moves slowly.

3.3 Equilibrium with Slow Moving Capital

Definition 2 *An equilibrium is a price system $(\mu_S, \sigma_{S_D}, \sigma_{S_Y})$ and a set of trading strategies $(u_1,\, u_2)$ of investors of type 1 and 2, respectively, such that (i) individual agents choose their optimal portfolio strategies and (ii) the stock market clears, that is $\forall t \in [0, \infty)$*

$$u_{1t} + u_{2t} = 0. \tag{25}$$

Appendix A proves the following theorem and corollaries 2 and 3 that describe the equilibrium

[6] The presence of the endowment process affects the risk premium and the volatility of the stock market return. It is easy to see from Appendix A that in the absence of the endowment process and capital delays the risk premium and the volatility are given by

$$\lim_{t\to\infty} E_0(\mu_{St} - rS_t) = \frac{r\gamma_1\gamma_2}{\gamma_1+\gamma_2}\sigma_{S_D} = 0.18,$$

$$\sigma_S = \sigma_{S_D} = 2.36.$$

It follows that adding an endowment process significantly increases both the stock market volatility and the risk premium. Still these changes are nearly not enough to explain the risk premium puzzle and the excess volatility puzzle.

Theorem 2 *Assume that H_2 and A_2^n, $n = 1, 2$ are positive, the nonlinear algebraic equation (A-33) has a solution C_0^{1*}, and the coefficient (A-37) is not zero. Then a linear equilibrium exists with the utility functions of investors, trading strategies and prices given by*[7]

$$\begin{aligned}
g^n &= A_0^n + A_1^n N_n + \frac{1}{2} A_2^n (N_n)^2 + (B_0^n + B_1^n N_n) Y + C_0^n Y^2, && (26)\\
u_n &= -\frac{A_1^n + A_2^n N_n + B_1^n Y}{2\alpha_n r \gamma_n}, && (27)\\
c_n &= -\frac{1}{\gamma_n}[\ln(r) + A_0^n + A_1^n N_n + \frac{1}{2} A_2^n (N_n)^2 + (B_0^n + B_1^n N_n) Y + C_0^n Y^2] \\
&+ r X_n, && (28)\\
S &= \frac{D}{r + \kappa_D} + H_0 + H_1 N_1 + H_2 Y, && (29)\\
\mu_S - rS &= R_0 + R_1 N_1 + R_2 Y, && (30)\\
\sigma_{S_D} &= \frac{\sigma_D}{r + \kappa_D}, && (31)\\
\sigma_{S_Y} &= \sigma_Y H_2, && (32)
\end{aligned}$$

where the coefficients are given by $C_0^1 = C_0^{1*}$ and by equations (A-10), (A-18), (A-26)–(A-32), (A-34)–(A-36).[8]

Apparently, the welfare and the trading strategy of an investor depend on the current level of the endowment volatility as well as on the trading history of investors. Contrary to the case with no allocation delays, investors in the given economy have to forecast the future possible trading shocks. Moreover, the returns follow 3–factor model with the factors being the dividend rate, the endowment volatility, and the stock holdings. It follows that the current allocations of investors will generally have a very strong impact on the conditional risk premium and the conditional Sharpe ratio.

The utility function of investor n reaches its maximum for a given value of the endowment volatility at allocation $\hat{N}_n$, where

$$\hat{N}_n = -\frac{A_1^n + B_1^n Y}{A_2^n}. \qquad (33)$$

The maximum of the utility function for investor n exists because A_2^n is assumed to be positive. It follows that the volume traded by an investor per time unit can be written as

$$u_n = -A_2^n \frac{N_n - \hat{N}_n}{2\alpha_n r \gamma^n}$$

and an investor buys the stock when $N_n < \hat{N}_n$ and sells it when $N_n > \hat{N}_n$. We also conclude that an investor will trade in the long run only if volatility of the endowment changes over time.

[7] Note that this economy may allow multiple equilibria. The issue of equilibrium multiplicity is not studied in this paper.

[8] These equations should be applied to find the coefficients in the following order: C_0^1, C_0^2, B_1^1, B_1^2, H_2, A_2^1, A_2^2, R_1, H_1, R_2, A_1^1, A_1^2, B_0^1, B_0^2, R_0, A_0^1, A_0^2, H_0.

Notice that coefficient A_0^n and discount parameter τ_n show up in the equilibrium solution only together and only in one equation for the equilibrium coefficients presented in Appendix A. Therefore, τ_n contributes only to the coefficient A_0^n and does not have any effects on the rest of coefficients in the linear equilibrium. The last conclusion implies that impatience of investors affect their welfare and consumption but has no impact on strategies and prices. We conclude that more impatient investors do not care to hold more liquid cash and less of the illiquid stock market in stationary economy. This result is at odds with the "clientele effect" according to which less impatient investors hold more liquid assets. See, for example, Amihud and Mendelson (1986) and Longstaff (2009) who considered the non–stationary finite horizon economies.

Risk Premium and Volatilities

The following corollary presents the long term expected stock allocation, rate of trading as well as the risk premium

Corollary 3 *The long–term expected allocation, rate of trading, and risk premium of the stock market are given by*

$$\lim_{t\to\infty} E_0(N_{nt}) = -\frac{A_1^n + B_1^n \bar{Y}}{A_2^n}, \tag{34}$$

$$\lim_{t\to\infty} E_0(u_t^n) = 0, \tag{35}$$

$$\lim_{t\to\infty} E_0(\mu_{St} - rS_t) = R_0 + R_2\bar{Y} - R_1\frac{A_1^1 + B_1^1\bar{Y}}{A_2^1}. \tag{36}$$

Following the analysis of Appendix A and using the calibration introduced above we find the equilibrium coefficients: $A_0^1 = -82.1652$, $A_0^2 = 30.0278$, $A_1^1 = -8.5477$, $A_1^2 = -11.4700$, $A_2^1 = 20.0177$, $A_2^2 = 20.0177$, $B_0^1 = 11.8423$, $B_0^2 = 15.9983$, $B_1^1 = 4.1561$, $B_1^2 = -4.1561$, $C_0^1 = -2.4917$, $C_0^2 = 3.4140$ and $R_0 = 217.3680$, $R_1 = -433.9016$, $R_2 = -188.3347$, $H_0 = 1046.4981$, $H_1 = 19.4820$, $H_2 = 185.6041$. These coefficients imply the following long–term expected allocation, the risk premium, and the Sharpe ratio of the stock market

$$\lim_{t\to\infty} E_0(N_{1t}) = 0.427, \tag{37}$$

$$\lim_{t\to\infty} E_0(\mu_{St} - rS_t) = 32.09, \tag{38}$$

$$\sigma_S = \sqrt{\frac{\sigma_D^2}{(r+\kappa_D)^2} + (H_2)^2\sigma_Y^2} = 139.22, \tag{39}$$

$$S_M = \lim_{t\to\infty} \frac{E_0(\mu_{St} - rS_t)}{\sigma_S} = 0.230. \tag{40}$$

The last results show that the presence of capital inertia causes a dramatic increase in the risk premium (close to eighty times) and volatility of stock market returns (close to thirty times). Moreover, it causes the Sharpe ratio to rise and makes it closer to its historical value. Here, like in the economy without delays in allocations, investor 2 trades in the direction of the market movement. However, in the economy with slow moving capital the impact of investor 2 on the volatility of the stock market return and its risk premium becomes much stronger since investor 1 faces more significant losses trying to hedge the endowment risk. Consequently, investor 2 becomes marginal, the volatility of stock returns rises dramatically and the risk premium follows to offset a higher risk exposure of investor 2.

If the endowment volatility undergoes a negative shock and the stock price decreases then investor 2 has a choice to buy the stock and enjoy the benefits of the substitution effect or she can sell shares and avail the wealth effect by saving on the value of her current holding. Similarly, if the endowment undergoes a positive shock and the stock price increases then investor 2 can sell shares and get an advantage from the substitution effect or she can purchase them and utilize the wealth effect since the price rises. Clearly, investor 2 chooses to neglect the substitution effect and take advantage of the wealth effect while on the other end of trades, investor 1 takes advantage of the substitution effect. The wealth effect defines a high volatility of the stock price in this economy.

By using the definition of the coefficients in the consumption volatility given by equation (16) we find from Theorem 2

$$\sigma_{C0} = r(\beta_1 + \beta_2)Y, \tag{41}$$

$$\sigma_{CD} = \frac{r\sigma_D}{r + \kappa_D}, \tag{42}$$

$$\sigma_{CY} = [rH_2 - \frac{1}{\gamma_1}(B_0^1 + B_1^1 N_1 + 2C_0^1 Y) - \frac{1}{\gamma_2}(B_0^2 + B_1^2 N_2 + 2C_0^2 Y)]\sigma_Y. \tag{43}$$

With the help from the last expressions and Monte–Carlo simulations we obtain:

Corollary 4 *The long–term volatility of the aggregate consumption rate is given by*

$$\lim_{t\to\infty} E_0[\sigma_{Ct}] = 0.724. \tag{44}$$

Comparison of volatilities of the aggregate consumption in the market with allocation delays and in the market without such delays suggests a significant increase of consumption volatility in the former case. Nonetheless, this increase is many times less substantial than the corresponding increase of the volatility of the stock market returns. The difference in the increases is due to smoothing the consumption of investors by hedging the endowment shocks rising from their position in the stock market.

Finally, we justify our choice of parameter α by finding a typical portfolio turnover in the economy. We evaluate the portfolio turnover in a steady state of the economy averaged over number of investors

$$\text{Portfolio Turnover } = \frac{1}{2}\lim_{t\to\infty}\frac{E_0\int_t^{t+1}|u_{1s}|ds}{E_0[N_{1t}]E_0[N_{2t}]}.$$

By using Monte–Carlo simulations we calculate that the portfolio turnover is 93.86% which is close to its historical value observed in the U.S. security markets.[9]

Price Overreaction

For a better intuition on the price dynamics in the economy with capital delays let us consider the following two realizations of the endowment volatility. In both cases we assume that $Y = \bar{Y}$ $(= 0)$ at time zero and the stock allocation of each investor is optimal ($N_n = \hat{N}_n$).

First, suppose that at time zero an endowment volatility instantly increases by a certain increment ΔY and then does not change. The optimal allocation of investor 1 becomes smaller and she starts selling the stock market at the maximal rate. In the meantime the stock market price undergoes a sharp increase at time zero since coefficient H_2 is positive, and then starts decreasing since coefficient H_1 is also positive and investor 1 sells shares. The price will continue to decrease until its long–rung limit corresponding to a new level of endowment. It follows that a good news results in the overreaction of the stock market price. Assuming that $S = \frac{D}{r+\kappa_D} D + H_0 + \Gamma$, the last conclusion is formally seen from the equation that we find for $\Gamma(t)$ under the described shock:

$$\Gamma(t) = \Gamma(0-) + H_2 \Delta Y - \frac{H_1 B_1^1 \Delta Y}{A_2^1} [1 - \exp(-\xi\, t)],$$

where $\Gamma(0-)$ is the price component Γ right before the jump nad $\xi = \frac{A_2^1}{2\alpha_1 r \gamma_1}$ is a half–life of the price impact. The coefficient $\frac{H_1 B_1^1}{A_2^1}$ in the last expression is positive implying that a negative news will also result in the price overreaction. The price overreaction to news is short–term since it is defined by coefficient ξ which appears to be very large. Moreover, it implies a negative contribution into a short–term predictability.[10] We conclude that the price overreaction to a news which is so often observed in practice does not have to imply irrationality of investors and follows from the slow movement of capital.

Now suppose that the endowment volatility undergoes changes which follow a sinusoid $Y = Y_A \sin(\omega t)$, $t \in [0, +\infty[$. It is straightforward to find the equation for the price component $\Gamma(t)$ and the rate of trading of investor 1:

$$\begin{aligned} \Gamma(t) &= \Gamma(0) + \frac{\varphi Y_A H_1}{\omega^2 + \xi^2}\left(\omega \cos(\omega t) - \xi \sin(\omega t) - \omega\, e^{-\xi\, t}\right) - H_1 \left[N_1(0) + \frac{A_1^1}{A_2^1}\right](1 - e^{-\xi\, t}) \\ &+ H_2 Y_A \sin(\omega t), \end{aligned} \tag{45}$$

$$u_1(t) = -\frac{\varphi Y_A \omega}{\omega^2 + \xi^2}\left(\omega \sin(\omega t) + \xi \cos(\omega t) - \xi\, e^{-\xi\, t}\right), \tag{46}$$

[9] The traditional choice for the portfolio turnover in the empirical literature is an annualized trading volume per number of outstanding shares: Portfolio Turnover^e $= \lim_{t\to\infty} E_0 \int_t^{t+1} |u_{1s}| ds$. It follows that under our calibration Portfolio Turnover^e $= 45.13\%$ which is close to its historical value over the past one hundred years.

[10] Clearly, the presence of overreaction contributes to predictability, however the presence of predictability does not imply over- or underreaction of prices. For example, predictability is significant in the economy with no capital inertia while the price overreaction is absent.

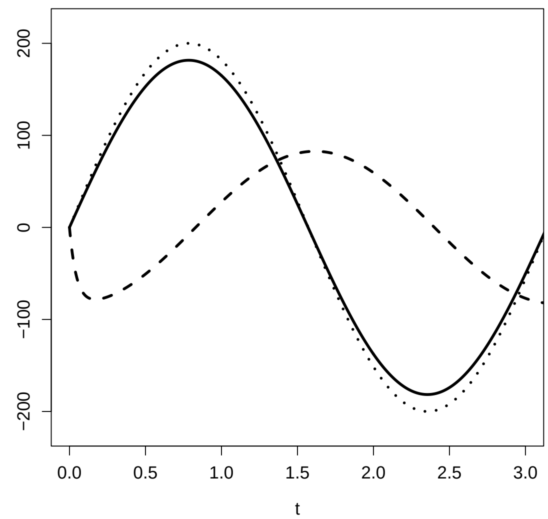

Figure 1: The figure shows the endowment volatility $Y(t)$ multiplied by 200 (dotted line), the price component $\Gamma(t)$-$\Gamma(0)$ (solid line), and trading rate $v_1(t)$ multiplied by 200 (dashed line) versus time t. We assume that $Y_a = 1$, $\omega = 2$, while the rest of parameters are $\gamma_1 = \gamma_2 = 5$, $\beta_1 = 2.0$, $\beta_2 = 0.0$, $r = 0.03$, $\tau_1 = \tau_2 = 0.05$, $\kappa_Y = 0.5$, $\bar{Y} = 0$, $\alpha = 3$, $\kappa_D = 0.5$, $\bar{D} = 1$, $\sigma_D = 1.25$, $\rho_{DY} = 0$, $\rho_{0D} = 0$, $\rho_{0Y} = 0$.

where $\varphi = \frac{B_1^1}{2\alpha_1 r\gamma_1}$. If time is sufficiently long then

$$\Gamma(t) \approx \Gamma(0) + \frac{\varphi Y_A H_1}{\omega^2 + \xi^2}[\omega\cos(\omega t) - \xi\sin(\omega t)] - \left[H_1 N_1(0) + \frac{H_1 A_1^1}{A_2^1}\right] + H_2 Y_A \sin(\omega t), \quad (47)$$

$$u_1(t) \approx -\frac{\varphi Y_A \omega}{\omega^2 + \xi^2}[\omega\sin(\omega t) + \xi\cos(\omega t)]. \quad (48)$$

Clearly, the times when the price and trading volume are maximal (minimal) are different from those of process Y. For example, if coefficient ξ is large, as it is under our calibration, and time is long, then $\Gamma(t)$ reaches its extremums when $tan^{-1}(\omega t) = \frac{H_1\varphi\omega}{H_2(\omega^2+\xi^2) - H_1\varphi\xi} \approx \frac{H_1\varphi\omega}{H_2(\omega^2+\xi^2)}$, while $v(t)$ reaches extremums when $tan^{-1}(\omega t) = \frac{\xi}{\omega}$. The first ratio is positive and small implying that price component Γ reaches its extremums slightly before the endowment volatility does, while the second ratio is much bigger implying that the trading volume reaches its extremums substantially before the price does. These conclusions are illustrated on Figure 1 which shows the magnified endowment volatility (dotted line), the price component Γ (solid line), and the magnified trading rate v_1 (dashed line). We conclude that the price changes precede the expected events. This lagging of expected news is characteristic for the economy with a capital inertia and is absent in the economy with no such inertia.

As claimed above, the price overreaction and the news lagging should affect the stock return predictability measured by the autocorrelation coefficient. Moreover, given that the trading volume significantly precede the changes in the stock price, the former could be used as a predictor for the price change. The following corollary presents the autocorrelation coefficients between the price changes and changes in the trading volume and the price.

Corollary 5 *The autocorrelation coefficient between the stock price change for time interval* $[t, t + dt]$ *and the stock price change for time interval* $[t', t' + dt']$*, where* $t' \geq t + dt$*, is given by*

$$\frac{Cov_t(dS_t, dS_{t'})}{\sigma_S^2\, dtdt'} = \left[\frac{\sigma_D^2}{(r+\kappa_D)^2} + (H_2)^2\sigma_Y^2\right]^{-1}\left[-\frac{\kappa_D\sigma_D}{r+\kappa_D}\, e^{-\kappa_D(t'-t)}\left(\frac{\sigma_D}{r+\kappa_D} + \rho_{YD}\sigma_Y H_2\right)\right.$$
$$\left. - \quad \sigma_Y\left(\frac{\rho_{DY}\sigma_D}{r+\kappa_D} + \sigma_Y H_2\right)e^{-\kappa_Y(t'-t)}\left(H_1\varphi + H_2\kappa_Y + [1 - e^{(\kappa_Y-\xi)(t'-t)}]\frac{\varphi H_1\xi}{\kappa_Y - \xi}\right)\right]. \quad (49)$$

The autocorrelation coefficient between the change in the trading rate during time interval $[t, t + dt]$ *and the stock price change during time interval* $[t', t' + dt']$ *is given by*

$$\frac{Cov_t(du_{1t}, dS_{t'})}{\varphi\sigma_Y\sigma_S\, dtdt'} = \left[\frac{\sigma_D^2}{(r+\kappa_D)^2} + (H_2)^2\sigma_Y^2\right]^{-1/2}\left(\frac{\kappa_D\sigma_D\rho_{DY}e^{-\kappa_D(t'-t)}}{r+\kappa_D}\right.$$
$$\left. + \quad e^{-\kappa_Y(t'-t)}\sigma_Y\left[H_1\varphi + H_2\kappa_Y + \frac{H_1\xi\varphi}{\kappa_Y - \xi}[1 - e^{(\kappa-\xi)(t'-t)}]\right]\right). \quad (50)$$

It follows that under our calibration

$$\frac{Cov_t(dS_t, dS_{t'})}{\sigma_S^2\, dtdt'} = -0.49e^{-0.5(t'-t)} - 0.50e^{-21.88(t'-t)}, \quad (51)$$
$$\frac{Cov_t(du_{1t}, dS_{t'})}{\varphi\sigma_Y\sigma_S\, dtdt'} = 0.49e^{-0.5(t'-t)} + 0.50e^{-21.88(t'-t)}. \quad (52)$$

Apparently, the predictability of the price changes is affected by delays in capital allocations mostly in the short run with the effect being represented by the second term on the right side of equation (51). This effect doubles the autocorrelation coefficient at times close to the initial time. The contribution of the capital inertia to predictability in the long run appears in the first term on the right side of equation (51) and is equal to 2%.[11]

The autocorrelation coefficient between the rate of trading of investor 1 and the change in the stock price given by equation (52) is positive since investor 1 trades against the market and the stock tends to mean–revert. This coefficient is very close in magnitude and has a very similar time–pattern as the autocorrelation coefficient between changes in the stock price. A significant value of this coefficient suggests that the trading volume can be used to predict the stock price changes.

[11]Alternatively to the autocorrelation coefficient between stock price changes, we could have considered the autocorrelation coefficient between dollar stock returns per time dt defined as $dR_t = dS_t + D_t dt$. The intuition behind our conclusions would remain unaffected. In particular, the principal contribution to the autocorrelation coefficient between stock returns from the price overreaction would be short–term and negative.

Sensitivity Analysis

In this subsection we consider the effects that α, σ_Y, and the difference between α_1 and α_2 have on the volatility, the risk premium, and predictability of the stock market returns.

Clearly, the characteristics of the stock returns are very sensitive to the cost of trading defined by coefficient α. Panel A of Figure 2 depicts the risk premium (the solid line) versus the natural logarithm of α. The risk premium is very close to its value in the economy with no delays in capital allocations when α is small and starts quickly increase after α exceeds 0.02. It becomes difficult for investors to trade the stock market. Yet investor 1 continue to trade for the purpose of hedging the endowment shocks. Meantime, investor 2 agrees to trade shares only for a substantially higher risk premium. The dashed line on Panel A of Figure 2 depicts the volatility of the stock return versus $\ln(\alpha)$. Apparently, the volatility is more sensitive to changes in α than the risk premium and starts quickly increase with α starting from $\alpha = 0.001$. Consequently, the Sharpe ratio falls with increasing α when this coefficient is small. The latter is due to an increased competition for price formation between the two investors. A negative impact on the risk premium coming from hedging by investor 1 attenuates the contribution to the risk premium resulting from the risk sharing by investor 2. Consequently, despite an increase in the risk premium, the Sharpe ratio actually decreases when α is small. This trend reverses at large α's where a bigger inertia of capital implies a higher Sharpe ratio in equilibrium. The Sharpe ratio multiplied by the factor of 50 is shown on Panel A of Figure 2 by the dotted line. Furthermore, we find that if α is large then the equilibrium coefficients behave as following: A_0^n, A_1^n, $A_2^n \sim \alpha$; $B_0^n, B_1^n \sim \sqrt{\alpha}$; $R_0, R_1, H_0, H_1 \sim \alpha$; $R_2, H_2 \sim \sqrt{\alpha}$, while C_0^n is independent from α. It follows that at large α the volatility of the stock returns is proportional to $\sqrt{\alpha}$ while the risk premium is proportional to α. The latter trends imply that the Sharpe ratio rises with α proportionally to $\sqrt{\alpha}$.

Panel B of Figure 2 depicts turnover of the investor's portfolio versus $\ln(\alpha)$. The portfolio turnover monotonically decreases with α showing empirically acceptable values even at very high levels of α. Furthermore, Panel C of Figure 2 shows coefficient ξ (solid line) and a magnified coefficient ψ (dashed line) versus $\ln(\alpha)$, where

$$\psi = \frac{\sigma_Y \varphi H_1 \xi}{\kappa_Y - \xi}\left(\frac{\rho_{DY}\sigma_D}{r+\kappa_D} + \sigma_Y H_2\right)\left[\frac{\sigma_D^2}{(r+\kappa_D)^2} + (H_2)^2\sigma_Y^2\right]^{-1}.$$

Coefficient ψ shows how strong the effect from the delays in capital allocations on the autocorrelation coefficient is, whereas coefficient ξ is a half–life of this effect. If the capital inertia is small then its effect on the autocorrelation coefficient is small and the current returns have a very short memory of the past ones (of the order of a few days). If the capital inertia increases then so does its impact on the magnitude of the price reversion and its time–span. Intuitively, the autocorrelation coefficient should be effected by the rate of trading which decrease will increase the magnitude of ψ and lower ξ. Moreover, this coefficient should be influenced by the volatility of the stock returns the rise of which should increase

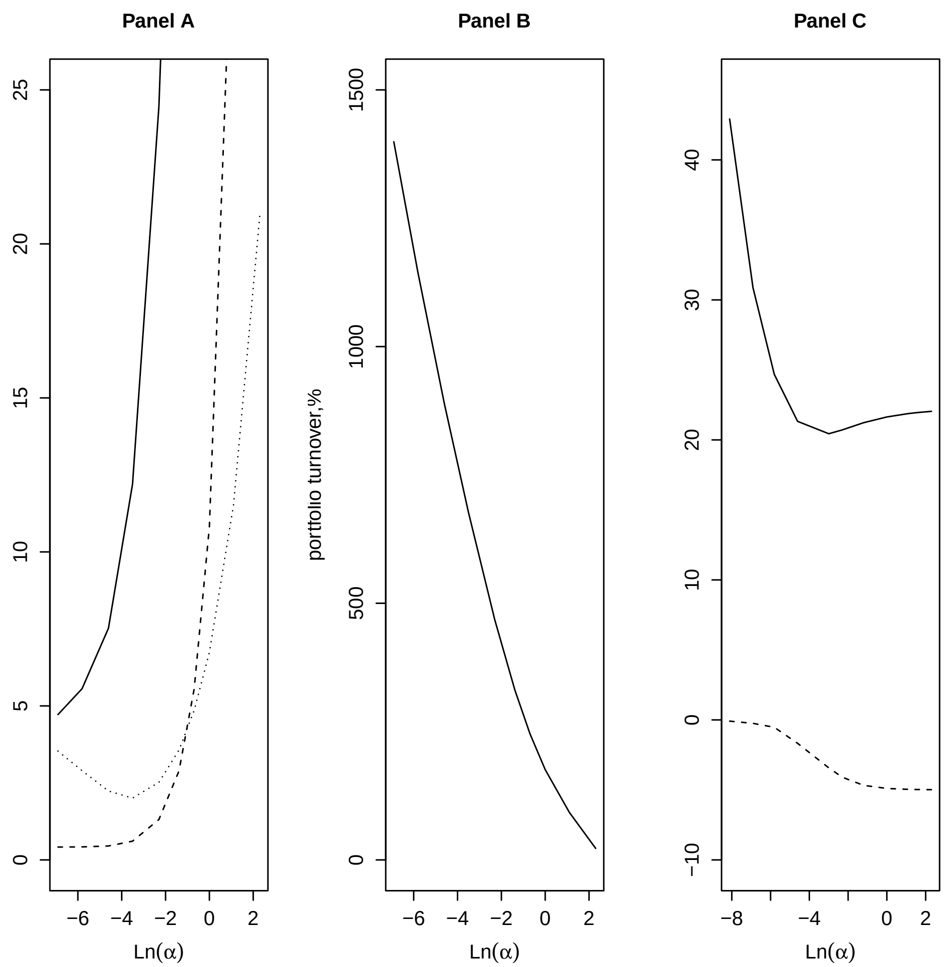

Figure 2: Panel A of this Figure shows the volatility of the stock return, (solid line), the risk premium (dashed line), and the Sharpe ratio multiplied by 50 (dotted line) versus $\ln(\alpha)$. Panel B depicts a portfolio turnover averaged over investors versus $\ln(\alpha)$. Panel C shows coefficient ξ (solid line) and coefficient ψ multiplied by 10 (dashed line) versus $\ln(\alpha)$. We assume that $\gamma_1 = \gamma_2 = 5$, $\beta_1 = 2.0$, $\beta_2 = 0.0$, $r = 0.03$, $\tau_1 = \tau_2 = 0.05$, $\kappa_Y = 0.5$, $\bar{Y} = 0$, $\sigma_Y = 0.75$, $\kappa_D = 0.5$, $\bar{D} = 1$, $\sigma_D = 1.25$, $\rho_{DY} = 0$, $\rho_{0D} = 0$, $\rho_{0Y} = 0$.

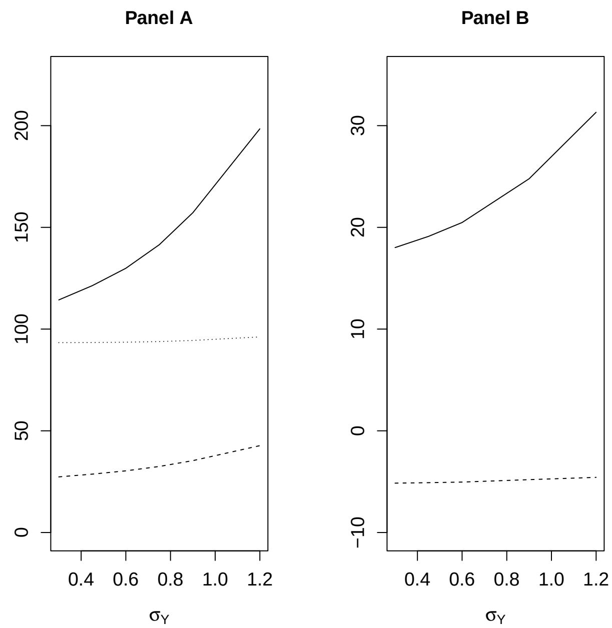

Figure 3: Panel A shows the volatility of the stock return, (solid line), the risk premium (dashed line), and the portfolio turnover (dotted line) versus σ_Y. Panel B shows coefficient ξ (solid line) and coefficient ψ multiplied by 10 (dashed line) versus σ_Y. We assume that $\gamma_1 = \gamma_2 = 5$, $\beta_1 = 2.0$, $\beta_2 = 0.0$, $r = 0.03$, $\tau_1 = \tau_2 = 0.05$, $\kappa_Y = 0.5$, $\bar{Y} = 0$, $\alpha = 3$, $\kappa_D = 0.5$, $\bar{D} = 1$, $\sigma_D = 1.25$, $\rho_{DY} = 0$, $\rho_{0D} = 0$, $\rho_{0Y} = 0$.

ξ and lower the magnitude of ψ. Panel C of Figure 2 shows that when the capital inertia becomes significant the time decay of the autocorrelation reaches the level of about two and a half weeks and then slowly rises with a further increase in the inertia. Both coefficients barely changes at high α due to the offsetting effects from the trading speed and volatility. We conclude based on Figure 2 that improvement in the capital mobility over years should results in a smaller volatility and a risk premium of stock returns as well as in their weaker predictability.

The risk premium and the volatility of the stock market return depend on trading incentives of investors which in turn arise from changes in volatility of the endowment. The latter is defined by the volatility of volatility, σ_Y. Given the stylized nature of our model and a broad interpretation of the endowment, the precise calibration of σ_Y lies beyond this study. Instead, we consider how this parameter affects the risk premium, volatility of the stock market returns, the turnover of investor's portfolio, and the autocorrelation coefficient. Figure 3 depicts these effects for the range of values of σ_Y at which linear equilibrium exists.[12]

Panel A of Figure 3 shows that the four–times increase in volatility of the endowment volatility results in only a few percent increase in the portfolio turnover. Apparently, a greater incentive to hedge for investor 1 is offset by the improved investment opportunities.

[12]We find that the linear equilibrium exists only if $\sigma_Y \geq 0.3$.

Table 1:

α_2	α_1	RP	σ_S	S_r	$E_0(\hat{N}_1)$	PT, %	ψ	ξ
0.3	3.0	955.07	266.77	3.58	0.694	43.85	-0.496	55.43
	1.0	77.31	103.94	0.744	0.610	88.08	-0.487	33.69
	0.3	3.42	44.36	0.077	0.429	287.99	-0.467	21.23
	0.1	0.463	11.58	0.040	0.453	433.25	-0.294	7.44
1.0	10.0	3145.71	484.15	6.50	0.694	23.99	-0.498	55.31
	3.0	189.01	170.32	1.110	0.599	52.87	-0.492	31.74
	1.0	10.851	80.51	0.135	0.428	161.29	-0.489	21.64
	0.3	1.069	40.96	0.026	0.293	485.21	-0.526	17.57
	0.1	0.499	15.13	0.033	0.414	498.80	-0.599	10.98
3.0	10.0	732.88	319.82	2.292	0.611	28.02	-0.495	33.28
	6.0	193.46	213.85	0.904	0.545	44.35	-0.493	26.70
	3.0	32.09	139.22	0.230	0.427	93.86	-0.496	21.88
	1.0	3.31	87.93	0.038	0.246	324.14	-0.541	21.20
	0.1	0.707	43.99	0.016	0.262	596.20	-0.987	35.67

The table reports the risk premium, RP, the volatility of return, σ_S, the Sharpe ratio, S_r, the long term expected stock holding of investor 1, $E_0(\hat{N}_1)$, the portfolio turnover, PT, and coefficients ξ and ψ for the different values of trading coefficients, α_1 and α_2. We assume that $\gamma_1 = \gamma_2 = 5$, $\beta_1 = 2.0$, $\beta_2 = 0.0$, $r = 0.03$, $\tau_1 = \tau_2 = 0.05$, $\kappa_Y = 0.5$, $\bar{Y} = 0$, $\sigma_Y = 0.75$, $\kappa_D = 0.5$, $\bar{D} = 1$, $\sigma_D = 1.25$, $\rho_{DY} = 0$, $\rho_{0D} = 0$, $\rho_{0Y} = 0$.

Meantime, the changes in the risk premium and volatility of stock returns are more significant but still much less pronounced than the variation in σ_Y. Consistently with these traits, our conclusion about relation between the moments of stock returns in the economies with and without delays in capital allocations remains intact for the shown range of σ_Y: These moments are much bigger in the economy with a slow capital movement. For example, if $\sigma_Y = 0.6$ then the risk premium and volatility of the stock market return are 0.42 and 2.55 in the economy with no allocation delays, versus staggering 30.14 and 128.92 in the economy with the delays.

Panel B of Figure 3 shows coefficients ξ (solid line) and ψ (dashed line) describing the impact of the capital inertia on the autocorrelation coefficient. Apparently, an increase in volatility has a minor effect on the size of the price reversal while its time decay undergoes a significant rise. The decay of the autocorrelation coefficient over time increases with higher σ_Y due to a quickly rising volatility of the stock returns.

The calibration of our numerical example assumes the heterogeneity between agents only in their endowments. Now we relax this assumption by allowing an additional heterogeneity in the agent's access to the capital mobility. Table 1 shows the risk premium, the volatility of the stock return, the Sharpe ratio, the expected share holdings, and the portfolio turnover for various combinations of coefficients in trading costs for the two agents. Apart from these coefficients, the rest of the calibration is the same as above. The reported results bring up a few observations. First, the heterogeneity in access to the capital mobility across agents has a very strong impact on equilibrium. Naturally, the strongest effect on prices comes from the access to the capital mobility by agent 1 who faces endowment shocks. The poorer this access is, the stronger should be the incentives for agent 2 to trade resulting in the higher risk premium and the higher Sharpe ratio. Furthermore, the poorer the access is, the stronger is the impact of agent 2 on price movements resulting in higher volatilities. Notice that the ratio of α's is often as important as the value of α_1 itself. In particular, for a fixed α_1 the impact of slow capital movement on prices rises when α_2 decreases even though the average α subsides. For example, Table 1 shows that if $\alpha_1 = 3$ and $\alpha_2 = 0.3$ the risk premium and the volatility of stock returns increase from the corresponding values in the economy with $\alpha_1 = \alpha_2 = 3$ by the factors of thirty and two, respectively. If α_2 decreases at fixed α_1 then the impact of investor 2 on prices becomes more pronounced and favorable for her implying the higher risk premium, the volatility, and the Sharpe ratio.

The interpretation of the calibration when an investor with an inferior access to the capital mobility also has a strong exposure to the endowment risk is straightforward. This investor represents individual investors who receive labor income, or business income, or have positions in the real estate market. These investors trade occasionally to increase their consumption and hedge an endowment risk. They face an opportunity cost for not trading frequently which is much more severe than for the other investors who trade professionally and make a living out of their trading.

We notice that the expected holdings of investor 1 do not depend on α_1 monotonically. This investor takes advantage of the substitution effect. Therefore, her expected holdings follow the trends in the Sharpe ratio: They increase with α_1 when the Sharpe ratio rise and fall when the Sharpe ratio decreases.[13] If α_1 is large investor 1 faces very high adjustment costs and tries to offset the impacts from endowment shocks by taking very long positions in a very cheap stock market. The limited ability to hedge endowment shocks are compensated by a high expected Sharpe ratio of her stock holdings. Lowering the cost of trading allows her to hedge these shocks better by trading faster and makes her reduce the expected stock holdings. Decreasing α_1 below α_2 causes investor 1 to raise her position in the stock market again. Here investor 1 becomes marginal, she moves the prices to weaken the unfavorable effects from endowment shocks allowing her to hold more shares.

The effect of heterogeneity in access to the capital mobility across investors have a very strong nonlinear effect on the autocorrelation coefficient showing in the coefficients ψ and ξ. First, an increase in α_1 above α_2 when the latter is fixed makes the impact of the capital inertia on the autocorrelation stronger and shorter. As the influence of agent 2 increases and she becomes marginal the lagged returns become correlated stronger but with a shorter memory. The former trend is due to a decreasing trading volume of the stock market while the latter trend is due to a quickly increasing volatility of the stock returns. Interestingly,

[13] The rise of the Sharpe ratio with decreasing α_1 is seen in Table 1 only for $\alpha_2 = 1$. It could be seen for the other values of α_2 when α_1 becomes very small.

coefficient ψ increases in magnitude when α_1 falls below α_2: As agent 1 becomes marginal, the prices tend to move more in reversal due to the substitution effect that drives trading of agent 1. Moreover, a very fast rate of trading starts to overcome the influence on coefficient ξ coming from volatility at very low α_1 causing this coefficient to increase when α_1 is much smaller than α_2 (see this coefficient when $\alpha_1 = 0.1$ and $\alpha_2 = 3$).

4 Conclusion

We consider an economy where heterogeneous investors trade the stock market and face limits in the capital mobility. This mobility is modeled by means of convex trading costs. We find a closed form solution of an equilibrium. By matching portfolio turnover with that observed empirically, we show that the capital inertia can easily have the first order effects on the risk premium and the volatility of the stock market. In particular, introduction of delays in capital allocations results in more than a thirtyfold increase in the risk premium and the volatility of the stock market returns. Finally, our model predicts price overreaction to both bad and good news at short horizons as well as the absence of the "clientele effect".

Appendix A

Proof of Theorem 2. The indirect utility function of an investor of type n, $V^n(X_n, N_n, Y)$, solves

$$\begin{aligned} 0 &= \max_{c_n \in R^+, u^n \in R} \Bigg\{ -\tau_n V^n + \frac{1}{2}\Big[(Y_n)^2 + 2Y_n N_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) + (N_n)^2\sigma_S^2\Big] V^n_{X_n X_n} \\ &+ \frac{1}{2}\sigma_Y^2 V^n_{YY} + \sigma_Y[Y_n\rho_{0Y} + N_n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]V^n_{X_n Y} \\ &+ u_n V^n_{N_n} + \mu_Y V^n_Y + \Big((\mu_S - rS)N_n - \alpha_n (u_n)^2 + rX_n - c_n\Big)V^n_{X_n} - \frac{1}{\gamma_n}\exp(-\gamma_n c_n)\Bigg\}. \end{aligned}$$

It follows that

$$c_n = -\frac{1}{\gamma_n}\ln(rV^n_{X_n}), \quad u_n = \frac{V^n_{N_n}}{2\alpha_n V^n_{X_n}}. \tag{A-1}$$

We conjecture that $V^n(X_n, N_n, Y) = -\frac{1}{\gamma_n}\exp[-\gamma_n r X_n + g^n(N_n, Y)]$ then

$$c_n = -\frac{1}{\gamma_n}[\ln(r) + g^n] + rX_n, \quad u_n = -\frac{g^n_N}{2\alpha_n r\gamma_n} \tag{A-2}$$

and g^n solves the following PDE

$$
\begin{aligned}
0 &= -r[\ln(r) + g^n] - \tau_n + r + \frac{(r\gamma_n)^2}{2}\Bigg[(Y_n)^2 + 2Y_n N_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_{jY}}) \\
&+ (N_n)^2\sigma_S^2\Bigg] - r\gamma_n(\mu_S - rS)N_n + \frac{1}{2}\sigma_Y^2[g^n_{YY} + (g^n_Y)^2] \\
&- \frac{(g^n_N)^2}{4\alpha_n r\gamma_n} + \mu_Y g^n_Y - \sigma_Y r\gamma_n[Y_n\rho_{0Y} + N_n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]g^n_Y. \qquad \text{(A-3)}
\end{aligned}
$$

Because the set of state variables for the stock price in this economy is D, N_1, Y we find from the Itô formula that $\sigma_{S_D} = S_D\sigma_D$ and $\sigma_{S_Y} = S_Y\sigma_Y$.

The drift of the stock returns is given by

$$
\mu_S = D + S_D\mu_D + S_{N_1}u_1 + S_Y\mu_Y + \frac{1}{2}\sigma_D^2 S_{DD} + \frac{1}{2}\sigma_Y^2 S_{YY} + \rho_{DY}\sigma_D\sigma_Y S_{DY}. \qquad \text{(A-4)}
$$

Assuming that $S(D,Y,N_1) = \frac{D}{r+\kappa_D} + h_j(Y,N_1)$, we find the PDE for h:

$$
\xi(Y,N_1) + rh = \frac{\kappa_D\bar{D}}{r+\kappa_D} + h_{N_1}u_1 + h_Y\mu_Y + \frac{1}{2}\sigma_Y^2 h_{YY}, \qquad \text{(A-5)}
$$

where ξ is the risk premium of the stock market and $\sigma_{S_D} = \frac{\sigma_D}{r+\kappa_D}$, $\sigma_{S_Y} = h_Y\sigma_Y$.

We conjecture the following solution of the equilibrium

$$
\begin{aligned}
g^n &= A_0^n + A_1^n N_n + \frac{1}{2}A_2^n(N_n)^2 + (B_0^n + B_1^n N_n)Y + C_0^n Y^2, \qquad \text{(A-6)} \\
h &= H_0 + H_1 N_1 + H_2 Y, \qquad \text{(A-7)} \\
\mu_S - rS &= R_0 + R_1 N_1 + R_2 Y, \qquad \text{(A-8)}
\end{aligned}
$$

where all coefficients are independent from N and Y. It follows that $\sigma_{S_Y} = \sigma_Y H_2$. We find from equation (A-3):

$$
\begin{aligned}
&- \tau_n - r[\ln(r) - 1 + A_0^n + A_1^n N_n + \frac{1}{2}A_2^n(N_n)^2 + (B_0^n + B_1^n N_n)Y + C_0^n Y^2] \\
&+ \frac{(r\gamma_n)^2}{2}\Bigg[(Y_n)^2 + 2Y_n N_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) + (N_n)^2\sigma_S^2\Bigg] \\
&- r\gamma_n(R_0 + R_1 N_1 + R_2 Y)N_n + \frac{1}{2}\sigma_Y^2[2C_0^n + (B_0^n + B_1^n N_n + 2C_0^n Y)^2] \\
&+ (B_0^n + B_1^n N_n + 2C_0^n Y)\Big(\mu_Y - \sigma_Y r\gamma_n[Y_n\rho_{0Y} + N_n(\rho_Y\sigma_{S_D} + \sigma_{S_Y})]\Big) \qquad \text{(A-9)} \\
&- \frac{(A_1^n + A_2^n N_n + B_1^n Y)^2}{4\alpha_n r\gamma_n}
\end{aligned}
$$

implying the following set of equations:

$$
\begin{aligned}
0 &= -rA_0^n + r[1-\ln(r)] - \tau_n + \frac{1}{2}\sigma_Y^2[2C_0^n + (B_0^n)^2] - \frac{(A_1^n)^2}{4\alpha_n r\gamma_n} + \kappa_Y \bar{Y} B_0^n, && \text{(A-10)}\\
0 &= -rA_1^1 - r\gamma_1 R_0 + \sigma_Y^2 B_0^1 B_1^1 - \frac{1}{2r\gamma_1}\frac{A_1^1 A_2^1}{\alpha_1} + \kappa_Y \bar{Y} B_1^1 && \text{(A-11)}\\
&- \sigma_Y r\gamma_1(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})B_0^1,\\
0 &= -rA_1^2 - r\gamma_2(R_0 + R_1) + \sigma_Y^2 B_0^2 B_1^2 - \frac{1}{2r\gamma_2}\frac{A_1^2 A_2^2}{\alpha_2} + \kappa_Y \bar{Y} B_1^2\\
&- \sigma_Y r\gamma_2(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})B_0^2, && \text{(A-12)}\\
0 &= -rA_2^1 + (r\gamma_1)^2\sigma_S^2 - 2r\gamma_1 R_1 + \sigma_Y^2 (B_1^1)^2 - \frac{(A_2^1)^2}{2\alpha_1 r\gamma_1}\\
&- 2\sigma_Y r\gamma_1(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})B_1^1, && \text{(A-13)}
\end{aligned}
$$

$$
\begin{aligned}
0 &= -rA_2^2 + (r\gamma_2)^2\sigma_S^2 + 2r\gamma_2 R_1 + \sigma_Y^2 (B_1^2)^2 - \frac{(A_2^2)^2}{2\alpha_2 r\gamma_2}\\
&- 2\sigma_Y r\gamma_2(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})B_1^2, && \text{(A-14)}\\
0 &= (2\sigma_Y^2 C_0^n - r - \kappa_Y - \sigma_Y r\gamma_n\beta_n\rho_{0Y})B_0^n - \frac{A_1^n B_1^n}{2\alpha_n r\gamma_n} + 2\kappa_Y C_0^n \bar{Y}, && \text{(A-15)}\\
0 &= -(r + 2\kappa_Y + 2\sigma_Y r\gamma_n\beta_n\rho_{0Y})C_0^n + \frac{(r\gamma_n\beta_n)^2}{2} + 2\sigma_Y^2 (C_0^n)^2 - \frac{(B_1^n)^2}{4\alpha_n r\gamma_n}, && \text{(A-16)}\\
0 &= -(r + \kappa_Y + \sigma_Y r\gamma_n\beta_n\rho_{0Y})B_1^n + (r\gamma_n)^2\beta_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) - r\gamma_n R_2\\
&+ 2\sigma_Y^2 B_1^n C_0^n - 2\sigma_Y r\gamma_n C_0^n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y}) - \frac{A_2^n B_1^n}{2\alpha_n r\gamma_n}. && \text{(A-17)}
\end{aligned}
$$

Similar, we find from equation (A-5):

$$
R_0 + R_1 N_1 + R_2 Y + r(H_0 + H_1 N_1 + H_2 Y) = \frac{\kappa_D \bar{D}}{r + \kappa_D} - H_1 \frac{A_1^1 + A_2^1 N_1 + B_1^1 Y}{2\alpha_1 r\gamma_1} + H_2 \mu_Y.
$$

implying the following relations:

$$
\begin{aligned}
R_0 + rH_0 &= \frac{\kappa_D \bar{D}}{r + \kappa_D} - H_1 \frac{A_1^1}{2\alpha_1 r\gamma_1} + \kappa_Y \bar{Y} H_2, && \text{(A-18)}\\
R_1 + rH_1 &= -H_1 \frac{A_2^1}{2\alpha_1 r\gamma_1}, && \text{(A-19)}\\
R_2 + rH_2 &= -H_1 \frac{B_1^1}{2\alpha_1 r\gamma_1} - \kappa_Y H_2. && \text{(A-20)}
\end{aligned}
$$

Now we consider the stock market clearing condition:

$$
\frac{A_1^1 + A_2^1 N_1 + B_1^1 Y}{\alpha_1\gamma_1} + \frac{A_1^2 + A_2^2 N_2 + B_1^2 Y}{\alpha_2\gamma_2} = 0,
$$

implying the following relations

$$\frac{A_1^1}{\alpha_1\gamma_1} + \frac{A_1^2 + A_2^2}{\alpha_2\gamma_2} = 0, \tag{A-21}$$
$$\frac{A_2^1}{\alpha_1\gamma_1} - \frac{A_2^2}{\alpha_2\gamma_2} = 0, \tag{A-22}$$
$$\frac{B_1^1}{\alpha_1\gamma_1} + \frac{B_1^2}{\alpha_2\gamma_2} = 0. \tag{A-23}$$

First, we use equations (A-16) to find $B_1^n(C_0^n)$:

$$B_1^n = (-1)^{n+1} 2\sqrt{\alpha_n r\gamma_n\left[2\sigma_Y^2(C_0^n)^2 - (r + 2\kappa_Y + 2\sigma_Y r\gamma_n\beta_n\rho_{0Y})C_0^n + \frac{(r\gamma_n\beta_n)^2}{2}\right]}, \tag{A-24}$$

where we set B_1^1 positive so that investor 1 sells stock shares when her endowment is excessive. Note that the last expression is well defined only if $C_0^n \leq C_0^{n-}$, $C_0^n \geq C_0^{n+}$, where

$$C_0^{n\pm} = \frac{r + 2\kappa_Y + 2\sigma_Y r\gamma_n\beta_n\rho_{0Y} \pm \sqrt{(r + 2\kappa_Y + 2\sigma_Y r\gamma_n\beta_n\rho_{0Y})^2 - (2r\sigma_Y\gamma_n\beta_n)^2}}{4\sigma_Y^2}.$$

Hence, from equations (A-23) and (A-24) we find

$$\begin{aligned} & 2\sigma_Y^2(C_0^2)^2 - (r + 2\kappa_Y + 2\sigma_Y r\gamma_2\beta_2\rho_{0Y})C_0^2 + \frac{(r\gamma_2\beta_2)^2}{2} \\ - & \frac{\gamma_2\alpha_2}{\gamma_1\alpha_1}\left[2\sigma_Y^2(C_0^1)^2 - (r + 2\kappa_Y + 2\sigma_Y r\gamma_1\beta_1\rho_{0Y})C_0^1 + \frac{(r\gamma_1\beta_1)^2}{2}\right] = 0. \end{aligned} \tag{A-25}$$

Or,

$$\begin{aligned} C_0^2 = (4\sigma_Y^2)^{-1}\Bigg[&(r + 2\kappa_Y + 2\sigma_Y r\gamma_2\beta_2\rho_{0Y}) + \Bigg\{(r + 2\kappa_Y + 2\sigma_Y r\gamma_2\beta_2\rho_{0Y})^2 - (2r\sigma_Y\gamma_2\beta_2)^2 \\ + & 8\sigma_Y^2\frac{\gamma_2\alpha_2}{\gamma_1\alpha_1}\left(2\sigma_Y^2(C_0^1)^2 - (r + 2\kappa_Y + 2\sigma_Y r\gamma_1\beta_1\rho_{0Y})C_0^1 + \frac{(r\gamma_1\beta_1)^2}{2}\right)\Bigg\}^{1/2}\Bigg], \end{aligned} \tag{A-26}$$

where we set a positive sign in front of the radical to make sure that investor 2, who buys stock when it is very expensive and increasing and sells it when it is very cheap and falling, has a low utility function at these states. Now we apply equations (A-13) and (A-14) to determine A_2^n and R_1 as functions of C_0^1 and H_2:

$$\begin{aligned} 0 &= \frac{(\alpha_1 + \alpha_2)}{2\alpha_1 r\gamma_1}(A_2^1)^2 + r(\alpha_1 + \alpha_2)A_2^1 - r^2\gamma_1\alpha_1(\gamma_1 + \gamma_2)\sigma_S^2 \\ &+ 2r(\gamma_1\alpha_1 - \gamma_2\alpha_2)\,\sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})B_1^1 - \sigma_Y^2\frac{\gamma_1\alpha_1^2 + \gamma_2\alpha_2^2}{\gamma_1\alpha_1}(B_1^1)^2, \end{aligned}$$

implying

$$A_2^1(C_0^1,H_2) = \alpha_1 r\gamma_1\Big\{-r+\Big[r^2+\frac{2}{\alpha_1 r\gamma_1(\alpha_1+\alpha_2)}\Big(2r\sigma_Y(\gamma_2\alpha_2-\gamma_1\alpha_1) \tag{A-27}$$

$$\times \; (\rho_{DY}\sigma_{S_D}+\sigma_{S_Y})B_1^1+r^2\gamma_1\alpha_1(\gamma_1+\gamma_2)\sigma_S^2+\sigma_Y^2\frac{\gamma_1\alpha_1^2+\gamma_2\alpha_2^2}{\gamma_1\alpha_1}(B_1^1)^2\Big)\Big]^{1/2}\Big\}$$

$$R_1(C_0^1,H_2) = -\frac{(A_2^1)^2}{4\alpha_1(r\gamma_1)^2}-\frac{A_2^1}{2\gamma_1}+\sigma_Y^2\frac{(B_1^1)^2}{2r\gamma_1}-\sigma_Y(\rho_{DY}\sigma_{S_D}+\sigma_{S_Y})B_1^1+\frac{1}{2}r\gamma_1\sigma_S^2, \tag{A-28}$$

where we set the sign in front of the radical positive so that A_2^1 is positive as well.

Next, we use equation (A-19) to find H_1 and equation (A-20) to find R_2:

$$H_1(C_0^1,H_2) = -\frac{2\alpha_1 r\gamma_1 R_1}{2\alpha_1 r^2\gamma_1+A_2^1}, \tag{A-29}$$

$$R_2(C_0^1,H_2) = -\frac{H_1B_1^1}{2\alpha_1 r\gamma_1}-(r+\kappa_Y)H_2. \tag{A-30}$$

Now, we use equations (A-17) for $n=1,2$ to find C_0^1 and H_2. First, we find H_2 in terms of C_0^1 and then we find C_0^1 numerically. From equations (A-17) we obtain

$$\Big[2\sigma_Y^2(C_0^1\alpha_1+C_0^2\alpha_2)-(\kappa_Y+\frac{r}{2})(\alpha_1+\alpha_2)-\sigma_Y\rho_{0Y}(r\gamma_1\beta_1\alpha_1+r\gamma_2\beta_2\alpha_2)\Big]\frac{B_1^1}{r\gamma_1}$$

$$-\; 2\alpha_1\sigma_Y(\rho_{DY}\sigma_{S_D}+\sigma_{S_Y})(C_0^1-C_0^2)+\alpha_1 r(\gamma_1\beta_1-\gamma_2\beta_2)(\rho_{0D}\sigma_{S_D}+\rho_{0Y}\sigma_{S_Y})$$

$$= \frac{B_1^1}{2r\gamma_1}(\alpha_1+\alpha_2)\Big[r^2-\frac{2}{\alpha_1 r\gamma_1(\alpha_1+\alpha_2)}\Big(2r(\gamma_1\alpha_1-\gamma_2\alpha_2)\,\sigma_Y(\rho_{DY}\sigma_{S_D}+\sigma_{S_Y})B_1^1$$

$$-\; r^2\gamma_1\alpha_1(\gamma_1+\gamma_2)\sigma_S^2-\sigma_Y^2\frac{\gamma_1\alpha_1^2+\gamma_2\alpha_2^2}{\gamma_1\alpha_1}(B_1^1)^2\Big)\Big]^{1/2}.$$

Or,

$$0 = \frac{(2r\gamma_1)^2}{[B_1^1(\alpha_1+\alpha_2)]^2}\Big(\Big[2\sigma_Y^2(C_0^1\alpha_1+C_0^2\alpha_2)-(\kappa_Y+\frac{r}{2})(\alpha_1+\alpha_2) \tag{A-31}$$

$$-\; \sigma_Y\rho_{0Y}(r\gamma_1\beta_1\alpha_1+r\gamma_2\beta_2\alpha_2)\Big]\frac{B_1^1}{r\gamma_1}-2\alpha_1\sigma_Y\rho_{DY}\sigma_{S_D}(C_0^1-C_0^2)+$$

$$+\; \alpha_1 r(\gamma_1\beta_1-\gamma_2\beta_2)\rho_{0D}\sigma_{S_D}+\alpha_1[r(\gamma_1\beta_1-\gamma_2\beta_2)\rho_{0Y}-2\sigma_Y(C_0^1-C_0^2)]H_2\sigma_Y\Big)^2$$

$$+\; \frac{2}{\alpha_1 r\gamma_1(\alpha_1+\alpha_2)}\Big(2r(\gamma_1\alpha_1-\gamma_2\alpha_2)\,\sigma_Y(\rho_{DY}\sigma_{S_D}+H_2\sigma_Y)B_1^1$$

$$-\; r^2\gamma_1\alpha_1(\gamma_1+\gamma_2)(\sigma_{S_D}^2+2\rho_{DY}\sigma_{S_D}\sigma_Y H_2+\sigma_Y^2H_2^2)-\sigma_Y^2\frac{\gamma_1\alpha_1^2+\gamma_2\alpha_2^2}{\gamma_1\alpha_1}(B_1^1)^2\Big)-r^2.$$

Now the solution of the last equations with respect to H_2 is obvious:

$$H_2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}, \tag{A-32}$$

where the sign in front of the radical is chosen to make H_2 positive and the expressions for coefficients a, b, and c follow from equation (A-31) and are not presented for the purpose of compactness.

Next, C_0^1 is calculated from equation (A-17):

$$(\alpha_1 + \alpha_2)R_2 = 2\sigma_Y^2 \left[\frac{\alpha_2 B_1^1}{r\gamma_1}(C_0^1 - C_0^2) - H_2(C_0^1\alpha_2 + C_0^2\alpha_1) \right], \quad \text{(A-33)}$$

where B_1^1 and C_0^2 are given by equations (A-24) and (A-26), respectively, while R_2 is defined from equation (A-30).

We have determined coefficients B_1^n, C_0^n, A_2^n, $n = 1, 2$, $H_2\,R_2$, and R_1. Next, we find the rest of coefficients. We use equation (A-21) to determine A_1^2 versus A_1^1:

$$A_1^2(A_1^1) = -\frac{\gamma_2\alpha_2}{\gamma_1\alpha_1}(A_2^1 + A_1^1).$$

Equations (A-15) suggests

$$B_0^n(A_1^1) \;=\; \frac{\frac{A_1^n B_1^n}{2\alpha_n r\gamma_n} - 2\kappa_Y \bar{Y} C_0^n}{2\sigma_Y^2 C_0^n - r - \kappa_Y - \sigma_Y \rho_{0Y} r\gamma_n\beta_n}. \quad \text{(A-34)}$$

Now we take advantage of linear equations (A-11) and (A-12) to determine A_1^1 and R_0:

$$R_0 \;=\; -\frac{A_1^1}{\gamma_1} + \frac{\sigma_Y^2}{r\gamma_1}B_0^1 B_1^1 - \frac{A_1^1 A_2^1}{2\alpha_1(r\gamma_1)^2} + \frac{\kappa_Y \bar{Y}}{r\gamma_1}B_1^1 - \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})B_0^1, \quad \text{(A-35)}$$

where

$$\begin{aligned} A_1^1 \;&=\; \left[\frac{A_2^1}{\gamma_1\alpha_1}\alpha_2 + \frac{(A_2^1)^2}{2(r\gamma_1\alpha_1)^2}\alpha_2 - R_1 - \kappa_Y \bar{Y}\frac{B_1^1}{r\gamma_1}\left(1 + \frac{\alpha_2}{\alpha_1}\right) \right. \quad \text{(A-36)} \\ &+\; \left. z_1[\sigma_Y^2 \frac{B_1^1}{r\gamma_1} - \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})] + z_2[\sigma_Y^2 \frac{B_1^1}{r\gamma_1} + \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})] \right] \pi^{-1}, \end{aligned}$$

and

$$\begin{aligned} \pi \;&=\; x_1[\sigma_Y^2 \frac{B_1^1}{r\gamma_1} - \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})] + x_2[\sigma_Y^2 \frac{B_1^1\alpha_2}{r\gamma_1\alpha_1} + \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})] \quad \text{(A-37)} \\ &-\; \frac{(\alpha_1 + \alpha_2)}{\gamma_1\alpha_1} - \frac{A_2^1}{2(r\gamma_1\alpha_1)^2}(\alpha_1 + \alpha_2), \quad x_n = \frac{(-1)^{n+1}B_1^n}{2\alpha_1 r\gamma_1(2\sigma_Y^2 C_0^n - r - \kappa_Y - \sigma_Y\rho_{0Y} r\gamma_n\beta_n)}, \\ z_1 \;&=\; \frac{2\kappa_Y \bar{Y} C_0^1}{2\sigma_Y^2 C_0^1 - r - \kappa_Y - \sigma_Y\rho_{0Y} r\gamma_1\beta_1}, \quad z_2 = \frac{\frac{A_2^1 B_1^2}{2\alpha_1 r\gamma_1} + 2\kappa_Y \bar{Y} C_0^2}{2\sigma_Y^2 C_0^2 - r - \kappa_Y - \sigma_Y\rho_{0Y} r\gamma_2\beta_2}. \end{aligned}$$

Finally, we find coefficients A_0^n and H_0 by using equations (A-10) and (A-18), respectively. Q.E.D.

Proof of corollary 3. We have

$$E_0(N_t^1) = \int_0^t E_0(u_s^1)ds = v_0^1 t + \int_0^t E_0(-\varphi Y_s + v_2^1 N^1)ds$$
$$= v_0^1 t + \int_0^t \{-\varphi[Y_0 e^{-\kappa_Y s} + \bar{Y}(1-e^{-\kappa_Y s})] + v_2^1 E_0(N_s^1)\}ds, \qquad \text{(A-38)}$$

where $v_0^1 = -\frac{A_1^1}{2\alpha_1 r\gamma_1}$, $v_1^1 = -\frac{B_1^1}{2\alpha_1 r\gamma_1}$, $v_2^1 = -\frac{A_2^1}{2\alpha_1 r\gamma_1}$. The last integral equation can be written as

$$X'(t) = v_0^1 + v_1^1[Y_0 e^{-\kappa_Y t} + \bar{Y}(1-e^{-\kappa_Y t})] + v_2^1 X(t), \qquad \text{(A-39)}$$

where $X(t) = E_0(N_{1t})$. The solution of homogeneous equation

$$X'(t) = v_2^1 X(t) \qquad \text{(A-40)}$$

is given by $X(t) = be^{v_2^1 t}$, where c is a constant. Now we assume that b depend on t and substitute the trial solution into equations (A-39): $b'(t)e^{v_2^1 t} = v_0^1 - \varphi[Y_0 e^{-\kappa_Y t} + \bar{Y}(1-e^{-\kappa_Y t})]$. It follows that $b'(t) = (v_0^1 - \varphi\bar{Y})e^{-v_2^1 t} - \varphi(Y_0 - \bar{Y})e^{-(\kappa_Y + v_2^1)t}$. We assume that $b(0) = b_0$ then it follows that

$$b(t) = b_0 - \varphi\frac{Y_0 - \bar{Y}}{v_2^1 + \kappa_Y}(1 - e^{-(v_2^1+\kappa_Y)t}) + \frac{v_0^1 - \varphi\bar{Y}}{v_2^1}(1 - e^{-v_2^1 t})$$

resulting in

$$X(t) = b_0 e^{v_2^1 t} - \varphi\frac{Y_0 - \bar{Y}}{v_2^1 + \kappa_Y}(e^{v_2^1 t} - e^{-\kappa_Y t}) + \frac{v_0^1 - \varphi\bar{Y}}{v_2^1}(e^{v_2^1 t} - 1) \qquad \text{(A-41)}$$

Equation (34) follows since $v_2^1 < 0$. Similar,

$$E_0(u_{1t}) = v_0^1 - \varphi E_0(Y_t) - \xi E_0(N_t^1) = v_0^1 - \varphi[Y_0 e^{-\kappa_Y t} + \bar{Y}(1-e^{-\kappa_Y t})]$$
$$+ \quad -\xi\left[c_0 e^{v_2^1 t} - \varphi\frac{Y_0 - \bar{Y}}{v_2^1 + \kappa_Y}(e^{v_2^1 t} - e^{-\kappa_Y t}) + \frac{v_0^1 - \varphi\bar{Y}}{v_2^1}(e^{v_2^1 t} - 1)\right] \qquad \text{(A-42)}$$

Taking the limit in the above equation results in

$$\lim_{t\to\infty} E_0(u_t^1) = v_0^1 + \bar{Y} - \varphi - (v_0^1 - \varphi\bar{Y}) = 0. \qquad \text{(A-43)}$$

Finally, equation (36) follows from equations (30) and (34). Q.E.D.

Proof of corollary 5

$$Cov_t(dS_t, dS_{t'}) = Cov_t\left(\frac{dD_t}{r+\kappa_D} + H_1 dN_t^1 + H_2 dY_t, \frac{dD_{t'}}{r+\kappa_D} + H_1 dN_{t'}^1 + H_2 dY_{t'}\right)$$

$$\begin{aligned}
&= Cov_t\left(\frac{\sigma_D dW_{Dt}}{r+\kappa_D} + H_2\sigma_Y dW_{Yt}, \frac{\kappa_D(\bar{D} - D_{t'})}{r+\kappa_D} + H_1 u_{t'}^1 + H_2\kappa_Y(\bar{Y} - Y_{t'})\right)dt' \\
&= -Cov_t\left(\frac{\sigma_D dW_{Dt}}{r+\kappa_D} + H_2\sigma_Y dW_{Yt}, \frac{\kappa_D D_{t'}}{r+\kappa_D} + H_1(\varphi Y_{t'} + \xi N_{t'}^1) + H_2\kappa_Y Y_{t'}\right)dt' \\
&= -Cov_t\left(\frac{\sigma_D dW_{Dt}}{r+\kappa_D} + H_2\sigma_Y dW_{Yt}, \frac{\kappa_D D_{t'}}{r+\kappa_D} + (H_1\varphi + H_2\kappa_Y)Y_{t'} + H_1\xi N_{t'}^1\right)dt'.
\end{aligned} \tag{A-44}$$

To find the last covariance we have to consider the following differential equations

$$\begin{aligned}
dD &= -\kappa_D D dt', \qquad D_k(t) = \sigma_D dW_{Dt}, \\
dY &= -\kappa_Y Y dt', \qquad Y(t) = \sigma_Y dW_{Yt}, \\
dN^1 &= -(\varphi Y + \xi N^1)dt', \quad N^1(t) = 0.
\end{aligned}$$

The solutions of these equations are straightforward:

$$D_{t'} = D_t \exp[-\kappa_D(t'-t)], \qquad D_t = \sigma_D dW_{Dt}, \tag{A-45}$$

$$Y_{t'} = Y_t \exp[-\kappa_Y(t'-t)], \qquad Y_t = \sigma_Y dW_{Yt}, \tag{A-46}$$

$$N^1(t') = \frac{\varphi Y_t}{\kappa_Y - \xi}[e^{-\kappa_Y(t'-t)} - e^{-\xi(t'-t)}]. \tag{A-47}$$

Substitution of equations (B-34), (B-35), and (A-47) into the last line of expression (A-44) results in

$$\begin{aligned}
& Cov_t(dS_t, dS_{t'})/dtdt' \\
&= -Cov_t\left(\frac{\sigma_D dW_{Dt}}{r+\kappa_D} + H_2\sigma_Y dW_{Yt}, \frac{\kappa_D\sigma_D e^{-\kappa_D(t'-t)}dW_{Dt}}{r+\kappa_D} + (H_1\varphi + H_2\kappa_Y)\right. \\
&\times \left. e^{-\kappa_Y(t'-t)}\sigma_Y dW_{Yt} + \sigma_Y H_1\xi\frac{\varphi}{\kappa_Y - \xi}[e^{-\kappa_Y(t'-t)} - e^{-\xi(t'-t)}]dW_{Yt}\right)/dt \\
&= -\frac{\kappa_D\sigma_D}{r+\kappa_D}e^{-\kappa_D(t'-t)}\left(\frac{\sigma_D}{r+\kappa_D} + H_2\sigma_Y\rho_{DY}\right) - \sigma_Y e^{-\kappa_Y(t'-t)}\left(\frac{\sigma_D\rho_{DY}}{r+\kappa_D} + H_2\sigma_Y\right) \\
&\times \left(H_1\varphi + H_2\kappa_Y + H_1\frac{\varphi\xi}{\kappa_Y - \xi}[1 - e^{(\kappa_Y - \xi)(t'-t)}]\right).
\end{aligned} \tag{A-48}$$

The derivation of the second correlation coefficient is similar:

$$Cov_t(du_{1t}, dS_{t'}) = -Cov_t\Bigg(\varphi dY_t + \xi dN_t^1, \frac{dD_{t'}}{r+\kappa_D} + H_1 dN_{t'}^1 + H_2 dY_{t'}\Bigg)$$

$$= \varphi\sigma_Y Cov_t\Bigg(dW_{Yt}, \frac{\kappa_D D_{t'}}{r+\kappa_D} + (H_1\varphi + H_2\kappa_Y)Y_{t'} + H_1\xi N_{t'}^1\Bigg)dt'$$

$$= \varphi\sigma_Y Cov_t\Bigg(dW_{Yt}, \frac{\kappa_D\sigma_D e^{-\kappa_D(t'-t)}dW_{Dt}}{r+\kappa_D} + (H_1\varphi + H_2\kappa_Y)e^{-\kappa_Y(t'-t)}\sigma_Y dW_{Yt}$$

$$+ \sigma_Y H_1\xi\frac{\varphi}{\kappa_Y - \xi}[e^{-\kappa_Y(t'-t)} - e^{-\xi(t'-t)}]dW_{Yt}\Bigg)dt'$$

$$= \varphi\sigma_Y\Bigg(\frac{\kappa_D\sigma_D\rho_{DY}e^{-\kappa_D(t'-t)}}{r+\kappa_D} + e^{-\kappa_Y(t'-t)}\sigma_Y\Bigg[H_1\varphi + H_2\kappa_Y + \frac{H_1\xi\varphi}{\kappa_Y - \xi}[1 - e^{(\kappa-\xi)(t'-t)}]\Bigg]\Bigg)dtdt'.$$

Q.E.D.

Appendix B

Proof of Theorem 1. It follows from portfolio selection problem (4)–(5) that investor n has to solve the following HJB equation for her value function $V^n(X_n, Y)$

$$0 = \max_{c_n\in R^+, N_n\in R}\Bigg\{-\frac{1}{\gamma_n}\exp(-\gamma_n c_n) - \tau_n V^n + \frac{1}{2}\Bigg[(Y_n)^2 + 2Y_n N_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y})$$

$$+ (N_n)^2\sigma_S^2\Bigg]V_{X_nX_n}^n + \frac{1}{2}\sigma_Y^2 V_{YY}^n + \sigma_Y[Y_n\rho_{0Y} + N_n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]V_{X_nY}^n + \mu_Y V_Y^n$$

$$+ \Bigg[(\mu_S - rS)N_n + rX_n - c_n\Bigg]V_{X_n}^n\Bigg\},$$

where $\sigma_S^2 = \sigma_{S_D}^2 + 2\rho_{DY}\sigma_{S_D}\sigma_{S_Y} + \sigma_{S_Y}^2$.

We conjecture that $V^n(X_n, Y) = -\frac{1}{\gamma_n}\exp[-\gamma_n r X_n + g^n(Y)]$. Then

$$c_n = -\frac{1}{\gamma_n}[\ln(r) + g^n] + rX_n$$

and g^n solves the following ODE

$$0 = -r[\ln(r) + g^n] - \tau_n + r + \frac{(r\gamma_n)^2}{2}\Bigg[(Y_n)^2 + 2Y_n N_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y})$$

$$+ (N_n)^2\sigma_S^2\Bigg] - r\gamma_n(\mu_S - rS)N_n + \mu_Y g_Y^n + \frac{1}{2}\sigma_Y^2[g_{YY}^n + (g_Y^n)^2]$$

$$- \sigma_Y r\gamma_n[Y_n\rho_{0Y} + N_n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]g_Y^n, \qquad \text{(B-1)}$$

where N_n is given by

$$N_n = \frac{(\mu_S - rS) + \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})g_Y^n - Y_n r\gamma_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y})}{r\gamma_n\sigma_S^2}. \quad \text{(B-2)}$$

The stock market clearing suggests that $N_1 + N_2 = 1$. Now let us consider a trial solution

$$g^n = A^n + B^n Y + C^n Y^2 \quad \text{(B-3)}$$

and assume that volatility of the stock returns is state independent, while

$$\mu_S - rS = R_0 + YR_2, \quad \text{(B-4)}$$

where A^n, B^n, C^n, R_0, R_2 are the coefficients to be determined. Similar, we assume that

$$N_n = G_1^n + YG_2^n \quad \text{(B-5)}$$

with G_1^n, G_2^n being constants.

Next find the stock price. The drift of the stock returns is given by

$$rS + R_0 + R_2 Y = D + S_D\mu_D + S_Y\mu_Y + \frac{1}{2}\sigma_D^2 S_{DD} + \frac{1}{2}\sigma_Y^2 S_{YY} + \rho_{DY}\sigma_D\sigma_Y S_{DY}. \quad \text{(B-6)}$$

We conjecture that

$$S = H_0 + \frac{D}{r+\kappa_D} + H_2 Y$$

where H_0, H_2 are constants. Then equation (B-6) becomes

$$r(H_0 + \frac{D}{r+\kappa_D} + H_2 Y) + R_0 + R_2 Y = D + \frac{\mu_D}{r+\kappa_D} + H_2\mu_Y. \quad \text{(B-7)}$$

Matching the coefficients results in

$$H_0 = \frac{1}{r}\left(-R_0 + \frac{\kappa_D\bar{D}}{r+\kappa_D} - \kappa_Y\bar{Y}\frac{R_2}{r+\kappa_Y}\right), \quad \text{(B-8)}$$

$$H_2 = -\frac{R_2}{r+\kappa_Y}. \quad \text{(B-9)}$$

It follows that $\sigma_{S_D} = \frac{\sigma_D}{r+\kappa_D}$, $\sigma_{S_Y} = H_2\sigma_Y = -\frac{R_2}{r+\kappa_Y}\sigma_Y$.

We conclude that the number of unknown coefficients to be determined is 12. From the stock market clearing condition we find $G_1^1 + YG_2^1 + G_1^2 + YG_2^2 = 1$. The latter relation suggests

$$G_1^2 = 1 - G_1^1 \quad \text{(B-10)}$$

$$G_2^2 = -G_2^1. \quad \text{(B-11)}$$

Then from equations (B-2) we find

$$\begin{aligned} 0 &= r\gamma_n\Big[(G_1^n + YG_2^n)\sigma_S^2 + Y\beta_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y})\Big] \\ &- (R_0 + YR_2) - \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})(B^n + 2C^nY). \end{aligned}$$

Matching the coefficients in the last equation provides the following equations

$$0 = r\gamma_n G_1^n\sigma_S^2 - R_0 - \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})B^n, \tag{B-12}$$

$$0 = r\gamma_n\Big[G_2^n\sigma_S^2 + \beta_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y})\Big] - R_2 - 2\sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})C^n. \tag{B-13}$$

Moreover, ODE's (B-1) for the two agents now can be written as

$$\begin{aligned} 0 &= -r[\ln(r) + A^n + B^nY + C^nY^2] - \tau_n + r + \frac{(r\gamma_n)^2}{2}\Big[(Y_n)^2 + 2Y_n(G_1^n + YG_2^n) \\ &\times (\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) + (G_1^n + YG_2^n)(G_1^n + YG_2^n)\sigma_S^2\Big] \\ &- r\gamma_n(R_0 + YR_2)(G_1^n + YG_2^n) + \frac{1}{2}\sigma_Y^2[2C^n + (B^n + 2C^nY)^2] \\ &+ \Big(\mu_Y - \sigma_Y r\gamma_n[Y_n\rho_{0Y} + (G_1^n + YG_2^n)(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]\Big)(B^n + 2C^nY). \end{aligned}$$

Matching the coefficients gives us 6 more equations:

$$\begin{aligned} A^n &= 1 - \ln(r) - \frac{\tau_n}{r} + \frac{r\gamma_n^2}{2}(G_1^n)^2\sigma_S^2 - \gamma_n R_0 G_1^n + \frac{1}{2r}\sigma_Y^2[2C^n + (B^n)^2] \\ &+ [\kappa_Y\bar{Y} - \sigma_Y r\gamma_n G_1^n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]\frac{B^n}{r}, \end{aligned} \tag{B-14}$$

$$\begin{aligned} 0 &= -rB^n + (r\gamma_n)^2\Big[\beta_n G_1^n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) + G_1^nG_2^n\sigma_S^2\Big] + 2\sigma_Y^2B^nC^n \\ &- r\gamma_n(R_0G_2^n + G_1^nR_2) + 2\Big(\kappa_Y\bar{Y} - \sigma_Y r\gamma_n G_1^n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})\Big)C^n \\ &- \Big(\kappa_Y + \sigma_Y r\gamma_n[\beta_n\rho_{0Y} + G_2^n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]\Big)B^n, \end{aligned} \tag{B-15}$$

$$\begin{aligned} 0 &= \frac{(r\gamma_n)^2}{2}\Big[(\beta_n)^2 + 2\beta_n G_2^n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) + (G_2^n)^2\sigma_S^2\Big] - r\gamma_n R_2 G_2^n + 2\sigma_Y^2(C^n)^2 \\ &- 2\Big(\kappa_Y + \sigma_Y r\gamma_n[\beta_n\rho_{0Y} + G_2^n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]\Big)C^n - rC^n. \end{aligned} \tag{B-16}$$

Notice that with the help of equation (B-13) the last equation can be rewritten as

$$0 = \frac{(r\gamma_n)^2}{2}\Big[(\beta_n)^2 - (G_2^n)^2\sigma_S^2\Big] + 2\sigma_Y^2(C^n)^2 - [r + 2(\kappa_Y + \sigma_Y r\gamma_n\beta_n\rho_{0Y})]C^n. \tag{B-17}$$

We use equations (B-13) to find $H_2(C^1, C^2)$:

$$H_2 \quad = \quad \frac{r\gamma_1\gamma_2(\beta_1+\beta_2)\rho_{0D}\sigma_{S_D} - 2\sigma_Y\rho_{DY}\sigma_{S_D}(\gamma_2C^1+\gamma_1C^2)}{2\sigma_Y^2(\gamma_2C^1+\gamma_1C^2) - (r+\kappa_Y)(\gamma_1+\gamma_2) - r\gamma_1\gamma_2(\beta_1+\beta_2)\rho_{0Y}\sigma_Y} \tag{B-18}$$

if $2\sigma_Y^2(\gamma_2C^1+\gamma_1C^2) - (r+\kappa_Y)(\gamma_1+\gamma_2) - r\gamma_1\gamma_2(\beta_1+\beta_2)\rho_{0Y}\sigma_Y \neq 0$.

The following solution of the problem applies to the case when $\rho_{DY} > 0$. Suppose that $2\sigma_Y^2(\gamma_2C^1+\gamma_1C^2) - (r+\kappa_Y)(\gamma_1+\gamma_2) - r\gamma_1\gamma_2(\beta_1+\beta_2)\rho_{0Y}\sigma_Y = 0$. Then we must have $r\gamma_1\gamma_2(\beta_1+\beta_2)\rho_{0D}\sigma_{S_D} - 2\sigma_Y\rho_{DY}\sigma_{S_D}(\gamma_2C^1+\gamma_1C^2) = 0$ which is possible only if

$$r\gamma_1\gamma_2\sigma_Y(\beta_1+\beta_2)(\rho_{0D} - \rho_{0Y}\rho_{DY}) = (r+\kappa_Y)(\gamma_1+\gamma_2)\rho_{DY}.$$

The latter equality implies that $H_2 = -\frac{\sigma_{SD}\rho_{DY}}{\sigma_Y} < 0$. Therefore, neither nominator nor denominator in (B-18) can be zero.

If $\rho_{DY} = 0$ then the nominator in equation (B-18) is zero if $(\beta_1+\beta_2)\rho_{0D} = 0$. In this case we find that

$$2\sigma_Y^2(\gamma_2C^1+\gamma_1C^2) - (r+\kappa_Y)(\gamma_1+\gamma_2) - r\gamma_1\gamma_2(\beta_1+\beta_2)\rho_{0Y}\sigma_Y = 0 \tag{B-19}$$

and equation (B-18) does not apply. The case of $\rho_{DY} = 0$ will be considered after equation (B-26) below.

Assume that $\rho_{DY} > 0$, equation (B-17) provides us with a quadratic equation for C^2 in terms of C^1:

$$\begin{aligned} 0 \quad &= \quad \frac{1}{2}[(\beta_1)^2 - (\beta_2)^2] + 2\sigma_Y^2\left[\frac{(C^1)^2}{(r\gamma_1)^2} - \frac{(C^2)^2}{(r\gamma_2)^2}\right] \\ &+ \quad [r + 2(\kappa_Y + \sigma_Y r\gamma_2\beta_2\rho_{0Y})]\frac{C^2}{(r\gamma_2)^2} - [r + 2(\kappa_Y + \sigma_Y r\gamma_1\beta_1\rho_{0Y})]\frac{C^1}{(r\gamma_1)^2}. \end{aligned} \tag{B-20}$$

The last quadratic equation can be easily solved to find $C^2(C^1)$:

$$\begin{aligned} C^2(C^1) \quad &= \quad \frac{1}{(2\sigma_Y)^2}\Bigg[r + 2(\kappa_Y + \sigma_Y r\gamma_2\beta_2\rho_{0Y}) + \Bigg\{[r + 2(\kappa_Y + \sigma_Y r\gamma_2\beta_2\rho_{0Y})]^2 \\ &+ \quad 8(\sigma_Y r\gamma_2)^2\left[\frac{1}{2}[(\beta_1)^2 - (\beta_2)^2] + 2\sigma_Y^2\frac{(C^1)^2}{(r\gamma_1)^2} - [r + 2(\kappa_Y + \sigma_Y r\gamma_1\beta_1\rho_{0Y})]\frac{C^1}{(r\gamma_1)^2}\right]\Bigg\}^{1/2}\Bigg], \end{aligned} \tag{B-21}$$

where, as in the case with capital delays, we set a positive sign in front of the radical to make sure that investor 2 has a low utility function when the stock is very expensive (Y is big and positive) and when the stock is very cheap (Y is negative and large in magnitude). Now we find from equation (B-13)

$$G_2^n \quad = \quad \frac{R_2 + 2\sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})C^n - r\gamma_n\beta_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y})}{r\gamma_n\sigma_S^2}. \tag{B-22}$$

Then we solve nonlinear equation (B-17) for G_2^n:

$$G_2^n = (-1)^n\frac{1}{r\gamma_n\sigma_S}\sqrt{(r\gamma_n\beta_n)^2 + 4\sigma_Y^2(C^n)^2 - 2[r + 2(\kappa_Y + \sigma_Y r\gamma_n\beta_n\rho_{0Y})]C^n}, \tag{B-23}$$

where we choose G_2^1 negative so that investor 1 decreases her holdings of the stock under a positive shock to endowment to hedge her endowment risk. It follows from equations (B-22) and (B-23) that

$$
\begin{aligned}
&(\sigma_{S_D}^2 + 2\rho_{DY}\sigma_{S_D}\sigma_Y H_2 + \sigma_Y^2 H_2^2)\Big[(r\gamma_n\beta_n)^2 + 4\sigma_Y^2(C^n)^2 - 2[r + 2(\kappa_Y + \sigma_Y r\gamma_n\beta_n\rho_{0Y})]C^n\Big] \\
= &\Big[H_2(2\sigma_Y^2 C^n - r\sigma_Y\gamma_n\beta_n\rho_{0Y} - r - \kappa_Y) + 2\sigma_Y\rho_{DY}\sigma_{S_D}C^n - r\gamma_n\beta_n\rho_{0D}\sigma_{S_D}\Big]^2. \qquad \text{(B-24)}
\end{aligned}
$$

Equation (B-24) is not singular only if C^n is outside of the range $]C_-^n, C_+^n[$, where

$$C_\pm^n = \frac{r + 2(\kappa_Y + \sigma_Y r\gamma_n\beta_n\rho_{0Y}) \pm \sqrt{(r + 2(\kappa_Y + \sigma_Y r\gamma_n\beta_n\rho_{0Y}))^2 - (2r\gamma_n\beta_n\sigma_Y)^2}}{4\sigma_Y^2}.$$

Equation (B-24) for $n = 1$ can be used to find H_2 in terms of C^1:

$$H_2 = \frac{-\phi \pm \sqrt{\phi^2 - \varphi\psi}}{\psi}, \qquad \text{(B-25)}$$

where

$$
\begin{aligned}
\psi &= \sigma_Y^2[(r\gamma_1\beta_1)^2 + 2rC^1] - (r\sigma_Y\gamma_1\beta_1\rho_{0Y} + r + \kappa_Y)^2, \\
\phi &= \sigma_Y\sigma_{S_D}(r\gamma_1\beta_1)^2(\rho_{DY} - \rho_{0D}\rho_{0Y}) - (r + \kappa_Y)r\gamma_1\beta_1\rho_{0D}\sigma_{S_D} \\
&+ 2\sigma_Y^2\sigma_{S_D}r\gamma_1\beta_1(\rho_{0D} - \rho_{0Y}\rho_{DY})C^1, \\
\varphi &= (2\sigma_Y\sigma_{S_D}C^1)^2(1 - \rho_{DY}^2) - 2\sigma_{S_D}^2[r + 2(\kappa_Y + \sigma_Y r\gamma_1\beta_1(\rho_{0Y} - \rho_{0D}\rho_{DY}))]C^1 \\
&+ (r\gamma_1\beta_1\sigma_{S_D})^2(1 - \rho_{0D}^2).
\end{aligned}
$$

The sign before the square root should set H_2 positive. Matching the last expression for H_2 with (B-18) and replacing C^2 in terms of C^1 provide us with a nonlinear equation for C^1:

$$\frac{r\gamma_1\gamma_2(\beta_1 + \beta_2)\rho_{0D}\sigma_{S_D} - 2\sigma_Y\rho_{DY}\sigma_{S_D}(\gamma_2 C^1 + \gamma_1 C^2(C^1))}{2\sigma_Y^2(\gamma_2 C^1 + \gamma_1 C^2(C^1)) - (r + \kappa_Y)(\gamma_1 + \gamma_2) - r\gamma_1\gamma_2(\beta_1 + \beta_2)\rho_{0Y}\sigma_Y} = \frac{-\phi \pm \sqrt{\phi^2 - \varphi\psi}}{\psi}. \qquad \text{(B-26)}$$

Now let us consider a special case when $\rho_{DY} = (\beta_1 + \beta_2)\rho_{0D} = 0$. By using equations (B-19) and (B-21) we find coefficients C^1, C^2:

$$
\begin{aligned}
C^1 &= \frac{[(r + \kappa_Y)(\gamma_1 + \gamma_2) + r\gamma_1\gamma_2\rho_{0Y}\sigma_Y(\beta_1 + \beta_2)][r\gamma_2 + \kappa_Y(\gamma_2 - \gamma_1) + r\gamma_1\gamma_2\rho_{0Y}\sigma_Y(\beta_1 - \beta_2)]}{2\sigma_Y^2 r\gamma_2(\gamma_1 + \gamma_2)} \\
&- \frac{r\gamma_1^2\gamma_2(\beta_1^2 - \beta_2^2)}{2(\gamma_1 + \gamma_2)}, \\
C^2 &= \frac{[(r + \kappa_Y)(\gamma_1 + \gamma_2) + r\gamma_1\gamma_2\rho_{0Y}\sigma_Y(\beta_1 + \beta_2)][r\gamma_1 + \kappa_Y(\gamma_1 - \gamma_2) + r\gamma_1\gamma_2\rho_{0Y}\sigma_Y(\beta_2 - \beta_1)]}{2\sigma_Y^2 r\gamma_1} \\
&- \frac{r\gamma_2^2\gamma_1(\beta_2^2 - \beta_1^2)}{2(\gamma_1 + \gamma_2)}.
\end{aligned}
$$

H_2 is found from equation (B-25). Moreover, in the special case when $\rho_{DY} = \rho_{0D} = 0$, H_2 becomes

$$H_2 = \sqrt{-\frac{\varphi}{\psi}},$$

where

$$\begin{aligned}
\psi &= \sigma_Y^2[(r\gamma_1\beta_1)^2 + 2rC^1] - (r\sigma_Y\gamma_1\beta_1\rho_{0Y} + r + \kappa_Y)^2, \\
\varphi &= \sigma_{S_D}^2[(2\sigma_Y C^1)^2 - 2[r + 2(\kappa_Y + \sigma_Y r\gamma_1\beta_1\rho_{0Y})]C^1 + (r\gamma_1\beta_1)^2].
\end{aligned}$$

Next, we utilize equations (B-12) to find G_1^1 and R_0 versus B^1 and B^2:

$$R_0 = \frac{r\gamma_1\gamma_2}{\gamma_1+\gamma_2}\sigma_S^2 - \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})\frac{\gamma_2 B^1 + \gamma_1 B^2}{\gamma_1+\gamma_2}, \tag{B-27}$$

$$G_1^1 = \frac{\gamma_2}{\gamma_1+\gamma_2} + \frac{\sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})(B^1 - B^2)}{r(\gamma_1+\gamma_2)\sigma_S^2}. \tag{B-28}$$

Hereafter, we use equations (B-12),(B-13) and rewrite the two joint linear equations (B-15) as

$$0 = -J(r\gamma_n\sigma_S)^2 G_1^n G_2^n + (2\sigma_Y^2 C^n - r - \kappa_Y - \rho_{0Y}\sigma_Y r\gamma_n\beta_n)B^n + 2\kappa_Y \bar{Y} C^n$$

and solve them to find B^1 and B^2:

$$B^1 = \frac{d_2\omega_{12} - d_1\omega_{22}}{\omega_{11}\omega_{22} - \omega_{12}\omega_{21}}, \tag{B-29}$$

$$B^2 = \frac{d_1\omega_{21} - d_2\omega_{11}}{\omega_{11}\omega_{22} - \omega_{12}\omega_{21}}, \tag{B-30}$$

where

$$d_n = 2\kappa_Y\bar{Y}C^n - \frac{\gamma_n}{\gamma_1+\gamma_2}(r\gamma_n\sigma_S)^2 G_2^n, \tag{B-31}$$

$$\omega_{nn} = 2\sigma_Y^2 C^n - r - \kappa_Y - \rho_{0Y}\sigma_Y r\gamma_n\beta_n - \frac{r(\gamma_n)^2}{\gamma_1+\gamma_2}\sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})G_2^n, \tag{B-32}$$

$$\omega_{nj} = \frac{r(\gamma_n)^2}{\gamma_1+\gamma_2}\sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})G_2^n, \quad n \neq j. \tag{B-33}$$

Finally, equations (B-14) are used to find A^1, A^2. Q.E.D.

Proof of corollary 2.

$$\begin{aligned}
Cov_t(dS_t, dS_{t'}) &= Cov_t\left(\frac{dD_t}{r+\kappa_D} + H_2 dY_t, \frac{dD_{t'}}{r+\kappa_D} + H_2 dY_{t'}\right) \\
&= Cov_t\left(\frac{\sigma_D dW_{Dt}}{r+\kappa_D} + H_2\sigma_Y dW_{Yt}, \frac{\kappa_D(\bar{D} - D_{t'})}{r+\kappa_D} + H_2\kappa_Y(\bar{Y} - Y_{t'})\right)dt' \\
&= Cov_t\left(\frac{\sigma_D dW_{Dt}}{r+\kappa_D} + H_2\sigma_Y dW_{Yt}, -\frac{\kappa_D D_{t'}}{r+\kappa_D} - H_2\kappa_Y Y_{t'}\right)dt'.
\end{aligned}$$

To find the last covariance we have to consider the following differential equations

$$\begin{aligned} dD &= -\kappa_D D dt, \qquad D(t) = \sigma_D dW_{Dt}, \\ dY &= -\kappa_Y Y dt, \qquad Y(t) = \sigma_Y dW_{Yt}. \end{aligned}$$

The solutions of these equations are straightforward:

$$D_{t'} = D_t \exp[-\kappa_D(t'-t)], \qquad D_t = \sigma_D dW_{Dt}, \tag{B-34}$$

$$Y_{t'} = Y_t \exp[-\kappa_Y(t'-t)], \qquad Y_t = \sigma_Y dW_{Yt}. \tag{B-35}$$

Substitution of equations (B-34) and (B-35) into the last line of expression (B-34) implies

$$\begin{aligned} & Cov_t(dS_t, dS_{t'})/dtdt' \\ = \ & Cov_t\left(\frac{\sigma_D dW_{Dt}}{r+\kappa_D} + H_2\sigma_Y dW_{Yt}, -\frac{\kappa_D\sigma_D e^{-\kappa_D(t'-t)} dW_{Dt}}{r+\kappa_D} - H_2\kappa_Y e^{-\kappa_Y(t'-t)}\sigma_Y dW_{Yt}\right)/dt \\ = \ & -\frac{\sigma_D}{r+\kappa_D}\left[\frac{\kappa_D\sigma_D}{r+\kappa_D}\, e^{-\kappa_D(t'-t)} + H_2\kappa_Y\sigma_Y\rho_{DY}\, e^{-\kappa_Y(t'-t)}\right] \\ - \ & H_2\sigma_Y\left[\frac{\kappa_D\sigma_D\rho_{DY}}{r+\kappa_D}\, e^{-\kappa_D(t'-t)} + H_2\kappa_Y\sigma_Y e^{-\kappa_Y(t'-t)}\right]. \end{aligned} \tag{B-36}$$

Q.E.D.

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