ON THE TERM STRUCTURE OF CALENDAR SWAP VOLATILITY AND SKEW IN THE CRUDE OIL FUTURES MARKET

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Abstract. An important class of exchange-traded derivatives are options on monthly calendar swaps on commodity futures. They are structured as Average Price Options (APO) on futures, with monthly averaging over the first nearby contract. In this paper we study the relative value of these instruments to vanilla options on commodity futures, focusing on the crude oil futures markets. We derive an analytical approximation for the implied volatility of forward start APOs in the local volatility model, which takes into account the skew in the futures markets. Using this result we perform an empirical study of the exchange-traded options on monthly calendar swaps on crude oil futures, comparing the theoretical prediction for the level and skew of the WTI monthly calendar swap options against market data. The empirical study suggests that the options on monthly calendar swaps trade at a small premium relative to other options on the underlying futures.

1. Introduction

Futures prices volatility is one of the most important features of the futures markets. An accurate understanding of the futures price volatilities and their dynamics plays an important role in the risk management of futures based strategies, in setting margin requirements and for pricing derivatives on futures. Commodity markets are prone to significant price swings, both in normal markets, and especially during financial crisis periods. An empirical study of the volatility for several commodity futures has been presented by Clewlow et al (2000) in [9].

There are two main measures of futures volatility: *realized price volatilities* (historical volatility), and *implied volatilities*, implied from the prices of options on futures. Implied volatilities are defined in terms of prices of European options on futures, and are used in model building to calibrate models which are used also for pricing American options and exotic derivatives on futures. For WTI crude oil futures, the implied volatilities have to be determined from American option prices, which are the most commonly traded instruments in these markets. Figure 1.1 shows a recent plot of the term structure of the implied volatilities of WTI futures, determined from the prices of exchange-traded American options on these futures (LO), produced by `QuikStrike` and available at `www.cmegroup.com`.

The most widely traded exotic options on commodity futures are the Average Price Options (APO), also known as Asian options. They are particularly popular in the energy

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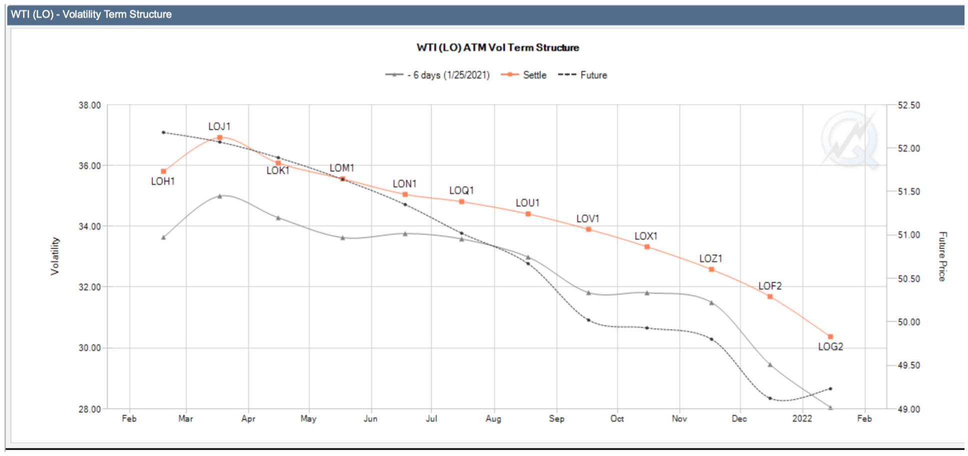

FIGURE 1.1. The term structure of WTI futures and of their implied volatilities determined from the American options (LO). The nearby option LOH1 expires on 17-Feb-21 (DTE 16). Market data as of 1-Feb-2021. (Source: QuikStrike)

futures markets, such as crude oil, heating oil, natural gas and power. The APOs are path-dependent options as their payoff depends on the average of the contract prices over a specified period, in contrast to the vanilla European and American options, for which the payoff depends only on the contract price at exercise date.

The APOs are popular in practice for two main reasons:

- Their payoff reflects more closely the structuring of typical energy transactions, which take place over a month via multiple deliveries. Such transactions are priced based on average rather than on terminal price.
- APOs are generally cheaper than the European vanilla options with the same expiry date, underlying and volatility, as the averaging has the effect of smoothing down price spikes. This makes them more attractive as risk management instruments.

The APOs can be also used to imply volatilities, similar to the more commonly used European and American volatilities. Industry practitioners use options on monthly calendar swaps and approximate them with European options, in the calibration of the futures volatility surface. For illustration we show in Figure 2.1 recent plots of the term structure of the implied volatility of options on monthly calendar futures (AO), produced by `QuikStrike` and available at `cmegroup.com`. These plots correspond to the same date 1-Feb-2021 as the plots in Figure 1.1.

Comparing the volatilities implied from American options in Fig. 1.1 and from options on calendar spreads in Fig. 2.1 they appear to have similar shapes. It is of interest to compare more closely these two measures of volatilities implied by these instruments, and examine their relative value. In this paper we would like to compare the implied futures volatilities determined from vanilla options and APOs.

Clearly, any analysis of this type is model dependent, as the relationship between these volatilities is fixed, once the dynamics of the futures process is specified. We will make the economical assumption that the futures price process follow a local volatility modification of the Black model, and use a novel analytical prediction for the APO implied volatility derived recently in [25]. This prediction expresses the APO implied volatility in terms of the usual implied volatility, under the assumption that the futures price process follows the local volatility model. We perform an empirical comparison of this theoretical result with data. The results could shed further light on market mechanisms impacting APOs pricing, such as relative market demand/offer for these instruments, and the use of the APOs for risk management purposes.

The paper is organized as follows. Section 2 describes the mechanics of the exchange-traded APOs, on the example of WTI crude oil AO options. Section 3 presents an analytical approximation for the APO implied volatility assuming that the futures follow a stochastic process described by the local volatility model. The special feature of exchange-traded APOs of changing underlying during the averaging period is handled by introducing a special basket approximation. In Section 4 we compare the theoretical results of Sec. 3 against the empirical market implied volatilities of the exchange-traded APOs. We predict the ATM volatility and skew of the APO volatility in terms of the implied volatility determined from the American options markets. Section 5 presents a discussion of the results and summarizes the conclusions of the study.

2. Exchange-traded Average Price Options on futures

We describe in this section the details of the payoff of the exchange traded Average Price Options (APO) on futures, with monthly arithmetic price averaging. Several APO's are traded on the CME Group Exchange[1] on WTI crude oil futures (AO), Brent Financial (BT), Dated Brent (Platts) Financial and Dubai Crude Oil (Platts) Financial.

For definiteness we will describe them on the example of the options on WTI crude oil futures, which are among the most liquid options listed on the CME Group Exchange.

The CME Group exchange lists Average Price Options, which are structured as European-style options on the WTI Financial Futures contracts (CS), also known as Calendar Swap Futures. The averaging style is arithmetic with daily frequency and the averaging period is monthly, at the settlement price of the first nearby futures contract. An example showing

[1] `www.cmegroup.com`.

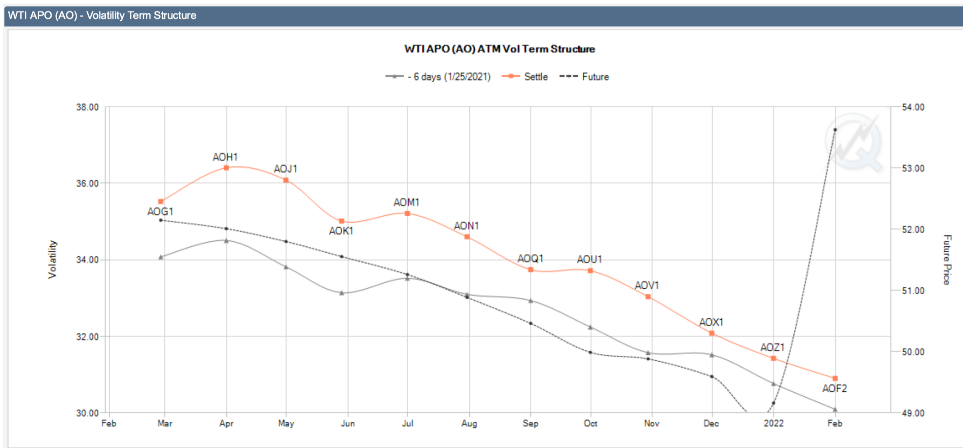

FIGURE 2.1. Term structure of WTI monthly calendar swaps and of their implied volatilities determined from the Average Price options (AO) prices. The front option AOG1 expires on 26-Feb-21 (DTE 25). Market data as of 1-Feb-2021. (Source: QuikStrike)

TABLE 1. The first few Average Price Options AO on CS futures (monthly calendar swap) and their underlying futures CL as traded on 22-May-2020. The time averaging period $[T_1, T_2]$ [months] is listed for each CS contract.

index	AO option	Underlying CL futures	T_1	T_2
1	Jul-20	(Aug-20, Sep-20)	1-Jul-20 ($1^1/4$)	31-Jul-20 ($2^1/4$)
2	Aug-20	(Sep-20, Oct-20)	1-Aug-20 ($2^1/4$)	31-Aug-20 ($3^1/4$)
3	Sep-20	(Oct-20, Nov-20)	1-Sep-20 ($3^1/4$)	30-Sep-20 ($4^1/4$)
4	Oct-20	(Nov-20, Dec-20)	1-Oct-20 ($4^1/4$)	31-Oct-20 ($5^1/4$)

in detail the averaging is discussed in Appendix A.1. These options are called WTI Average Price Options and are denoted AOF1 (Jan-21), AOG1 (Feb-21), AOH1(Mar-21), etc.

Table 1 lists some of the first few AO Average Options and the corresponding WTI Calendar Swap futures (CS) which are their underlyings. See Fig. 2.1 for the term structure of the CS futures and the implied volatilities of the AO options as of 1-Feb-21.

These options are fixed-strike forward start Asian options which pay at time T_2 the amount

$$
\begin{aligned}
\text{Pay}_C &= \max(A_{T_1,T_2} - K, 0) \text{ (call options)} \\
\text{Pay}_P &= \max(K - A_{T_1,T_2}, 0) \text{ (put options)}
\end{aligned}
\tag{1}
$$

where $A_{T_1,T_2} = \frac{1}{n}\sum_{i=1}^{n} S_i$ is the daily average price of the underlying S_i over the averaging period $[T_1, T_2]$. Both call and put options are traded. The dates T_1, T_2 are the first and the last day of the contract month, respectively. The asset price S_i is the settlement price of the nearby crude futures contract on day t_i.

3. Pricing of the exchange-traded Average Price Options on futures

Pricing options on futures requires that a stochastic model is specified for the dynamics of the futures prices. Several types of models are used in the literature for this purpose. Spot models describe the futures dynamics and the term structure of the futures curve as following from the dynamics of the stochastic spot price and of the convenience yield. This modeling approach is closer to economic fundamentals. Typical models of this type are the Gabillon (1991) [16] and Schwartz (1997) [26] models. See also Gibson and Schwartz (1990) [18] for an earlier version of such models.

For the purpose of pricing derivatives on futures, it is more convenient to adopt models where the currently observed futures curve is an input to the model. In these models the futures curve is treated as the underlying stochastic variable, with a specified dynamics. This class of models are known as the Markov futures models, and are inspired by the HJM models used for yield curve modeling. Examples of models of this type are the Cortazar and Schwartz (1994) [11] model and the Clewlow and Strickland (1999) [7] model. Andersen (2010) [1] gives an overview with aspects of practical use for derivatives pricing. Textbook treatments are given by Clewlow, Strickland [8], Geman [17] and Eydeland [13].

In the simplest formulation a typical Markov futures model specifies the dynamics of the futures prices $F_t(T)$ with delivery date T as a stochastic process in the risk-neutral measure

$$\frac{dF_t(T_i)}{F_t(T_i)} = \alpha(T_i)[\sigma_s e^{-\beta_s(T_i-t)} dW_s(t) + \sigma_\ell e^{-\beta_\ell(T_i-t)} dW_\ell(t)] \tag{2}$$

where the short and long drivers $dW_s(t), dW_\ell(t)$ are correlated with correlation ρ. The Schwartz (1997) two-factor model [26] can be put into such a form, see the Appendix of [7]. This dynamics can be also obtained by a PCA analysis of the futures curve dynamics.

This model generates log-normal dynamics and thus cannot accommodate commodity futures smiles. Smile effects can be taken into account either by adding stochastic volatility as for example in the Eydeland, Geman (1998) model, similar to the Heston model, by adding a regime switching model [1] or by introducing a local volatility model. See Section 2.5 in [1] for a discussion of local volatility modeling for commodity futures. The simplest approximation of this type is a local volatility model modification of the classical Black model for futures dynamics [5] in the risk-neutral measure

$$dF_t(T_i) = \sigma(t, T_i, F_t(T_i))F_t(T_i)dW_t \tag{3}$$

where $\sigma(t, T_i, F)$ is a local volatility function, and the initial condition is given by the current futures price $F_0(T_i) = F(T_i)$. This model reduces to the Black model [5] in the limit of a constant local volatility $\sigma(t, T, F) = \sigma$. Although the model (3) does not have mean reversion and thus cannot be expected to describe satisfactorily the joint dynamics of the futures contracts, it should be a good approximation for pricing derivatives on one or two futures contracts, which is the case we consider here.

As discussed in the previous section, exchange traded APOs are fixed strike forward start Asian options, with payoffs of the form (1). Several valuation approaches for forward start Asian options have been proposed in the literature. Assuming that the underlying asset follows a geometric Brownian motion $dS_t = \sigma S_t dW_t + (r-q)S_t dt$ (Black-Scholes model), an analytical approximation based on a Taylor expansion in maturity has been proposed by Bouaziz, Briys and Crouhy (1994) [6]. Their result has been corrected by Tsao, Chang and Lin (2003) [28] who identified a missing term in the result of [6]. Both these works consider so-called *floating-strike* forward start Asian options, which are defined by a payoff at time T_2 of the form

$$\mathrm{Pay}_{C-FS} = \max(\kappa S_{T_2} - A_{T_1,T_2}, 0)\,, \quad \mathrm{Pay}_{P-FS} = \max(A_{T_1,T_2} - \kappa S_{T_2}, 0)\,. \tag{4}$$

This is different from the payoff of the exchange-traded APOs, see Eq. (1).

Vanmaele et al. (2006) [29] considered forward starting Asian options with discrete time averaging under the Black-Scholes model, and derived precise upper and lower bounds on the prices of these instruments. They considered the fixed-strike payoff required here. We compared the analytical approach discussed below on their benchmark cases, see the discussion below.

Both [28] and [29] assume a log-normal dynamics for the underlying futures, and thus do not capture the skew effects in the APO pricing. We present next a simple analytical approximation which captures both the level and the skew of the futures implied volatility.

3.1. **Analytical approximation for the forward start APOs.** Pricing of forward start Asian option prices under the local volatility model (3) has been studied using a short maturity (equivalently, small volatility) expansion in [25]. In order to make the presentation self-contained we give a brief summary of this approach. Under risk-neutral valuation, option prices are given by the discounted expectation of the payoff in the risk-neutral measure. Explicitly, for call options

$$C(K, T_1, T_2) = e^{-rT_2}\mathbb{E}\left[\left(\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} S_t dt - K\right)^+\right]\,. \tag{5}$$

and for put options

$$P(K, T_1, T_2) = e^{-rT_2}\mathbb{E}\left[\left(K - \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} S_t dt\right)^+\right]\,. \tag{6}$$

Reference [25] derived the leading asymptotics of these option prices in the limit $T_1, T_2 \to 0$ at fixed ratio $\tau = T_1/T_2$, under the assumption that the underlying asset S_t follows a local volatility model. This asymptotics is most conveniently expressed in terms of the *equivalent* *log-normal implied volatility*, or *equivalent Black-Scholes volatility* of a forward start Asian option. See Section 3.4 in [25] for more details. We give a brief summary of the relevant results below.

Denote $\Sigma_{\rm LN}(K, T_1, T_2)$ the *equivalent log-normal implied volatility* of a forward start Asian option with strike K and averaging period $[T_1, T_2]$. This is defined as that volatility which reproduces the price of the Asian option when substituted in the Black-Scholes formula for an European option with maturity T_2 on an underlying with forward price

$$A(T_1, T_2) = \frac{S_0}{(r-q)(T_2 - T_1)} \left(e^{(r-q)T_2} - e^{(r-q)T_1}\right) \tag{7}$$

which is assumed to follow Black-Scholes dynamics with constant volatility $\Sigma_{\rm LN}$, and risk-free rate and dividend yield r, q. In the model (3) Specializing to the dynamics (3) where futures are martingales, we obtain $A(T_1, T_2) = S_0$ by taking $r - q \to 0$.

Considering for example a forward start Asian call option, its price can be written in Black-Scholes form as

$$C(K, T_1, T_2) = C_{\rm BS}(S_0, K, \Sigma_{\rm LN}(K, T_1, T_2), T_2) \tag{8}$$

with

$$C_{\rm BS}(S_0, K, \Sigma_{\rm LN}, T) = e^{-rT}[A(T_1, T_2)N(d_1) - KN(d_2)] \tag{9}$$

$$d_{1,2} = \frac{1}{\Sigma_{\rm LN}\sqrt{T}} \left(\log \frac{A}{K} \pm \frac{1}{2}\Sigma_{\rm LN}^2 T\right) . \tag{10}$$

It is convenient to consider an expansion in log-strike $x = \log(K/S_0)$. The first two terms are the level and skew

$$\Sigma_{\rm LN}(K, T_1, T_2) = \Sigma_A(T_1, T_2) + x s_A(T_1, T_2) + O(x^2) . \tag{11}$$

A similar expansion can be written for the implied volatility of the European options on futures

$$\Sigma_{\rm BS}(K, T) = \sigma_{\rm ATM} + x s_E + O(x^2) \tag{12}$$

The short maturity asymptotics of the option prices derived in [25] translates into an exact result for the equivalent log-normal implied volatility of a forward start Asian option in the limit $T_{1,2} \to 0$ at fixed ratio $\tau = T_1/T_2$. The precise statement is Proposition 3.17 in [25], which expresses $\Sigma_{\rm LN}$ in terms of a *rate function* $\mathcal{I}_{\rm fwd}(K, S_0, \tau)$ given by the solution of a variational problem in Theorem 2.2 of [25]. The rate function $\mathcal{I}_{\rm fwd}(K, S_0, \tau)$ simplifies in the particular case of the Black-Scholes model as in Theorem 3.7 of [25]. The Black-Scholes model is analyzed in detail in [25] and the properties of the rate function for this case are studied in various regimes of small and large strikes in Section 3 of [25].

We are interested here in the general case of the local volatility model. For this case the rate function is given in Proposition 2.5 of [25] and is expressed as an extremal problem over a simpler rate function $\mathcal{I}(x, K)$ related to Asian options with averaging over $[0, T]$. Here we simplify this result further and put into a form which contains only market observable quantities.

The following result gives a relation between the ATM level and skew of the equivalent log-normal volatility of a forward start Asian option and the corresponding parameters of the European implied volatility.

Proposition 3.1. *Assume that the futures prices follow a local volatility model (3) where the local volatility function $\sigma(t, T, x)$ satisfies the technical conditions in Eq. (2.7) of* [25]*. Define the volatility level Σ_A and skew s_A of a forward start Average Price Option with averaging period $[T_1, T_2]$ as in (11). In the short maturity limit, these parameters are related to the corresponding parameters $\sigma_{\rm ATM}, s_E$ of the vanilla implied volatility for maturity T_2, defined as in (12), as*

$$\Sigma_A = \sigma_{ATM}\sqrt{\frac{1+2\tau}{3}} \tag{13}$$

$$s_A = \sqrt{\frac{1+2\tau}{3}}\left\{\frac{1}{10}\sigma_{ATM}\frac{(1-\tau)^2}{(1+2\tau)^2} + \frac{3}{5}s_E\frac{2-4\tau+17\tau^2}{(1+2\tau)^2}\right\} \tag{14}$$

where $\tau = T_1/T_2 < 1$ is the ratio between the start and end dates of the averaging period.

Proof. The proof is given in the Appendix. □

The relation (13) can be formulated equivalently as follows: the price of an ATM forward start Asian option with averaging period $[T_1, T_2]$ and payout at T_2 is equal to the price of an European option with volatility $\sigma_{\rm ATM}$ and maturity $T_1 + \frac{1}{3}(T_2 - T_1)$. This follows from writing this equation as $\sigma^2_{\rm ATM}[T_1 + \frac{1}{3}(T_2 - T_1)] = \Sigma_A^2 T_2$. This reproduces a rule of thumb for pricing forward start Asian options used by practitioners, which is thus seen to follow as an exact asymptotic result in the short maturity limit.

The relation (14) extends this result to the skew of an ATM forward start Asian option. This is seen to consist of one component proportional to the ATM vanilla implied volatility, and a component proportional to the vanilla skew. In the Black-Scholes model the European options skew vanishes $s_E = 0$ and the result (14) reduces to

$$\Sigma_{\rm LN}(K, S_0, \tau) = \sigma_{ATM}\sqrt{\frac{1+2\tau}{3}}\left(1 + \frac{(1-\tau)^2}{10(1+2\tau)^2}x + O(x^2)\right) \tag{15}$$

This reproduces the result quoted in Remark 3.15 in [25] in the Black-Scholes model.

Consider also the limiting case of an Asian option with averaging period $[0, T]$. For this case we have $\tau = 0$ and the expression (14) reproduces the result in Remark 20 in [24] for the level and skew of an Asian option with averaging starting at time zero

$$\Sigma_{ATM}|_{\tau=0} = \frac{1}{\sqrt{3}}\sigma_{ATM} \tag{16}$$

$$s_A|_{\tau=0} = \frac{1}{\sqrt{3}}\left(\frac{1}{10}\sigma_{ATM} + \frac{6}{5}s_E\right) . \tag{17}$$

Finally, in the $\tau \to 1$ limit the forward start Asian option approaches an European option. The relations (14) reduce to the expected limiting values for this case

$$\Sigma_{ATM}|_{\tau=1} = \sigma_{ATM}\,, \quad s_A|_{\tau=1} = s_E\,. \tag{18}$$

3.2. **Basket approximation for two-futures averages.** The result of the previous section cannot be applied directly to exchange-traded APOs. Their payoffs are linked to the monthly average of the nearby contract, which switches once during the averaging period. This feature makes them in effect two-futures derivatives. Consider as an illustration the average price option AO-Jul-20. This is a forward start Asian option, with averaging period $[T_1, T_2]$ starting on 1-Jul-20 and ending on 31-Jul-20. The averaging is performed over the settlement prices of the CL-Aug-20 contract until 21-Jul-20, and over the settlement prices of the CL-Sep-20 contract for the remainder of the averaging period.

We propose to approximate the actual average over the first nearby contract A_{T_1,T_2} with the time-average of a basket of the two futures contracts with uniform composition throughout the averaging interval and weights proportional to the number of days each asset contributes to the average.

We state this approximation in a more general way as follows.

Definition 3.1 (Basket approximation for two-futures APOs)**.** *Assume that $S_{1,2}(t)$ are martingales in the risk neutral measure, thus $S_i(0) = \mathbb{E}[S_i(t)]$ for any $t > 0$. Under the basket approximation the underlying A of the APO is approximated with $\tilde{A}$, defined as below.*

$$A = \frac{1}{n}\sum_{i=1}^{n} X(t_i)\,, \qquad \tilde{A} := \frac{1}{n}\sum_{i=1}^{n} Y(t_i) \tag{19}$$

where

$$X(t_i) = \begin{cases} S_1(t_i) & ,1 \le i \le n_1 \\ S_2(t_i) & ,n_1 + 1 \le i \le n_1 + n_2 \end{cases} \tag{20}$$

and

$$Y(t_i) = w_1 S_1(t_i) + w_2 S_2(t_i)\,, \quad w_{1,2} = \frac{n_{1,2}}{n_1 + n_2} \tag{21}$$

This approximation assumes that $S_1(t)$ is traded for the entire duration of the averaging period. In practice the first nearby contract stops trading shortly after n_1. For the purpose of applying this approximation we will assume that both contracts are traded for the entire averaging period.

We note that the expectations of $A, \tilde{A}$ are equal

$$\mathbb{E}[A] = \mathbb{E}[\tilde{A}] = w_1 S_1(0) + w_2 S_2(0)\,. \tag{22}$$

This property follows immediately from the martingale property of the assets $S_{1,2}(t)$.

The equality (22) implies that the prices of two futures contracts which deliver A and $\tilde{A}$ at maturity are equal. In particular, the price of the CS contract is reproduced exactly in terms of the prices of the futures contracts which contribute to the average

$$F_i = w_1 F_1^{(i)} + w_2 F_2^{(i)} \tag{23}$$

with i the CS contract month index, and $F_{1,2}^{(i)}$ are the current prices of the futures which participate in the averaging process for the given month.

What is the error introduced by substituting $A \to \tilde{A}$ on option prices? We give next an upper bound on the error of the basket approximation for option prices.

Proposition 3.2. *Denote $C(K,T) = \mathbb{E}[(A-K)^+]$ and $\tilde{C}(K,T) = \mathbb{E}[(\tilde{A}-K)^+]$ the (undiscounted) prices of Asian call options with payoffs linked to the averages $A, \tilde{A}$ respectively.*

The approximation error of using $\tilde{A}$ instead of A when pricing Asian options is bounded as

$$|\tilde{C}(K,T) - C(K,T)| \leq \mathbb{E}|Z| \leq \sqrt{Var(Z)} \tag{24}$$

where

$$Z := \tilde{A} - A = \frac{1}{n}\sum_{i=1}^{n} q_i (S_1(t_i) - S_2(t_i)) \tag{25}$$

with time-dependent weights

$$q_i = \begin{cases} -w_2 & , 1 \leq i \leq n_1 \\ +w_1 & , n_1 + 1 \leq n_1 + n_2 \end{cases} \tag{26}$$

A similar result holds for put options. The random variable Z has zero mean $\mathbb{E}[Z] = 0$.

Proof. We have

$$\begin{aligned} \tilde{A} - A &= \frac{1}{n}\sum_{i=1}^{n_1}\left(\frac{n_1}{n_1+n_2}S_1(t_i) - S_1(t_i) + \frac{n_2}{n_1+n_2}S_2(t_i)\right) \\ &+ \frac{1}{n}\sum_{i=n_1+1}^{n_1+n_2}\left(\frac{n_2}{n_1+n_2}S_2(t_i) - S_2(t_i) + \frac{n_1}{n_1+n_2}S_1(t_i)\right) \\ &= \frac{1}{n}\sum_{i=1}^{n_1}\frac{n_2}{n_1+n_2}(S_2(t_i) - S_1(t_i)) + \frac{1}{n}\sum_{i=n_1+1}^{n_1+n_2}\frac{n_2}{n_1+n_2}(S_1(t_i) - S_2(t_i)) \end{aligned} \tag{27}$$

which reproduces the result in (25).

Furthermore, we have

$$|\mathbb{E}[(\tilde{A}-K)^+]-\mathbb{E}[(A-K)^+]| \leq \mathbb{E}[|(\tilde{A}-K)^+-(A-K)^+|] \tag{28}$$
$$\leq \mathbb{E}[|\tilde{A}-A|]=\mathbb{E}[|Z|]$$

The second inequality in (24) follows by the Jensen's inequality.

□

The L_1 norm of Z is bounded from above as

$$\mathbb{E}|Z| \leq \frac{1}{n}\sum_{i=1}^{n}|q_i||S_1(t_i)-S_2(t_i)| \leq \frac{1}{n}\sum_{i=1}^{n}|S_1(t_i)-S_2(t_i)| \tag{29}$$

If both $S_{1,2}$ are traded for the entire averaging period, this expectation can be expressed as the sum of ATM call and put Asian options on the spread between the two assets (ATM averaged straddle on spread)

$$C_{ATM}(K=F,T)+P_{ATM}(K=F,T)=\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[|S_1(t_i)-S_2(t_i)|] \tag{30}$$

The basket approximation is amenable to analytical treatment since we have powerful asymptotic methods for studying the distribution of a basket of assets. In the local volatility model the short maturity asymptotics of baskets was studied by Avellaneda et al. [3], Henry-Labordére [20] and and by Bayer and Laurence [4]. The latter authors derived also $O(T)$ corrections to the short maturity asymptotics, and demonstrated the good numerical performance of the asymptotic formulas. Tail asymptotics of baskets of log-normally distributed assets have been studied by Gulisashvili and Tankov [19].

We give a brief summary of the short maturity asymptotics for baskets, following [3] and [4], and extract from their results an analytical result for the ATM skew of the basket implied volatility for a basket of two assets following correlated local volatility models.

Define the basket of two assets $B = w_1S_1 + w_2S_2$. Assume that the two assets are martingales and follow correlated local volatility models $dS_i(t)/S_i(t) = \sigma_i(S_i)dW_t^{(i)}$ with $\mathrm{corr}(dW_t^{(1)}, dW_t^{(2)}) = \rho$.

Proposition 3.3. *Denote $\sigma_{BS}(z,T)$ the implied volatility of options on the basket with strike K and log-strike $z=\log(K/B(0))$ with $B_0 = w_1S_1(0)+w_2S_2(0)$. The basket implied volatility is expanded around the ATM point $z=0$ as*

$$\sigma_{BS}^{(B)}(z,T)=\sigma_B+s_Bz+O(z^2)+O(T) \tag{31}$$

The parameters σ_B, s_B can be expressed in terms of the basket components' ATM implied volatilities and skews $\sigma_{1,2}, s_{1,2}$, defined by expanding the implied volatilities of the two assets as $\sigma_{\rm BS}^{(i)}(x)=\sigma_i+s_ix+O(x^2)$ with $x=\log(K/S_i(0))$.

The leading order short maturity basket ATM implied volatility is

$$\sigma_B^2 = p_1^2\sigma_1^2 + p_2^2\sigma_2^2 + 2\rho p_1 p_2 \sigma_1\sigma_2 \tag{32}$$

where $p_i = w_1 S_i(0)/B(0)$.

The ATM skew of the basket implied volatility is

$$s_B(\rho) = \frac{1}{2\sigma_B^3}\left(\kappa_0 + \kappa_1\rho + \kappa_2\rho^2\right) \tag{33}$$

where

$$\kappa_0 = 2p_1^3(\sigma_1^3 s_1) + p_1p_2\left(p_1\sigma_1^2 - p_2\sigma_2^2\right)^2 + 2p_2^3(\sigma_2^3 s_2) \tag{34}$$

$$\kappa_1 = -2p_1p_2\sigma_1\sigma_2\left\{(p_1-p_2)[p_1\sigma_1^2 - p_2\sigma_2^2] - 2p_1\sigma_1 s_1 - 2p_2\sigma_2 s_2\right\} \tag{35}$$

$$\kappa_2 = p_1p_2\sigma_1\sigma_2\left\{(p_1-p_2)^2\sigma_1\sigma_2 + 2p_2\sigma_2 s_1 + 2p_1\sigma_1 s_2\right\}\,. \tag{36}$$

Proof. The ATM basket implied volatility (32) is obtained by taking the ATM point in Eq. (18) of [3]. We extracted the result for the basket skew (33) from the results in Section 3.1 of [4] by expanding in the log-strike $z = \log(K/B_0)$ and keeping only the terms of $O(z)$. □

Remark 3.1. *i) The ATM skew (33) has the expected limiting behavior in the limit of a single-asset basket: for example as $p_2 \to 0$ it approaches $s_B \to s_1$ (the basket skew approaches the skew of the component S_1).*

ii) In the limit when the two basket components satisfy the Black-Scholes equation $s_{1,2} = 0$, the basket skew does not vanish in general. Assuming furthermore that the two components have the same volatility $\sigma_1 = \sigma_2 = \sigma$, the basket skew becomes

$$s_B = \sigma\frac{p_1p_2(p_1-p_2)^2}{2(p_1^2+p_2^2+2\rho p_1p_2)^{3/2}}(1-\rho)^2\,, \tag{37}$$

which is in general non-zero and positive. The skew vanishes in the limit when the components are perfectly correlated $\rho = 1$, and in the limit of equal-value baskets $p_1 = p_2$.

Remark 3.2. *The ATM skew of the basket implied volatility simplifies greatly in the perfectly correlated limit $\rho = 1$, and becomes*

$$s_B(\rho = 1) = \frac{1}{\sigma_B}\left(p_1\sigma_1 s_1 + p_2\sigma_2 s_2 + \frac{1}{2}p_1p_2(\sigma_1-\sigma_2)^2\right) \tag{38}$$

where $\sigma_{1,2}, s_{1,2}$ are the ATM volatilities and skews, respectively, of the two basket components.

The skew of a basket of two Black-Scholes components with different volatilities is always positive, and increases with the difference between the volatilities of the components. When applied to monthly calendar options, this implies that a term structure of the volatility (of either sign) induces a positive contribution to the APO skew.

4. Empirical analysis

In this section we apply the theoretical results of Sec. 3 for the ATM volatility and skew of the exchange-traded Average Price Options on WTI crude oil futures, and compare the results against empirical market data.

4.1. **Market data.**

The tests will require two market data sets, which we describe briefly below.

- *Futures data (CL data)*: WTI futures settlement prices (CL) and their volatility data (ATM vol and skew). The volatility data is extracted from American option (LO) implied volatilities around the ATM point, at moneyness $97.5\%, 100\%, 102.5\%$
- *Average Price Options data (AO data)*: settlement prices of monthly calendar swap (CS) on crude oil futures and implied volatility of Average Price Options (AO) on the CS calendar swaps.

The ATM implied volatilities of the basket components $\sigma_{1,2}$ are directly available in the market as the LO implied volatilities at 100% moneyness. The skews $s_{1,2}$ are estimated from the 102.5% and 97.5% moneyness volatility points (the moneyness is relative to the CS futures price) as

$$s_i = \frac{\sigma_{102.5,i} - \sigma_{97.5,i}}{\log(102.5/97.5)}\,, \quad i = 1, 2\,. \tag{39}$$

Once the components' implied volatility level and skew $\sigma_{1,2}, s_{1,2}$ are determined, we proceed in two steps:

i) Compute the ATM level and skew of the equivalent basket consisting of the two futures contributing to the monthly average. The basket has weights w_i given by (21). The ATM level and skew of the equivalent basket σ_B, s_B are computed from Eqs. (32) and (33), respectively. For simplicity we assume perfectly correlated nearby contracts $\rho = 1$ and use the simpler result (38) for the basket skew.

ii) Use the parameters computed in step (i) as inputs for the computation of the Asian implied volatility $\sigma_B \to \sigma_{ATM}, s_B \to s_E$ using Equations (13) and (14), respectively. In this step we use the basket approximation of Sec. 3.2 to approximate the average over the nearby futures contract prices with an average over an equivalent basket of the two futures.

A detailed example of the calculation is shown in Appendix A.2 for the AO-Jul-20 contract as of 22-May-2020.

Using the underlying data we compute the predictions for the ATM volatility and skew of the AO options, using the two-step process outlined at the end of the previous section: i) compute the volatility σ_B and skew s_B of the equivalent basket of futures for each APO

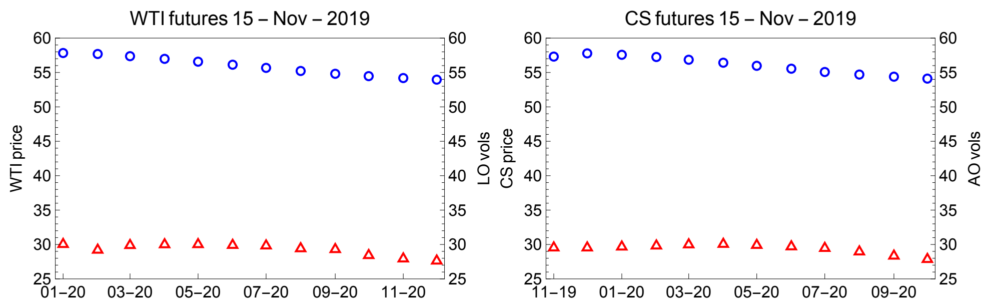

FIGURE 4.1. The term structure of WTI futures and of their implied volatilities determined from the American options (LO) (left) and of the Calendar Swap futures (CS) and their implied volatilities determined from the Average Price Options (AO) (right). Futures prices are shown as circles and volatilities as triangles. Market data as of 15-Nov-2019.

using (32) and (33), ii) substitute these values into Eqs. (13) and (14) to obtain Σ_A, s_A. These predictions are compared against the Average Price Options market data.

The empirical analysis was performed for two dates: 15-Nov-2019 and 22-May-2020. We chose these dates because they represent typical snapshots of the crude oil markets before and after the Covid-19 crisis. The term structure of the WTI futures prices and volatilities on these dates are shown in Fig. 4.1 (15-Nov-19) and Fig. 4.2 (22-May-20), respectively.

4.2. **Results.** The results of the empirical analysis for WTI crude oil futures are shown in Table 2 (15-Nov-2019) and Table 3 (22-May-2020).

TABLE 2. Predicted and observed ATM volatility and skew for the Average Price Options (APO) on WTI crude futures. The last two columns show the ATM level and skew of the equivalent basket of futures. Pricing date 15-Nov-2019.

Contract	**APO vol Σ_A**		**APO skew s_A**		**Basket params**	
CS	predict	observed	predict	obs	σ_B	s_B
Dec-20	22.16	29.53	-16.46	-40.59	29.73	-40.86
Jan-20	25.22	29.66	-22.55	-36.59	29.45	-37.28
Feb-20	26.92	29.81	-23.50	-31.99	29.92	-32.90
Mar-20	27.69	29.98	-23.07	-29.39	30.00	-29.69
Apr-20	28.09	30.05	-21.80	-26.79	29.97	-26.67
May-20	28.30	29.90	-21.47	-25.39	29.87	-25.39
Jun-20	28.32	29.69	-22.62	-25.79	29.67	-26.11
Jul-20	28.19	29.46	-23.14	-26.99	29.37	-26.24
Aug-20	27.94	28.95	-21.95	-25.39	28.97	-24.55
Sep-20	27.33	28.34	-22.46	-28.39	28.24	-24.83
Oct-20	27.00	27.85	-20.51	-25.19	27.82	-22.47
Nov-20	26.54	27.47	-17.64	-21.59	27.28	-19.18
Dec-20	25.97	26.92	-17.27	-24.59	26.64	-18.65

TABLE 3. Predicted and observed ATM volatility and skew for the Average Price Options (APO) on WTI crude futures. The last two columns show the ATM level and skew of the equivalent basket of futures. Date 22-May-2020.

Contract	**APO vol Σ_A**		**APO skew s_A**		**Basket params**	
CS	predict	observed	predict	obs	σ_B	s_B
Jul-20	49.72	56.19	-25.37	-35.59	59.27	-45.10
Aug-20	46.86	50.41	-24.83	-37.59	52.56	-35.88
Sep-20	43.87	47.01	-23.97	-32.19	47.77	-31.39
Oct-20	42.30	44.55	-22.47	-30.79	45.27	-27.79
Nov-20	41.50	43.71	-22.20	-30.19	43.91	-26.45
Dec-20	40.76	43.17	-25.98	-28.99	42.77	-30.15
Jan-21	40.68	43.27	-26.34	-28.39	42.43	-29.99
Feb-21	40.41	42.65	-23.96	-27.19	41.96	-26.88
Mar-21	39.44	41.69	-21.53	-25.19	40.79	-23.87
Apr-21	39.05	40.92	-21.79	-22.99	40.26	-23.92
May-21	38.26	40.12	-20.27	-21.80	39.35	-22.07

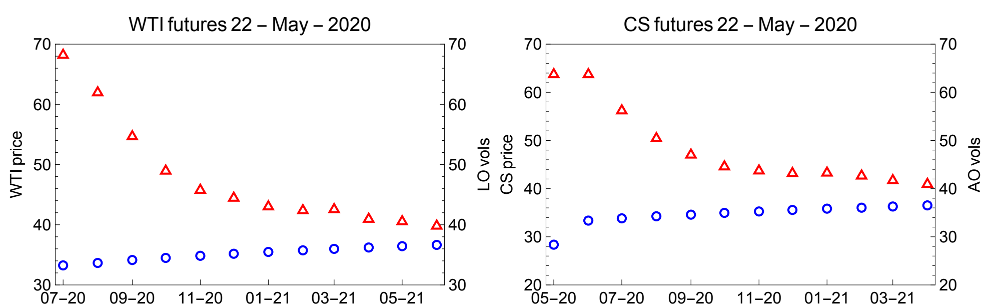

FIGURE 4.2. Left: term structure of WTI futures prices (circles) and of their implied volatilities (triangles) of the American options (LO); right: same for the Calendar Swap futures prices (CS) (circles) and their implied volatilities determined from the Average Price Options (AO) (triangles). Market data as of 22-May-2020.

The results in Tables 2 and 3 show that the APOs with monthly averaging are underpriced compared to the theoretical predictions. This is apparent in the lower values for the predicted APO volatilities Σ_A compared with the observed volatilities. Interestingly, they are priced closer to *European* basket options on the two futures participating in the averaging. This follows from the close values of the observed Σ_A and predicted basket ATM volatility σ_B. This may reflect market usage of the options on monthly calendar swap as indicators of volatility at the end of the averaging period, similar to an European option.

A similar result is obtained for the Asian skew s_A: the observed value is smaller (in absolute value) than the theoretical prediction. However in all cases it is closer to the skew s_B of a basket European option on the equivalent basket of futures. Again this is indicative of market participants pricing the exchange traded APOs as basket options on the two futures contributing to the average.

5. Conclusions and discussion

In this paper we study the pricing of exchange-traded Average Price Options on futures. These options are fixed-strike forward start Asian options with monthly averaging period over the nearby contract. The nearby contract changes during the averaging period, which renders these options two-futures derivatives. Our treatment takes into account this technical complication by introducing a basket approximation: the exchange-traded APO is approximated with a forward-start Asian option on a basket of the two futures, weighted by the number of days they contribute to the average. We give an upper bound on the error of the basket approximation in terms of the prices of calendar spread options.

We propose an analytical approximation for the volatility level and skew of exchange-traded Average Price Options on futures, which becomes exact in the short maturity/volatility limit. The inputs to this approximation are the market data on vanilla options on futures contracts underlying the APO.

The results of Section 3.2 can be used also for general Asian options on baskets of futures. These are popular products traded in the OTC markets. The analytical approximation presented here could be used to obtain an approximation of the APO implied volatility for such an Asian basket option around the ATM point.

We presented an empirical analysis of the relation between APOs and American options volatilities on the case of the crude oil West Texas Intermediate (WTI) futures, which are among the most liquidly traded commodity futures.

The results of the empirical analysis suggest that the options on monthly calendar swaps generally trade at a premium relative to the American options on futures, although this premium decreased somewhat for the second date considered (May 2020). We estimated this premium as the difference between the observed market implied APO volatility and the theoretical prediction from the American options on futures. A similar difference appears in the skew of the APOs: the market implied skew is smaller (in absolute value) than the theoretically predicted skew.

We considered several possible explanations for these differences:

i) The results of Proposition 3.1 assume continuous time averaging, while the actual averaging is performed only over the business days of the contract month. Furthermore, the averaging times are irregular due to the weekend gaps.

In Appendix B we tested the impact of the discrete averaging by comparing the prediction (13) with the results of a MC simulation of forward start Asian options with daily averaging with one month averaging periods. The impact of the continuous time averaging assumption is less than 1-2% in all cases.

ii) The assumption of perfectly correlated nearby contracts $\rho = 1$. This is expected to be a good approximation as the correlations of consecutive futures contracts are close to 100%. As mentioned this assumption gives an upper bound on the prices. The impact of this approximation is estimated in Appendix B using the results of Proposition 3.3. Varying the correlation ρ in the range 0.7-1.0 decreases the predicted APO volatility by less than 1%.

iii) The impact of the basket approximation, which accounts for the nearby futures switch during the averaging month. Under this approximation the APO is treated as a forward start basket Asian option, with weights proportional to the number of days each of the two futures contributes to the monthly average. The impact of this approximation is bounded in Proposition 3.2. In Appendix B we estimated the impact of this approximation in the Black model, where the futures follow correlated geometric Brownian motions. The impact on the APO volatility was found to be small for perfectly correlated contracts $\rho = 1$ (less than 1%), and somewhat larger as the correlation decreases.

We observe surprisingly good agreement of the observed APO volatility and skew with the corresponding parameters of an European basket option on a basket of the futures contributing to the average, and paying at the end of the APO contract month. This suggests that market participants effectively price the exchange-traded APOs as European options on a basket of the two contributing nearby futures.

Finally there is the possibility of a structural explanation, which could be related to the use of the APOs for risk management purposes. Any asymmetry in demand for APO calls vs puts would impact the skew of these options. For example, higher buy demand for calls would have a positive contribution to the skew. We plan to pursue further research along this direction.

Appendix A. Appendix: Details of the analysis

A.1. **Details of averaging for the exchange traded APOs.** Consider for definiteness the Average Price option AO Jul-20. The underlying of this option is the Monthly Calendar Swap futures contract CS Jul-20. This contract is settled at maturity (31-Jul-20) in the amount of the arithmetic average of the front CL contract on each trading day of the month of July 2020.

During July 2020 the front WTI contract is one of the two futures:

- **CL Aug-20**: trades from 1-Jul-20 until 21-Jul-20, for a total of 14 trading days (3-Jul-20 is a holiday).
- **CL Sep-20**: trades all month long. However this is the front contract only from 22-Jul-20 until 31-Jul-20, for a total of 8 trading days.

At any time prior to the start of the averaging period $t < T_1$, the price of the CS contract is related to the prices of the two contributing CL contracts, as

$$CS_{\mathrm{Jul-20}} = \frac{14}{22} \cdot CL_{\mathrm{Aug-20}} + \frac{8}{22} \cdot CL_{\mathrm{Sep-20}} \,. \tag{40}$$

This equation expresses the averaging rule for the CS contract, in terms of the nearby CL contract on each day during the averaging month. Substituting the CL settlement prices gives $\frac{14}{22} \times 33.65 + \frac{8}{22} \times 34.14 = 33.83$ which agrees with the price of the Jul-20 CS contract.

Data was sourced from Bloomberg and all prices are settlement prices. The Bloomberg codes of the relevant instruments are shown in Table 4.

Table 4. Bloomberg codes for the crude oil futures and the monthly average price options on these futures.

Instrument	Bloomberg code	Exchange symbol
WTI futures	`CLA Comdty`	`CL`
WTI APOs	`G9A Comdty`	`AO`

A.2. **Details of the empirical analysis.** We give in this Appendix a sample computation of the level and skew of the APO volatility, implied from the corresponding parameters obtained from American options on futures. Table 5 shows the details of computation for the AO-Jul-20 contract as of 22-May-2020.

Appendix B. Error analysis

Time discretization error. Real world APOs have daily averaging over business days. We study here the accuracy of approximating the discrete time averaging with continuous time averaging, assuming that the underlying asset follows the Black-Scholes model.

Table 5. Sample computation of the averaged volatilities and skews, and their use for the prediction of the APO volatility level and skew. Weights $w_1 = \frac{14}{22}, w_2 = \frac{8}{22}$. The averaging period is $[T_1, T_2]$ with $T_1 = 1.25, T_2 = 2.25$ months, and thus $\tau = T_1/T_2 = 0.556$. Market data as of 22-May-2020.

		Implied volatility (LO)					Basket equivalent		
CL	Price	97.50%	100.00%	102.50%	s_E	Month	B_0	σ_B	s_B
Aug-20	33.65	63.26	61.95	60.81	-49.0	Jul-20	33.83	59.17	-44.73
Sep-20	34.14	55.67	54.65	53.78	-37.80				

		APO volatility (AO)			
CS	Price	97.50%	100.00%	102.50%	s_A (obs)
Jul-20	33.83	57.17	56.19	55.39	-35.59

	APO vol Σ_A		APO skew s_A			
			$(s_A)_\sigma$	$(s_A)_s$	s_A	
CS	predict	obs	predict			obs
Jul-20	49.64	56.19	0.22	-25.59	-25.37	-35.59

Consider the test case of a forward start Asian option with $T_2 = 120$ days, $T_2 - T_1 = 30$ days and strike $K = \{90, 100, 110\}$ on an asset with $S_0 = 100, r = 0.0, \sigma = \{0.2, 0.4\}$. The number of averaging time steps is taken as $m = \{30, 60, 90\}$.

The results of pricing the APOs by MC simulation with $N_{MC} = 10^6$ paths are shown in Table 6. The last row gives the result from the asymptotic expansion described in Sec. 3.1. This was obtained from Eq. (8) with a linear approximation for the log-normal equivalent volatility $\Sigma_{LN} = \Sigma_A + s_A x$. For the case considered here $\tau = T_1/T_2 = 3/4 = 0.75$ we have $\Sigma_A = \sigma\sqrt{5/6}$. We note good convergence to the continuous time limit, and the difference between the asymptotic result and the price with daily averaging $m = 30$ is below 1-2% in all cases.

Table 6. Comparing the discrete time arithmetic option MC simulation with the continuous time approximation based on the short maturity expansion. The MC used $N_{MC} = 10^6$ paths, and the errors are in units of 10^{-4}.

	$\sigma = 0.2$			$\sigma = 0.4$		
m	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
30	10.8178(93)	4.1826(65)	1.0874(34)	13.9084(170)	8.3530(139)	4.6600(106)
60	10.8125(93)	4.1781(65)	1.0858(34)	13.8972(170)	8.3456(139)	4.6548(106)
90	10.8138(93)	4.1741(65)	1.0828(33)	13.8938(170)	8.3362(139)	4.6441(106)
∞	10.8163	4.1744	1.0811	13.8973	8.3374	4.6427

Impact of the correlation assumption $\rho = 1$**.** The empirical analysis assumed that the consecutive futures contributing to the averaging over a given contract month are perfectly correlated $\rho = 1$. The impact of this assumption can be estimated in the basket approximation, using the explicit results for the basket ATM volatility and skew given in Proposition 3.3.

We estimate numerically the impact of varying the correlation for the Jul-20 APO considered in detail in Section A.2. The parameters of this option are $T_2 = 2.25$ months, $S_1(0) = 33.65, S_2(0) = 34.14$, weights $w_1 = \frac{14}{22}, w_2 = \frac{8}{22}$ and volatilities $\sigma_1 = 61.95\%, \sigma_2 = 54.65\%$. Under the basket approximation the underlying of the option is the basket of two CL futures Aug-20 and Sep-20 with forward price $F_{CS} = w_1 S_1(0) + w_2 S_2(0) = 33.83$.

We would like to study the sensitivity of the basket parameters σ_B, s_B and the Asian volatility parameters Σ_A, s_A as the correlation ρ between the Aug-20 and Sep-20 WTI contracts is varied between 0 and 1. Table 7 gives the results for these parameters following from Propositions 3.3 and 3.1.

The volatility parameters Σ_A, σ_B increase with the correlation and reach maximal values at $\rho = 1$. The perfectly correlated case $\rho = 1$ corresponds to a basket of co-monotonic assets. This agrees with the result of [29] that for baskets with positive weights, the co-monotonic joint distribution gives an upper bound on the basket price.

A similar dependence is observed for the skews s_B, s_A which increase (in absolute value) with ρ.

Impact of the basket approximation. In the basket approximation, the switch of the nearby futures contract during the averaging period is approximated by averaging over a basket of the two contributing futures, with weights proportional to the number of days they contribute to the monthly average. The impact of this approximation on APO prices is bounded by (24) as

$$\Delta C \leq \sqrt{Var(Z^2)}\,, \tag{41}$$

Table 7. The impact of changing the correlation ρ between the Aug-20 and Sep-20 CL futures contracts on the parameters of the exchange traded Jul-20 APO.

ρ	σ_B	s_B	Σ_A	s_A
0.0	44.05	-35.72	36.96	-20.12
0.1	45.80	-36.65	38.43	-20.64
0.2	47.48	-37.60	39.84	-21.17
0.3	49.11	-38.55	41.21	-21.71
0.4	50.69	-39.51	42.53	-22.25
0.5	52.22	-40.46	43.81	-22.78
0.6	53.70	-41.41	45.06	-23.31
0.7	55.15	-42.34	46.27	-23.84
0.8	56.55	-43.27	47.45	-24.36
0.9	57.93	-44.19	48.61	-24.88
1.0	59.27	-45.10	49.73	-25.39

with Z the random variable defined in (25).

We will estimate this bound in the Black model where the futures prices $S_{1,2}(t)$ are correlated geometric Brownian motions. Using discrete time averaging, the variance of Z can be computed in closed form with the result

$$Var(Z) = \frac{1}{n^2}\sum_{i=1}^{n} q_i^2 M_i^{(1)} + \frac{2}{n^2}\sum_{1=i<j}^{n} q_i q_j M_{ij}^{(2)} \tag{42}$$

where

$$\begin{aligned} M_i^{(1)} &:= \mathbb{E}[(S_1(t_i) - S_2(t_i))^2] \qquad (43)\\ &= S_1^2(0)e^{\sigma_1^2 t_i} - 2S_1(0)S_2(0)e^{\rho\sigma_1\sigma_2 t_i} + S_2^2(0)e^{\sigma_2^2 t_i} \\ M_{ij}^{(2)}|_{i<j} &:= \mathbb{E}[(S_1(t_i) - S_2(t_i))(S_1(t_j) - S_2(t_j))] \qquad (44)\\ &= S_1^2(0)e^{\sigma_1^2 t_i} - 2S_1(0)S_2(0)e^{\rho\sigma_1\sigma_2 t_i} + S_2^2(0)e^{\sigma_2^2 t_i} \end{aligned}$$

The APO price impact can be translated into an impact on the APO volatility which is given to a good approximation by

$$\Delta C = F_{\rm CS}\Delta\Sigma_A\sqrt{\frac{T_2}{2\pi}} \tag{45}$$

We show in Table 8 numerical results for the upper bound on the APO volatility $\Delta\Sigma_A$ for several maturities, corresponding to the pricing date 22-May-2020. These results are obtained with daily averaging. These results suggest that the impact of the basket approximation is minimal in the perfectly correlated case $\rho = 1$ and increases as the two futures become less correlated. Also, the impact decreases with the maturity of the APO. In the perfectly correlated case $\rho = 1$ the impact is less than 1% for all maturities.

Table 8. Upper bound on the impact of the basket approximation of the APO vol $\Delta\Sigma_A$ (in %) estimated in the Black model using equation (45) for several values of the correlation parameter ρ. Pricing date 22-May-2020.

APO month	$\rho = 1.0$	$\rho = 0.95$	$\rho = 0.9$
Jul-20	0.93	2.60	3.56
Aug-20	0.64	1.94	2.66
Sep-20	0.29	1.49	2.08
Oct-20	0.09	1.27	1.78
Nov-20	0.10	1.14	1.60
Dec-20	0.03	1.04	1.46

Appendix C. Proof of Proposition 3.1

The starting point is Proposition 2.5 in [25] which we reproduce here for convenience of reference. This expresses the rate function $\mathcal{I}_{\rm fwd}(S_0, K, \tau)$ as the solution of an extremal problem.

Proposition C.1 (Prop. 2.5 in [25])**.** *Assuming that the asset price S_t follows the local volatility model*

$$\frac{dS_t}{S_t} = \sigma(S_t) dW_t + (r-q)dt\,, \tag{46}$$

the rate function of the forward start Asian option is given by the extremum problem

$$\mathcal{I}_{\rm fwd}(S_0, K, \tau) = \inf_{c\in\mathbb{R}} \left(\frac{1}{2} c^2 \tau + \frac{1}{1-\tau} \mathcal{I}(S_0 e^{F^{-1}(c\tau)}, K) \right) \tag{47}$$

where $\mathcal{I}(x, K)$ is given by the variational problem

$$\mathcal{I}(x, K) = \inf_{\varphi\in A(K/x,0)} \left\{ \frac{1}{2} \int_0^1 \left(\frac{\varphi'(u)}{\sigma(S_0 e^{\varphi(u)})} \right)^2 du \right\} \tag{48}$$

and $F(x)$ is defined by

$$F(x) = \int_0^x \frac{dz}{\sigma(S_0 e^z)}\,. \tag{49}$$

The solution of the variational problem for $\mathcal{I}(S_0, K)$ was given as an expansion in log-strike $\log(K/S_0)$ in Proposition 14 in [24]. The function $F^{-1}(z)$ appearing in (49) is denoted as $Y(z)$ in the statement of Proposition 14 in [24] and its expansion is given in equation (38) of [24] which we reproduce below

$$F^{-1}(z) = z + \frac{b_2}{b_1^2} z^2 + \frac{b_3}{b_1^3} z^3 + O(z^4)\,. \tag{50}$$

The first three coefficients $b_{1,2,3}$ are given in (44)-(46) of the same paper. They are expressed in terms of $\sigma(S_0)$ and its derivatives.

Using this result we obtain the following expansion of the rate function for forward start Asian options in the local volatility model.

Proposition C.2. *The first two terms in the series expansion of the rate function $\mathcal{I}_{\rm fwd}(S_0, K, \tau)$ given by Proposition 2.5 in [25] in powers of log-strike $x = \log(K/S_0)$ are*

$$\begin{aligned} \mathcal{I}_{\rm fwd}(S_0, K, \tau) = \frac{1}{\sigma^2(S_0)} &\left\{ \frac{3}{2} \frac{x^2}{1+2\tau} \right. \\ &\left. + \left[-\frac{3}{10} \frac{(1-\tau)^2}{(1+2\tau)^3} - \frac{9}{10} S_0 \frac{\sigma'(S_0)}{\sigma(S_0)} \frac{2-4\tau+17\tau^2}{(1+2\tau)^3} \right] x^3 + O(x^4) \right\} \end{aligned} \tag{51}$$

Proof. The proof is based on expanding the minimizer c in the extremal problem (47) in powers of log-strike $x = \log(K/S_0)$. Denote the function to be minimized as

$$\Lambda(c) := \frac{1}{2}c^2\tau + \frac{1}{1-\tau}\mathcal{I}(S_0 e^{F^{-1}(c\tau)}, K) \tag{52}$$

The rate function $\mathcal{I}(x, K)$ is expressed as a series expansion in $\log(K/x)$ in equation (37) of [24]. Using this result we have

$$\mathcal{I}(z, K) = \frac{1}{\sigma(S_0)^2}\left\{\frac{3}{2}\left(\log\frac{K}{S_0} - F^{-1}(z)\right)^2 + O\left(\log\frac{K}{S_0} - F^{-1}(z)\right)^3\right\}. \tag{53}$$

Consider first the ATM point $x = 0$. The function to be minimized has the form $\Lambda(c) \sim c^2 + O(c^3)$ which has an infimum at $c = 0$. Thus we seek the solution as an expansion of the form

$$c = f_1 x + f_2 x^2 + O(x^3)$$

The extremal condition $\Lambda'(c) = 0$ gives a sequence of equations for the coefficients f_i. The first coefficient is $f_1 = \frac{3}{1+2\tau}$. Substituting the result back into the extremal problem (47) gives the expansion (51) of the rate function in powers of log-strike x. □

For practical application to pricing forward start Asian options, it is convenient to express the rate function $\mathcal{I}_{\rm fwd}(S_0, K, \tau)$ as an equivalent log-normal volatility $\Sigma_{\rm LN}(S_0, K, \tau)$. We obtain the result

$$\begin{aligned}\Sigma_{\rm LN}(S_0, K, \tau) &= \frac{x}{\sqrt{2\mathcal{I}_{\rm fwd}(S_0, K, \tau)}} = \sigma(S_0)\sqrt{\frac{1+2\tau}{3}} \\ &\times\left\{1 + \left[\frac{1}{10}\frac{(1-\tau)^2}{(1+2\tau)^2} + \frac{3}{10}S_0\frac{\sigma'(S_0)}{\sigma(S_0)}\frac{2-4\tau+17\tau^2}{(1+2\tau)^2}\right]x + O(x^2)\right\}\end{aligned} \tag{54}$$

In the Black-Scholes model where $\sigma(S)$ is a constant, this reduces to the simpler result in Remark 3.15 of [25].

The local volatility $\sigma(S)$ is not a market observable. Its ATM value and skew are related to the corresponding quantities of the European implied volatility as

$$\sigma_{\rm ATM} = \sigma(S_0), \quad s_E = \frac{1}{2}S_0\sigma'(S_0). \tag{55}$$

The ATM level and skew of the log-normal equivalent volatility of the Asian options are defined as

$$\Sigma_{\rm A} = \Sigma_{LN}(S_0, S_0, \tau), \quad s_A = S_0\frac{d}{dK}\Sigma_{LN}(S_0, K, \tau)|_{K=S_0} \tag{56}$$

They can be read off as the coefficients in the expansion (54). Substituting $\sigma(S_0)$ and $\sigma'(S_0)$ from (55) gives the result 13) and (14), which completes the proof of the stated result.

References

[1] L. Andersen (2010), Markov models for commodity futures: Theory and practice, *Quantitative Finance* **10**(8), 831-854.

[2] Arguin, L.P., N.-L. Liu and T.-H. Wang. (2018) Most-likely-path in Asian option pricing under local volatility models, *IJTAF* **21**(5), August 2018.

[3] M. Avellaneda, D. Boyer-Olson, J. Busca and P. Friz (2002), Reconstructing volatility, *Risk* October 2002, 91.

[4] C. Bayer and P. Laurence (2004) Asymptotics beats Monte Carlo: The case of correlated local vol baskets, *Comm. Pure and Applied Mathematics* 67(10), 1618-1657.

[5] Black, F. (1976). The pricing of commodity contracts, *Journal of Financial Economics* **3**, 167-179.

[6] Bouaziz, L., E. Briys and M. Crouhy (1994). The pricing of forward-starting Asian options. *Journal of Banking and Finance.* **18**, 823-839.

[7] Clewlow, L. and C. Strickland (1999), A multi-factor model for energy derivatives risk management, Working paper, School of Finance and Economics, University of Technology, Sydney

[8] Clewlow, L. and C. Strickland (2000), Energy Derivatives: Pricing and Risk Management, London, Lacima Publications

[9] Clewlow, L., B. Kang and C.S. Nikitopoulos (2014), On the volatility of commodity futures prices, in *Nonlinear Economic Dynamics and Financial Modelling*, R. Dieci, X.Z. He and C. Hommes (Eds.), p.315, Springer, New York.

[10] Cortazar, G. and E.S. Schwartz (1994) The valuation of commodity contingent claims, *The Journal of Derivatives* **1**(4) 27-39

[11] Cortazar, G. and Eduardo S. Schwartz (2003) Implementing a stochastic model for oil futures prices. *Energy Economics* **25**(3) (2003)

[12] Curran, M. (1994). Valuing Asian options and portfolio options by conditioning on the geometric mean price. *Management Science.* **40**, 1705-1711.

[13] Eydeland, A. and K. Wolyniec. (2003). *Energy and power risk management - New developments in modeling, pricing and hedging*, Wiley and Sons, Hoboken.

[14] Foschi, P., Pagliarani, S. and A. Pascucci. (2013). Approximations for Asian options in local volatility models. *Journal of Computational and Applied Mathematics.* **237**, 442-459.

[15] Fusai, G. and A. Roncoroni (2008), *Implementing models in quantitative finance: Methods and cases*, Springer Finance Series, Springer, New York.

[16] Gabillon, J. (1991) The term structures of oil futures prices. Oxford Institute for Energy Studies, Working paper (1991)

[17] Geman, H. (2005). Commodities and Commodity Derivatives: Modeling and Pricing for Agricultural, Metals and Energy, Wiley Finance Series, Wiley and Sons, Hoboken.

[18] Gibson, R. and E. Schwartz (1990). Stochastic convenience yield and the pricing of oil contingent claims, *Journal of Finance* **45**, 959-976.

[19] A. Gulisashvili and P.Tankov (2013) Tail behavior of sums and differences of log-normal random variables, *Bernoulli* **22**(1) 444-493.

[20] P. Henry-Labordère(2008) *Analysis, Geometry and Modeling in Finance. Advanced Methods in Option Pricing*, Chapman and Hall/CRC, Financial Mathematics Series

[21] Jamshidian, F. (1991). Commodity option valuation in the Gaussian futures term structure model, *Review of Futures Markets* **10**, 61-86.

[22] Miltersen, K. and E. Schwartz (1998). Pricing of options on commodity futures with stochastic term structures of convenience yield and interest rates, *Journal of Financial and Quantitative Analysis* **33**, 33-59.

[23] Miltersen, K. (2003) Commodity price modeling that matches current observables, *Quantitative Finance* **3**, 51-58.

[24] Pirjol, D. and L. Zhu. (2016). Short maturity Asian options in local volatility models. *SIAM Journal on Financial Mathematics.* **7**, 947-992.

[25] Pirjol, D., J. Wang and L. Zhu, Short maturity forward start Asian options in Local Volatility models, *Applied Mathematical Finance* **26**(3), 187-221 (2019).

[26] Schwartz, E.S. (1997) The stochastic behavior of commodity prices: implications for valuation and hedging. *J. Finance* **52**(3), 923–973

[27] Kenichiro Shiraya, Akihiko Takahashi, Pricing Multiasset Cross-Currency Options, Journal of Futures Markets, 10.1002/fut.21590, 34, 1, (1-19), (2012).

[28] Tsao, C.-Y., Chuang-Chang Chang and Chung-Gee Lin. (2003) Analytic approximation formulae for pricing forward-starting Asian options. *The Journal of Futures Markets* **23**, 487-516.

[29] Vanmaele, M., Deelstra, G., Liinev, J., Dhaene, J. and Goovaerts, M.J. (2006). Bounds for the price of discrete arithmetic Asian options, *J. Comp. Appl. Math.* **185**, 51-90.

[30] Vecer, J. (2001). A new PDE approach for pricing arithmetic average Asian options. *J. Comp. Finance* **4**(4), 105-113.

[31] Vecer, J. (2002). Unified Asian pricing. *Risk.* **15**, 113-116.

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