Information, Insider Trading, Regulation, Executive Reload Stock Options and Incentives

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Latest Revision: March 9th, 2018

[Preliminary draft, please do not quote.]

Abstract.

Executives, typically, have insider information. Consequently, conventional executive stock options (ESOs) may fail in aligning executive and shareholder interests. We examine this issue by introducing a theoretical model that studies how insider portfolio optimization can reduce the effectiveness of firms' executive incentive schemes. The study provides policy implications for the regulation required to maintain the effectiveness of such schemes. We offer a proposal for granting executive stock reload options, in combination with insiders' blackout trading restriction. We show how this proposal might be calibrated to prevent harmful effects of insider trading on the one hand, while allowing fairness to executives on the other. We, thus, suggest a welfare-improving mechanism.

Keywords: Executive Stock Options, Insider, Constrained Portfolio Optimization, Non-Hedgeable Non-Transferable, Reload, enlarged filtration

JEL Codes: G11, G13, C02, C61

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[d] We thank Ben Goldys, Marek Rutkowski for helpful discussions.

1. Introduction

The possession of material non-public information (henceforth insider information) can influence insiders' trading from two segments: arbitrage and portfolio optimization. We claim that the regulators should prohibit insider arbitrage and allow, to some extent, insider portfolio optimization. Accordingly, this research studies how insiders' optimization can reduce the effectiveness of firms' executive incentive schemes. We provides policy implications for the regulation required to maintain the effectiveness of such schemes. We offer a proposal for granting executive stock reload options, in combination with insiders' blackout trading restriction. We show how this proposal might be calibrated to prevent harmful effects of insider trading on the one hand, while allowing fairness to executives on the other.

The regulators should prohibit insider arbitrage and allow, to some extent, insider portfolio optimization for two reasons: First, although both types of insider trading cause injustices in the secondary market and, accordingly, jeopardize the fundamental operation of the primary market as a resource allocation optimizer, it is relatively easy for regulators to detect and monitor insider arbitrage. This is because the attainment of infinite wealth through arbitrage (i.e., replicating a risk-free zero coupon bond at a price that is strictly less than the initial price of that bond) requires insiders to take infinite financial position(s)[1]; while portfolio optimization implies only a limited difference between insiders' and outsiders' optimal portfolio processes. Second, although insider trading facilitates rapid price discovery and enhances market efficiency, the price change caused by insider arbitrage is a one-off instant occurrence, and insiders are the only beneficiaries; whereas insider portfolio optimization is a sustained information release process, and there is time for a co-sharing of profit between insiders and outsiders.

The Securities Exchange Commission (SEC) promulgated provisions against fraud [See Section 10(b) and Rule 10b-5 under Securities Exchange Act of 1934, Trading Sanctions Act of 1984 (ITSA), see also Bainbridge 2013] that insiders are prohibited from selling or purchasing firm stock unpremeditatedly, or according to a predetermined plan written after the insider became aware of the information. We think that Rule 10b-5 takes effect only for insider arbitrage, because, first, it is very hard to detect and prove that an insider trader intended to use privileged information for personal gain unless the trading volume is abnormally high, as in the case of an infinite leverage required by arbitrage; second, it is almost impossible to identify at what time, and to what degree of accuracy, the insider information is acknowledged by insiders. We will

[1] Assume insiders know the stock spot price for a future time T (no matter greater than, equal to, or less than the current spot one), they can exhaust the benefit of receiving option premium by shorting infinite units of European options (call or put) with maturity set as T and strike price equal to the future spot price, because the option will never be exercised. Or, insiders can arbitrage through buying infinite shares of firm stock at infinite leverage, holding the stocks till T, and using the proceeds of selling those stocks to pay for the principal and interest.

demonstrate in Subsections 2.4 and 2.5 that insiders can still achieve infinite derived utility through portfolio optimization based on the noisy insider information they possess. The insiders who possess noisy information can set up a Rule 10b5-1 trading plan, under which to buy or sell shares at a predetermined time on a scheduled basis, to avoid accusations of insider trading. The SEC cannot verify, at the time the plan is initiated, if insiders had noisy insider information or had no material information at all.

Hence, we believe Rule 10b-5 illegalizes insider arbitrage, while Rule 10b5-1 legalizes insider portfolio optimization. Note that insiders must do the initial filing with SEC on form 3 to state the ownership of firm securities, and must also report the changes on Form 4 (or Form 5 for deferred reporting), so that outsiders can benefit from the information disclosed, while insiders conduct insider trading and fulfil their filing responsibility.

Does Rule 10b5-1 violate the justice principle, while legalizing insider portfolio optimization? To achieve infinite derived utility through portfolio optimization with finite initial wealth, the insiders are required to rebalance the portfolio with infinite frequency as the insider information disclosure time approaches. Section 306(a) and Regulation Blackout Trading Restriction (BTR) under Sarbanes-Oxley Act of 2002, prohibit corporate executives and rank-and-file employees from engaging in transactions during a blackout period in order to invalidate the necessary condition for insiders to achieve infinite derived utility - the infinite rebalancing frequency in the neighborhood of the disclosure time. For example, a blackout period might be enacted as beginning two weeks prior to the end of a fiscal quarter and ending upon completion of one full trading day after the public announcement of earnings for that quarter.

Furthermore, insider liability allows the executives to exercise the option through an “intro-company” approach (See Nathan and Hoffman 2013), i.e., executives provide value to the company in the form of cash or stock shares, in exchange for shares. Any other approach involving contemporaneous sale into the market is prohibited, e.g., the “Broker-assisted cashless” exercise, whereby, at the time of exercise, some or all of the exercised shares are sold into the market, the requisite amount of the sale proceeds are used to pay the company for the exercise, and the holder keeps the net proceeds and any unsold shares. Even taking the “intro-company” approach, the executives cannot sell the resulting shares during the blackout period.

Furthermore, Statement of Financial Accounting Standards No. 123 (revised 2004) [FAS123(R)], paragraph B73 to B79 precludes executives from transferring vested equity share options to third parties; paragraph B80 to B82 outlines the inability to sell shares of the issuer’s stock that executives do not own, i.e., non-transferable and non-hedgeable (NTNH). Therefore, those insider trading regulations impact on the insider executives’ portfolio optimization through revising the portfolio constraints from NTNH only into a combination of the insider trading prohibition and NTNH constraints. Because the “intro-company” exercise requires the executives to optimally keep cash or stock as reservation in advance, the potential insider liability ties the executives to extra portfolio constraints.

The contribution of this research includes: first we decompose insiders' derived utility improvement into substantial and perceived components, where the former is caused by insiders' ability to choose the optimal portfolio process conditioning on an enlarged information set, and the latter is due to insiders and outsiders differing perceptions, but the perception itself does not make any difference to the realized terminal wealth of a traditional portfolio optimization problem, where the assets are all primary assets. Hence, the perceived derived utility change has been neglected in the literature. However, insider executives usually hold NTNH ESOs. We claim that, although insiders and outsiders holding differing perceptions will observe the same sequence of realised stock prices, the perceived component affects an insider's choice of the optimal exercise time (or the optimal exercise rate if partial exercise is allowed) of the ESOs they have. Because ESOs are NTNH, exercising ESOs will relax the portfolio constraints, which will, in turn, affect the insider's optimal constrained portfolio process and, accordingly, cause substantial impacts.

Second, we provide direct insight into the mechanism by which insider portfolio optimization reduces the effectiveness of firms' incentives. We prove that, conditional on the insider executives' ability to rebalance their portfolios at any frequency in the neighbourhood of the disclosure time, they can achieve infinite derived utilities using finite initial outside wealth[2], even if the non-public information about the future risk source is noisy, with any level of information quality and, meantime, the insider executives bear costs of holding a constrained portfolio[3]. This result has practical significance. As is known to all, in reality, an accurate forecast in stock markets can never exist, even though insiders sometimes know a future event, such as a merger or acquisition, will happen for certain.

Third, we provide policy implications for the regulations required to maintain the effectiveness of firms' incentives to executives. The SEC, who monitor and regulate blackouts on a macroscale, aims its regulations towards eliminating the inequitable distribution of wealth caused by insiders' information advantage. It is the firm's responsibility to stipulate a particular length of blackout period suitable for reducing the impact of insider trading on the effectiveness of firm-granted incentives. We define the lower bound of blackout as the longest that makes the insider executives exercise all the American ESOs at the time ESOs are just past their vesting period; and the upper bound as the one equating the insider's information superiority and the

[2] We assume that executives' portfolios consist of a number of shares of NTNH ESOs and a sub-portfolio composed of primary assets only. We name the sub-portfolio *outside wealth*.

[3] In practice, as infinite rebalancing frequency does exist, this could result in the high abnormal profits that we see insiders achieve.

stricter portfolio-constraint's inferiority brought about by the blackout. Any blackout equal to or shorter than the lower bound is considered to be inadequate, and results in the invalidation of the incentivizing mechanism. Any blackout longer than the upper bound is considered to be excessive, and will result in insiders being worse off than outsiders owing to the non-trading prohibition, and that will exacerbate price discontinuity and harm market efficiency. We then determine the range of required blackout length, within which, the blackouts are considered to be applicable, and insider executives are allowed to enjoy limited information advantages but still perform in whatever situation results in the most benefit to the shareholders.

Fourth, based on the theoretical results derived, we provide further policy recommendations that if the insider information is about an idiosyncratic risk, then the executives, directors and employees at all levels should only be forbidden to trade on their firm stocks; otherwise, if the information is about systematic risk, the SEC should take the role of legislators to enact laws preventing insiders from making any alterations to their retirement or investment plans during blackout.

Fifth, we claim that finding a fixed blackout window that works across all firm insiders is always an issue, hence, it is critical that firms develop other incentivizing schemes, superior to restricted stocks and ESOs, for boosting the effectiveness of the incentives. 1) In reality, an applicable blackout might not exist, because the derived lower bound can be greater than the upper bound. 2) Even if it exists, the boundary varies with the individual, and, by definition, depends on insiders' attributes, such as the information possessed (type and quality), total wealth and wealth composition, and the constraints binding portfolios. Hence, we should not equalize the treatment of executives and rank-and-file employees with respect to their ability to engage in transactions during a blackout. 3) Even for a particular executive, "applicable blackout" is not a static concept. As the executive's total wealth and portfolio constraints change dynamically, a fixed blackout window for a particular insider might switch among different states (inadequate, applicable, and excessive) from time to time. 4) Job termination might reduce the portfolio holding period, and might extend the blackout period equivalently from an inadequate one to an applicable one, hence, ESOs can provide different incentives for different groups of insider executives jointly categorized by the adequacy of the existing blackout period and the foreseeability of job termination. 5) We also document a *tolerance effect* of ESOs brought by the executives' insider trading, and find that a stronger incentive might result in the failure of the incentive mechanism. Owing to the above five reasons, if, in practice, a firm can only mandate a unique predetermined blackout applied across all corporate insiders for an event, then the development of incentives alternative to ESOs is critical.

Sixth, we provide a policy recommendation that executive reload stock options (RSOs), which fell out of favour around 2006, should be reinstated, as they strategically align the long-term interests of shareholders and executives better than do ESOs, and they will not bestow excessively lucrative shares (as a money pump) upon executives: a frequent criticism. We prove that the upper bound of the firm cost of granting an RSO is just the firm stock price, and this result holds when the recipients of RSOs are insiders. To demonstrate this, we develop a pricing methodology for RSOs, considering executives' insider trading, information quality and the portfolio constraints caused by the covenants applied to RSOs and the prevailing security regulations. The Accounting Standards Board (FASB) continues to believe that, owing to the pricing difficulty, subsequent granting of reload options pursuant to that provision should be accounted for as a separate award when the reload options are granted [See FAS123(R) paragraph 24 to 26, see also Saly et al 1999] That is shown to be not necessarily true.

Our results are mostly consistent with, and provide theoretical foundations for, the following empirical research progression. Roulstone (2003) found insider trading laws increase executive compensation and equity-based incentives. Denis and Xu (2013) showed that those results are robust to alternative definitions of insider trading restrictions and enforcement, and to panel regressions with country-fixed effects. Henderson (2011) further studied the same relationship, but focused on Rule 10b5-1, and isolated the potential profits from portfolio optimization and the informed trading. The evidence suggests that executives whose trading freedom increased using Rule 10b5-1 trading plans experienced reductions in other forms of pay, to offset the potential gains from trading. Carpenter and Remmers (2001) and Aboody et al (2008) examined whether insiders use private information to time the exercise of their ESOs. Fu and Ligon (2010) investigated whether insider information motivates executives' early exercise upon vesting. Bettis et al (2005), taking executives' insider role into account, calibrated Carpenter (1998)'s utility-based model creatively to get ESO values and incentives. Then, based on these, documented the impact on insiders' exercise behavior. Brooks et al (2010) found the most informed executives tended to exercise early, and the operating performance of firms following exercises motivated by private information was significantly worse than that of firms in which the exercises were not motivated by private information.

The mathematical achievements in this field that we thankfully employed in solving insiders' asset pricing include that of Pikovsky and Karatzas (1996), who applied 'enlargement-of-filtration' techniques to allow the anticipative feature of portfolio optimization, based on which, and combining with the work of Karatzas and Kou (1996), a pricing framework for insiders' contingent claims under constraints can be created. Colwell et al (2015) relaxed certain technical assumptions to enable portfolio optimization subject to NTNH constraints, so that the framework could be used directly in pricing insiders' NTNH ESOs or, with some modifications, in pricing insiders' equity-based compensation, reflecting various features, such as reload.

The paper proceeds as follows. Section 2 studies how insider trading invalidates firms' incentive schemes, through the portfolio optimization approach. Section 3 demonstrates that RSOs are more efficient incentives to align insider executives' long term interests with those of shareholders. Section 4 concludes.

2. Insider portfolio optimization

Pikovsky and Karatzas (1996) applied enlarged-filtration techniques to allow the anticipative feature of portfolio optimization. Combined with Cvitanić and Karatzas (1992)'s constrained portfolio optimization approach, this methodology enables us to take executives' insider role into account when pricing NTNH ESOs under insider trading regulations.

2.1. Initial enlargement of filtration

First, we brief the model setting of original market, $\mathcal{M}$, where there is a traded bond, whose price evolves according the differential equation,

$$dS_0(t) = S_0(t)r(t)dt, \;\; S_0(0) = 1, \tag{1}$$

where $r(t)$ is risk-free rate. Primary asset prices $S_i(t), i = 1, \dots, d$ follow the dynamics of

$$dS_i(t) = S_i(t)\left[b_i(t)dt + \textstyle\sum_{j=1}^{d} \sigma_{i,j}(t)dW_j(t)\right], \;\; S_i(0) = s_i, \;\;\; j = 1,2, \dots, d, \tag{2}$$

where $W(t) = (W_1(t), \dots, W_d(t))^\top$ is a d-dimensional standard Brownian motion in $\Re^d$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and we denote by $\{\mathcal{F}_t\}$ the $\mathbb{P}$-augmentation of the natural filtration $\mathcal{F}_t^W = \sigma(W(s); 0 \le s \le t)$, with time span $[0, T]$ for some finite $T > 0$. $\sigma(t) \triangleq \left(\sigma_{i,j}(t)\right)$ is a $d \times d$ volatility matrix, and $b(t) \triangleq \left(b_i(t)\right)$ is a $d \times 1$ drift rate vector. $\sigma(t)$ and $b(t)$are assumed to be progressively measurable with respect to $\{\mathcal{F}_t\}$. The market-price-of- risk vector is $\theta(t) = \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}]$.

Throughout Section 2, we assume an initially enlarged filtration model, and use an $\mathcal{F}_{T^*}$-measurable random variable G *to* represent the insider information that the executives possess throughout from the beginning. The probability space is $(\Omega, \mathcal{G}, \mathbb{P})$, where the probability measure $\mathbb{P}$ is unchanged, and the filtration is enlarged from $\mathcal{F}$ to $\mathcal{G} = \{\mathcal{G}_t\}_{0 \le t \le T^*}$ with $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(G)$, and insider executives can dynamically choose portfolios from the class of $\mathcal{G}$-adapted processes. The corresponding Girsanov theorem for $(\Omega, \mathcal{G}, \mathbb{P})$ is as stated below.

Lemma 1: $W(t)$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, then $\widetilde{W}(t) = W(t) - \int_0^t a(s)ds$ is a Brownian motion on $(\Omega, \mathcal{G}, \mathbb{P})$, where $a(t)$ is uniquely determined by $dq_t^y / q_t^y = a(t)dW(t)$, and $q_t^y \equiv \frac{P(G \in dy | \mathcal{F}_t)}{dy}$.

Proof. See Appendix A.

Corollary 1: On $\mathcal{G}_t$, $\forall t \in [0, T^*)$, $Z(t) \equiv \left.\frac{d\mathbb{Q}^{\mathbb{G}}}{d\mathbb{P}}\right|_{\mathcal{G}_t} = e^{-\int_0^t \Theta(s)d\widetilde{W}(s) - \frac{1}{2}\int_0^t (\Theta(s))^2 ds}$, with $\Theta(t) = \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}] + a(t)$ is the Radon-Nikodym derivative changing the probability measure from $\mathbb{P}$ to $\mathbb{Q}^{\mathbb{G}}$, under which the discounted stock price $e^{-\int_0^t r(s)ds} S(t)$ is a martingale.

Proof. See Appendix B.

For $d = 1$, if insiders know the exact value of $W(T^*)$ beforehand, i.e., $\forall t \in [0, T^*)$, $G = W(T^*)$, then many sample paths will be ruled out, and the remaining possible ones represent a Brownian bridge. For general d-dimensional case, without generality, we assume $G = W_1(T^*)$, the compensation process vector $a(t) = \left(\frac{W_1(T^*)-W_1(t)}{T^*-t}, 0, \ldots, 0\right)'$, see Pikovsky and Karatzas (1996).

2.2. Insider and outsider's initial wealth

Possessing insider information of firm stock affects the insiders' trading in two ways: insider arbitrage and portfolio optimization, and results in insider arbitrage price and insider fair price of asset, respectively. In practice, Rule 10b-5 illegalize insider arbitrage, and Rule 10b-5.1 legalize insider portfolio optimization, hence the fair price is the focus of this study.

Note that insiders can always choose to neglect the insider information to sell the assets at the market price, when insiders know they are actually worth less based on the information they possess. Hence the fair price of any contingent claim on firm stock is stated in Lemma 2. We denote $H_0^a(t) \triangleq \exp\left\{-\int_0^t r(s)\,ds\right\}\exp\left\{-\int_0^t [\Theta(s)]^\top dW(s) - \frac{1}{2}\int_0^t \|\Theta(s)\|^2 ds\right\}$, and $\Theta(s) \triangleq \sigma^{-1}(s)[b(s) - r(s)] + a(s)$,

Lemma 2: At any time, $t \in [0, \bar{T}]$, insider's fair price of European contingent claim with maturity $\bar{T}$, is $p^{\mathbb{G}}(t) = \max\left(p^a(t), p^0(t)\right)$, where $B(\bar{T})$ is the payoff of contingent claim with maturity $\bar{T} \leq T^*$, and

$$p^a(t) = \mathbb{E}^{\mathbb{P}}[H_0^a(T)B(T)|\mathcal{G}_t]/H_0^a(t),\ \ p^0(t) = p^a(t)|_{a=0}. \tag{3}$$

Proof. It is obvious from Corollary 1 and Theorem 7.4 in Karatzas and Kou (1996).

Lemma 2 implies that the replicating position and correspondingly the prices of derivatives can be different to the insiders and outsiders, however due to the replicating position of underlying is constant one, the firm stock price to the insiders and outsiders are the same, and it is the market stock price[4] $S(t)$. Taking $G = W(T^*)$ as an example, even if we assume $\theta(s) = \sigma^{-1}(s)[b(s) - r(s)]$ is deterministic, because $a(s) = \left(\frac{W_1(T^*)-W_1(s)}{T^*-s}, 0, \ldots, 0\right)'$, $\Theta(s)$ is $\mathcal{G}_s$-progressively measurable. According to Clark-Ocone formula [see Ocone and Karatzas 1991, Eq (2.20)], the replicating position of option on stock from the insiders is different from what Black-Scholes suggests, hence we denote them as $\Phi^{\mathbb{G}}$ and $\Phi^{\mathbb{F}}$, respectively. If $\theta(s)$ is $\mathcal{F}_t$- progressively measurable, then the replicating position of a derivative from the insiders and outsiders, and the

[4] Assume an insider is holding one share of firm stock currently worth \$5, and she knows that the stock price will be \$10 in one year, and its present value is \$9. She can liquidate the stock by borrowing \$9 now, holds the stock she has for one year, sell the stock for \$10 in one year, and use the proceed to pay for the principal and interest of the borrowing. Through this way, equivalently, she sells the stock at \$9 today. However, this strategy requires the insider to be able to trade at T^*. Throughout the study, we assume that insiders' trading time interval $[0, T^*)$ does not include T^*. Therefore, the firm stock price to insiders and outsiders are the same, and equal to the market spot price.

one suggested by Black-Scholes, are mutually different. We denote $x_t^{\mathbb{G}}$ and $x_t^{\mathbb{F}}$ as the different values of the same initial total wealth x_t (including firm granted ESOs and outside wealth) to the insiders and outsiders, respectively.

2.3. Insider unconstrained portfolio optimization without considering regulations

In this subsection, we do not consider any portfolio constraints, and assume the ESOs that insider executives have are all European. Assume a $ln(\cdot)$ utility function, for a terminal time, $T \in (0, T^*)$, we define the derived utility increment due to the insider information as $\Delta J(x_t, t, T) \equiv J^{\mathbb{G}}(x_t, t, T) - J^{\mathbb{F}}(x_t, t, T)$, where $J^{\mathbb{G}}(x_t, t, T) \equiv \underset{\pi \in \mathcal{A}_0^{\mathbb{G}}(x_t,t,T)}{\text{esssup}} \mathbb{E}^{\mathbb{P}}\left[U\left(X^{x_t,\pi}(T)\right)|\mathcal{G}_t\right]$, $J^{\mathbb{F}}(x_t, t, T) \equiv \underset{\pi \in \mathcal{A}_0^{\mathbb{F}}(x_t,t,T)}{\text{esssup}} \mathbb{E}^{\mathbb{P}}\left[U\left(X^{x_t,\pi}(T)\right)|\mathcal{F}_t\right]$. $\mathcal{A}_0^{\mathbb{G}}(x_t, t, T)$ and $\mathcal{A}_0^{\mathbb{F}}(x_t, t, T)$ are defined as the sets of admissible portfolio processes from time t to T, with initial total wealth being x_t, ensuring that the portfolio wealth process, denoted by $X^{x_t,\pi}(s)$, is finite for $s \in [t, T]$ and satisfies $\mathbb{E}^{\mathbb{P}}\left[U^-\left(X^{x_t,\pi}(T)\right)|\mathcal{G}_t\right] < \infty$ and $\mathbb{E}^{\mathbb{P}}\left[U^-\left(X^{x_t,\pi}(T)\right)|\mathcal{F}_t\right] < \infty$, respectively, where $U^-(x) \triangleq -\min(U(x), 0)$.

Lemma 3: Under $ln(\cdot)$ utility assumption, if $\int_0^T \mathbb{E}^{\mathbb{P}} \left|\left[\sigma^{-1}(t)[b(t) - r(t)\mathbf{1}]\right]^\top a(t)\right| du < \infty$, $\int_0^T \mathbb{E}^{\mathbb{P}}[\|a(t)\|^2]\, du < \infty$, $\int_0^T \mathbb{E}^{\mathbb{P}} \left|r(u) + \frac{1}{2}\|\sigma^{-1}(u)[b(u) - r(u)\mathbf{1}]\|^2\right| du < \infty$, then

$$\Delta J(x_t, t, T) = ln\left(x_t^{\mathbb{G}}\right) - ln\left(x_t^{\mathbb{F}}\right) + \int_t^T \left(\mathbb{E}^{\mathbb{P}}\left[r(u) + \frac{1}{2}\|\Theta(u)\|^2|\mathcal{G}_t\right]\right) du - \int_t^T \left(\mathbb{E}^{\mathbb{P}}\left[r(u) + \frac{1}{2}\|\theta(u)\|^2|\mathcal{F}_t\right]\right) du, \tag{4}$$

where $\Theta(u) = \sigma^\top(t)\pi^{\mathbb{G}*}(t)$ and $\theta(u) = \sigma^\top(t)\pi^{\mathbb{F}*}(t)$ represent the risk prices for insiders and outsiders, respectively; $\pi^{\mathbb{F}*}(t) = [\sigma(t)\sigma^\top(t)]^{-1}[b(u) - r(u)\mathbf{1}]$ and $\pi^{\mathbb{G}*}(t) = \pi^{\mathbb{F}*}(t) + [\sigma^\top(t)]^{-1}a(u)$ are outsiders' and insiders' optimal portfolio process, then equivalently

$$\Delta J(x_t, t, T) = \Delta J_0 + \Delta J_1 + \Delta J_2 + \Delta J_3, \tag{5}$$

where

$$\begin{aligned}
&\Delta J_0 \equiv ln\left(x_t^{\mathbb{G}}\right) - ln\left(x_t^{\mathbb{F}}\right),\\
&\Delta J_1 \equiv \frac{1}{2}\int_t^T (\mathbb{E}^{\mathbb{P}}[\|a(u)\|^2|\mathcal{G}_t])du,\\
&\Delta J_2 \equiv \int_t^T \mathbb{E}^{\mathbb{P}}\left[\left[\sigma^{-1}(u)[b(u) - r(u)\mathbf{1}]\right]^\top a(u)|\mathcal{G}_t\right] du,\\
&\Delta J_3 \equiv \int_t^T \left(\mathbb{E}^{\mathbb{P}}\left[r(u) + \frac{1}{2}\|\sigma^{-1}(u)[b(u) - r(u)\mathbf{1}]\|^2|\mathcal{G}_t\right] - \mathbb{E}^{\mathbb{P}}\left[r(u) + \frac{1}{2}\|\sigma^{-1}(u)[b(u) - r(u)\mathbf{1}]\|^2\right|\mathcal{F}_t\right]\right) du.,
\end{aligned} \tag{6}$$

Proof. See Appendix C.

We filter out the perceived component inherent in $\Delta J(x_t, t, T)$ by adding an $\mathbb{E}^{\mathbb{P}}(\cdot|\mathcal{F}_t)$ operator and, then, we get the substantial derived utility improvement $\mathbb{E}^{\mathbb{P}}(\Delta J(x_t, t, T)|\mathcal{F}_t)$, which is composed of ΔJ_0, an improvement due to insiders' information advantage enabling them to liquidate the under-valued ESOs in their initial wealth package at a price greater than their market value, and $\Delta J_i, i = 1,2$, owing to insiders' ability to choose an optimal portfolio process from a class of $\mathcal{G}$-adapted processes, conditioning on an enlarged information set.

The literature often neglects the perceived component, ΔJ_3, because it is caused by insiders' and outsiders' differing perceptions, but the perception itself does not make any difference to the realized terminal wealth or the derived utility. For example, assuming: $W_1(T^*) = y$, then insiders' expectation on $W_1(T^*)$ at time zero is: $\mathbb{E}^{\mathbb{P}}\left[\left(W_1(T^*)\right)\middle|\mathcal{G}_0\right] = y$; outsiders' expectation is $\mathbb{E}^{\mathbb{P}}\left[\left(W_1(T^*)\right)\middle|\mathcal{F}_0\right] = 0$, but the realized $W_1(T^*)$ is y, no matter at what time this fact is acknowledged by the investors.

Remark 1: The fact that the initial total wealth of insiders and outsiders is different, is often neglected by the existing literature. As insider executives usually hold ESOs, possessing insider information will enable them to liquidate the options equivalently, using different hedging positions (which is the opposite of the replicating position that we discussed in Subsection 2.2). Also, to obtain ΔJ_0, insiders are required to update the hedging position dynamically, hence, such improvement is substantial. Even if we incorporate the NTNH constraints later in Section 2.5 and then, because the non-hedgeability, ΔJ_0 disappears, the fact that ESOs can be of more value to insiders, implies that NTNH ESOs are taking a greater proportion in insiders' portfolios, which, in turn, introduces stricter portfolio constraints. Hence, distinguishing between the ESO price to insiders and that to outsiders is critical.

Remark 2: ΔJ_1 is determined by the squared norm of compensating process, $\|a(u)\|^2$, which implies a symmetrically benefiting component to the insiders, whether the insider information is good (e.g., $G = W_1(T^*) > 0$) or bad (e.g., $G = W_1(T^*) < 0$) news.

Remark 3: ΔJ_2 is determined by a compensating process, $a(u)$. Becuase $\mathbb{E}^{\mathbb{P}}[a(u)|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[dW(u)|\mathcal{F}_t] - \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\widetilde{W}(u)|\mathcal{G}_t\right]|\mathcal{F}_t\right] = 0$, if, for any time, $u \in [t, T]$, $a(u)$ is independent of $\mathcal{F}_u$, then ΔJ_2 becomes zero; otherwise, good news and bad news will cause an asymmetric impact on insiders' optimal derived utility.

Remark 4: Whether $a(u)$ is independent of $\mathcal{F}_u$, is determined by the type of insider information, such as terminal value of risk source, upper bound at fixed time, local time at fixed time, last zero before a fixed time, first hitting time, and their combinations, etc. See page 34, Mansuy and Yor (2006). Denote $(W(s);\ s \geq 0)$ as a standard Brownian motion and $\forall t \in [0, T^*]$, $\mathcal{S}(t) = \sup_{s\in[0,t]}(W(s))$, if $G = \mathcal{S}(T^*)$, then $a(t) = -\frac{1}{\sqrt{T-t}}\varphi\left(\frac{\mathcal{S}(t)-W(t)}{\sqrt{T-t}}\right)1_{\mathcal{S}(t)=\mathcal{S}(T^*)} + ((\mathcal{S}(T^*) - W(t))/T - t)1_{\mathcal{S}(t)<\mathcal{S}(T^*)}$, where $\varphi(x) = e^{-x^2/2}/\int_0^x e^{-u^2/2}du$. In that case, $\mathbb{E}^{\mathbb{P}}(\Delta J_2|\mathcal{F}_t) \neq 0$, ΔJ_2 reflects the substantial impact, which is asymmetrically determined by the good news or bad news.

Remark 5: We claim that if the NTNH constraints and the American feature of ESOs are considered, the perceived component ΔJ_3 will affect insiders' decisions on choosing the optimal exercise time (or optimal exercise rate if partial exercise is allowed) of the ESOs they have.

Because ESOs are NTNH, exercising ESOs will relax the portfolio constraints, which will, in turn, alter the insider's optimal constrained portfolio process and, accordingly, cause substantial impacts.

Corollary 2: If $G = W_1(T^*)$ and $\mathbb{E}^{\mathbb{P}}\left[\left(\frac{b_1(u)-r(u)}{\sigma_{1,1}(u)}\right)\middle|\,\mathcal{G}_t\right]$ is not a function of u, then as $T \to T^*$, $\mathbb{E}^{\mathbb{P}}[\Delta J(x_t, t, T)|\mathcal{F}_t] \to \infty$.

Proof. See Appendix D.

Corollary 2 suggests that the insider executives can achieve infinite derived utilities using finite initial outside wealth through the insiders' portfolio process, if they know the value of the risk source determining the future stock price. That explains how insider portfolio optimization causes firms' incentives to executives lose effectiveness.

However, an accurate forecast in the stock market almost never exists, even though the insiders sometimes know a future event, e.g., a merger and acquisition, will happen for certain. Also, the insider executives bear costs of holding a constrained portfolio. Therefore, in practice, will insiders be able to achieve an infinite derived utility? In Subsections 2.4 and 2.5, for the first time in the literature, we document a counterintuitive result: that insiders can achieve an infinite derived utility with noisy insider information about $W(T^*)$ and with NTNH constraints binding their portfolios. That implies the necessity of introducing blackout regulation.

2.4. Weakly progressive enlargement of filtration

In this subsection, we consider a case more close to reality: that insiders observe a mixture of true information and noise over time, rather than that they know the terminal value of a risk source in advance, accurately. We model the noise as an additional risk source and, artificially, make the imaginary primary asset driven by this risk source non-tradable. We name this setting *weakly progressive enlargement of filtration*[5].

Specifically, we assume $W(t)$ is a $(d+1)$-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. The primary asset prices, $S_i(t), i = 1, \dots, d+1$, follow the dynamics of

$$dS_i(t) = S_i(t)\left[b_i(t)dt + \sum_{j=1}^{d+1} \sigma_{i,j}(t)dW_j(t)\right],\ \ S_i(0) = s_i,\ \ j = 1,2, \dots, d+1, \tag{7}$$

Without loss of generality, $S_1(t)$ is fully determined by $W_1(t)$. We achieve this, by setting the submatrix $\left(\sigma_{i,j}(t)\right)$ for $i, j \le d$ to be the lower unit triangular of the Cholesky decomposition of variance-covariance matrix of the primary asset's return vector $(dS_i(t)/S_i(t))$ for $i \le d$. The purpose of introducing $\{W_{d+1}(t)\}$ is to add a noise component to the insider information regarding the risk source, $\{W_1(t)\}$, to reflect the weakly progressive enlargement feature; hence, we make the $(d+1)^{th}$ primary asset non-tradable artificially. We also set $\sigma_{i,j}(t) = 0$ for $i = d +$

[5] We call it "weak" to differentiate it from the usual meaning of progressive enlargement of $\mathbb{F}$, where $G(t)$ is a $\mathbb{G}$-stopping time.

1 and $j \neq d+1$, to make the return of the imaginary $(d+1)^{th}$ primary asset independent of any other tradable primary asset $i \leq d$.

At time t, the executives can only observe $G(t)$, the mixture of true information $W_1(T^*) - W_1(t)$ and noise $W_{d+1}(T^*) - W_{d+1}(t)$ on top of the realized value of $W_1(t)$, i.e., $G(t) \equiv W_1(t) + \lambda[W_1(T^*) - W_1(t)] + \sqrt{1-\lambda^2}[W_{d+1}(T^*) - W_{d+1}(t)]$, with constant $\lambda \in [0,1]$. They cannot separate the true information from the insider information that they possess. A greater λ indicates the higher quality of insider information that the executives possess. By construction, insider information is unbiased, i.e., $\forall i = 1, \dots, d$, $\forall t$, $\mathbb{E}^{\mathbb{P}}[G(t)|W(t)] = W_1(t)$, and the inaccuracy of insider information is reduced at a speed proportional to the remaining time, up to the date at which the information will be disclosed, i.e., $Var^{\mathbb{P}}[G(t)|W(t)] = T^* - t$. Then, at time t, the value of any Borel function $f(G(t), W(t))$ is known by the insider executives. Namely, holding the information of $G(t)$ or $f(G(t), W(t))$ is indifferent. Among those functions, we define $G \equiv \lambda W_1(t) + \sqrt{1-\lambda^2} W_{d+1}(t) + [G(t) - W_1(t)] = \lambda W_1(T^*) + \sqrt{1-\lambda^2} W_{d+1}(T^*)$. In this case, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(G(t)) = \mathcal{F}_t \vee \sigma(G)$, and G is $\mathcal{G}_0$ measurable.

Remark 6: Hence, although the weak progressive enlargement of filtration problem can model the case where executives learn information gradually, rather than acquire full information from the beginning, essentially, it can still be categorized as falling under the general concept of initial enlargement of filtration, and the technique discussed in Subsection 2.1-2.3 can still be employed.

Corollary 3: $W(t)$ is a $(d+1)$-dimensioanl Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, then $\widetilde{W}(t) = W(t) - \int_0^t a(s)ds$, where $a_1(t) = \lambda\left(\lambda[W_1(T^*) - W_1(t)] + \sqrt{1-\lambda^2}[W_{d+1}(T^*) - W_{d+1}(t)]\right)/(T^* - t)$, $a_{d+1}(t) = \sqrt{1-\lambda^2}\left(\lambda[W_1(T^*) - W_1(t)] + \sqrt{1-\lambda^2}[W_{d+1}(T^*) - W_{d+1}(t)]\right)/(T^* - t)$ and $a_i(t) = 0$, for $i = 2,3, \dots, d$, is a $(d+1)$-dimensioanl Brownian motion on $(\Omega, \mathcal{G}, \mathbb{P})$, where $\mathcal{G} = \{\mathcal{G}_t\}$, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(G)$ and $G \equiv \lambda W_1(T^*) + \sqrt{1-\lambda^2} W_{d+1}(T^*)$.

Proof. See Appendix E.

Without loss of generality, we set $\sigma_{d+1,d+1}(t) \equiv \sqrt{1-\lambda^2}\sigma_{1,1}/(1-\lambda)$, $\omega_1 \equiv \lambda$, and $\omega_{d+1} \equiv 1-\lambda$, where ω_1 and ω_{d+1} represent the proportion assigned to the 1^{st} and the $(d+1)^{th}$ asset, within the combination of two, respectively. Then,

$$\begin{aligned} &\omega_1[dS_1(t)/S_1(t)] + \omega_{d+1}[dS_{d+1}(t)/S_{d+1}(t)] = [\lambda b_1(t)dt + (1-\lambda)b_{d+1}(t)dt] + \\ &\sigma_{1,1}(t)\left[\lambda dW_1(t) + \sqrt{1-\lambda^2} dW_{d+1}(t)\right]. \end{aligned} \tag{8}$$

Then according to Corollary 2, knowing $G \equiv \lambda W_1(T^*) + \sqrt{1-\lambda^2} W_{d+1}(T^*)$, i.e., the future risk source determining a portfolio, which is never rebalanced after being initially constructed, will enable the insider to achieve infinite derived utility.

Next, we will consider the non-tradability of $S_{d+1}(t)$ to reflect the noisy insider information that insiders possess. We claim that, as the portfolio holding period T approaches T^*, the substantial derived utility improvement owing to noisy insider information still goes to infinity, even when the following security regulations are applied: First, the executives must hold the equity-based firm-granted incentives non-transferred. Second, a 'no short selling of firm stock' constraint is applied, to avoid negative signalling in order to keep the incentivizing power of firm grants, and to remove any possibility of insiders' initial wealth increase should they liquidate their derivatives equivalently. Third, a "short-equivalent position" is forbidden, i.e., adopting trading strategies whose net replicating position on firm stock is a short position, is not allowed.

2.5. Insider constrained portfolio optimization considering security regulations

We employ the equivalence of solving the *constrained optimization problem* in the original market $\mathcal{M}$ and the *unconstrained problem* in an auxiliary market, $\mathcal{M}_{\hat{v}}$, which is first developed in Cvitanić and Karatzas (1992), assuming the portfolio is composed of primary assets only, and then extend the methodology using Colwell et al (2015)'s approach, which considered the executives' total wealth $\mathbb{X}(t)$, including n shares of ESOs and outside wealth $X(t)$, using a replication argument to translate portfolios with NTNH derivatives into portfolios of primary assets (only) with stochastic portfolio constraints.

In particular, we let $\mathcal{K} \triangleq \{K(t,\omega); (t,\omega) \in [0,T] \times \Omega\}$ be a family of closed, convex, nonempty subsets of $\mathcal{R}^d$; The corresponding family of support functions is $\delta\big(v(t)\big) \equiv \delta\big(v(t,\omega)\big|K(t,\omega)\big) \triangleq \left\{\sup\limits_{\varrho\in K}\big(-\varrho^\top v(t)\big): \Re^d \to \Re\cup\{+\infty\}; (t,\omega) \in [0,T] \times \Omega\right\}$. The effective domain of the support function is $\widetilde{K}(t,\omega) \triangleq \{v(t) \in \Re^d;\ \delta(v(t)|K) < \infty\}$. In the auxiliary market, $\mathcal{M}_{\hat{v}}$, the risk-free rate is $r_{\hat{v}}(t) = r(t) + \delta(\hat{v}(t))$; the vector drift rate is $b_{\hat{v}}(t) = b(t) + \hat{v}(t) + \delta(\hat{v}(t))\mathbf{1}$, where $\mathbf{1} \triangleq (1\ 1 \dots 1)^\top$is $d \times 1$,

$$\hat{v} \triangleq \underset{v\in\mathfrak{D}_t}{\operatorname{argmin}}\,\tilde{J}(y_t,t,T;v) = \underset{v\in\mathfrak{D}_t}{\operatorname{argmin}}\,\mathbb{E}\big[\widetilde{U}(y_t H_v(T)/H_v(t))|\mathcal{F}_t\big], \tag{9}$$

where $\mathfrak{D}_t = \mathfrak{D}_t(K) \triangleq \left\{v \equiv \big(v_1(t,\omega),\dots,v_d(t,\omega)\big)^\top \in \mathcal{H};\ \mathbb{E}\int_t^T \delta\big(v(s)\big)ds < \infty\right\}$, and $\mathcal{H}$ denote the Hilbert space of $\{\mathcal{F}_t\}$-progressively measurable processes v with values in $\Re^d$, and with the inner product $< v_1, v_2 >\triangleq \mathbb{E}\int_0^T (v_1(t))^\top v_2(t)dt$. $\widetilde{U}(y) \triangleq \max\limits_{x>0}[U(x) - xy] = U(I(y)) - yI(y)$, $U: (0,\infty) \to \Re$ being a strictly increasing, strictly concave, utility function of class C^1, satisfying $U'(0+) \triangleq \lim\limits_{x\downarrow 0} U'(x) = \infty$, $U'(\infty) \triangleq \lim\limits_{x\to\infty} U'(x) = 0$, $I(\cdot)$ is the inverse function of $U'(\cdot)$. $H_v(t) \triangleq \exp\left\{-\int_0^t r_v(s)\,ds\right\}\exp\left\{-\int_0^t [\theta_v(s)]^\top dW(s) - \frac{1}{2}\int_0^t \|\theta_v(s)\|^2 ds\right\}$, where $\theta_v(s) \triangleq \sigma^{-1}(s)[b_v(s) - r_v(s)]$. $y_t = y \times H_v(t)$, where $y \equiv \mathcal{Y}_v(x)$ is the inverse function of $x = \mathcal{X}_v(y) \triangleq \mathbb{E}\big[H_v(T) \times I\big(yH_v(T)\big)\big]$.

Assume the primary asset prices $S_i(t), i = 1,\dots,d+1$, follow the dynamics of Eq. (7) in the *weakly progressive enlargement of filtration* setting with portfolio constraints $K(t,\omega) = \big[\big(n - \widehat{N}(t)\big)\Phi^{\mathbb{G}} S_1(t)/\mathbb{X}(t),\infty\big) \times (-\infty,\infty)^{d-1} \times [0,0]$, then the adjusted drift rates of $S_i(t), i = 1,\dots,d$, and $S_{d+1}(t)$ in the equivalent unconstrained auxiliary market $\mathcal{M}_v$ on $(\Omega,\mathcal{G},\mathbb{P})$, are $b_i(t) +$

$\sigma_{i,1}(t)a_1(t) + v_i^{\mathbb{G}}(t) + \delta\left(v^{\mathbb{G}}(t)\right)$ and $b_{d+1}(t) + \sigma_{d+1,d+1}(t)a_{d+1}(t) + v_{d+1}^{\mathbb{G}}(t) + \delta\left(v^{\mathbb{G}}(t)\right)$, respectively, where $v^{\mathbb{G}}(t) \equiv \left(v_1^{\mathbb{G}}(t), v_2^{\mathbb{G}}(t), \dots v_{d+1}^{\mathbb{G}}(t)\right)$ is implied from Eq. (9), with filtration $\mathcal{F}_t$ being revised into $\mathcal{G}_t$ and, correspondingly, the drift rate vector $b(t)$ being revised into $b(t) + \sigma(t)a(t)$.

Proposition 1: Assume the *usual security regulation* applies, i.e., with $K(t,\omega) = \left[\left(n - \widehat{N}(t)\right)\Phi^{\mathbb{G}} S_1(t)/\mathbb{X}(t), \infty\right) \times (-\infty,\infty)^{d-1} \times [0,0]$, for $t \le T$, then under the $\ln(\cdot)$ utility assumption,

$$v_1^{\mathbb{G}}(t) = \max\left(\frac{(n - \widehat{N}(t))\Phi^{\mathbb{G}} S_1(t)/\mathbb{X}(t) - \sigma_{1,1}^{-1}(t)a_1(t) - \sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(t)\sum_{j=1}^{i}\sigma_{i,j}^{-1}(t)[b_j(t) - r(t)]\right)}{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(t)\right)^2}, 0\right). \tag{10}$$

and

$$v_{d+1}^{\mathbb{G}}(t) = r(t) - b_{d+1}(t) - \sigma_{d+1,d+1}(t)a_{d+1}(t). \tag{11}$$

For $i = 2, \dots, d$, $v_i^{\mathbb{G}}(t) = 0$. By definition, $\widetilde{K} = [0,\infty) \times \{0\}^{d-1} \times (-\infty,\infty)$, $\delta\left(v^{\mathbb{G}}(t)\right) \equiv -\left(n - \widehat{N}(t)\right)\Phi^{\mathbb{G}} S_1(t)/\mathbb{X}(t) \times v_1^{\mathbb{G}}(t)$ on $\widetilde{K}$.

Proof. See Appendix F.

Corollary 4: Under the $\ln(\cdot)$ utility assumption, with constraints, $K(t,\omega) = \left[\left(n - \widehat{N}(t)\right)\Phi^{\mathbb{F}} S_1(t)/\mathbb{X}(t), \infty\right) \times (-\infty,\infty)^{d-1} \times [0,0]$, for $t \le T$,

$$v_1^{\mathbb{F}}(t) = \max\left(\frac{(n - \widehat{N}(t))\Phi^{\mathbb{F}} S_1(t)/\mathbb{X}(t) - \sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(t)\sum_{j=1}^{i}\sigma_{i,j}^{-1}(t)[b_j(t) - r(t)]\right)}{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(t)\right)^2}, 0\right), \tag{12}$$

and

$$v_{d+1}^{\mathbb{F}}(t) = r(t) - b_{d+1}(t). \tag{13}$$

For $i = 2, \dots, d$, $v_i^{\mathbb{F}}(t) = 0$. By definition, $\widetilde{K} = [0,\infty) \times \{0\}^{d-1} \times (-\infty,\infty)$, $\delta\left(v^{\mathbb{F}}(t)\right) \equiv -\left(n - \widehat{N}(t)\right)\Phi^{\mathbb{F}} S_1(t)/\mathbb{X}(t) \times v_1^{\mathbb{F}}(t)$ on $\widetilde{K}$.

Proof. Directly proved from Proposition 1 by changing filtration into $\mathcal{F}_t$ and setting $a(t) = \mathbf{0}$.

Proposition 2: Assume the *usual security regulation* applies, then under $\ln(\cdot)$ utility assumption, with constraints, $K(t,\omega) = \left[\left(n - \widehat{N}(t)\right)\Phi^{\mathbb{G}} S_1(t)/\mathbb{X}(t), \infty\right) \times (-\infty,\infty)^{d-1} \times [0,0]$, $t \le T$, then $\mathbb{E}^{\mathbb{P}}[\Delta J(x_t, K(t,\omega), \lambda, t, T)|\mathcal{F}_t] = \lambda \mathbb{E}^{\mathbb{P}}\left[1 - \frac{[\sigma_{11}^{-1}(u)]^2}{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\right)^2}|\mathcal{F}_t\right] ln\left(\frac{T^*-t}{T^*-T}\right) + C \to \infty$, as $T \to T^*$, if $\exists\, i = 2, \dots, d, s.t. \sigma_{i,1}(u) \neq 0$, where C is a finite value.

Proof. See Appendix G.

Remark 7: Note that even if the insider information is bad news which, e.g., indicates a very low stock price, and the insiders have to obey the no-short selling constraints, counterintuitively, by knowing that information, the insiders can still achieve an infinite substantial expected derived utility improvement. The reason is that the firm stock return can be correlated with other stocks' returns, and the insiders can adjust the position on other stocks to benefit from the insider information.

Proposition 2 implies the importance of applying blackout. Considering the case that the noisy insider information is about the future value of a risk source, as the portfolio holding period T approaches T^*, the substantial expected derived utility improvement goes to infinity[6] at a speed proportional to the information quality λ. That sheds light on two important facts: first, a blackout regulation is necessary. Without it, noisy insider information at any level of quality invalidates firm incentives for executives the same way as accurate insider information does. Second, the better the quality of the insider information, the longer is the blackout period required to effectively prevent insiders from obtaining a certain high level of expected derived utility improvement. Then, for a given executive's attributes, how can the blackout regulatory schemes be calibrated to prevent the harmful effects of insider trading on the one hand, while delivering fairness to executives on the other?

2.6. Inadequacy and excessiveness of blackout

We define the lower bound of blackout as the longest one making insider executives' optimal exercise rate of the firm-granted American ESOs 100% at the time ESOs have just passed their vesting period; any blackout period equal to, or shorter than, the lower bound is considered to be inadequate, and results in invalidating the incentivizing mechanism. We define the upper bound of blackout as the one that equates the insiders' inferiority brought about by the stricter portfolio constraints during the blackout, and their information superiority. Any blackout period longer than that is considered to be excessive, and results in insiders being worse off than outsiders, owing to the non-trading prohibition, and that will exacerbate price discontinuity and harm market efficiency.

2.6.1. Lower bound of blackout

Assume the incentives granted by the firm are NTNH American ESOs. The lower bound of blackout is that making a 100% lump-sum exercise arrive at the time ESOs are just past their vesting period.

Specifically, assuming the portfolio includes NTNH American ESOs with a continuous partial exercise policy, then it is obvious that the exercise rate process $\dot{n}(t)$ determines the remaining NTNH ESOs un-exercised, which constitutes the portfolio constraints. $\dot{n}(t)$ belongs to the set $\mathcal{N}(t)$, the collection of all feasible $\{\mathcal{G}_t\}$-progressively measurable exercise rates, i.e. $\mathcal{N}(t) \triangleq \{\dot{n}(t): \dot{n}(t) = 0 \text{ for } t < t_{vo},\ \dot{n}(t) \geq 0 \text{ for } t \geq t_{vo}, \text{and } \widehat{N}(t) \leq n \text{ for } t \geq 0\}$, where $\widehat{N}(t) \triangleq \int_0^t \dot{n}(s)ds$ is the accumulated number of ESOs that have been exercised at time t, n is the number of ESOs granted at $t = 0$, t_{vo} is the vesting end of ESOs. According to the replication argument

[6] A similar result was given in Pikovsky and Karatzas (1996), see page 1103. However, we argue, under Pikovsky and Karatzas (1996)'s setting, the outsiders can filter the true information from observing insiders' trading activity just after a few periods.

in Colwell et al (2015), the constraint of the portfolio process is, $\pi(t) \in K(t,\omega) = \left[\left(n - \widehat{N}(t)\right)\Phi^{\mathbb{G}} S_1(t)/\mathbb{X}(t), \infty\right) \times (-\infty,\infty)^{d-1}$. Then,

$$\dot{n}^*(t) \equiv \underset{\dot{n}\in\mathcal{N}(t)}{\text{argesssup}}\, J^{\mathbb{G}}\left(x_t, K(t,\omega), \lambda, t, T\right) \tag{14}$$

Remark 8: The perceived component ΔJ_3 is determined by the exercise rate, $\dot{n}(t)$, which varies the portfolio constraints $K(t,\omega)$ and further influences insiders' optimal portfolio process $\pi^*(t) \in K(t,\omega)$ from the class of $\mathcal{G}$-adapted processes. Again, we state the importance of keeping ΔJ_3 in the objective function of insiders' optimization, when portfolio process and ESOs exercise rate are both control variables.

Proposition 2 in Colwell et al (2015) gave the first order condition (FOC) of the (viscosity) solution of $\dot{n}^*(t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$. As the proof for space $(\Omega, \mathcal{F}, \mathbb{P})$ in Colwell et al (2015) goes through for $(\Omega, \mathcal{G}, \mathbb{P})$, the FOC considering the insider information, is

$$-\left(BS(t) - B(t)\right)\frac{\partial \mathbb{J}^{\mathbb{G}}\left(\mathbb{X}^{*,\dot{n}}(t), \widehat{N}(t), S(t), \lambda, t, T\right)}{\partial \mathbb{X}^{*,\dot{n}}(t)} + \frac{\partial \mathbb{J}^{\mathbb{G}}\left(\mathbb{X}^{*,\dot{n}}(t), \widehat{N}(t), S(t), \lambda, t\, T\right)}{\partial \widehat{N}(t)} = 0. \tag{15}$$

where $\mathbb{J}^{\mathbb{G}}\left(\mathbb{X}^{*,\dot{n}^*}(t), \widehat{N}(t), \lambda, t, T\right) = \underset{\dot{n}\in\mathcal{N}(t)}{\text{esssup}}\, J^{\mathbb{G}}\left(\mathbb{X}^{*,\dot{n}}(t), K(t,\omega), \lambda, t, T\right)$.

Proposition 3: Assume $\partial^2 \mathbb{J}^{\mathbb{G}}\left(\mathbb{X}^{*,\dot{n}}(t), \widehat{N}(t), \lambda, t\, T\right) / \left(\partial \widehat{N}(t)\right)^2 < 0$, then $\exists!\, B^L$, s.t. $\forall T \in [T^* - B^L, T^*]$, $\dot{n}^*(t_{vo}) \equiv arg\, \text{esssup}_{\dot{n}\in\mathcal{N}(t)} J^{\mathbb{G}}\left(x_{t_{vo}}, K(t_{vo},\omega), \lambda, t_{vo}, T\right) = 100\%$, and B^L is an increasing function of n, the total number of ESOs initially granted.

Proof. See Appendix H.

Proposition 3 implies that B^L is the lower bound of the blackout period; any blackout period equal to or shorter than that is considered to be inadequate, and may result in invalidating the incentivizing mechanism.

2.6.2. Upper bound of blackout

Assume the incentives granted by the firm are NTNH American ESOs. Then the upper bound of blackout is the one making insider executives' substantial improvements on the expected derived utility equal to zero, compared to the outsiders', while we assume that the outsiders can freely trade till T^*, and the terminal time of insiders' portfolios is just before the blackout starts.

Proposition 4: If $G = W_1(T^*)$, then $\exists!\, B^U$, s.t. at $T = T^* - B^U$, $\mathbb{E}^{\mathbb{P}}\left[\mathbb{J}^{\mathbb{G}}\left(x_t, \widehat{N}(t), \lambda, t\, T\right) \middle| \mathcal{F}_t\right] = J^{\mathbb{F}}(x_t, t, T^*)$.

Proof. See Appendix I.

Proposition 4 implies that B^U is the upper bound of the blackout period; any blackout equal to or longer than that, is considered to be excessive, and may result in insiders' being even worse off than outsiders. That hampers the information flow to the public which, in turn, exacerbates price discontinuity and harms market efficiency.

2.7. Legislators and regulations

Naturally we are interested in the question of what occasions (i.e., what major changes to a firm plan), should trigger the SEC's prohibition of certain trading activities during the blackout? Proposition 5 below, claims that, if the risk that comprises the insider information is idiosyncratic, then the executives, directors and employees at all levels should be forbidden to trade only on their firm stocks. If it is systematic, such as information about a big merger and acquisition under negotiation, technological advances, or corporate reorganization, which may also be material to other firms, then the SEC should take the role of legislators, and enact laws preventing insiders from making any alterations to their retirement or investment plans during blackout.

Proposition 5: $\forall G$ and the corresponding $a(u)$, neglecting the information quality consideration, under the *usual security regulations*, i) if $\forall\, i = 2, \dots, d,\ \sigma_{i,1}(u) = 0$, and only the primary asset $i = 1$ is non-tradable for $u \in [t, T^*]$, then $a(u) + \sigma^{-1}(u)v^{\mathbb{G}}(u) = \sigma^{-1}(u)v^{\mathbb{F}}(u)$, and $\mathbb{E}^{\mathbb{P}}[\Delta J(x_t, K(t,\omega), t, T)|\mathcal{F}_t] \equiv \mathbb{E}^{\mathbb{P}}\big[J^{\mathbb{G}}(x_t, K(t,\omega), t\, T) - J^{\mathbb{F}}(x_t, t, T)\big|\mathcal{F}_t\big] = 0$. ii) If $\exists\, i = 2, \dots, d, s.t. \sigma_{i,1}(u) \neq 0$, and only the primary asset $i = 1$ is non-tradable for $u \in [t, T^*]$, then $a(u) + \sigma^{-1}(u)v^{\mathbb{G}}(u) \neq \sigma^{-1}(u)v^{\mathbb{F}}(u)$, and $\mathbb{E}^{\mathbb{P}}[\Delta J(x_t, K(t,\omega), t, T)|\mathcal{F}_t] \to \infty$, as $T \to T^*$. iii) If $\exists\, i = 2, \dots, d, s.t. \sigma_{i,1}(u) \neq 0$, and all primary assets are $i = 1, \dots d$ non-tradable for $u \in [t, T^*]$, then $a(u) + \sigma^{-1}(u)v^{\mathbb{G}}(u) = \sigma^{-1}(u)v^{\mathbb{F}}(u)$, and $\mathbb{E}^{\mathbb{P}}[\Delta J(x_t, K(t,\omega), t, T)|\mathcal{F}_t] = 0$.

Proof. See Appendix J.

Remark 9: $a(u) + \sigma^{-1}(u)v^{\mathbb{G}}(u) = \sigma^{-1}(u)v^{\mathbb{F}}(u)$ represents the balance between the disadvantage of portfolio constraints brought about by insider regulations, and the advantage of possessing insider information. Proposition 5 iii) states that insiders cannot make further substantial improvement on their derived utility, compared with the case that they do not have insider information, after a blackout period starts, because the insiders cannot construct a better portfolio process based on the insider information they possess while the wealth is frozen during the blackout period.

Proposition 5 iii) provides a theoretical foundation for applying blackout periods to universally prevent insiders from getting any substantial improvements on the expected derived utility. This result holds for any type of insider information that the insiders might have[7], and for any risk correlation pattern of the primary assets.

3. Blackout, job termination and RSO incentives

In this section, we claim that finding a fixed blackout window that works across all firm insiders is always an issue, hence, it is critical for firms to develop other incentivizing schemes, superior to restricted stocks and ESOs, for boosting the effectiveness of the incentives.

[7] See Remark 4, there are many types of insider information, each generating a correspondingly initially enlarged filtration with a compensation process $a(u)$ for $u \in [t, T^*)$. Proposition 5 iii) is derived without specifying the format of $a(u)$, hence, the result holds for any type of insider information that executives have.

First, an applicable blackout, i.e., $B \in (B^L, B^U]$ might not always exist[8], as $B^L < B^U$ is a possibility. From the analysis in Subsection 2.6, the boundary of blackout varies with the individual, and, by definition, depends on insiders' attributes, such as the information possessed (type and quality), total wealth and wealth composition, and the constraints binding the portfolio. That provides a theoretical ground for the fact that the blackout should be mandated by the firm, rather than by SEC and other organizations that monitor and regulate trading-related activities, in particular, prohibiting insider trading during the blackout. Also, firms should not equalize the treatment of executives and rank-and-file employees with respect to their ability to engage in transactions during a blackout.

Second, even for a particular executive, "applicable blackout" is not a static concept. As an executive's total wealth and portfolio constraints change dynamically, a fixed blackout window for a particular insider might switch between different states (inadequate, applicable, and excessive) from time to time.

Third, job termination may reduce the portfolio holding period, and may equivalently extend the blackout period from an inadequate one into an applicable one, hence ESOs can provide differing incentives for different groups of insider executives jointly categorized by the adequacy of the existing blackout period and the foreseeability of job termination.

Due to the above five reasons, if, in practice, a firm can mandate only a unique predetermined blackout applied across all corporate insiders for an event, then developing incentives alternative to ESOs is critical.

3.1. Advantage of granting RSOs to insiders who plan the future

So far, we assume the executives have the insight, but are incapable of impacting the future risk source. In that case, ESOs provide a short-term incentive and motivate insider executives to boost the current spot price, to achieve a higher derived utility. We've shown that an inadequate blackout invalidates the incentives of firm granted American ESOs.

In the following subsections, we assume the executives have the ability to determine or at least influence the future risk source. Then the firm will have the motivation to grant more ESOs to better align the interests of executives and shareholders in the long run. We will show that, even though the blackout as initially set is adequate, it can become inadequate as the firm grants more ESOs which, in turn, invalidates both the long-term and the short-term incentives[9] of ESOs.

[8] In that case, if we let $B = B^U$, and give fairness a higher priority, then blackout B is, by definition, inadequate.

[9] We measure the short-term incentive of ESOs using the first order derivative of $V^{\mathbb{G}}(x_t)$ with respect to $S_1(t)$, and the long-term incentive of ESOs using the first order derivative of $V^{\mathbb{G}}(x_t)$ with respect to $S_1(T^*)$. We do not use subjective price sensitivity, because, with NTNH constraints, the objective of the optimal exercise policy problem and the portfolio optimization problem is to maximize expected utility generated by terminal total wealth, rather than to maximize the subjective price of ESOs. Those two coincide only when the portfolio is unconstrained.

We pictorially call it the *tolerance effect* of the ESOs, and offer a scheme for granting executive stock reload options of the firm's stock as an alternative long term incentive.

3.2. Tolerance effect of ESOs

The optimal portfolio in $\mathcal{A}^{\mathbb{G}}(x_t, t, T)$ is $\pi^* = [\sigma^\top(t)\sigma(t)]^{-1}[b(t) - r(t)1] + \sigma(t)^{-1}a(t)$. Because for $i = 2, \dots d$, $a_i(t) = 0$, $\sigma_{1,d+1}^{-1}(t) = 0$, and by construction $\pi_{d+1}{}^* = 0$, then only $\pi_1{}^*$ and the proportion assigned to riskfree asset $\left(1 - \sum_{i=1}^{d} \pi_i{}^*\right)$ are affected by $a_1(t) = \lambda\big(\lambda[W_1(T^*) - W_1(t)] + \sqrt{1-\lambda^2}[W_{d+1}(T^*) - W_{d+1}(t)]\big)/(T^* - t)$. In particular, $\pi_1{}^*$ is an increasing function of $W_1(T^*)$, hence, the higher the terminal value of the firm stock, the greater the proportion the executives should optimally hold. If the terminal value of firm stock is low enough to make $\pi_1{}^*$ negative, then the non-hedgeable (i.e., no short selling) constraint prevents the executives from trading optimally as if there is no such constraint. Then, the executives will have the motivation to boost the future terminal value of stock, i.e., $S_1(T^*)$ [or, equivalently, $W_1(T^*)$] to make the optimal firm stock proportion $\pi_1{}^*$ positive, in order to rid themselves of the constraint, to get more wealth, and improve the expected derived utility.

Following the above analysis, we further claim that firms have the motivation to provide stronger incentives to tighten the constraints. If the constraint is stricter, e.g., a grant of non-transferable ESOs, in combination with non-hedgeable constraints, it will result in the opportunity set of π_1 being $[\zeta, \infty)$, where ζ is a positive constant. Then, the insider executives are motivated to boost $S_1(T^*)$ till the optimal portfolio $\pi_1{}^* \geq \zeta$, then they can escape from the NTNH constraints. In other words, a stronger incentive comes from stricter NTNH constraints by granting more ESOs.

Proposition 3 claims that the lower bound of blackout, preventing the executives' from exercising all the ESOs at the time ESOs have just passed their vesting period, is an increasing function of the total number of ESOs initially granted. Then, even though the firm set an adequate blackout, such a blackout can become inadequate, as the firm granted more ESOs to the executives later. We call this phenomenon that a stronger incentive might result in leading to the failure of the incentive mechanism, the *tolerance effect* of ESOs, brought about by the executives' insider trading.

3.3. RSO pricing

RSO, invented by Frederic W, Cook and Co in 1987, is an NTNH American call option that grants additional options upon exercising the initial one. The option holder pays the strike price in stock already in possession, rather than in cash (stock-for-stock). Meanwhile, a new strike is set to be the market value of the underlying stock at the time the option is being exercised. In this subsection, we claim that the optimal exercise policy of RSOs with infinite reload is to exercise the options whenever the stock price is creating a historical record new high, and that policy is robust to the amount of the information that the insider possesses. At that point, the granting of RSOs is a more efficient approach to refreshing the incentives.

In particular, we first neglect the insider trading, vesting period, the NTNH constraints, and RSOs' stock-for-stock feature to establish the pricing and exercise policy of RSOs with infinite-reloads. We then add those aforementioned features back to reflect the real-life situation.

Theorem 1: The firm cost of an at-the-money RSO with infinite reloads is

$$P(S_1(t), t, t^0) = \mathbb{E}^{\mathbb{P}}\left[\int_t^T \frac{H_0(u)}{H_0(t)} \frac{S_1(t^0)}{S_1(u)} \left(dS_1(u)\right)^+ \mathbf{1}_{\{S_1(u)=A(u)\}} \middle| \mathcal{F}_t\right], \tag{16}$$

where $H_0(u) \equiv H_0^a(u)|_{a=0}$, $A(u) \equiv \max(S_1(s), 0 \leq s \leq u)$, and t^0 is the RSO granting time, at which the strike price of RSO was initially set as $S_1(t^0)$.

Proof. See Appendix K.

The optimal reload policy of RSOs with infinite reloads is to exercise and reload the option immediately after the stock price passes through the historical maximal price from below to a new high, and to stop exercising the option if the stock price goes down until it rebounds to hit the historical record high. Hence holding RSOs and reloading them with an optimal reload policy is equivalent to accumulating the instant payoff realized at any appropriate reloading time. Therefore, without considering the present value discount, the realized cash payoff at time t is the historical record high [denoted as $A(t)$], which is a non-decreasing envelope[10], See Figure 1.

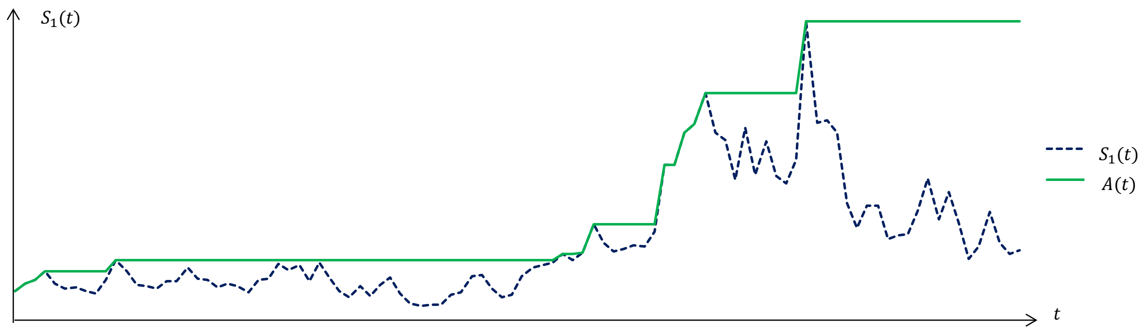

Figure 1. RSO exercise policy envelop. As the number of reload opportunities approaches infinity, its randomness, in terms of the optimal reload policy, is weakening; the certainty is being enhanced. The reload time is still being determined by the firm stock price; hence, it is still a stopping time. However, determining the reload time is not a free boundary problem anymore. The reload option is more like a barrier option, and the pricing will be a lot easier.

3.4. Insider RSO pricing with stock-for-stock and NTNH constraints

In this subsection, we incorporate in Eq. (16) the features that have been neglected previously. According to Frederic W. Cook and Co. (1998), in terms of the guidelines for reload program design, a reload option should be granted only when an employee has exercised a vested

[10] The idea is similar to Snell envelope, but in our case, it is forwardly rather than backwardly created.

stock option using already-owned "mature" shares (stock-for-stock)[11]. In their research, that was true in 37 of 40 companies (92%). The other three provided reloads for all the shares exercised, regardless of whether the exercise was financed with cash or mature shares. Meanwhile they are not aware of any company that allows the exercise and reload of an unvested stock option.

From now on, we assume that 1) Executives have insider information regarding the terminal values of driving Brownian motion. 2) The option holder pays the strike price in mature stock already in his/her possession, rather than in cash. 3) The employee is always holding enough mature shares to pay for the exercise price (We are not assuming the employee can borrow the necessary shares). 4) An option can be exercised only after its predetermined vesting period. 5) Executives are prohibited from short selling the firm stocks and transferring the options. 6) Executives are not allowed to sell the shares they own during the "blackout period". Please note that Exercising RSOs through "stock-for-stock" belongs to the "intro-company" approach, which does not involve contemporaneous sale into the market and, hence, is allowed during the "blackout" (See Nathan and Hoffman 2013)[12].

We claim that, if the executives have insider information (of any type and level of quality) and the portfolios are subject to NTNH and "stock-for-stock" constraints, the optimal reload policy derived in Section 3.3 still holds.

Theorem 2: Insider executives should exercise the NTNH infinite reload options with stock-for-stock conventions, if and only if, the firm stock price is creating a historical record new high.

Proof. See Appendix L.

The exercise of American ESOs (RSOs with infinite reloads) is determined backwardly (forwardly), and is affected by (robust to) the portfolio constraints, as well as to the insider information that executives possess. Granting ESOs successively, induces executives' successive

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[11] Stock-for-stock - the exercise of a stock option using mature shares of company stock to satisfy all or part of the exercise price. This is typically a "paper" transaction only; without actually surrendering stock certificates, but merely "attesting to" or certifying ownership of them. The company then issues only the incremental shares (that is, the number of shares normally received from the option exercise, less the shares deemed exchanged in the exercise). RSOs are received for the same number of shares used in the stock-for-stock exercise. Shares that are generally either acquired in the open market or through a company plan, such as a stock option exercise or share grant, and held for a minimum specified period of time, such as six months, to avoid "pyramiding" (exercising an entire option with only one owned share, tantamount to a stock SAR), which triggers compensation expense for the gain at exercise under Accounting Principles Board Opinion 25 (APB 25).

[12] Dybvig and Loewenstein (2003), Dai and Kwok (2008) and Bélanger and Forsyth (2008) all unnecessarily imposed a strict assumption that the executives keep the new stocks, so that continuous dividend yields could be received. Some earlier studies, e.g., Ingersoll (2007) analysed the various forms of reload options; Dai and Kwok (2005) constructed the corresponding numerical algorithm for pricing reload options with time vesting being considered; Brenner et al (2000) studied the pricing of options, whose exercise price could be reset; Saly et al (1999) employed a binomial model to value reload options. But, none of them considered the non-transferability and non-hedgeability of the reload option.

short-term performance, which can be weakened owing to the tolerance effect caused by the insider trading. Granting long-term RSOs with infinite reloads, incentivises insider executives' long-term performance.

Corollary 5: Under the "stock-for-stock" convention, the firm cost (objective price) of at-the-money NTNH RSO with infinite reloads is

$$\tilde{P}(S_1(t), t,\ t^0) = P(S_1(t), t,\ t^0). \tag{17}$$

Proof. See Appendix M.

Corollary 6: The upper bound, at any time $\boldsymbol{t}$, of the firm cost (objective price) of granting an at-the-money NTNH RSO with infinite reloads is the initial granting time spot stock price, i.e.,

$$\tilde{P}(S_1(t), t,\ t^0) \le S_1(\ t^0), \tag{18}$$

Proof. See Appendix N.

RSOs fell out of favour around 2006, and firms gradually stopped granting RSOs thereafter for two reasons: the pricing difficulty, and claim that RSOs bestow too many lucrative shares (as a money pump) to executives.

First, due to the extensive use of equity-based compensation, optional reporting of their fair value, reflecting the grant-date share price, and other pertinent factors, e.g. volatility, restrictions and conditions inherent, became mandatory by the Financial Accounting Standard Board (FASB) in 2004. Although much progress has been made on ESO pricing, considering NTNH constraints, and adding the reload feature, will escalate the pricing difficulty, and the Financial Accounting Standards Board (FASB) continues to believe that the reload term makes it impossible to estimate a reasonably fair value of options at the grant date. Subsequent granting of reload options pursuant to that provision should be accounted for as a separate award when the reload options are granted [See FAS123(R) paragraph 24 to 26, see also Saly et al 1999]. Corollary 5 gives the firm cost of RSOs, taking all the aforementioned factors into account.

Second, RSOs used to be blamed for bestowing too many lucrative shares to the executives. This project endeavours to test the truth or falsehood of that claim from a whole new perspective by taking executives' insider trading into account. Corollary 6 claims that the firm cost of granting one at-the-money NTNH RSO with infinite reloads is no more than granting one share of firm stock. Hence, the claim that RSOs are money pumps for executives is groundless.

4. Conclusion

Insider trading is a fairly controversial activity in academic discussions. It is unethical because it gives unfair advantages to insiders, enabling them to profit from non-public information, through either insider arbitrage or insider portfolio optimization; yet it is ethical in the sense that it facilitates rapid price discovery and makes the market more efficient.

The price change caused by insider arbitrage is a one-off instant occurrence but, although information is transferred to the public in the most efficient way through arbitrage, the profits all accrue to insiders, and there is little profit space left for the outsiders; whereas insider portfolio optimization is a sustained information release process and, in that case, there is a co-sharing of profit space for both insiders and outsiders. Hence, we interpret that the SEC-promulgated Rule 10b-5 is designed to illegalize insider trading, and Rule 10b5-1, as well as the compulsory filing regulations, are designed to legalize insider portfolio optimization.

To prevent insiders from obtaining infinite derived utility (often seen as abnormally large profit in practice), a blackout trading restriction is necessary, even when the information about the future risk source is a mixture of the true information and noise over time. Meanwhile the portfolios that the insider executives hold are binding, owing to the NTNH constraints. We demonstrate that, on the one hand, an over-short blackout will induce the insider executives to exercise all American ESOs optimally once the ESOs pass their vesting period, and this, in turn, invalidates the incentivizing scheme; on the other hand, an excessively long blackout will result in insiders' being worse off than the outsiders owing to the non-trading prohibition. Therefore, we suggest a range of applicable blackouts, which allow insider executives to enjoy limited information advantages, while motivating those executives to favour actions that result in the most benefit to shareholders.

Specifically, trading on firm stock should be forbidden if the non-public information is about an idiosyncratic risk. However, if the information concerns a systematic risk, all the portfolio positions should be free from alteration during the blackout.

Unfortunately, an applicable blackout might not exist in all cases. Even when it does exist, the boundary for the blackout applicable is determined by each individual executive's attributes. Also, because each executive's attributes change dynamically, a fixed blackout might switch among the three states (inadequate, applicable, and excessive) from time to time. Hence, even though firms, rather than the SEC, take responsibility for stipulating a particular length of blackout, enacting a fixed blackout and making it applicable across all corporate insiders is always an issue; thus, the incentivizing scheme of ESOs granted by the firm, in a general sense, will be invalidated and weakened inevitably.

Furthermore, this study claims a tolerance effect of ESOs, i.e., that insider executives have the motivation to push up the future terminal value of stock in order to rid themselves of the constraint, and that results in stricter NTNH constraints associated with firms' stronger incentives, requiring a longer blackout, and making the current one inadequate.

Fortunately, we find RSOs with infinite reloads are superior to ESOs as incentivizing schemes for boosting the effectiveness of incentives. Thus, RSOs can better align the long-term interests of shareholders and executives, strategically. We claim that the optimal exercise policy of RSOs is to exercise and reload the option whenever the stock price is creating a historical record high. We also find that the firm cost of an RSO is bounded from above by the firm stock price, and that fact can be used to negate the claim that RSOs are money pumps for executives. The exercise policy and the upper bound claims are both robust to changes in portfolio constraints, information types and levels of information quality, and are irrelevant to the adequacy of blackout.

The project provides policy recommendations to many areas of finance. The benefit to policy makers could be: 1) Legislators, state governments, entities enacting security regulations can benefit from an empirically testable theory to value the harmful effects of insider trading, and use the results as the basis for policy debates as well as to evaluate the adequacy of regulations and penalties for achieving justice in the market. 2) Corporate boards of directors and HR departments can benefit from designing the most efficient tools to incentivize their executives. 3) Executives, senior staff and public investors also benefit by optimizing their wealth packages.

Appendix A

Proof of Lemma 1.

Assume $\widetilde{W}(t)$ is a Brownian motion on $(\Omega, \mathcal{G}, \mathbb{P})$, we change the measure from $\mathbb{P}$ to $\mathbb{P}^{\mathbb{G}}$, so that $W(t)$ becomes a Brownian motion on $(\Omega, \mathcal{G}, \mathbb{P}^{\mathbb{G}})$, then

$$\frac{d\mathbb{P}^{\mathbb{G}}}{d\mathbb{P}}\bigg|_{\mathcal{G}_t} = exp\left\{-\int_0^t a(s)\, d\widetilde{W}(s) - \frac{1}{2}\int_0^t [a(s)]^2\, ds\right\}, \tag{A. 1}$$

Define $p_t^y \equiv \frac{P(G_t \in dy | \mathcal{F}_t)}{P(G_t \in dy)}$, then according to Amendinger (2000), $\frac{d\mathbb{P}^{\mathbb{G}}}{d\mathbb{P}}\Big|_{\mathcal{G}_t} = \frac{1}{p_t^y}$. From Eq. (A. 1), we have $d\frac{1}{p_t^y}/\frac{1}{p_t^y} = -a(t)d\widetilde{W}(t)$. Then, from Itō's lemma, $dp_t^y/p_t^y = [a(t)]^2dt + a(t)d\widetilde{W}_t = [a(t)]^2dt + a(t)[dW(t) - a(t)dt] = a(t)dW(t)$. We define $q_t^y \equiv \frac{P(G_t \in dy | \mathcal{F}_t)}{dy}$, then $dp_t^y/p_t^y = dq_t^y/q_t^y$, because $P(G_t \in dy) = \mathbb{E}\left(\mathbf{1}_{\{G_t \in dy\}}\right)$ is a constant with respect to $\mathcal{F}_t$, and scaling by a constant does not change the dynamic of dp_t^y/p_t^y. So, dq_t^y/q_t^y uniquely determines the compensating process $a(t)$.

This completes the proof of Lemma 1.

□

Appendix B

Proof of Corollary 1.

We assume $dS(t)/S(t) = b(t)dt + \sigma(t)dW(t)$, where $S(t)$ and $b(t)$ are $d \times 1$ vectors, $\sigma(t)$ is a $d \times d$ matrix, $W(t)$ is a $d \times 1$ Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. According to Lemma 1, $dS(t)/S(t) = \big(b(t) + \sigma(t)a(t)\big)dt + \sigma(t)d\widetilde{W}(t)$, where $\widetilde{W}(t)$ is $d \times 1$ Brownian motion on $(\Omega, \mathcal{G}, \mathbb{P})$. We change measure to make the discounted stock price a martingale on $(\Omega, \mathcal{G}, \mathbb{Q}^{\mathbb{G}})$,

$$\frac{d\mathbb{Q}^{\mathbb{G}}}{d\mathbb{P}}\bigg|_{\mathcal{G}_t} = Z(t) = e^{-\int_0^t \Theta^{\top}(s)d\widetilde{W}(s) - \frac{1}{2}\int_0^t \|\Theta(s)\|^2 ds}, \quad 0 \le t < T^*. \tag{A. 2}$$

Then

$$\Theta(t) = \sigma^{-1}(t)[b(t) + \sigma(t)a(t) - r(t)\mathbf{1}] = \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}] + a(t). \tag{A. 3}$$

It is easy to check that $dS(t)/S(t) = \left(b(t) + \sigma(t)a(t)\right)dt + \sigma(t)d\widetilde{W}(t) = \left(b(t) + \sigma(t)a(t)\right)dt - \sigma(t)\Theta(t)dt + \sigma(t)\left(\Theta(t)dt + d\widetilde{W}(t)\right) = r(t)\mathbf{1}dt + \sigma(t)d\widehat{W}(t)$.

This completes the proof of Corollary 1.

□

Appendix C

Proof of Lemma 3**.**

Taking the notations in Appendix B, we solve $dS(t)/S(t) = r(t)\mathbf{1}dt + \sigma(t)d\widehat{W}(t)$, and get $X^{x_t,\pi}(T) = x_t^{\mathbb{G}}\exp\left[\int_t^T\left(r(u) - \frac{1}{2}\|\sigma^\top(u)\pi(u)\|^2\right)du + \int_t^T\pi^\top(u)\sigma(u)d\widehat{W}(u)\right]$. Insiders' portfolio optimization is,

$$J^{\mathbb{G}}(x_t,t,T) \equiv \underset{\pi\in\mathcal{A}_0^{\mathbb{G}}(x_t,t,T)}{\operatorname{esssup}} \mathbb{E}^{\mathbb{P}}\left[U\left(X^{x_t,\pi}(T)\right)|\mathcal{G}_t\right]. \tag{A. 4}$$

Under $ln(\cdot)$ utility assumption, we have

$$J^{\mathbb{G}}(x_t,t,T) = \underset{\pi\in\mathcal{A}_0^{\mathbb{G}}(x_t,t,T)}{\operatorname{esssup}} \mathbb{E}^{\mathbb{P}}\left[\left(ln\left(x_t^{\mathbb{G}}\right) + \int_t^T\left(r(u) + \frac{1}{2}\|\Theta(u)\|^2\right)du - \frac{1}{2}\int_t^T\|\Theta(t) - \sigma^\top(u)\pi(u)\|^2du + \int_t^T\pi^\top(u)\sigma(u)d\widetilde{W}(u)\right)|\mathcal{G}_t\right]. \tag{A. 5}$$

Because $\widetilde{W}(t)$ is a martingale on $(\Omega,\mathcal{G},\mathbb{P})$, $\mathbb{E}^{\mathbb{P}}\left[\int_t^T\pi^\top(u)\sigma(u)d\widetilde{W}(u)\,|\mathcal{G}_t\right] = 0$. Then, the optimal portfolio process is $\pi^{*\mathbb{G}}(t) = \underset{\pi\in\mathcal{A}_0^{\mathbb{G}}(x_t,t,T)}{\operatorname{argmin}} \|\Theta(t) - \sigma^\top(u)\pi(u)\|^2 = [\sigma^\top(t)]^{-1}\Theta(t)$. Substituting $\pi^{*\mathbb{G}}(t)$ and $\Theta(t)$ into Eq. (A. 5), we get

$$J^{\mathbb{G}}(x_t,t,T) = ln\left(x_t^{\mathbb{G}}\right) + \mathbb{E}^{\mathbb{P}}\left[\int_t^T\left(r(u) + \frac{1}{2}\|\sigma^{-1}(u)[b(u) - r(u)\mathbf{1}]\|^2 + \left[\sigma^{-1}(u)[b(u) - r(u)\mathbf{1}]\right]^\top a(u) + \frac{1}{2}\|a(u)\|^2\right)du\middle|\mathcal{G}_t\right]. \tag{A. 6}$$

and

$$J^{\mathbb{G}}(x_t,t,T) = ln\left(x_t^{\mathbb{G}}\right) + \mathbb{E}^{\mathbb{P}}\left[\int_t^T\left(r(u) + \frac{1}{2}\|\Theta(u)\|^2\right)du\middle|\mathcal{G}_t\right] = ln\left(x_t^{\mathbb{G}}\right) + \mathbb{E}^{\mathbb{P}}\left[\int_t^T\left(r(u) + \frac{1}{2}\left\|\sigma^\top(t)\pi^{*\mathbb{G}}(t)\right\|^2\right)du\middle|\mathcal{G}_t\right]. \tag{A. 7}$$

It is the existing knowledge that

$$J^{\mathbb{F}}(x_t,t,T) = ln(x_t^{\mathbb{F}}) + \mathbb{E}^{\mathbb{P}}\left[\int_t^T\left(r(u) + \frac{1}{2}\|\sigma^{-1}(u)[b(u) - r(u)\mathbf{1}]\|^2\right)du\middle|\mathcal{F}_t\right]. \tag{A. 8}$$

Because $\int_0^T\mathbb{E}^{\mathbb{P}}\left|\left[\sigma^{-1}(t)[b(t) - r(t)\mathbf{1}]\right]^\top a(t)\right|du < \infty$, and $\int_0^T\mathbb{E}^{\mathbb{P}}[\|a(t)\|^2]\,du < \infty$, then

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T\left(\left[\sigma^{-1}(u)[b(u) - r(u)\mathbf{1}]\right]^\top a(u) + \frac{1}{2}\|a(u)\|^2\right)du\middle|\mathcal{G}_0\right] = \int_0^T\left(\mathbb{E}^{\mathbb{P}}\left[\left(\left[\sigma^{-1}(u)[b(u) - r(u)\mathbf{1}]\right]^\top a(t)\right)|\mathcal{G}_0\right] + \frac{1}{2}\mathbb{E}^{\mathbb{P}}[(\|a(u)\|^2)|\mathcal{G}_0]\right)du. \tag{A. 9}$$

Then, due to $W(t)$'s Markov property,

$$\Delta J(x_t,t,T) = \Delta J_0(x_t,t,T) + \Delta J_1(x_t,t,T) + \Delta J_2(x_t,t,T) + \Delta J_3(x_t,t,T). \tag{A. 10}$$

This completes the proof of Lemma 3.

$\square$

Appendix D

Proof of Corollary 2.

If $G(t) = W_1(T^*)$, then $a(t) = \left(\frac{W_1(T^*)-W_1(t)}{T^*-t}, 0, \dots, 0\right)^\top$, and

$$\Delta J_1(x_t,t,T) + \Delta J_2(x_t,t,T) = \mathbb{E}^{\mathbb{P}}\left[\int_t^T \left(\frac{W_1(T^*)-W_1(u)}{T^*-u}, 0, \dots, 0\right) \times [\sigma^{-1}(u)[b(u) - r(u)\mathbf{1}]]du \middle| \mathcal{G}_t\right] + \frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[\int_t^T \left(\frac{W_1(T^*)-W_1(t)}{T^*-t}\right)^2 du \middle| \mathcal{G}_t\right]. \tag{A. 11}$$

Because the diagonal entries of the inverse matrix of lower triangular matrix are the inverses of the corresponding entries of the lower triangular matrix. We have

$$\Delta J_1(x_t,t,T) + \Delta J_2(x_t,t,T) = \mathbb{E}^{\mathbb{P}}\left[\int_t^T \left(\frac{b_1(u)-r(u)}{\sigma_{1,1}(u)}\right) \times \left(\frac{W_1(T^*)-W_1(u)}{T^*-u}\right) du \middle| \mathcal{G}_t\right] + \frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[\int_t^T \left(\frac{W_1(T^*)-W_1(t)}{T^*-t}\right)^2 du \middle| \mathcal{G}_t\right]. \tag{A. 12}$$

According to Bayesian formula,

$$\begin{aligned} &\mathbb{P}(W_1(u) - W_1(t) = x | W_1(T^*) - W_1(t) = y) \\ &= \frac{\mathbb{P}(W_1(T^*) - W_1(t) = y = y | W_1(u) - W_1(t) = x)\mathbb{P}(W_1(u) - W_1(t) = x)}{\mathbb{P}(W_1(T^*) - W_1(t) = y)} \\ &= \frac{\frac{1}{\sqrt{2\pi(T^*-u)}} e^{-\frac{(y-x)^2}{2(T^*-u)}} \frac{1}{\sqrt{2\pi(u-t)}} e^{-\frac{x^2}{2(u-t)}}}{\frac{1}{\sqrt{2\pi(T^*-t)}} e^{-\frac{y^2}{2(T^*-t)}}} \\ &= \frac{1}{\sqrt{2\pi}\sqrt{\frac{(T^*-u)(u-t)}{T^*-t}}} e^{-\frac{\left(x-\frac{y(u-t)}{T^*-t}\right)^2}{2\frac{(u-t)(T^*-u)}{(T^*-t)}}}. \end{aligned} \tag{A. 13}$$

Then, we have,

$$\mathbb{E}^{\mathbb{P}}[(W_1(u) - W_1(t))|\mathcal{G}_t] = \frac{[W_1(T^*) - W_1(t)](u-t)}{T^* - t}, \tag{A. 14}$$

and

$$\mathbb{E}^{\mathbb{P}}\left[(W_1(u) - W_1(t))^2 \middle| \mathcal{G}_t\right] = \frac{(u-t)(T^*-u)}{(T^*-t)} + \left(\frac{[W_1(T^*) - W_1(t)](u-t)}{T^*-t}\right)^2. \tag{A. 15}$$

Then, we get

$$\mathbb{E}^{\mathbb{P}}\left[\left(\frac{W_1(T^*)-W_1(u)}{T^*-u}\right)\middle| \mathcal{G}_t\right] = \frac{W_1(T^*)-W_1(t)}{T^*-u} - \frac{1}{T^*-u}\mathbb{E}^{\mathbb{P}}[(W_1(u) - W_1(t))|\mathcal{G}_t] = \frac{W_1(T^*)-W_1(t)}{T^*-u} - \frac{1}{T^*-u}\left[\frac{[W_1(T^*)-W_1(t)](u-t)}{T^*-t}\right] = \frac{W_1(T^*)-W_1(t)}{T^*-t}. \tag{A. 16}$$

and

$$\int_t^T \mathbb{E}^{\mathbb{P}}\left[\left(\left|\frac{W_1(T^*)-W_1(u)}{T^*-u}\right|\right)\middle| \mathcal{G}_t\right] du \leq= |W_1(T^*) - W_1(t)| \times \left(\frac{1}{T^*-u} + \frac{u-t}{(T^*-u)(T^*-t)}\right) \times (T - t) < \infty. \tag{A. 17}$$

We also get,

$$\mathbb{E}^{\mathbb{P}}\left[\left(\frac{W_1(T^*)-W_1(u)}{T^*-u}\right)^2\middle|\mathcal{G}_t\right]=\frac{1}{(T^*-u)^2}\left(\mathbb{E}^{\mathbb{P}}\left[\left(\left(W_1(T^*)-W_1(t)\right)-\left(W_1(u)-W_1(t)\right)\right)^2\middle|\mathcal{G}_t\right]\right)=\frac{1}{(T^*-u)^2}\left(\mathbb{E}^{\mathbb{P}}\left[\left(W_1(T^*)-W_1(t)\right)^2\middle|\mathcal{G}_t\right]-2\mathbb{E}^{\mathbb{P}}\left[\left(W_1(T^*)-W_1(t)\right)\left(W_1(u)-W_1(t)\right)\middle|\mathcal{G}_t\right]+\mathbb{E}^{\mathbb{P}}\left[\left(W_1(u)-W_1(t)\right)^2\middle|\mathcal{G}_t\right]\right)=\frac{1}{(T^*-u)^2}\left(\left(W_1(T^*)-W_1(t)\right)^2-2\left(W_1(T^*)-W_1(t)\right)^2\frac{(u-t)}{T^*-t}+\frac{(u-t)(T^*-u)}{(T^*-t)}+\left(\frac{[W_1(T^*)-W_1(t)](u-t)}{T^*-t}\right)^2\right)=\left(\frac{W_1(T^*)-W_1(t)}{T^*-t}\right)^2+\frac{(u-t)}{(T^*-u)(T^*-t)}. \tag{A. 18}$$

and

$$\int_t^T\mathbb{E}^{\mathbb{P}}\left[\left(\frac{W_1(T^*)-W_1(u)}{T^*-u}\right)^2\middle|\mathcal{G}_t\right]du=\left(\frac{W_1(T^*)-W_1(t)}{T^*-t}\right)^2\times(T-t)+\int_t^T\frac{(u-t)}{(T^*-u)(T^*-t)}du=\left(\frac{W_1(T^*)-W_1(t)}{T^*-t}\right)^2\times(T-t)+\frac{1}{T^*-t}\int_0^{(T-t)}\frac{(u-t)}{((T^*-t)-(u-t))}d(u-t)=\left(\frac{W_1(T^*)-W_1(t)}{T^*-t}\right)^2\times(T-t)+\frac{1}{T^*-t}\int_0^{(T-t)}\left(\frac{s-(T^*-t)+(T^*-t)}{((T^*-t)-s)}\right)ds=\left(\frac{W_1(T^*)-W_1(t)}{T^*-t}\right)^2\times(T-t)-\frac{T-t}{T^*-t}+\frac{1}{T^*-t}\int_{(T^*-T)}^{(T^*-t)}\left(\frac{T^*-t}{o}\right)do=\left(\frac{W_1(T^*)-W_1(t)}{T^*-t}\right)^2\times(T-t)-\frac{T-t}{T^*-t}+\ln\left(\frac{T^*-t}{T^*-T}\right)<\infty. \tag{A. 19}$$

Because Eq. (A. 17) and Eq. (A. 19), we have

$$\Delta J_1(x_t,t,T)+\Delta J_2(x_t,t,T)=\mathbb{E}^{\mathbb{P}}\left[\int_t^T\left(\frac{b_1(u)-r(u)}{\sigma_{1,1}(u)}\right)\times\left(\frac{W_1(T^*)-W_1(u)}{T^*-u}\right)du\middle|\mathcal{G}_t\right]+\frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[\int_t^T\left(\frac{W_1(T^*)-W_1(t)}{T^*-t}\right)^2du\middle|\mathcal{G}_t\right]=\int_t^T\mathbb{E}^{\mathbb{P}}\left[\left(\frac{b_1(u)-r(u)}{\sigma_{1,1}(u)}\right)\times\left(\frac{W_1(T^*)-W_1(u)}{T^*-u}\right)\middle|\mathcal{G}_t\right]du+\frac{1}{2}\int_t^T\mathbb{E}^{\mathbb{P}}\left[\left(\frac{W_1(T^*)-W_1(t)}{T^*-t}\right)^2\middle|\mathcal{G}_t\right]du. \tag{A. 20}$$

From Eq. (A. 16) and Eq. (A. 18), neither $\mathbb{E}^{\mathbb{P}}\left[\left(\frac{W_1(T^*)-W_1(u)}{T^*-u}\right)\middle|\mathcal{G}_t\right]$ nor $\mathbb{E}^{\mathbb{P}}\left[\left(\frac{W_1(T^*)-W_1(u)}{T^*-u}\right)^2\middle|\mathcal{G}_t\right]$ is a function of $W(u)$, Because $b(u), r(u), \sigma(u)$ are $\mathcal{F}_u$-measurable, and $W(u)$ is Markov, $\left(\frac{b_1(u)-r(u)}{\sigma_{1,1}(u)}\right)$ can be represented as a function of $W(u)$. Hence, $\frac{b_1(u)-r(u)}{\sigma_{1,1}(u)}$ and $\left(\frac{W_1(T^*)-W_1(u)}{T^*-u}\right)$ are independent under $(\Omega,\mathcal{G},\mathbb{P})$. If $\mathbb{E}^{\mathbb{P}}\left[\left(\frac{b_1(u)-r(u)}{\sigma_{1,1}(u)}\right)\middle|\mathcal{G}_t\right]$ is not a function of u, then substitute Eq. (A. 16) and Eq. (A. 18) into Eq. (A. 20), we get

$$\Delta J_1(x_t,t,T)+\Delta J_2(x_t,t,T)=\mathbb{E}^{\mathbb{P}}\left[\left(\frac{b_1(u)-r(u)}{\sigma_{1,1}(u)}\right)\middle|\mathcal{G}_t\right]\frac{[W_1(T^*)-W_1(t)](T-t)}{T^*-t}+\frac{1}{2}\left(\frac{W_1(T^*)-W_1(t)}{T^*-t}\right)^2\times(T-t)-\frac{1}{2}\frac{T-t}{T^*-t}+\frac{1}{2}\ln\left(\frac{T^*-t}{T^*-T}\right)=\frac{T-t}{T^*-t}\left(\mathbb{E}^{\mathbb{P}}\left[\left(\frac{b_1(u)-r(u)}{\sigma_{1,1}(u)}\right)\middle|\mathcal{G}_t\right][W_1(T^*)-W_1(t)]+\frac{1}{2}\frac{[W_1(T^*)-W_1(t)]^2}{T^*-t}-\frac{1}{2}\right)+\frac{1}{2}\ln\left(\frac{T^*-t}{T^*-T}\right)\to\infty,\ \text{As}\ T\to T^*. \tag{A. 21}$$

This completes the proof of Corollary 2.

□

Appendix E

Proof of Corollary 3.

Under the weakly progressive enlargement of filtration setting,

$$q_t^y \equiv \frac{P(G\in dy|\mathcal{F}_t)}{dy} = P\{\lambda W_1(T^*) + \sqrt{1-\lambda^2}W_{d+1}(T^*) \in dy|W(t)\} = \frac{1}{\sqrt{2\pi(T^*-t)}} exp\left\{-\frac{\left[y-\left(\lambda W_1(t)+\sqrt{1-\lambda^2}W_{d+1}(t)\right)\right]^2}{2(T^*-t)}\right\}, \tag{A. 22}$$

Using Itô's formula,

$$dq_t^y = \frac{\partial q_t^y}{\partial t}dt + \sum_{i=1}^d \frac{\partial q_t^y}{\partial W_i(t)} dW_i(t) + \frac{1}{2}\sum_{i=1}^d \frac{\partial^2 q_t^y}{\partial [W_i(t)]^2}dt, \tag{A. 23}$$

Taking derivative of q_t^y *w.r.t* $W_i(t)$,

$$\frac{\partial q_t^y}{\partial W_1(t)} = q_t^y * \left(\frac{\lambda\left[y-\left(\lambda W_1(t)+\sqrt{1-\lambda^2}W_{d+1}(t)\right)\right]}{(T^*-t)}\right), \tag{A. 24}$$

$$\frac{\partial q_t^y}{\partial W_i(t)} = 0, i = 2,3, \ldots d, \tag{A. 25}$$

$$\frac{\partial q_t^y}{\partial W_{d+1}(t)} = q_t^y * \left(\frac{\sqrt{1-\lambda^2}\left[y-\left(\lambda W_1(t)+\sqrt{1-\lambda^2}W_{d+1}(t)\right)\right]}{(T^*-t)}\right), \tag{A. 26}$$

Thanks to Jeanblanc (2010)'s Proposition 2.3.2 (Jacod's Criterion) on page 26, we have $\widetilde{W}(t) = W(t) - \int_0^t a(s)ds$, where $a(s)$ is a vector, and $\langle q^y, W_i\rangle_t = \int_0^t q_s^y a_i(s)ds$. Therefore,

$$a_i(t) = \begin{cases} \frac{\lambda[\lambda(W_1(T^*)-W_1(t))+\sqrt{1-\lambda^2}(W_{d+1}(T^*)-W_{d+1}(t))]}{(T^*-t)} & , \ i = 1 \\ 0 & , \ i = 2,3, \ldots, d, \\ \frac{\sqrt{1-\lambda^2}[\lambda(W_1(T^*)-W_1(t))+\sqrt{1-\lambda^2}(W_{d+1}(T^*)-W_{d+1}(t))]}{(T^*-t)} & , \ i = d+1 \end{cases} \tag{A. 27}$$

This completes the proof of Corollary 3.

□

Appendix F

Proof of Proposition 1.

If $K(t,\omega) = \left[\left(n - \widehat{N}(t)\right)\Phi^{\mathbb{G}}S_1(t)/\mathbb{X}(t), \infty\right) \times (-\infty,\infty)^{d-1} \times [0,0]$, then, by definition, $\widetilde{K} = [0,\infty) \times \{0\}^{d-1} \times (-\infty,\infty)$, $\delta(v(t)) \equiv -\left(n - \widehat{N}(t)\right)\Phi^{\mathbb{G}}S_1(t)/\mathbb{X}(t) \times v_1(t)$ on $\widetilde{K}$. To differentiate the $v(t)$ under $\mathcal{G}$ and $\mathcal{F}$, we denote $v^{\mathbb{G}}(t)$ as the $v(t)$ on $(\Omega, \mathcal{G}, \mathbb{P})$, and $v^{\mathbb{F}}(t)$ as the $v(t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$.

On $(\Omega, \mathcal{G}, \mathbb{P})$, the price of risk vector is $\Theta(t) = \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}] + a(t) = \sigma^{-1}(t)[b(t) + \sigma(t)a(t) - r(t)\mathbf{1}]$, equivalently the drift rate vector becomes $b(t) + \sigma(t)a(t)$. Then the vector $v^{\mathbb{G}}(t) \equiv \left(v_1^{\mathbb{G}}(t), v_2^{\mathbb{G}}(t), \ldots v_{d+1}^{\mathbb{G}}(t)\right)$ in the duality problem of constrained portfolio optimization is $v^{\mathbb{G}}(t) = \underset{v\in\widetilde{K}}{\operatorname{argmin}}[2\delta(v) + \|\Theta(t) + \sigma^{-1}(t)v\|^2]$ for every $t \in [0,T]$, see Eq. (11.4) in Cvitanić and Karatzas (1992). Specifically $v^{\mathbb{G}}(t) = \underset{v=(v_1,v_2,\ldots v_{d+1})\in\widetilde{K}}{\operatorname{argmin}}\left[-\frac{2\left(n-\widehat{N}(t)\right)\Phi^{\mathbb{G}}S_1(t)v_1}{\mathbb{X}(t)} + \|\sigma^{-1}(t)[b(t) + \sigma(t)a(t) - r(t)\mathbf{1} + v]\|^2\right]$. Denote $\sigma_{i,j}^{-1}(t)$ as the element in row i and column j of the inverse matrix of $\left(\sigma_{i,j}(t)\right)$, we have that the objective is $-\frac{2(n-\hat{N}(t))\Phi^{\mathbb{G}}S_1(t)v_1(t)}{\mathbb{X}(t)}+\left(a_1(t)+\sigma_{1,1}^{-1}(t)[b_1(t)-r(t)+v_1(t)]\right)^2+\sum_{i=2}^{d}\left(\sigma_{i,1}^{-1}(t)[b_1(t)-r(t)+v_1(t)]+\sum_{j=2}^{i}\sigma_{i,j}^{-1}(t)\left[b_j(t)-r(t)\right]\right)^2=$ $-\frac{2(n-\hat{N}(t))\Phi^{\mathbb{G}}S_1(t)v_1(t)}{\mathbb{X}(t)}+\left(\sigma_{1,1}^{-1}(t)\right)^2\left(v_1(t)\right)^2+2\sigma_{1,1}^{-1}(t)\left(a_1(t)+\sigma_{1,1}^{-1}(t)[b_1(t)-r(t)]\right)v_1(t)+\left(a_1(t)+\sigma_{1,1}^{-1}(t)[b_1(t)-r(t)]\right)^2+\sum_{i=2}^{d}\left(\sigma_{i,1}^{-1}(t)v_1(t)+\sum_{j=1}^{i}\sigma_{i,j}^{-1}(t)\left[b_j(t)-r(t)\right]\right)^2=$ $-\frac{2(n-\hat{N}(t))\Phi^{\mathbb{G}}S_1(t)v_1(t)}{\mathbb{X}(t)}+\left(\sigma_{1,1}^{-1}(t)\right)^2\left(v_1(t)\right)^2+2\sigma_{1,1}^{-1}(t)\left(a_1(t)+\sigma_{1,1}^{-1}(t)[b_1(t)-r(t)]\right)v_1(t)+\left(a_1(t)+\sigma_{1,1}^{-1}(t)[b_1(t)-r(t)]\right)^2+\sum_{i=2}^{d}\left(\left(\sigma_{i,1}^{-1}(t)\right)^2\left(v_1(t)\right)^2+2v_1(t)\sigma_{i,1}^{-1}(t)\sum_{j=1}^{i}\sigma_{i,j}^{-1}(t)\left[b_j(t)-r(t)\right]+\left(\sum_{j=1}^{i}\sigma_{i,j}^{-1}(t)\left[b_j(t)-r(t)\right]\right)^2\right)=\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(t)\right)^2\left(v_1(t)\right)^2+2\left(-\frac{(n-\hat{N}(t))\Phi^{\mathbb{G}}(t)}{\mathbb{X}(t)}+\sigma_{1,1}^{-1}(t)a_1(t)+\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(t)\sum_{j=1}^{i}\sigma_{i,j}^{-1}(t)\left[b_j(t)-r(t)\right]\right)\right)v_1(t)+\left(a_1(t)+\sigma_{1,1}^{-1}(t)[b_1(t)-r(t)]\right)^2+\sum_{i=2}^{d}\left(\left(\sum_{j=1}^{i}\sigma_{i,j}^{-1}(t)\left[b_j(t)-r(t)\right]\right)^2\right)$,

Then, we have $v_i^{\mathbb{G}}(t)=0$, for $i=2,\dots d$, and

$$v_1^{\mathbb{G}}(t)=\max\left(\frac{(n-\hat{N}(t))\Phi^{\mathbb{G}}S_1(t)/\mathbb{X}(t)-\sigma_{1,1}^{-1}(t)a_1(t)-\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(t)\sum_{j=1}^{i}\sigma_{i,j}^{-1}(t)\left[b_j(t)-r(t)\right]\right)}{\sqrt{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\right)^2}},0\right), \tag{A. 28}$$

and

$$v_{d+1}^{\mathbb{G}}(t)=r(t)-b_{d+1}(t)-\sigma_{d+1,d+1}(t)a_{d+1}(t). \tag{A. 29}$$

This completes the proof of Proposition 1.

□

Appendix G

Proof of Proposition 2.

In general case, $\sigma(t)$ is not diagonal, i.e., primary assets are correlated, then $\|\sigma^{-1}(u)[b(u)+v(u)-r(u)1]+a(u)\|^2=\left(\sigma_{11}^{-1}(u)[b_1(u)+v_1(u)-r(u)]+a_1(u)\right)^2+(\sigma_{21}^{-1}(u)[b_1(u)+v_1(u)-r(u)]+\sigma_{22}^{-1}(u)[b_2(u)-r(u)])^2+(\sigma_{31}^{-1}(u)[b_1(u)+v_1(u)-r(u)]+\sigma_{32}^{-1}(u)[b_2(u)-r(u)]+\sigma_{33}^{-1}(u)[b_3(u)-r(u)])^2+\cdots+\left(\sigma_{d1}^{-1}(u)[b_1(u)+v_1(u)-r(u)]+\sigma_{d2}^{-1}(u)[b_2(u)-r(u)]+\cdots+\sigma_{dd}^{-1}(u)[b_d(u)-r(u)]\right)^2+\left(\sigma_{d+1,d+1}^{-1}(u)[b_{d+1}(u)+v_{d+1}(u)-r(u)]+a_{d+1}(u)\right)^2=\left(\sigma_{11}^{-1}(u)[b_1(u)+v_1(u)-r(u)]+a_1(u)\right)^2+\sum_{i=2}^{d}\left(\sigma_{i1}^{-1}(u)[b_1(u)+v_1(u)-r(u)]+\sum_{j=2}^{i}\sigma_{ij}^{-1}(u)\left[b_j(u)-r(u)\right]\right)^2+\left(\sigma_{d+1,d+1}^{-1}(u)[b_{d+1}(u)+v_{d+1}(u)-r(u)]+a_{d+1}(u)\right)^2=\left(A^{\mathbb{G}}v_1(u)+B^{\mathbb{G}}\right)^2+C^{\mathbb{G}}+\left(\sigma_{d+1,d+1}^{-1}(u)[b_{d+1}(u)+v_{d+1}(u)-r(u)]+a_{d+1}(u)\right)^2$, where $A^{\mathbb{G}}=$

$\sqrt{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\right)^2}$, $B^{\mathbb{G}} = \frac{\sigma_{1,1}^{-1}(u)a_1(u)+\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\sum_{j=1}^{i}\sigma_{i,j}^{-1}(u)[b_j(u)-r(u)]\right)}{\sqrt{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\right)^2}}$, $C^{\mathbb{G}} = \left(\sigma_{11}^{-1}(u)[b_1(u) - r(u)] + a_1(u)\right)^2 + \sum_{i=2}^{d}\left(\sum_{j=1}^{i}\sigma_{ij}^{-1}(u)\left[b_j(u)-r(u)\right]\right)^2 - \left(B^{\mathbb{G}}\right)^2$.

In likely manner, $\|\sigma^{-1}(u)[b(u)+v(u)-r(u)1]\|^2 = (\sigma_{11}^{-1}(u)[b_1(u)+v_1(u)-r(u)])^2 + \sum_{i=2}^{d}\left(\sigma_{i1}^{-1}(u)[b_1(u)+v_1(u)-r(u)] + \sum_{j=2}^{i}\sigma_{ij}^{-1}(u)\left[b_j(u)-r(u)\right]\right)^2 + \left(\sigma_{d+1,d+1}^{-1}(u)[b_{d+1}(u) + v_{d+1}(u)-r(u)]\right)^2 = \left(A^{\mathbb{F}}v_1(u)+B^{\mathbb{F}}\right)^2 + C^{\mathbb{F}} + \left(\sigma_{d+1,d+1}^{-1}(u)[b_{d+1}(u)+v_{d+1}(u)-r(u)]\right)^2$,

where $A^{\mathbb{F}} = \sqrt{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\right)^2}$, $B^{\mathbb{F}} = \frac{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\sum_{j=1}^{i}\sigma_{i,j}^{-1}(u)[b_j(u)-r(u)]\right)}{\sqrt{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\right)^2}}$, $C^{\mathbb{F}} = (\sigma_{11}^{-1}(u)[b_1(u) - r(u)])^2 + \sum_{i=2}^{d}\left(\sum_{j=1}^{i}\sigma_{ij}^{-1}(u)\left[b_j(u)-r(u)\right]\right)^2 - \left(B^{\mathbb{F}}\right)^2$.

We then have $C^{\mathbb{G}} - C^{\mathbb{F}} = \left[\left(\sigma_{11}^{-1}(u)[b_1(u)-r(u)] + a_1(u)\right)^2 - (\sigma_{11}^{-1}(u)[b_1(u)-r(u)])^2\right] + \left[\frac{\left(\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\sum_{j=1}^{i}\sigma_{i,j}^{-1}(u)[b_j(u)-r(u)]\right)\right)^2}{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\right)^2} - \frac{\left(\sigma_{1,1}^{-1}(u)a_1(u)+\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\sum_{j=1}^{i}\sigma_{i,j}^{-1}(u)[b_j(u)-r(u)]\right)\right)^2}{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\right)^2}\right] = \left[1 - \frac{[\sigma_{11}^{-1}(u)]^2}{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\right)^2}\right]\left(a_1(u)\right)^2 + 2\sigma_{1,1}^{-1}(u)\left[[b_1(u)-r(u)] - \frac{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\sum_{j=1}^{i}\sigma_{i,j}^{-1}(u)[b_j(u)-r(u)]\right)}{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\right)^2}\right]a_1(u)$, and

$\mathbb{E}^{\mathbb{P}}\left[C^{\mathbb{G}} - C^{\mathbb{F}}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[1 - \frac{[\sigma_{11}^{-1}(u)]^2}{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\right)^2}|\mathcal{F}_t\right] \times \mathbb{E}^{\mathbb{P}}\left[\left(a_1(u)\right)^2|\mathcal{F}_t\right]$.

If no short-selling on S_1 and non-tradable on S_{d+1} applies, and the initial total wealth x_t is cash, then, $\forall u \in [t,T]$, $K(u,\omega) = [0,\infty) \times (-\infty,\infty)^{d-1} \times [0,0]$. By definition, in this case, $\widetilde{K} = (0,\infty) \times \{0\}^{d-1} \times (-\infty,\infty)$, $\delta\left(v^{\mathbb{G}}(u)\right) \equiv 0$, and $\delta\left(v^{\mathbb{F}}(u)\right) \equiv 0$. $v^{\mathbb{G}}(u) = \underset{v\in\widetilde{K}}{\operatorname{argmin}}[2\delta(v) + \|\Theta_v(u)\|^2] = \underset{v\in\widetilde{K}}{\operatorname{argmin}}[\|\sigma^{-1}(u)[b(u)+v(u)-r(u)\mathbf{1}] + a(u)\|^2]$, $v^{\mathbb{F}}(u) = \underset{v\in\widetilde{K}}{\operatorname{argmin}}[2\delta(v) + \|\theta_v(u)\|^2] = \underset{v\in\widetilde{K}}{\operatorname{argmin}}[\|\sigma^{-1}(u)[b(u)+v(u)-r(u)\mathbf{1}]\|^2]$ for every $t \in [0,T]$. Then, on $(\Omega,\mathcal{F},\mathbb{P})$, we have $v_i^{\mathbb{G}}(u) = 0$, for $i = 2, \dots d$, and

$$v_1^{\mathbb{G}}(u) = \max\left(\frac{-\sigma_{1,1}^{-1}(u)a_1(u)-\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\sum_{j=1}^{i}\sigma_{i,j}^{-1}(u)[b_j(u)-r(u)]\right)}{\sqrt{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\right)^2}}, 0\right), \tag{A. 30}$$

and

$$v_{d+1}^{\mathbb{G}}(t) = r(t) - b_{d+1}(t) - \sigma_{d+1,d+1}(t)a_{d+1}(t). \tag{A. 31}$$

On $(\Omega,\mathcal{F},\mathbb{P})$, we have $v_i^{\mathbb{F}}(u) = 0$, for $i = 2, \dots d$, and

$$v_1^{\mathbb{F}}(u) = \max\left(\frac{-\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\sum_{j=1}^{i}\sigma_{i,j}^{-1}(u)[b_j(u)-r(u)]\right)}{\sqrt{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\right)^2}}, 0\right), \tag{A. 32}$$

and

$$v_{d+1}^{\mathbb{F}}(t) = r(t) - b_{d+1}(t). \tag{A. 33}$$

We have the following inequalities:

$\left(\sigma_{d+1,d+1}^{-1}(u)\left[b_{d+1}(u) + v_{d+1}^{\mathbb{G}}(t) - r(u)\right] + a_{d+1}(u)\right)^2 - \left(\sigma_{d+1,d+1}^{-1}(u)\left[b_{d+1}(u) + v_{d+1}^{\mathbb{F}}(t) - r(u)\right]\right)^2 \geq 0$, if $-\sigma_{d+1,d+1}(t)a_{d+1}(t) \leq 0$, i.e., $a_{d+1}(t) \geq 0$. If only $a_{d+1}(t)$ is finite, the value of such discrepancy is finite. Of course we only interested in the case that $a_{d+1}(t) = \frac{\sqrt{1-\lambda^2}[\lambda(W_1(T^*)-W_1(t))+\sqrt{1-\lambda^2}(W_{d+1}(T^*)-W_{d+1}(t))]}{(T^*-t)}$ is finite.

$$\mathbb{E}^{\mathbb{P}}\left[C^{\mathbb{G}} - C^{\mathbb{F}}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[1 - \frac{[\sigma_{11}^{-1}(u)]^2}{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\right)^2}|\mathcal{F}_t\right] \times \mathbb{E}^{\mathbb{P}}\left[\left(a_1(u)\right)^2|\mathcal{F}_t\right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[1 - \frac{[\sigma_{11}^{-1}(u)]^2}{\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\right)^2}|\mathcal{F}_t\right]$$

$$\times\ \mathbb{E}^{\mathbb{P}}\left[\left(\lambda\left(\lambda[W_1(T^*) - W_1(t)] + \sqrt{1-\lambda^2}[W_{d+1}(T^*) - W_{d+1}(t)]\right)\right.\right.$$

$$\left.\left./(T^*-t)\right)^2|\mathcal{F}_t\right]$$

$\int_t^T \mathbb{E}^{\mathbb{P}}\left[C^{\mathbb{G}} - C^{\mathbb{F}}|\mathcal{F}_t\right]du \to \infty$, as $T \to T^*$, and $\int_t^T \mathbb{E}^{\mathbb{P}}\left[\left(A^{\mathbb{G}}v_1^{\mathbb{G}}(u) + B^{\mathbb{G}}\right)^2 - \left(A^{\mathbb{F}}v_1^{\mathbb{F}}(u) + B^{\mathbb{F}}\right)^2|\mathcal{F}_t\right]du \geq -\int_t^T \mathbb{E}^{\mathbb{P}}\left[\left(A^{\mathbb{F}}v_1^{\mathbb{F}}(u) + B^{\mathbb{F}}\right)^2|\mathcal{F}_t\right]du$. If $G = \lambda W_1(T^*) - W_1 + \sqrt{1-\lambda^2}W_{d+1}(T^*) - W_{d+1}(t)$ then, we have $\mathbb{E}^{\mathbb{P}}(\Delta J(x_t, K(t,\omega))|\mathcal{F}_t) = \frac{1}{2}\int_t^T \mathbb{E}^{\mathbb{P}}\left[\left\|\Theta_{v^{\mathbb{G}}}(u)\right\|^2 - \left\|\theta_{v^{\mathbb{G}}}(u)\right\|^2|\mathcal{F}_t\right]du = \frac{1}{2}\int_t^T \mathbb{E}^{\mathbb{P}}\left[\left\|\sigma^{-1}(u)\left[b(u) + v^{\mathbb{G}}(u) - r(u)\mathbf{1}\right] + a(u)\right\|^2 - \left\|\sigma^{-1}(u)[b(u) + v^{\mathbb{F}}(u) - r(u)\mathbf{1}]\right\|^2|\mathcal{F}_t\right]du = \frac{1}{2}\int_t^T \mathbb{E}^{\mathbb{P}}\left[\left(A^{\mathbb{G}}v_1^{\mathbb{G}}(t) + B^{\mathbb{G}}\right)^2 - \left(A^{\mathbb{F}}v_1^{\mathbb{F}}(t) + B^{\mathbb{F}}\right)^2|\mathcal{F}_t\right] + \mathbb{E}^{\mathbb{P}}\left[C^{\mathbb{G}} - C^{\mathbb{F}}|\mathcal{F}_t\right]du \to \infty$, as $T \to T^*$, if $\exists\ i = 2, \ldots, d, s.t. \sigma_{i,1}(u) \neq 0$, otherwise $\mathbb{E}^{\mathbb{P}}(\Delta J(x_t, K(t,\omega))|\mathcal{F}_t)$ is finite, as $T \to T^*$.

So far, we haven't consider the fact that if $\exists\ i = 2, \ldots, d$, such that the initial total wealth includes derivatives underlying on $S_i(t)$ and $\sigma_{i,1}(u) \neq 0$, even though the constraint, no short selling of $S_1(t)$, applies, the insiders know insider information of $S_i(t)$, although the information is not accurate, it affects the value of initial total wealth to the insiders, as the portfolio are not binded to the no short selling of $S_i(t)$. However, even though $\mathbb{E}^{\mathbb{P}}\left(\Delta J_0\left(x_t, K(t,\omega)\right)\middle|\mathcal{F}_t\right) \neq 0$, $\mathbb{E}^{\mathbb{P}}\left(\Delta J_0\left(x_t, K(t,\omega)\right)\middle|\mathcal{F}_t\right)$ is finite. Again, as the portfolio are not binded to the no short selling of $S_i(t)$, even though the hedging position of derivatives underlying on $S_i(t)$ to insiders and outsider are different, portfolio constraints will not be affected. Hence $\mathbb{E}^{\mathbb{P}}(\Delta J(x_t, K(t,\omega))|\mathcal{F}_t) \to \infty$, as $T \to T^*$, if $\exists\ i = 2, \ldots, d, s.t. \sigma_{i,1}(u) \neq 0$ still holds, if the that the initial total wealth includes derivatives underlying on $S_i(t)$, $\sigma_{i,1}(u) \neq 0$ and $i \neq 1$.

According to Remark 2, if insider holds non-transferable and non-hedgeable equity-based compensations, and $\exists\, i = 2, \dots, d, s.t. \sigma_{i,1}(u) \neq 0$, then $\mathbb{E}^{\mathbb{P}}\left[\Delta J\left(x_t, K(t,\omega)\right)\middle|\mathcal{F}_t\right] \to \infty$, as $T \to T^*$.

This completes the proof of Proposition 2.

□

Appendix H

Proof of Proposition 3.

In order to prove $\exists B^L$, s.t. $\forall T \in [T^* - B^L, T^*]$, the $\dot{n}$ solving Eq. (14) at time t_{vo} is 100%, it is sufficient to prove $\exists T^B$, s.t. $\forall\ T \in [T^B, T^*\ ]$ and $\forall \dot{n}(t) \in [0\%, 100\%)$, $\frac{\partial \mathbb{J}^{\mathbb{G}}\left(\mathbb{X}^{*,\dot{n}}(t), \widehat{N}(t), S(t), \lambda, t\ T, T^*\right)}{\partial \widehat{N}(t)} > \frac{\left(BS(t) - B(t)\right)\partial \mathbb{J}^{\mathbb{G}}\left(\mathbb{X}^{*,\dot{n}}(t), \widehat{N}(t), S(t), \lambda, t, T, T^*\right)}{\partial \mathbb{X}^{*,\dot{n}}(t)}$. Because $\mathbb{E}^{\mathbb{P}}\left[r(u) + \delta(v^{\mathbb{G}}(u)) + \frac{1}{2}\|\Theta_v(u)\|^2 | \mathcal{G}_t\right]$ and $\partial \mathbb{E}^{\mathbb{P}}\left[r(u) + \delta(v^{\mathbb{G}}(u)) + \frac{1}{2}\|\Theta_v(u)\|^2 | \mathcal{G}_t\right] / \partial \widehat{N}(t)$ are both continuous over a region in the form $[t, T] \times [0\%, 100\%]$, then $LHS = \frac{\partial \mathbb{J}^{\mathbb{G}}\left(\mathbb{X}^{*,\dot{n}}(t), \widehat{N}(t), S(t), \lambda, t\ T, T^*\right)}{\partial \widehat{N}(t)} = \int_t^T \frac{\partial \mathbb{E}^{\mathbb{P}}\left[r(u) + \delta(v^{\mathbb{G}}(u)) + \frac{1}{2}\|\Theta_v(u)\|^2 | \mathcal{G}_t\right]}{\partial \widehat{N}(t)} du = \int_t^T \frac{\partial \mathbb{E}^{\mathbb{P}}\left[\left(A^{\mathbb{G}}(u) v_1^{\mathbb{G}}(u) + B^{\mathbb{G}}(u)\right)^2 | \mathcal{G}_t\right]}{\partial \widehat{N}(t)} du + \int_t^T \frac{\partial \mathbb{E}^{\mathbb{P}}\left[-\left(n - \widehat{N}(u)\right)\Phi^{\mathbb{G}}(u) S_1(u) / \mathbb{X}(u) v_1^{\mathbb{G}}(u) | \mathcal{G}_t\right]}{\partial \widehat{N}(t)} du = \frac{1}{\sum_{i=1}^d \left(\sigma_{i,1}^{-1}(u)\right)^2} \times \int_t^T \left[\lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}^{\mathbb{P}}[A - B + C - D | \mathcal{G}_t]}{\varepsilon}\right]$, where

$$A = \left(\max\left(\begin{matrix} \left(n - \widehat{N}(t) - \varepsilon\right)\Phi^{\mathbb{G}}(t) S_1(t)/\mathbb{X}(t), \\ \left(\sigma_{1,1}^{-1}(u) a_1(u) + \sum_{i=1}^d \left(\sigma_{i,1}^{-1}(u) \sum_{j=1}^i \sigma_{i,j}^{-1}(u)\left[b_j(u) - r(u)\right]\right)\right) \end{matrix}\right)\right)^2,$$

$$B = \left(\max\left(\begin{matrix} \left(n - \widehat{N}(t)\right)\Phi^{\mathbb{G}}(t) S_1(t)/\mathbb{X}(t), \\ \left(\sigma_{1,1}^{-1}(u) a_1(u) + \sum_{i=1}^d \left(\sigma_{i,1}^{-1}(u) \sum_{j=1}^i \sigma_{i,j}^{-1}(u)\left[b_j(u) - r(u)\right]\right)\right) \end{matrix}\right)\right)^2,$$

$$C = \left(n - \widehat{N}(t)\right)\Phi^{\mathbb{G}}(u) S_1(u)/\mathbb{X}(u) \max\left(\begin{matrix} \left(n - \widehat{N}(t)\right)\Phi^{\mathbb{G}}(t) S_1(t)/\mathbb{X}(t) - \sigma_{1,1}^{-1}(t) a_1(t) \\ - \sum_{i=1}^d \left(\sigma_{i,1}^{-1}(t) \sum_{j=1}^i \sigma_{i,j}^{-1}(t)\left[b_j(t) - r(t)\right]\right), 0 \end{matrix}\right),$$

$$D = \left(n - \widehat{N}(t) - \varepsilon\right)\Phi^{\mathbb{G}}(t) S_1(t)/\mathbb{X}(t) \max\left(\begin{matrix} \left(n - \widehat{N}(t) - \varepsilon\right)\Phi^{\mathbb{G}}(t) S_1(t)/\mathbb{X}(t) - \sigma_{1,1}^{-1}(t) a_1(t) \\ - \sum_{i=1}^d \left(\sigma_{i,1}^{-1}(t) \sum_{j=1}^i \sigma_{i,j}^{-1}(t)\left[b_j(t) - r(t)\right]\right), 0 \end{matrix}\right).$$

and $\Phi^{\mathbb{G}}(t)$ is the Clark-Ocone hedge ratio of European call option. Note that $a_1(t)$ is a function of λ.

$$LHS = \frac{1}{\sum_{i=1}^d \left(\sigma_{i,1}^{-1}(u)\right)^2} \times$$

$$\int_t^T \left[\lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}^{\mathbb{P}}\left[\begin{matrix} (-\varepsilon)\Phi^{\mathbb{G}}(t) S_1(t)/\mathbb{X}(t)\left[\sigma_{1,1}^{-1}(t) a_1(t) + \sum_{i=1}^d \left(\sigma_{i,1}^{-1}(t) \sum_{j=1}^i \sigma_{i,j}^{-1}(t)\left[b_j(t) - r(t)\right]\right)\right] \\ \times 1_{\left(n - \widehat{N}(t)\right)\Phi^{\mathbb{G}}(t) S_1(t)/\mathbb{X}(t) > \left(\sigma_{1,1}^{-1}(u) a_1(u) + \sum_{i=1}^d \left(\sigma_{i,1}^{-1}(u) \sum_{j=1}^i \sigma_{i,j}^{-1}(u)\left[b_j(u) - r(u)\right]\right)\right)} \end{matrix}\middle| \mathcal{G}_t\right]}{\varepsilon}\right] = \frac{1}{\sum_{i=1}^d \left(\sigma_{i,1}^{-1}(u)\right)^2} \times$$

$$\int_t^T \lim_{\varepsilon \downarrow 0} \mathbb{E}^{\mathbb{P}}\left[\begin{matrix} -\Phi^{\mathbb{G}}(t) S_1(t)/\mathbb{X}(t)\left[\sigma_{1,1}^{-1}(t) a_1(t) + \sum_{i=1}^d \left(\sigma_{i,1}^{-1}(t) \sum_{j=1}^i \sigma_{i,j}^{-1}(t)\left[b_j(t) - r(t)\right]\right)\right] \\ \times 1_{\left(n - \widehat{N}(t)\right)\Phi^{\mathbb{G}}(t) S_1(t)/\mathbb{X}(t) > \left(\sigma_{1,1}^{-1}(u) a_1(u) + \sum_{i=1}^d \left(\sigma_{i,1}^{-1}(u) \sum_{j=1}^i \sigma_{i,j}^{-1}(u)\left[b_j(u) - r(u)\right]\right)\right)} \end{matrix}\middle| \mathcal{G}_t\right] du.$$

We assume that $\frac{\partial^2 \mathbb{J}^{\mathbb{G}}(\mathbb{X}^{*,\dot{n}}(t),\widehat{N}(t),S(t),\lambda,t\,T,T^*)}{\left(\partial \widehat{N}(t)\right)^2} < 0$, then we only to prove for $\widehat{N}(t) = 100\%$, LHS $>$ $\frac{BS(t)-B(t)}{\mathbb{X}^{*,\dot{n}}(\mathrm{t})}$, namely,

$$\int_t^T \mathbb{E}^{\mathbb{P}}\left[\left.\begin{array}{c} -\Phi^{\mathbb{G}}(d_1)S_1(u)/\mathbb{X}(u)\left[\sigma_{1,1}^{-1}(u)a_1(u) + \sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\sum_{j=1}^{i}\sigma_{i,j}^{-1}(u)\left[b_j(u)-r(u)\right]\right)\right] \\ \times 1_{\left(\sigma_{1,1}^{-1}(u)a_1(u)+\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(u)\sum_{j=1}^{i}\sigma_{i,j}^{-1}(u)\left[b_j(u)-r(u)\right]\right)\right)<0} \end{array}\right| \mathcal{G}_t\right] du >$$

$\frac{(BS(t)-B(t))\sum_{i=1}^{d}\left(\sigma_{i,1}^{-1}(t)\right)^2}{\mathbb{X}^{*,\dot{n}}(t)}$. LHS is an increasing function of T, hence we can find a T satisfying the above inequality, then the existence of B^L is proved. As when $\widehat{N}(t) = 100\%$, $\mathbb{X}^{*,\dot{n}}(t) = x_t + nB(t)$, then B^L is a function of n. If n is greater， in order to satisfy the inequality, a shorter T is required, then a longer B^L is required.

This completes the proof of Proposition 3.

□

Appendix I

Proof of Proposition 4.

If $G = W_1(T^*)$, then $\mathbb{E}^{\mathbb{P}}\left[\mathbb{J}^{\mathbb{G}}\left(x_t,\widehat{N}(t),\lambda,t\,T\right)\middle|\mathcal{F}_t\right] \to \infty$, as $T \to T^*$ even though the NTNH American options are neglected. If we can show that $\mathbb{E}^{\mathbb{P}}\left[\mathbb{J}^{\mathbb{G}}\left(x_t,\widehat{N}(t),\lambda,t\,T\right)\middle|\mathcal{F}_t\right]$ is an monotonously increasing function of T, then there exists a unique T making $\mathbb{E}^{\mathbb{P}}\left[\mathbb{J}^{\mathbb{G}}\left(x_t,\widehat{N}(t),\lambda,t\,T\right)\middle|\mathcal{F}_t\right] = J^{\mathbb{F}}(x_t,t,T^*)$ holds. Next, we want to show if $T_1 < T_2$, then $\mathbb{E}^{\mathbb{P}}\left[\mathbb{J}^{\mathbb{G}}\left(x_t,\widehat{N}(t),\lambda,t,T_1\right)\middle|\mathcal{F}_t\right] < \mathbb{E}^{\mathbb{P}}\left[\mathbb{J}^{\mathbb{G}}\left(x_t,\widehat{N}(t),\lambda,t,T_2\right)\middle|\mathcal{F}_t\right]$. If insiders are allowed to hold the portfolio till T_2, then optimally holding it till T_1 and then exercise all the options and liquate all the assets and deposit proceeds in a savings account to earn risk free rate is just one choice in the opportunity set. That choice being opitmal requires insider optimal portfolio process $\pi^{\mathbb{G}*}(t) = [\sigma(t)\sigma^{\top}(t)]^{-1}\left[b(t)+\sigma(t)a(t)+v^{\mathbb{G}}(t)-r(t)\mathbf{1}\right] \equiv 0$, for any $t \in [T_1,T_2]$. Even we assume $\sigma(t), b(t), r(t)$ are all constants. As $a(t) = \left(\frac{W_1(T^*)-W_1(t)}{T^*-t},0,\ldots,0\right)'$, which depends on $W_1(t)$, then for $\forall t \in [T_1,T_2], \pi^{\mathbb{G}*}(t) \equiv 0$ cannot hold. Then, $\mathbb{E}^{\mathbb{P}}\left[\mathbb{J}^{\mathbb{G}}\left(x_t,\widehat{N}(t),\lambda,t,T_1\right)\middle|\mathcal{F}_t\right] < \mathbb{E}^{\mathbb{P}}\left[\mathbb{J}^{\mathbb{G}}\left(x_t,\widehat{N}(t),\lambda,t,T_2\right)\middle|\mathcal{F}_t\right]$.

This completes the proof Proposition 4.

□

Appendix J

Proof of Proposition 5 i).

If S_1 is non-tradable from time t to T, then $K(u,\omega) = \left[\frac{\phi_1(t)S_1(u)}{\mathbb{X}(u)},\frac{\phi_1(t)S_1(u)}{\mathbb{X}(u)}\right] \times (-\infty,\infty)^{d-1}$, $\forall u \in [t,T]$, which implies buying/selling or exercising options to make alterations to the positions on S_1 is not allowed. The position changes purely due to the change of $\frac{S_1(u)}{\mathbb{X}(u)}$. By definition, in this case, $\widetilde{K}=(-\infty,\infty)\times\{0\}^{d-1}$, $\delta\left(v^{\mathbb{G}}(u)\right)\equiv-\frac{\phi_1(t)S_1(u)}{\mathbb{X}(u)}\times v_1^{\mathbb{G}}(u)$ on $\widetilde{K}$, and $\delta\left(v^{\mathbb{F}}(u)\right)\equiv-\frac{\phi_1(t)S_1(u)}{\mathbb{X}(u)}\times v_1^{\mathbb{F}}(u)$ on $\widetilde{K}$. $v^{\mathbb{G}}(u)=\underset{v\in\widetilde{K}}{\operatorname{argmin}}[2\delta(v)+\|\Theta_v(u)\|^2]=\underset{v\in\widetilde{K}}{\operatorname{argmin}}[2\delta(v)+\|\sigma^{-1}(u)[b(u)+v(u)-r(u)\mathbf{1}]+a(u)\|^2]$, $v^{\mathbb{F}}(u)=\underset{v\in\widetilde{K}}{\operatorname{argmin}}[2\delta(v)+\|\theta_v(u)\|^2]=\underset{v\in\widetilde{K}}{\operatorname{argmin}}[2\delta(v)+\|\sigma^{-1}(u)[b(u)+v(u)-r(u)\mathbf{1}]\|^2]$ for every $u\in[t,T]$.

If $\sigma(u)$ is diagonal, i.e., primary assets are independent, then $\|\sigma^{-1}(u)[b(u)+v(u)-r(u)\mathbf{1}]+a(u)\|^2=\left(\sigma_{11}^{-1}(u)[b_1(u)+v_1(u)-r(u)]+a_1(u)\right)^2+\sum_{i=2}^{d}\left(\sigma_{ii}^{-1}(u)[b_i(u)-r(u)]\right)^2=\left(\sigma_{11}^{-1}(u)v_1(u)+\sigma_{11}^{-1}(u)[b_1(u)-r(u)]+a_1(u)\right)^2+C^{\mathbb{G}}=\left(A^{\mathbb{G}}v_1(u)+B^{\mathbb{G}}\right)^2+C^{\mathbb{G}}$, where $A^{\mathbb{G}}=\sigma_{11}^{-1}(u)$, $B^{\mathbb{G}}=\sigma_{11}^{-1}(u)[b_1(u)-r(u)]+a_1(u)$, and $C^{\mathbb{G}}=\sum_{i=2}^{d}\left(\sigma_{ii}^{-1}(u)[b_i(u)-r(u)]\right)^2$. $\|\sigma^{-1}(u)[b(u)+v(u)-r(u)\mathbf{1}]\|^2=\left(A^{\mathbb{F}}v_1(u)+B^{\mathbb{F}}\right)^2+C^{\mathbb{F}}$, where $A^{\mathbb{F}}=\sigma_{11}^{-1}(u)$, $B^{\mathbb{F}}=\sigma_{11}^{-1}(u)[b_1(u)-r(u)]$. Because $C^{\mathbb{G}}=C^{\mathbb{F}}$, $\min_{v_1\in(-\infty,\infty)}\left(A^{\mathbb{G}}v_1(u)+B^{\mathbb{G}}\right)^2+C^{\mathbb{G}}=\min_{v_1\in(-\infty,\infty)}\left(A^{\mathbb{F}}v_1(u)+B^{\mathbb{F}}\right)^2+C^{\mathbb{F}}$. Because $A^{\mathbb{G}}=A^{\mathbb{F}}$, then adding $2\delta(v)=-2\frac{\phi_1(t)S_1(u)}{\mathbb{X}(u)}\times v_1(u)$ does not change the shape of parabola, but shift the parabola to the right by the same distance, then we have $\left(A^{\mathbb{G}}v_1^{\mathbb{G}}(u)+B^{\mathbb{G}}\right)^2+C^{\mathbb{G}}=\left(A^{\mathbb{F}}v_1^{\mathbb{F}}(u)+B^{\mathbb{F}}\right)^2+C^{\mathbb{F}}$, where $v_1^{\mathbb{G}}(u)=\underset{v_1\in(-\infty,\infty)}{\operatorname{argmin}}\left[2\delta(v)+\left(A^{\mathbb{G}}v_1(u)+B^{\mathbb{G}}\right)^2+C^{\mathbb{G}}\right]$ and $v_1^{\mathbb{F}}(u)=\underset{v_1\in(-\infty,\infty)}{\operatorname{argmin}}\left[2\delta(v)+\left(A^{\mathbb{F}}v_1(u)+B^{\mathbb{F}}\right)^2+C^{\mathbb{F}}\right]$, namely $\left\|\sigma^{-1}(u)\left[b(u)+v^{\mathbb{G}}(u)-r(u)1\right]+a(u)\right\|^2=\left\|\sigma^{-1}(u)[b(u)+v^{\mathbb{F}}(u)-r(u)1]\right\|^2$. Thus, $\mathbb{E}^{\mathbb{P}}(\Delta J(x_t,K(t,\omega))|\mathcal{F}_t)=\frac{1}{2}\int_t^T\mathbb{E}^{\mathbb{P}}[\|\Theta_v(u)\|^2-\|\theta_v(u)\|^2|\mathcal{F}_t]\,du=0$. And $\sigma^{-1}(u)v^{\mathbb{G}}(u)+a(u)=\sigma^{-1}(u)v^{\mathbb{F}}(u)$ holds for any $a(u)$, and correspondingly for any type of insider information.

Proof of Proposition 5 ii).

If $\sigma(u)$ is not diagonal, i.e., primary assets are correlated, then $\|\sigma^{-1}(u)[b(u)+v(u)-r(u)1]+a(u)\|^2=\left(\sigma_{11}^{-1}(u)[b_1(u)+v_1(u)-r(u)]+a_1(u)\right)^2+(\sigma_{21}^{-1}(u)[b_1(u)+v_1(u)-r(u)]+\sigma_{22}^{-1}(u)[b_2(u)-r(u)])^2+(\sigma_{31}^{-1}(u)[b_1(u)+v_1(u)-r(u)]+\sigma_{32}^{-1}(u)[b_2(u)-r(u)]+\sigma_{33}^{-1}(u)[b_3(u)-r(u)])^2+\cdots+\left(\sigma_{d1}^{-1}(u)[b_1(u)+v_1(u)-r(u)]+\sigma_{d2}^{-1}(u)[b_2(u)-r(u)]+\cdots+\sigma_{dd}^{-1}(u)[b_d(u)-r(u)]\right)^2=\left(\sigma_{11}^{-1}(u)[b_1(u)+v_1(u)-r(u)]+a_1(u)\right)^2+\sum_{i=2}^{d}\left(\sigma_{i1}^{-1}(u)[b_1(u)+v_1(u)-r(u)]+\sum_{j=2}^{i}\sigma_{ij}^{-1}(u)\left[b_j(u)-r(u)\right]\right)^2=\left(A^{\mathbb{G}}v_1(u)+B^{\mathbb{G}}\right)^2+C^{\mathbb{G}}$, where $A^{\mathbb{G}}=\sqrt{\sum_{i=1}^{d}[\sigma_{i1}^{-1}(u)]^2}$, $B^{\mathbb{G}}=\left[\sigma_{11}^{-1}(u)\left(\sigma_{11}^{-1}(u)[b_1(u)-r(u)]+a_1(u)\right)+\sum_{i=2}^{d}\left(\sigma_{i1}^{-1}(u)\sum_{j=1}^{i}\sigma_{ij}^{-1}(u)\left[b_j(u)-r(u)\right]\right)\right]/A^{\mathbb{G}}$, $C^{\mathbb{G}}=\left(\sigma_{11}^{-1}(u)[b_1(u)-r(u)]+a_1(u)\right)^2+\sum_{i=2}^{d}\left(\sum_{j=1}^{i}\sigma_{ij}^{-1}(u)\left[b_j(u)-r(u)\right]\right)^2-\left(B^{\mathbb{G}}\right)^2$. And

$\|\sigma^{-1}(u)[b(u)+v(u)-r(u)1]\|^2 = \left(A^{\mathbb{F}}v_1(u)+B^{\mathbb{F}}\right)^2 + C^{\mathbb{F}}$, where $A^{\mathbb{F}} = \sqrt{\sum_{i=1}^{d}[\sigma_{i1}^{-1}(u)]^2}$, $B^{\mathbb{F}} = \left[\sum_{i=1}^{d}\left(\sigma_{i1}^{-1}(u)\sum_{j=1}^{i}\sigma_{ij}^{-1}(u)\left[b_j(u)-r(u)\right]\right)\right]/A^{\mathbb{F}}$, $C^{\mathbb{F}} = \sum_{i=1}^{d}\left(\sum_{j=1}^{i}\sigma_{ij}^{-1}(u)\left[b_j(u)-r(u)\right]\right)^2 - \left(B^{\mathbb{F}}\right)^2$

Because $A^{\mathbb{G}} = A^{\mathbb{F}}$, then adding $2\delta(v) = -2\frac{\phi_1(t)S_1(u)}{\mathbb{X}(u)} \times v_1(u)$ to the objective function does not change the shape of the parabola, but shift the parabola to the right by the same distance only. Because $C^{\mathbb{G}} = \min_{v_1\in(-\infty,\infty)}\left(A^{\mathbb{G}}v_1(u)+B^{\mathbb{G}}\right)^2 + C^{\mathbb{G}} \neq C^{\mathbb{F}} = \min_{v_1\in(-\infty,\infty)}\left(A^{\mathbb{F}}v_1(u)+B^{\mathbb{F}}\right)^2 + C^{\mathbb{F}}$, we have $\left(A^{\mathbb{G}}v_1^{\mathbb{G}}(u)+B^{\mathbb{G}}\right)^2 + C^{\mathbb{G}} \neq \left(A^{\mathbb{F}}v_1^{\mathbb{F}}(u)+B^{\mathbb{F}}\right)^2 + C^{\mathbb{F}}$, where $v_1^{\mathbb{G}}(u) = \operatorname*{argmin}_{v_1\in(-\infty,\infty)}\left[2\delta(v) + \left(A^{\mathbb{G}}v_1(u)+B^{\mathbb{G}}\right)^2 + C^{\mathbb{G}}\right]$ and $v_1^{\mathbb{F}}(u) = \operatorname*{argmin}_{v_1\in(-\infty,\infty)}\left[2\delta(v) + \left(A^{\mathbb{F}}v_1(u)+B^{\mathbb{F}}\right)^2 + C^{\mathbb{F}}\right]$, namely $\left\|\sigma^{-1}(u)\left[b(u)+v^{\mathbb{G}}(u)-r(u)\mathbf{1}\right]+a(u)\right\|^2 \neq \left\|\sigma^{-1}(u)[b(u)+v^{\mathbb{F}}(u)-r(u)\mathbf{1}]\right\|^2$. As $\|\Theta_v(u)\|^2 - \|\theta_v(u)\|^2 = C^{\mathbb{G}} - C^{\mathbb{F}} = \left(\sigma_{11}^{-1}(u)[b_1(u)-r(u)]+a_1(u)\right)^2 - (\sigma_{11}^{-1}(u)[b_1(u)-r(u)])^2 - \left(\left(B^{\mathbb{G}}\right)^2 - \left(B^{\mathbb{F}}\right)^2\right) = \left(a_1(u)\right)^2 + 2\sigma_{11}^{-1}(u)[b_1(u)-r(u)]a_1(u) - \left(\left[\left(\sigma_{11}^{-1}(u)a_1(u)\right)^2 + 2\sigma_{11}^{-1}(u)a_1(u)(\sigma_{11}^{-1}(u)[b_1(u)-r(u)])\right]/\sum_{i=1}^{d}[\sigma_{i1}^{-1}(u)]^2\right)$, Thus, $\mathbb{E}^{\mathbb{P}}(\Delta J(x_t,K(t,\omega))|\mathcal{F}_t) = \frac{1}{2}\int_t^T \mathbb{E}^{\mathbb{P}}[\|\Theta_v(u)\|^2 - \|\theta_v(u)\|^2|\mathcal{F}_t]\,du = \frac{1}{2}\int_t^T \mathbb{E}^{\mathbb{P}}\left[\left(a_1(u)\right)^2 + \left[\left(\sigma_{11}^{-1}(u)a_1(u)\right)^2\right]/\sum_{i=1}^{d}[\sigma_{i1}^{-1}(u)]^2\,|\mathcal{F}_t\right]du \to \infty$, as $T \to T^*$.

Proof of Proposition 5 iii).

Second, if $S_i, i = 1,2,\ldots d$ are all non-tradable from time t to T, then $K(u,\omega) = \left[\frac{\phi_1(t)S_1(u)}{\mathbb{X}(u)}, \frac{\phi_1(t)S_1(u)}{\mathbb{X}(u)}\right] \times \left[\frac{\phi_2(t)S_2(u)}{\mathbb{X}(u)}, \frac{\phi_2(t)S_2(u)}{\mathbb{X}(u)}\right] \times \ldots \times \left[\frac{\phi_d(t)S_d(u)}{\mathbb{X}(u)}, \frac{\phi_d(t)S_d(u)}{\mathbb{X}(u)}\right]$, $\forall u \in [t,T]$, which implies the investment plan cannot be changed since t onwards. $\phi_i(t)$ represents the number of shares of primary asset i that the executive holds at time t. By definition, in this case, $\widetilde{K} = (-\infty,\infty)^d$, $\delta\left(v^{\mathbb{G}}(u)\right) \equiv -\frac{\phi_1(t)S_1(u)}{\mathbb{X}(u)} \times v_1^{\mathbb{G}}(u) - \frac{\phi_2(t)S_2(u)}{\mathbb{X}(u)} \times v_2^{\mathbb{G}}(u) \ldots - \frac{\phi_d(t)S_d(u)}{\mathbb{X}(u)} \times v_d^{\mathbb{G}}(u)$ on $\widetilde{K}$, and $\delta\left(v^{\mathbb{F}}(u)\right) \equiv -\frac{\phi_1(t)S_1(u)}{\mathbb{X}(u)} \times v_1^{\mathbb{F}}(u) - \frac{\phi_2(t)S_2(u)}{\mathbb{X}(u)} \times v_2^{\mathbb{F}}(u) \ldots - \frac{\phi_d(t)S_d(u)}{\mathbb{X}(u)} \times v_d^{\mathbb{F}}(u)$ on $\widetilde{K}$. Because the vector $v^{\mathbb{G}}(u) = \operatorname*{argmin}_{v\in\widetilde{K}}[2\delta(v) + \|\Theta_v(u)\|^2] = \operatorname*{argmin}_{v\in\widetilde{K}}[2\delta(v) + \|\sigma^{-1}(u)[b(u)+v(u)-r(u)1]+a(u)\|^2]$, $v^{\mathbb{F}}(u) = \operatorname*{argmin}_{v\in\widetilde{K}}[2\delta(v) + \|\theta_v(u)\|^2] = \operatorname*{argmin}_{v\in\widetilde{K}}[2\delta(v) + \|\sigma^{-1}(u)[b(u)+v(u)-r(u)1]\|^2]$ for every $t \in [0,T]$.

We consider the general case that $\sigma(t)$ is not diagonal. We claim that there exists a unique vector $v^{\mathbb{G}}(u) \equiv \left(v_1^{\mathbb{G}}(u), v_2^{\mathbb{G}}(u), \ldots, v_d^{\mathbb{G}}(u)\right)$ minimizing $2\delta(v) + \|\sigma^{-1}(u)[b(u)+v(u)-r(u)1]+a(u)\|^2$ to 0. $2\delta(v) + \|\sigma^{-1}(u)[b(u)+v(u)-r(u)1]+a(u)\|^2 = \sum_{i=1}^{d} m_i^{\mathbb{G}}$, where $m_1^{\mathbb{G}} = \left[\left(\sigma_{11}^{-1}(u)[b_1(u)+v_1(u)-r(u)]+a_1(u)\right)^2 - 2\frac{\phi_1(t)S_1(u)}{\mathbb{X}(u)} \times v_1(u)\right]$, and $m_i^{\mathbb{G}} =$

$\left(\sum_{j=1}^{i}\sigma_{ij}^{-1}(u)\left[b_j(u)+v_j(u)-r(u)\right]\right)^2-2\frac{\phi_j(t)S_j(u)}{\mathbb{X}(u)}\times v_i(u)$, for $i=2,\dots d$. If $v_1^{\mathbb{G}}(u)$ solves $\mathcal{m}_1^{\mathbb{G}}(v_1(u))=0$, then substitute $v_1^{\mathbb{G}}(t)$ into $\mathcal{m}_2^{\mathbb{G}}$, and get $v_2^{\mathbb{G}}(u)$ by solving $\mathcal{m}_2^{\mathbb{G}}(v_2^{\mathbb{G}}(u))=0$. In likely manner, for $i=2,\dots d-1$, we substitute $v_i^{\mathbb{G}}(t)$ into $\mathcal{m}_{i+1}^{\mathbb{G}}$, and get $v_{i+1}^{\mathbb{G}}(u)$ by solving $\mathcal{m}_{i+1}^{\mathbb{G}}(v_{i+1}^{\mathbb{G}}(u))=0$. $\sum_{i=1}^{d}\mathcal{m}_i^{\mathbb{G}}$ can be written into a sum of squared items, $\sum_{i=1}^{d}\mathcal{m}_i^{\mathbb{G}}=0$ has achieved its possible lower bound. Hence it must have been minimized.

We also claim that there exists a unique vector $v^{\mathbb{F}}(u)\equiv\left(v_1^{\mathbb{F}}(u),v_2^{\mathbb{F}}(u),\dots,v_d^{\mathbb{F}}(u)\right)$ minimizing $2\delta(v)+\|\sigma^{-1}(u)[b(u)+v(u)-r(u)1]\|^2$ to 0. $2\delta(v)+\|\sigma^{-1}(u)[b(u)+v(u)-r(u)1]\|^2=\sum_{i=1}^{d}\mathcal{m}_i^{\mathbb{F}}$, where $\mathcal{m}_1^{\mathbb{F}}=\left[(\sigma_{11}^{-1}(u)[b_1(u)+v_1(u)-r(u)])^2-2\frac{\phi_1(t)S_1(u)}{\mathbb{X}(u)}\times v_1(u)\right]$, and for $i=2,\dots d$, $\mathcal{m}_i^{\mathbb{F}}=\left(\sum_{j=1}^{i}\sigma_{ij}^{-1}(u)\left[b_j(u)+v_j(u)-r(u)\right]\right)^2-2\frac{\phi_j(t)S_j(u)}{\mathbb{X}(u)}\times v_i(u)$. If $v_1^{\mathbb{F}}(u)$ solves $\mathcal{m}_1^{\mathbb{F}}(v_1(u))=0$, then substitute $v_1^{\mathbb{F}}(t)$ into $\mathcal{m}_2^{\mathbb{F}}$, and get $v_2^{\mathbb{F}}(u)$ by solving $\mathcal{m}_2^{\mathbb{F}}(v_2^{\mathbb{F}}(u))=0$. In likely manner, for $i=2,\dots d-1$, we substitute $v_i^{\mathbb{F}}(t)$ into $\mathcal{m}_{i+1}^{\mathbb{F}}$, and get $v_{i+1}^{\mathbb{F}}(u)$ by solving $\mathcal{m}_{i+1}^{\mathbb{F}}(v_{i+1}^{\mathbb{F}}(u))=0$. $\sum_{i=1}^{d}\mathcal{m}_i^{\mathbb{F}}$ can be written into a sum of squared items, $\sum_{i=1}^{d}\mathcal{m}_i^{\mathbb{F}}=0$ has achieved its possible lower bound. Hence it must have been minimized.

Denote $\|\sigma^{-1}(u)[b(u)+v(u)-r(u)1]+a(u)\|^2\equiv\sum_{i=1}^{d}\mathcal{n}_i^{\mathbb{G}}$, where $\mathcal{n}_1^{\mathbb{G}}=\left(\sigma_{11}^{-1}(u)[b_1(u)+v_1(u)-r(u)]+a_1(u)\right)^2$, and $\mathcal{n}_i^{\mathbb{G}}=\left(\sum_{j=1}^{i}\sigma_{ij}^{-1}(u)\left[b_j(u)+v_j(u)-r(u)\right]\right)^2$ for $i=2,\dots d$. $\|\sigma^{-1}(u)[b(u)+v(u)-r(u)1]\|^2\equiv\sum_{i=1}^{d}\mathcal{n}_i^{\mathbb{F}}$, where, for $i=1,\dots d$, $\mathcal{n}_i^{\mathbb{F}}=\left(\sum_{j=1}^{i}\sigma_{ij}^{-1}(u)\left[b_j(u)+v_j(u)-r(u)\right]\right)^2$. $\mathcal{n}_1^{\mathbb{G}}\equiv\left(A_1^{\mathbb{G}}v_1(u)+B_1^{\mathbb{G}}\right)^2+C_1^{\mathbb{G}}$, where $A_1^{\mathbb{G}}=\sigma_{11}^{-1}(u)$, $C_1^{\mathbb{G}}=0$; $\mathcal{n}_1^{\mathbb{F}}\equiv\left(A_1^{\mathbb{F}}v_1(u)+B_1^{\mathbb{F}}\right)^2+C_1^{\mathbb{F}}$ where $A_1^{\mathbb{F}}=\sigma_{11}^{-1}(u)$, $C_1^{\mathbb{F}}=0$. As $A_1^{\mathbb{G}}=A_1^{\mathbb{F}}$, $C_1^{\mathbb{G}}=C_1^{\mathbb{F}}$, and $\mathcal{m}_1^{\mathbb{G}}(v_1^{\mathbb{G}}(u))=0$, $\mathcal{m}_1^{\mathbb{F}}(v_1^{\mathbb{F}}(u))=0$, then $\left(\sigma_{11}^{-1}(u)\left[b_1(u)+v_1^{\mathbb{G}}(u)-r(u)\right]+a_1(u)\right)^2=\left(\sigma_{11}^{-1}(u)\left[b_1(u)+v_1^{\mathbb{F}}(u)-r(u)\right]\right)^2$.

Substituting $v_1^{\mathbb{G}}(u)$ into $\mathcal{n}_2^{\mathbb{G}}$, we have $\mathcal{n}_2^{\mathbb{G}}(v_2(u))=\left(\sigma_{21}^{-1}(u)\left[b_1(u)+v_1^{\mathbb{G}}(u)-r(u)\right]+\sigma_{22}^{-1}(u)[b_2(u)+v_2(u)-r(u)]\right)^2\equiv\left(A_2^{\mathbb{G}}v_2(u)+B_2^{\mathbb{G}}\right)^2+C_2^{\mathbb{G}}$, which is a parabola of $v_2(u)$ with $A_2^{\mathbb{G}}=\sigma_{22}^{-1}(u)$, $C_2^{\mathbb{G}}=0$. Substituting $v_1^{\mathbb{F}}(u)$ into $\mathcal{n}_2^{\mathbb{F}}$, we have $\mathcal{n}_2^{\mathbb{F}}(v_2(u))=\left(\sigma_{21}^{-1}(u)\left[b_1(u)+v_1^{\mathbb{F}}(u)-r(u)\right]+\sigma_{22}^{-1}(u)[b_2(u)+v_2(u)-r(u)]\right)^2\equiv\left(A_2^{\mathbb{F}}v_2(u)+B_2^{\mathbb{F}}\right)^2+C_2^{\mathbb{F}}$, which is a parabola of $v_2(u)$ with $A_2^{\mathbb{F}}=\sigma_{22}^{-1}(u)$, $C_2^{\mathbb{F}}=0$. As $A_2^{\mathbb{G}}=A_2^{\mathbb{F}}$, $C_2^{\mathbb{G}}=C_2^{\mathbb{F}}$, and $\mathcal{m}_2^{\mathbb{G}}(v_2^{\mathbb{G}}(u))=0$, $\mathcal{m}_2^{\mathbb{F}}(v_2^{\mathbb{F}}(u))=0$, then $\left(\sigma_{21}^{-1}(u)\left[b_1(u)+v_1^{\mathbb{G}}(u)-r(u)\right]+\sigma_{22}^{-1}(u)\left[b_2(u)+v_2^{\mathbb{G}}(u)-r(u)\right]\right)^2=\left(\sigma_{21}^{-1}(u)\left[b_1(u)+v_1^{\mathbb{F}}(u)-r(u)\right]+\sigma_{22}^{-1}(u)\left[b_2(u)+v_2^{\mathbb{F}}(u)-r(u)\right]\right)^2$.

In likely manner, for $i=2,\dots d-1$, substituting $v_1^{\mathbb{G}}(u)$,…, $v_i^{\mathbb{G}}(u)$ into $\mathcal{n}_{i+1}^{\mathbb{G}}$, and substituting $v_1^{\mathbb{F}}(u)$, …, $v_i^{\mathbb{F}}(u)$ into $\mathcal{n}_{i+1}^{\mathbb{F}}$, we get $\mathcal{n}_{i+1}^{\mathbb{G}}(v_{i+1}(u))\equiv\left(A_{i+1}^{\mathbb{G}}v_{i+1}(u)+B_{i+1}^{\mathbb{G}}\right)^2+C_{i+1}^{\mathbb{G}}$, and $\mathcal{n}_{i+1}^{\mathbb{F}}(v_{i+1}(u))\equiv\left(A_{i+1}^{\mathbb{F}}v_{i+1}(u)+B_{i+1}^{\mathbb{F}}\right)^2+C_{i+1}^{\mathbb{F}}$. As $i=2,\dots d-1$, $A_{i+1}^{\mathbb{G}}=A_{i+1}^{\mathbb{F}}$, $C_{i+1}^{\mathbb{G}}=C_{i+1}^{\mathbb{F}}$, and

$m_{i+1}^{\mathbb{G}}(v_{i+1}^{\mathbb{G}}(u)) = 0$, $m_{i+1}^{\mathbb{F}}(v_{i+1}^{\mathbb{F}}(u)) = 0$, then the value of $\sum_{i=1}^{d} n_i^{\mathbb{G}}$ at $v^{\mathbb{G}}(u)$ is equal to the value of $\sum_{i=1}^{d} n_i^{\mathbb{F}}$ at $v^{\mathbb{F}}(u)$, i.e., $\mathbb{E}^{\mathbb{P}}[\Delta J(x_t, K(t,\omega))|\mathcal{F}_t] = \int_t^T \left(\frac{1}{2}\mathbb{E}^{\mathbb{P}}[\|\Theta_v(u)\|^2 - \|\theta_v(u)\|^2|\mathcal{F}_t]\right) du = 0$, and $a(u) + \sigma^{-1}(u)v^{\mathbb{G}}(u) = \sigma^{-1}(u)v^{\mathbb{F}}(u)$ holds for any $a(u)$, and correspondingly for any type of insider information.

This completes the proof of Proposition 5.

□

Appendix K

Proof of Theorem 1.

By definite of RSO, martingale approach implies that the price of RSO with n reloads can be expressed as

$$R^n(t, S_t, K) = \underset{\tau_n \in \Gamma}{\operatorname{esssup}} \, \mathbb{E}\left[H_0(\tau_n)\left(S_{\tau_n} - K + \frac{K}{S_{\tau_n}} R^{n-1}(\tau_n, S_{\tau_n}, S_{\tau_n})\right)^+ \middle| \mathcal{F}_t\right]. \tag{A. 34}$$

where Γ is the class of $\{\mathcal{F}_t\}$ stopping times. According to the optimal stopping time theory, the optimal exercise time is determined as follows,

$$\tau_n = \inf\left\{u{:}\ R^n(u, S_u, K) = S_u - K + \frac{K}{S_u} R^{n-1}(u, S_u)\right\}. \tag{A. 35}$$

The intuition is that If $R^n(u, S_u, K)$ is bounded, by the fact that $R^n(u, S_u, K)$ is a nondecreasing function of n when $n \to \infty$, the sequence must be convergence, namely, $R^n(u, S_u, K) \to R^\infty(u, S_u, K),\ for\ \forall u, K, when\ n \to \infty$.

$$\tau_\infty = \inf\left\{u{:}\ R^\infty(u, K) = S_u - K + \frac{K}{S_u} R^\infty(u, S_u)\right\}. \tag{A. 36}$$

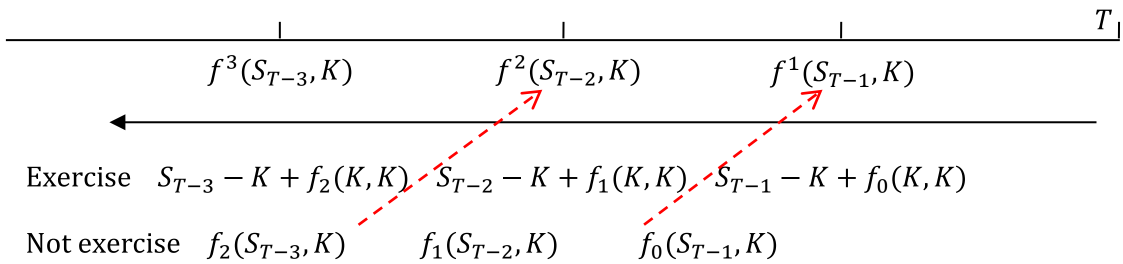
Obviously, $S_u = K$, is a solution, and if there is no other solution, then

$$\tau_\infty = \inf\{u{:}\ S_u = K\}. \tag{A. 37}$$

Then how do we prove there is no other solution satisfying,

$$R^n(u, S_u, K) + K = S_u + \frac{K}{S_u} R^n(u, S_u, S_u), when\ n \to \infty. \tag{A. 38}$$

Graph 1



To answer the above question we need to reconstruct the model. See Graph 1, we assume the option can only be reloaded on the grid, and study the optimal exercise problem backwardly. Please note the difference between the superscripts and the subscripts. The superscripts stand for how many times the option can be reloaded on the grid point including the current time. The subscripts stand for how many times the option can be reloaded on the grid point excluding the current time. Here we are considering the infinite reload case. For example, assume at time $T-3$, before making decision to exercise the option, the value of option is denoted as $f^3(S_{T-3},K)$. If it is optimal to reload it, then the instant payoff is $S_{T-3}-K+f_2(K,K)$, otherwise the value is denoted as $f_2(S_{T-3},K)$. If at $T-3$, it is optimal not to reload it, then at $T-2$, $f_2(S_{T-3},K)$ becomes $f^2(S_{T-2},K)$. The stike price does not change, because only reloading option will reset the strike price. Eventually, at time T, if $S_T>K$, then exercise the option, otherwise not to exercise it.

Lemma A.1: At $\mathbf{T-1}$, if $\mathbf{S_{T-1}} > (=,<)\mathbf{K}$, then $\mathbf{S_{T-1}-K+f_0(K,K)} > (=,<)\mathbf{f_0(S_{T-1},K)}$. That is, the option holder should reload the option, if it is in the money; and give up the reload opportunity, if the option is out of money. Hence we have

$$f^1(S_{T-1},K)=(S_{T-1}-K+f^0(K,K))1_{\{S_{T-1}>K\}}+f_0(S_{T-1},K)1_{\{S_{T-1}\le K\}}. \quad (A.39)$$

Proof of Lemma A.1.

Note that $f_0(S,K)$ is a plain vanilla European call option. We know that for any S, $\frac{\partial f_0(S_{T-1},K)}{\partial S_{T-1}}\le 1$, and so by the Mean Value Theorem, $\frac{f_0(S_{T-1},K)-f_0(K,K)}{S_{T-1}-K}\le 1$. Thus, if $S_{T-1}-K>0$, then $f_0(S_{T-1},K)-f_0(K,K)\le S_{T-1}-K$, while if $S_{T-1}-K<0$, then $f_0(S_{T-1},K)-f_0(K,K)\ge S_{T-1}-K$. Hence, at $T-1$, the option holder will reload the option if the option is in the money and give up the reload opportunity if the option is out of money. Then,

$$f^1(S_{T-1},K)=(S_{T-1}-K+f^0(K,K))1_{\{S_{T-1}>K\}}+f_0(S_{T-1},K)1_{\{S_{T-1}\le K\}}. \quad (A.40)$$

This completes the proof of Lemma A.1.

To reconcile with the fact that changing monetary unit, will not change the actual asset price, then $\frac{f_n(S_u,S_u)}{S_u}=\frac{f_n(K,K)}{K}$. This result will simply the following analysis.

Lemma A.2: $\forall n$, at $T-n$, if $S_{T-n}>(=,<)K$, then $S_{T-n}-K+f_{n-1}(K,K)>(=,<)\, f_{n-1}(S_{T-n},K)$.

Proof of Lemma A.2.

If the option is exercised at time $T-1$, then

$$f^1(S_{T-1},K)=(S_{T-1}-K+f^0(K,K))1_{\{S_{T-1}>K\}}+f_0(S_{T-1},K)1_{\{S_{T-1}\le K\}}. \quad (A.41)$$

otherwise if we do not exercise, then

$$f^1(S_{T-1},K_{T-1})=f_0(S_{T-1},K_{T-1}). \quad (A.42)$$

Because $\frac{K}{S_{T-1}}f_0(S_{T-1},S_{T-1})$ is a constant, we can see that in the case where we exercise,

$$\frac{\partial}{\partial S_{T-1}}f^1(S_{T-1},K_{T-1})=1+0=1. \quad (A.43)$$

In the case where we do not exercise, it is well known that $\frac{\partial}{\partial S_{T-1}}f_0(S_{T-1},K)<1$ for European call options. So, in either case, $\frac{\partial}{\partial S_{T-1}}f^1(S_{T-1},K)\le 1$.

Next, by the usual Cox-Ross-Rubinstein methodology,

$$f_1(S_{T-2}, K) = e^{-rh}\big(pf^1(uS_{T-2}, K) + (1-p)f^1(dS_{T-2}, K)\big). \tag{A. 44}$$

where r is the risk-free rate. Differentiating with respect to S_{T-2}, we see that

$$\begin{aligned}
&\frac{\partial}{\partial S_{T-2}} f_1(S_{T-2}, K_{T-2}) \\
&\qquad = e^{-rh}\left(p\frac{\partial}{\partial S_{T-2}} f^1(uS_{T-2}, K) + (1-p)\frac{\partial}{\partial S_{T-2}} f^1(dS_{T-2}, K)\right) \\
&\qquad = e^{-rh}\left(pu\frac{\partial}{\partial S} f^1(S, K) + (1-p)d\frac{\partial}{\partial S} f^1(S, K)\right) \\
&\qquad \leq e^{-rh}(pu + (1-p)d) = 1,
\end{aligned} \tag{A. 45}$$

where the last equality holds from the fact that $p = \frac{e^{rh}-d}{u-d}$, $u = e^{\sigma\sqrt{h}}$ and $d = u^{-1}$. We have proved the case for n = 1.

By the inductive step, we assume that in the case where we do not exercise, we have

$$\frac{\partial}{\partial S_{T-n}} f^n(S_{T-n}, K) \leq 1. \tag{A. 46}$$

It follows from the previous argument, by the usual Cox-Ross-Rubinstein methodology,

$$\begin{aligned}
&\frac{\partial}{\partial S_{T-(n+1)}} f_n\big(S_{T-(n+1)}, K\big) = e^{-rh}\Bigg(p\frac{\partial}{\partial S_{T-(n+1)}} f^n\big(uS_{T-(n+1)}, K\big) + (1- \\
&p)\frac{\partial}{\partial S_{T-(n+1)}} f^n\big(dS_{T-(n+1)}, K\big)\Bigg) = e^{-rh}\Bigg(pu\frac{\partial}{\partial S_{T-(n+1)}} f^n\big(S_{T-(n+1)}, K\big) + (1- \\
&p)d\frac{\partial}{\partial S_{T-(n+1)}} f^n\big(S_{T-(n+1)}, K\big)\Bigg) \leq e^{-rh}(pu + (1-p)d) = 1.
\end{aligned} \tag{A. 47}$$

Next we prove,

$$\frac{\partial}{\partial S_{T-(n+1)}} f^{n+1}\big(S_{T-(n+1)}, K\big) \leq 1. \tag{A. 48}$$

If we exercise at date $T-(n+1)$, then

$$f^{n+1}\big(S_{T-(n+1)}, K\big) = S_{T-(n+1)} - K + \frac{K}{S_{T-(n+1)}} f_n\big(S_{T-(n+1)}, S_{T-(n+1)}\big), \tag{A. 49}$$

while if we do not exercise, then

$$f^{n+1}\big(S_{T-(n+1)}, K\big) = f_n\big(S_{T-(n+1)}, K\big). \tag{A. 50}$$

hence in the case where we exercise,

$$\frac{\partial}{\partial S_{T-(n+1)}} f^{n+1}\big(S_{T-(n+1)}, K\big) = 1 + 0 = 1. \tag{A. 51}$$

In the case where we do not exercise, it is well known that $\frac{\partial}{\partial S_{T-(n+1)}} f_n\big(S_{T-(n+1)}, K\big) \leq 1$ for European call options. So, in either case, $\frac{\partial}{\partial S_{T-(n+1)}} f^{n+1}\big(S_{T-(n+1)}, K\big) \leq 1$.

Next, by the usual Cox-Ross-Rubinstein methodology,

$$
\begin{aligned}
&\frac{\partial}{\partial S_{T-(n+2)}} f_{n+1}\left(S_{T-(n+2)}, K\right) \\
&\qquad = e^{-rh}\left(p \frac{\partial}{\partial S_{T-(n+2)}} f^{n+1}\left(u S_{T-(n+2)}, K\right)\right. \\
&\qquad \left. + (1-p) \frac{\partial}{\partial S_{T-(n+2)}} f^{n+1}\left(d S_{T-(n+2)}, K\right)\right) \\
&\qquad = e^{-rh}\left(pu \frac{\partial}{\partial S_{T-(n+2)}} f^{n+1}\left(S_{T-(n+2)}, K\right) + (1\right. \\
&\qquad \left. - p) d \frac{\partial}{\partial S_{T-(n+2)}} f^{n+1}\left(S_{T-(n+2)}, K\right)\right) \leq e^{-rh}(pu + (1-p)d) \\
&\qquad = 1.
\end{aligned}
\tag{A. 52}
$$

This completes the proof of Lemma A.2.

Hence the executive will reload the option if it is in the money, and drop the current reload opportunity if it is at the money or out of money. Please note that the strike price will be revised to the current stock price after each reload, hence it is not the constant K all the time. The current strike price is the historical highest stock price. Take limit of the reload time n into infinity, we get that the at-the-money RSO price with infinite reloads as shown in Eq. (16).

This completes the proof of Theorem 1.

□

Appendix L

Proof of Theorem 2.

After considering executives' insider role and NTNH "stock-for-stock" constraints, the risk free rate and drift rate of stock price dynamic becomes a $\boldsymbol{\mathcal{G}_t}$-measurable random variable, and the volatility is unchanged. However, the analogous proofs of Lemma A.1 and Lemma A.2 go through, because even though we replace $\boldsymbol{r}$ in Eq. (A. 45) into $\boldsymbol{r} + \boldsymbol{\delta}(\widehat{\boldsymbol{v}}(\boldsymbol{t}))$, the last inequality in Eq. (A. 45) still holds, $\boldsymbol{\partial f_0(S_{T-1}, K)/\partial S_{T-1} \leq 1}$ still holds, then by induction, the optimal exercise policy is unchanged.

Intuitively, the optimal exercise policy for unconstrained reload option (grid model) is to exercise and reload whenever the option is in the money, otherwise the option will be devalued. Reloading option will relax the constraints. Hence this is not the same as the constrained American option problem, there is no trade-off between losing time value and relaxing the constraints, for early exercise; Under NTNH constraints and vesting period requirement, reloading option by exercising the option dominates non-reloading or partial reloading the option, from both perspectives of avoid devaluation and constraints relaxation.

This completes the proof of Theorem 2.

□

Appendix M

Proof of Corollary 5.

From Theorem 2, considering executives' insider role and NTNH "stock-for-stock" constraints, the exercise policy is to exercise the option whenever the option is in the money, then the payoffs over time are unchanged. To value the firm cost, the stochastic discount factor should be the market one, i.e., $H_0(t)$ (see Colwell et al 2015).

This completes the proof of Corollary 5.

□

Appendix N

Proof of Corollary 6.

We rewrite Eq. (16) into $P(S_1(t), t, t^0) = \frac{S_1(t^0)}{S_1(t)} P(S_1(t), t, t)$. It is obvious that $P(S_1(t), t, t) \to 0$ as $S_1(t^0) \to 0$, and by Eq. (A. 52), for any $t^0 \leq t$, we have $\frac{\partial P(S_1(t), t, t^0)}{\partial S_1(t)} \leq 1$. Then $P(S_1(t), t, t^0) \leq S_1(t)$, and $P(S_1(t), t, t^0) \leq S_1(t^0)$. According to Corollary 5, $\tilde{P}(S_1(t), t, t^0) \leq S_1(t^0)$.

This completes the proof of Corollary 6.

□

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