Modeling VXX under jump diffusion with stochastic long-term mean

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We develop a model for the VXX, the most actively traded VIX futures exchange-traded note (ETN), using Duffie, Pan, and Singleton's (2000) affine jump diffusion, where the volatility process has jumps and a stochastic long-term mean. We calibrate the model parameters using the VIX term structure data and show that our model provides the theoretical link between the VIX, VIX futures and the VXX. Our model can be used for pricing VIX futures, the VXX and other short-term VIX futures exchange-traded products (ETPs). Our model could be extended to price options on the VXX and other short-term VIX futures ETPs.

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1 Introduction

In this paper we extend the model of Gehricke and Zhang (2018), by including jumps and letting the long-term mean of volatility be mean-reverting. This results in a better fit to the VIX term structure and the VXX, the most actively traded VIX futures exchange-traded note (ETN), compared with the nested models.[1] We calibrate to the VIX term structure and then explore the fit of the model for the VXX. The model also performs well for other short-term VIX futures exchange-traded products (ETPs), which dominate the market, and could be extended for VIX futures ETP option pricing.

VIX index exposure first became accessible to investors in 2004, when VIX futures contracts were launched by the Chicago Board Options Exchange (CBOE), followed in 2006 by VIX options. More recently, since 2009, VIX futures ETPs have been heavily traded. The VXX was the first ETP tracking the short-term VIX futures index (SPVXSTR), which represents the return on a portfolio of VIX futures that is rebalanced to achieve an almost constant one-month maturity. Since 2009 the number of other VIX futures ETPs has been rapidly growing, but with first mover advantage the VXX has been the largest and most heavily traded VIX futures ETP throughout this period. In this paper we model and fit the short-term VIX futures ETPs, while calibrating to the VIX term structure.

In figure 1, we can see that the market capitalization has grown to around $4 Billion and the average daily dollar trading volume is around $ 2 billion. On some

[1] The first nested model is the Heston (1993) model, as used by Gehricke and Zhang (2018). The second nested model is the floating θ model, as presented in appendix A.6 equations (55), (56) and (57). The last nested model is equivalent to the full model, as presented in equations (3), (4) and (5), but where $\kappa = \kappa_V = \kappa_\theta$.

days the ETPs are traded so heavily that the dollar trading volume is several multiples of the market capitalization, meaning the market can turn over several times in a day.

The long-exposure ETPs were initially marketed as diversification tools for equity portfolios, due to the negative correlation between the VIX and the S&P 500; however, several studies have shown that they are not useful for diversification (Alexander et al., 2016; Deng, McCann, and Wang, 2012; Hancock, 2013). The reason why these products are not good for diversification is due to their underperformance relative to the VIX index, which is an empirical fact in contrast to the common misconception that investing in the VXX is like investing in the VIX index. In October of 2017 Wells Fargo was ordered to pay remunerations of $3.4 million to investors because they were advising them to invest in VIX futures ETPs as hedging tools (Banerji, 2017).

The under (out) performance of the short-term long- (short-) exposure VIX futures ETPs is well documented in the literature, and can be seen in table 1. Alexander and Korovilas (2013), Liu and Dash (2012) and Whaley (2013) suggest that the usually contango (upward-sloping) VIX futures term structure is the driver of the underperformance of the VXX. Gehricke and Zhang (2018) are the first to model the VXXs price while accounting for the dynamic relationships between the SPX, VIX index, VIX futures and the ETN price. They show that the underperformance of the VXX relative to the VIX index is mainly due to the roll yield, which measures the effect of rebalancing from the nearest to the second nearest futures contract. We confirm this finding with our extended model. The roll yield will be negative (positive) when the VIX futures term structure is in contango (backwardation). They show that the negative roll yield is driven by the market price of variance risk, on aggregate.

Their result is consistent with that of Eraker and Wu (2017), who show that the underperformance of the SPVXSTR index is driven by the variance risk premium. The market price of variance risk and the variance risk premium are two closely related concepts, which Zhang and Huang (2010) show are almost proportional to each other. Eraker and Wu (2017) show, in a consumption-based equilibrium setting, that the underlying driver of the negative variance risk premium is investor risk aversion.

Recently, there have been several news releases (Jakab, 2018; Zuckermann and Fletcher, 2018; Burger, 2018) on the sudden collapse of the XIV ETN, February 2018, which tracked the inverse performance of the SPVXSTR index. This event was caused by a spike in the VIX which led to massive losses for the XIV, and even further selling pressure compounded this effect, providing more evidence that the negative returns to the SPVXSTR index (and ETPs which track it) in normal times are a premium for the risk of a spike in volatility.

A feasible motivation for investors to trade these products is short-term hedging against volatility spikes or speculation, which are both in line with the high trading activity of these products. Another explanation could be the ease of access to this market; any investors can easily invest in VIX futures ETPs, as they are traded on stock exchanges. This is a dangerous setting, as these are highly complicated derivative instruments which are not yet fully understood by academics or practitioners, let alone retail investors. The VIX futures ETP market is also a way that some mutual funds, which are restricted from investing in traditional derivatives, can enter volatility positions.

In this paper our focus is on developing a new model for the VXX, building on the work of Gehricke and Zhang (2018), which accounts for the relationship between the S&P 500, the VIX term structure, VIX futures and the VXX. We calibrate our model to the VIX term structure, which allows us to fit the VXX time series well. We compare the fit of our model with two simpler nested models and that of Gehricke and Zhang (2018), showing that our full model is better at fitting the short or full VIX term structure. We provide two different formulas for modeling the VXX and other short-term VIX futures ETPs, one idealistic (daily rebalanced with constant maturity) and one realistic (continuously rebalanced with varying maturity) model. We show that the realistic model outperforms the idealistic in fitting the VXX time series.

We find that calibrating to the first three-points of the VIX term structure results in a better fit to the VXX time series, compared with calibrating to the full VIX term structure. This is intuitive, as the VXX and other short-term VIX futures ETPs, represent exposure to short-term volatility and should not be affected by market expectations on longer-term volatility. When we calibrate to the short VIX term structure, the mean-reverting speed of the long-term mean level of the instantaneous variance κ_θ is small, showing that the mean reversion characteristic does not contribute much for modeling shorter-dated volatility, and its derivatives.

Our model can also be used to price other short term VIX futures ETPs. Short-term VIX futures ETPs dominate the VIX futures ETP market, as shown in figure 1. It shows that the short-term VIX futures ETPs consistently make up around 80% and almost 100% of the VIX futures ETP market size and trading volume, respectively.

Several authors have studied the daily and intra-daily price discovery dynamics between the VIX index and its futures (Shu and Zhang, 2012; Frijns, Tourani-Rad, and Webb, 2016; among others) concluding that, at the intraday level, VIX futures lead the VIX index. Bordonado et al. (2017) indirectly show that VIX futures ETNs lead VIX futures exchange-traded funds (ETFs). Bollen, O'Neill, and Whaley (2017) show that the VXX leads the VIX futures in price discovery and that VIX futures lead VIX options in price discovery. While, Gehricke and Zhang (2017) show that all of the VIX futures ETNs lead the VIX futures and that no single ETN leads the others consistently. Combining the results we could say that VIX futures ETNs lead VIX futures, which in turn lead the VIX index. This further highlights the importance of understanding and accurately pricing the VIX futures ETPs, as these products drive the market for volatility trading.

Our paper is also related to the vast literature on volatility derivative pricing. Many papers have studied different model settings for pricing VIX futures (Zhang and Zhu, 2006; Zhang, Shu, and Brenner, 2010; Lu and Zhu, 2010; Dupoyet, Daigler, and Chen, 2011; Zhu and Lian, 2012; Huskaj and Nossman, 2013). Lin (2007) derives and studies the pricing performance of closed form VIX futures pricing formulas under several different affine dynamics. Although Lin (2007) does not test the out-of-sample pricing ability of our model, the study finds that out of the models tested the stochastic volatility with jumps in the volatility (SVVJ) performs best for short-term (< 60 days) VIX futures contracts, which is the model closest to ours. Developing models for VIX options has also been the focus of several articles (Wang and Daigler, 2011; Chung, Tsai, Wang, and Weng, 2011; Cont and Kokholm, 2013; Lian and Zhu, 2013; Bardgett, Gourier, and Leippold, 2014; Papanicolaou and Sircar, 2014). Luo and Zhang (2012) study the VIX term structure; that is, they calculate different maturity VIX indices and examine their properties. They also provide an affine model for the VIX term structure with jumps in the stock price and a stochastic long-term mean of volatility. They calibrate their model using a similar calibration method to that in our study. Bao, Li, and Gong (2012) price VXX options by assuming an affine structure for the VXX and its volatility. This approach ignores the relationship between the SPX, VIX term structure, VIX futures and the VXX. Our model can be used to price VXX options; while accounting for these relationships.

The rest of this paper is organized as follows. Section 2 summarizes the methodology for calculating the short-term VIX futures index that the ETPs are tracking, the SPVXSTR index. Then section 3 outlines the model set-up and derives formulas for the VIX index, VIX futures, the VXX and other short-term VIX futures ETPs. Next in section 4 we calibrate our model and nested models to the short and full VIX term structure and compare their fit. In section 5 we examine the fit to the VXX and other short-term VIX futures ETP time series, comparing the performance of the short and full VIX term structure calibration and the discrete and idealistic model. Finally, in section 6 we conclude.

2 VIX futures indices

The underlying indices of the short-term VIX futures ETPs are either the S&P 500 short-term VIX futures Total Return Index (SPVXSTR) or the S&P 500 short-term VIX futures Excess Return Index (SPVXSER). The SPVXSTR index is calculated as:

$$SPVXSTR_t = SPVXSTR_{t-1}(1 + CDR_t + TBR_t), \tag{1}$$

and the SPVXSER index is the same but without the interest return on the underlying futures position, TBR_t. The contract daily return (CDR_t) of the futures position is given by:

$$CDR_t = \frac{w_{1,t-1}F_t^{T_1} + w_{2,t-1}F_t^{T_2}}{w_{1,t-1}F_{t-1}^{T_1} + w_{2,t-1}F_{t-1}^{T_2}} - 1, \tag{2}$$

where $F_t^{T_i}$ is the current price of the i-th maturing VIX futures contract and $w_{i,t-1}$ is the weight of the position invested in the i-th maturing VIX futures contract the preceding business day. The weights are calculated such that the futures position has a maturity of one month, which fluctuates around 30 days.[2]

3 Model

3.1 Model dynamics

We extend the model of Gehricke and Zhang (2018), who use a Heston (1993) framework, by adding a jump component in the instantaneous variance process and making the long-term mean level of the instantaneous variance stochastic. For the ex-dividend stock price and its variance, under the risk-neutral probability measure, we adopt the following dynamics:

$$dS_t = rS_t dt + \sqrt{V_t} S_t dB_{1,t} \tag{3}$$

$$dV_t = \kappa_V(\theta_t - V_t)dt + \sigma_V\sqrt{V_t}dB_{2,t} + ydN_t - \lambda E^Q[y]dt \tag{4}$$

$$d\theta_t = \kappa_\theta(\bar{\theta} - \theta_t)dt + \sigma_\theta\sqrt{\theta_t}dB_{3,t}, \tag{5}$$

where θ_t is the effective long-term mean level of V_t, which is the instantaneous variance of the SPX.[3] Here, r is the risk free rate, θ_t is the effective long-term mean level of

[2] For further details on the calculation of the weights and indices please see Gehricke and Zhang (2018) and/or S&P Dow Jones Indices (2012).

[3] If jumps are included in the stock price process then V_t can be seen as the instantaneous squared VIX, as in Luo and Zhang (2012), and all following results are identical.

V_t. κ_V and κ_θ are the mean-reverting speeds of V_t and θ_t, respectively. The effective long-term mean of θ_t is given by $\bar{\theta}$. The volatility of the variance is given by σ_V, while σ_θ measures the volatility of θ_t. $B_{1,t}$, $B_{2,t}$ and $B_{3,t}$ are three standard Brownian motions that describe the diffusive randomness in S_t, V_t and θ_t, respectively. The Brownian motions $B_{1,t}$ and $B_{2,t}$ are correlated by a constant correlation coefficient ρ, while $B_{3,t}$ is independent of the other two. Also, dN_t is a Poisson process with arrival intensity λ and jump size y. The jump size y can be any independently distributed random variable.

The jump component allows more flexibility in the modeling of the density of the instantaneous variance V_t. Bardgett, Gourier, and Leippold (2014) find that mean-reverting volatility and jumps in volatility are important in capturing volatility smiles in both the SPX and VIX markets. Our jump term is compensated, which keeps the VIX formula and estimation simple compared with models that do not compensate the jump component (Luo and Zhang, 2012). Also, Lin (2007) shows that for VIX futures with maturity of less than 60 days the SVVJ (stochastic volatility with jumps in volatility) model outperforms the SV (stochastic volatility, i.e. Heston, 1993), SVJ (stochastic volatility with jumps in the stock price) and SVJJ (stochastic volatility with jumps in equity and volatility) models.

Our model differs from the standard SVVJ model as the long-term mean level of the instantaneous variance θ_t is stochastic and mean-reverting. Implementing a stochastic θ_t allows for more realistic transient changes in the VIX and VIX futures term structures (Zhang and Huang, 2010; Zhang, Shu, and Brenner, 2010; (Zhang, Zhen, Sun, and Zhao, 2017)).

In order to derive formulas for the VIX index term structure and VIX futures prices, we will later need the first two central moments of V_s and θ_s, where $s > t$. These are given in Lemma 1 below.

Lemma 1. *The risk-neutral first and second central moments of θ_t and V_t can be derived as:*

$$
\begin{aligned}
E_t^Q[\theta_s] &= e^{-\kappa_\theta(s-t)}\theta_t + (1 - e^{-\kappa_\theta(s-t)})\bar{\theta}, && (6)\\
E_t^Q[(\theta_s - E_t^Q[\theta_s])^2] &= \frac{\sigma_\theta^2 \theta_t}{\kappa_\theta}(e^{-\kappa_\theta(s-t)} - e^{-2\kappa_\theta(s-t)}) \\
&\quad + \frac{\sigma_\theta^2 \bar{\theta}}{2\kappa_\theta}(1 - 2e^{-\kappa_\theta(s-t)} + e^{-2\kappa_\theta(s-t)}) && (7)
\end{aligned}
$$

$$
\begin{aligned}
E_t^Q[V_s] &= e^{-\kappa_V(s-t)}V_t + \frac{\kappa_V}{\kappa_V - \kappa_\theta}\left(e^{-\kappa_\theta(s-t)} - e^{-\kappa_V(s-t)}\right)\theta_t \\
&\quad + \left(1 - e^{-\kappa_V(s-t)} - \frac{\kappa_V}{\kappa_V - \kappa_\theta}(e^{-\kappa_\theta(s-t)} - e^{-\kappa_V(s-t)})\right)\bar{\theta} && (8)\\
E_t^Q[(V_s - E_t^Q[V_s])^2] &= X + Y. && (9)
\end{aligned}
$$

where

$$
\begin{aligned}
X &= \frac{\kappa_V^2 \sigma_\theta^2}{(\kappa_V - \kappa_\theta)^2}\Bigg[\Bigg(\frac{e^{-\kappa_\theta(s-t)} - e^{-2\kappa_\theta(s-t)}}{\kappa_\theta} - \frac{2e^{-\kappa_\theta(s-t)}(1 - e^{-\kappa_V(s-t)})}{\kappa_V} + \frac{e^{-\kappa_\theta(s-t)} - e^{-\kappa_V(s-t)}}{2\kappa_V - \kappa_\theta}\Bigg)\theta_t \\
&\quad + \Bigg(\frac{(1 - e^{-\kappa_\theta(s-t)})^2}{2\kappa_\theta} - \frac{2(1 - e^{-(\kappa_\theta - \kappa_V)(s-t)})}{\kappa_\theta + \kappa_V} + \frac{2(e^{-\kappa_\theta(s-t)} - e^{-(\kappa_\theta - \kappa_V)(s-t)})}{\kappa_V} \\
&\quad + \frac{1 - e^{-2\kappa_V(s-t)}}{2\kappa_V} - \frac{e^{-\kappa_\theta(s-t)} e^{-2\kappa_V(s-t)}}{2\kappa_V - \kappa_\theta}\Bigg)\bar{\theta}\Bigg] && (10)\\
Y &= \frac{\sigma_V^2}{\kappa_V}\left(e^{-\kappa_V(s-t)} - e^{-2\kappa_V(s-t)}\right)V_t + \frac{\lambda}{2\kappa_V}\left(1 - e^{-2\kappa_V(s-t)}\right)E_t[y^2] \\
&\quad + \frac{\kappa_V \sigma_V^2}{\kappa_V - \kappa_\theta}\left(\frac{e^{-\kappa_\theta(s-t)} - e^{-\kappa_V(s-t)}}{2\kappa_V - \kappa_\theta} - \frac{e^{-\kappa_V(s-t)} - e^{-2\kappa_V(s-t)}}{\kappa_V}\right)\theta_t \\
&\quad + \sigma_V^2\left(\frac{(1 - e^{-\kappa_V(s-t)})^2}{2\kappa_V} - \frac{\kappa_V}{\kappa_V - \kappa_\theta}\left[\frac{e^{-\kappa_\theta(s-t)} - e^{-\kappa_V(s-t)}}{2\kappa_V - \kappa_\theta} - \frac{e^{-\kappa_V(s-t)} - e^{-2\kappa_V(s-t)}}{\kappa_V}\right]\right)\bar{\theta} && (11)
\end{aligned}
$$

The proofs for the moments of θ_t and V_t can be found in appendix A.1 and A.2, respectively.

Remark 1. The coefficients in front of θ_t, $\bar{\theta}$ in equation (6) can only take on values between 0 and 1 for any value of s between 0 and ∞. The sum of the coefficients is equal to 1. Therefore we can say that $E_t^Q[\theta_s]$ is a weighted average of θ_t and $\bar{\theta}$, where the two coefficients mentioned are the weights.

Remark 2. The coefficients in front of V_t, θ_t and $\bar{\theta}$ in equation (8) can only take on values between 0 and 1 for any value of s between 0 and ∞. The sum of the three coefficients is equal to 1. Therefore we can say that $E_t^Q[V_s]$ is a weighted average of V_t, θ_t and $\bar{\theta}$, where the three coefficients mentioned are the weights.

3.2 VIX term structure

The VIX index measures the markets expectation of 30-day implied volatility. However, the VIX methodology can be applied to essentially any maturity, and the CBOE has recently started to report several S&P 500 implied volatility indices with different maturities. From the CBOE website we can get implied volatility time series for any maturity index.

Carr and Wu (2009) show that the VIX index is equivalent to the 30-day variance swap rate which is equal to the risk-neutral conditional expectation of variance over the next 30 days, when the stock price is modeled without jumps. Therefore the VIX squared with any maturity, T_i, is given by:

$$\left(\frac{VIX_t^{T_i}}{100}\right)^2 = E_t^Q\left[\frac{1}{\tau_i}\int_t^{t+\tau_i} V_s ds\right] = \frac{1}{\tau_i}\int_t^{t+\tau_i} E_t^Q[V_s]ds \tag{12}$$

where τ_i is the time to maturity in years. We can interchange the expectation and integral (justified by Tonelli's theorem) to get the second equality. Using lemma 1 we get the following proposition.

Proposition 1. *The VIX index, under our model dynamics, is given by:*

$$\frac{VIX_t}{100} = \sqrt{A\ V_t + B\theta_t + (1 - A - B)\bar{\theta}}, \tag{13}$$

where

$$\begin{aligned} A &= \frac{1-e^{-\kappa_V \tau_0}}{\kappa_V \tau_0}, \\ B &= \frac{\kappa_V (1-e^{-\kappa_\theta \tau_0})}{\kappa_\theta \tau_0 (\kappa_V - \kappa_\theta)} - \frac{(1-e^{-\kappa_V \tau_0})}{\tau_0 (\kappa_V - \kappa_\theta)}. \end{aligned}$$

and $\tau_0 = 30/365$ *can be replaced by any maturity in order to model the VIX term structure.*

The proof for this proposition is provided in appendix A.3.

Remark 3. The coefficients in front of V_t, θ_t and $\bar{\theta}$ in equation (13) can again only take on values between 0 and 1 for any value of s between 0 and ∞ and the sum of the three coefficients is equal to 1. Therefore we can say that the VIX index (VIX term structure) is the square root of a weighted average of V_t, θ_t and $\bar{\theta}$, where the three coefficients mentioned are the weights.

3.3 VIX futures

Now that we have a formula for the VIX index, equation (13), we can derive a VIX futures price formula. The VIX futures price is given by the conditional risk-neutral expectation of the VIX index at the futures contract's maturity T:

$$\frac{F_t^T}{100} = E_t^Q \left(\frac{VIX_T}{100} \right) \tag{14}$$

Plugging in our formula for the VIX index from proposition 1 we get:

$$\frac{F_t^T}{100} = E_t^Q \left[\left(A\, V_T + B\theta_T + (1-A-B)\bar{\theta} \right)^{\frac{1}{2}} \right]. \tag{15}$$

Lemma 2. *The risk-neutral conditional expectation of the VIX squared, or a VIX* *squared futures price, is given by:*

$$\begin{aligned}
E_t^Q\left[\left(\frac{VIX_T}{100}\right)^2\right] &= AE_t^Q[V_T] + BE_t^Q[\theta_T] + (1-A-B)\bar{\theta} \\
&= A\bigg[e^{-\kappa_V(T-t)}V_t + \frac{\kappa_V}{\kappa_V-\kappa_\theta}\left(e^{-\kappa_\theta(T-t)} - e^{-\kappa_V(T-t)}\right)\theta_t \\
&\quad + \left(1 - e^{-\kappa_V(T-t)} - \frac{\kappa_V}{\kappa_V-\kappa_\theta}\left(e^{-\kappa_\theta(T-t)} - e^{-\kappa_V(T-t)}\right)\right)\bar{\theta}\bigg] \\
&\quad + B\left[e^{-\kappa_\theta(T-t)}\theta_t + (1-e^{-\kappa_\theta(T-t)})\bar{\theta}\right] + (1-A-B)\bar{\theta} \\
&\quad = CV_t + D\theta_t + (1-C-D)\bar{\theta}, \qquad (16)
\end{aligned}$$

where

$$\begin{aligned}
C &= Ae^{-\kappa_V(T-t)}, \\
D &= A\frac{\kappa_V}{\kappa_V-\kappa_\theta}\left(e^{-\kappa_\theta(T-t)} - e^{-\kappa_V(T-t)}\right) + Be^{-\kappa_\theta(T-t)}.
\end{aligned}$$

To approximate the expectation of the non-linear VIX equation, equation (15), we follow a similar methodology as Zhang, Shu, and Brenner (2010). We expand the square root form equation (15) using the two variable Taylor expansion near the points $E_t^Q[V_T]$ and $E_t^Q[\theta_T]$, which results in proposition 2 below.

Proposition 2. *The VIX futures price can be approximated by:*

$$\begin{aligned}
\frac{F_t^T}{100} &= \left(CV_t + D\theta_t + (1-C-D)\bar{\theta}\right)^{\frac{1}{2}} - \frac{1}{8}\left(CV_t + D\theta_t + (1-C-D)\bar{\theta}\right)^{-\frac{3}{2}} A^2\left(X\big|_{s=T} + Y\big|_{s=T}\right) \\
&\quad - \frac{1}{8}\left(CV_t + D\theta_t + (1-C-D)\bar{\theta}\right)^{-\frac{3}{2}} B^2 \\
&\quad \times\left[\frac{\sigma_\theta^2\theta_t}{\kappa_\theta}\left(e^{-\kappa_\theta(T-t)} - e^{-2\kappa_\theta(T-t)}\right) + \frac{\sigma_\theta^2\bar{\theta}}{2\kappa_\theta}\left(1 - 2e^{-\kappa_\theta(T-t)} + e^{-2\kappa_\theta(T-t)}\right)\right], \qquad (17)
\end{aligned}$$

where T is the maturity of the VIX futures contract.[4]

The proof for this proposition is given in the appendix A.4.

[4]An analytical formula for the VIX futures price can be found by using the technique developed by Duffie, Pan, and Singleton (2000) for affine jump diffusion models. A comparative stude between our approximate formula and the analytical formula is left for further research.

As in Gehricke and Zhang (2018) we can further approximate the short-term, up to 60 days maturity, VIX futures price by removing the convexity adjustments in equation (17). This is justified as they do not have much impact at such short maturities. Therefore the short-term VIX futures price can be approximated by:[5]

$$\frac{F_t^T}{100} = \left(CV_t + D\theta_t + (1-C-D)\bar{\theta}\right)^{\frac{1}{2}}. \tag{18}$$

3.4 Idealistic model

We term one model the idealistic model and another the realistic model, and examine the fit of each one. The idealistic model is presented here where the maturity of the SPVXSTR is assumed to be a constant 30 days and its futures position is rebalanced daily. However, the weighted average maturity of the SPVXSTR's underlying futures position actually fluctuates between 27 and 37 calendar days. The index is also rebalanced daily rather than continuously. The realistic model takes these factors into account by using the short-term VIX futures pricing formula, equation (18), in the SPVXSTR methodology with the fluctuating time to maturity and daily rebalancing. The realistic model will be presented in section 3.5.

We now derive a model for pricing the VXX and other short-term VIX futures ETPs assuming the underlying futures position rebalances continuously. To do this we first take the natural logarithm of equation (18):

$$\ln\left(\frac{F_t^T}{100}\right) = \frac{1}{2}\ln\left(CV_t + D\theta_t + (1-C-D)\bar{\theta}\right) \tag{19}$$

[5]Please note that the jump parameters have dropped out in our formula for the short-term VIX futures and the VXX models. For long-term VIX futures and the mid-term VIX futures ETPs the jump parameters could play an important role. Our setup with jump diffusion allows for these more general cases. The formulas for these will be presented in future research.

Using Ito's lemma we can model the change in log short-term VIX futures price (continuous return) as:

$$\begin{aligned} d\ln F_t^T &= \frac{\partial \ln F_t^T}{\partial V_t} dV_t + \frac{1}{2}\frac{\partial^2 \ln F_t^T}{\partial V_t^2}(dV_t)^2 + \frac{\partial \ln F_t^T}{\partial \theta_t} d\theta_t \\ &\quad + \frac{1}{2}\frac{\partial^2 \ln F_t^T}{\partial \theta_t^2}(d\theta_t)^2 + \frac{\partial^2 \ln F_t^T}{\partial V_t \theta_t} dV_t d\theta_t + \frac{\partial \ln F_t^T}{\partial t} dt. \end{aligned} \tag{20}$$

where the partial derivatives are given in appendix A.5.

The SPVXSTR index is rebalanced daily to maintain a VIX futures position with one-month maturity. The contract daily return (CDR_t) of the SPVXSTR's underlying VIX futures position can therefore be modeled as the log return of the short-term VIX futures price, equation (20), with a maturity of 30 days, given by:

$$\begin{aligned} CDR_t = d\ln F_t^T\Big|_{T=t+\tau_0} &= \frac{\partial \ln F_t^T}{\partial V_t}\Big|_{T=t+\tau_0} dV_t + \frac{1}{2}\frac{\partial^2 \ln F_t^T}{\partial V_t^2}\Big|_{T=t+\tau_0}(dV_t)^2 \\ &\quad + \frac{\partial \ln F_t^T}{\partial \theta_t}\Big|_{T=t+\tau_0} d\theta_t + \frac{1}{2}\frac{\partial^2 \ln F_t^T}{\partial \theta_t^2}\Big|_{T=t+\tau_0}(d\theta_t)^2 \\ &\quad + \frac{\partial^2 \ln F_t^T}{\partial V_t \theta_t}\Big|_{T=t+\tau_0} dV_t d\theta_t + \frac{\partial \ln F_t^T}{\partial t}\Big|_{T=t+\tau_0} dt. \end{aligned} \tag{21}$$

The SPVXSTR return, as shown in equation (1), consists of the CDR_t and a risk-free return earned on the notional of the underlying VIX futures position. The VXX return is equal to the SPVXSTR return less an investor fee, accounting for this and plugging in the partial derivatives leads to the VXX model in proposition 3 below.

Proposition 3. *We can model the log return of the VXX as follows:*

$$\begin{aligned} d\ln VXX_t &= CDR_t + (r_t - 0.0089)\ dt \\ &= d\ln F_t^{t+\tau_0} + RY_t dt + (r_t - 0.0089)\ dt, \end{aligned} \tag{22}$$

where

$$\begin{aligned}
d\ln F_t^{t+\tau_0} &= \frac{1}{2}\frac{C_{\tau_0}}{E}dV_t - \frac{1}{4}\frac{C_{\tau_0}^2}{E^2}(dV_t)^2 + \frac{1}{2}\frac{D_{\tau_0}}{E}d\theta_t - \frac{1}{4}\frac{D_{\tau_0}^2}{E^2}(d\theta_t)^2 - \frac{1}{2}\frac{C_{\tau_0}D_{\tau_0}}{E^2}dV_t d\theta_t, \\
RY_t &= \frac{1}{2}\frac{\kappa_V C_{\tau_0}V_t + G_{\tau_0}\theta_t - \big(\kappa_V C_{\tau_0} + G_{\tau_0}\big)\bar{\theta}}{E}, \\
C_{\tau_0} &= C\big|_{T=t+\tau_0}, \\
D_{\tau_0} &= D\big|_{T=t+\tau_0}, \\
E_{\tau_0} &= C_{\tau_0}V_t + D_{\tau_0}\theta_t + (1 - C_{\tau_0} - D_{\tau_0})\bar{\theta}, \\
G_{\tau_0} &= \frac{\kappa_V}{\kappa_V - \kappa_\theta}\left(A\kappa_\theta e^{-\kappa_\theta\tau_0} - \kappa_V C_{\tau_0}\right) + \kappa_\theta B e^{-\kappa_\theta\tau_0},
\end{aligned}$$

where r is the risk free return, the investor fee is 0.89% per annum, $F_t^{t+\tau_0}$ is the price of the constant 30-day-to-maturity VIX futures contract and RY_t is the roll yield of the VIX futures position underlying the SPVXSTR index.[6]

Our model for the VXX return presented in proposition 3, confirms the finding of Gehricke and Zhang (2018) that the underperformance of the VXX, relative to the constant 30-day-to-maturity VIX futures price and the closely related VIX index, is driven by the VXX's roll yield. The difference here is the expressions for the roll yield and the 30-day-to-maturity VIX futures price.

The short-term VIX futures ETFs will try to match the leveraged daily return of the SPVXSTR, i.e. two times leveraged SPVXSTR for the UVXY, by holding a daily rebalanced replicating portfolio of VIX futures, swaps and money market instruments, where as the ETNs just promise to pay the final indicative value at maturity or upon early redemption.[7] The effect of this is that both types of ETPs should track their

[6]Empirically the 30-day-to-maturity VIX futures price can be calculated by linear interpolation, as in Zhang, Shu and Brenner (2010) and Gehricke and Zhang (2018).

[7]An ETN is a non-securitized debt obligation, like a zero coupon bond, that has a indicative value based on the value of some underlying benchmark and contract specifications.

indicative values fairly well.[8] Therefore if we can model one of the short-term ETPs, we can model them all.

Any non-dynamic VIX futures ETP that tracks either total return or excess return of the short-term VIX futures index can be modeled by:

$$\begin{aligned} d\ln ETP_{i,t} &= L_i\, CDR_t + (r_t - fee_t)dt \\ &= L_i\big(d\ln F_t^{t+\tau_0} + RY_t dt\big) + (r_t - fee_t)dt \end{aligned} \tag{23}$$

where $ETP_{i,t}$ is the price of ETP i at time t, fee_i is its investor fee (expense ratio) and L_i is its leverage.

In figure 1 we can see that the short-term VIX futures ETPs make up about 80% of the total VIX futures ETP market capitalization and almost 100% of the total daily dollar trading volume, and have done so historically. Therefore our model is useful for the most important part of the VIX futures ETP market.[9]

3.5 Realistic model

Our model in the previous section and that of Gehricke and Zhang (2018) assume that the underlying futures position of the SPVXSTR is rebalanced continuously and that the weighted maturity resultant from the rebalancing is always 30 days. Gehricke and Zhang (2018) show that using continuous rebalancing to approximate the daily rebalancing only has a small impact on the time series of the SPVXSTR index. However, the authors also show that the weighted maturity of the index can fluctuate between 27 and 37 days. These two effects combined could lead to substantial errors

[8] Although, Gehricke and Zhang (2017) show that they do not track their indicative values perfectly, especially at the intraday level.

[9] To model the longer term VIX futures ETPs we would not recommend removing the convexity adjustment for the VIX futures formula, as in equation (18), but our methodology can be followed using the full approximation to arrive at a VIX futures ETP model

when comparing the calibrated model implied and market VXX price time series. To account for these effects we model the VXX discretely, as shown in proposition 4 below.

Proposition 4. *We can model the VXX discretely by using the model implied nearest and second nearest VIX futures time series, as follows:*

$$VXX_t = VXX_{t-1}\left(\frac{wF^{T_1}_{imp,t} + (1-w)F^{T_2}_{imp,t}}{wF^{T_1}_{imp,t-1} + (1-w)F^{T_2}_{imp,t-1}} + \frac{r_t - 0.0089}{365}\right), \tag{24}$$

where w is the weight in the nearest maturity VIX futures contract at time t, calculated using the SPVXSTR methodology. $F^{T_1}_{imp,t}$ and $F^{T_2}_{imp,t}$ are the model implied VIX futures prices using the short-term VIX futures formula, equation (18)*.*

Again, we can extend this model for any short-term (non-dynamic) VIX futures ETP by accounting for the difference in fees and leverage as follows:

$$ETP_{i,t} = ETP_{i,t-1}\left(1 + L_i\left[\frac{wF^{T_1}_{imp,t} + (1-w)F^{T_2}_{imp,t}}{wF^{T_1}_{imp,t-1} + (1-w)F^{T_2}_{imp,t-1}} - 1\right] + \frac{r_t - fee}{365}\right). \tag{25}$$

4 Calibration

4.1 Method

We estimate model parameters using the daily term structure of the VIX index. The VIX term structure data are obtained from the CBOE. Every day we use all option expirations available, each representing a different maturity VIX index, for a daily sample from November 24, 2010 to June 22, 2017.[10] We need to estimate the κ_V, κ_θ and $\bar{\theta}$ structural parameters as well as the daily V_t and θ_t parameters. To do this

[10]We call the different maturity implied volatility calculated using the VIX methodology, different maturity VIX indices throughout this paper.

we modify the efficient two-step iterative estimation of Christoffersen et al. (2009), also used by Luo and Zhang (2012) and Zhang, Zhen, Sun, and Zhao (2017), among others.

Since we have three structural parameters the two step procedure runs into some non-convergence issues, therefore we estimate the $\bar{\theta}$ parameter as the mean of the daily θ_t, estimated from the floating θ model.[11] Our calibration procedure is then as follows:

Step one: We set our $\bar{\theta}$ equal to the mean of the daily θ_t values from the floating θ model calibration.

Step two: We obtain the time series of $\{V_t, \theta_t\}$ for $t = 1, 2, ...T$ for a given parameter set $\{\kappa_V, \kappa_\theta, \bar{\theta}\}$, by solving T minimizations of the daily sums of squared errors as follows:

$$\{\hat{V}_t, \hat{\theta}_t\} = \min \sum_{i=1}^{n_t} \left(VIX_{t,\tau_i}^{KTM} - VIX_{t,\tau_i}^{MI}\right)^2, \qquad t = 1, 2, ...T, \tag{26}$$

where VIX_{t,τ_i}^{MKT} is the market value of the τ_i maturity VIX index and VIX_{t,τ_i}^{MI} is the implied τ_i maturity VIX index value using our model, equation (13), on day t. Here, n_t is the number of maturities on day t and T is the total number of days in the sample.

Step three: We estimate the parameter set $\{\kappa_V, \kappa_\theta\}$, using the daily $\hat{V}_t, \hat{\theta}_t$, estimated from step two, by minimizing the total SSE as:

$$\{\hat{\kappa_V}, \hat{\kappa_\theta}\} = \min \sum_{t=1}^{T} \sum_{i=1}^{n_t} \left(VIX_{t,\tau_i}^{MKT} - VIX_{t,\tau_i}^{MI}\right)^2. \tag{27}$$

Steps two and three are then repeated until there is no further significant decrease in the error.

[11] The floating θ model dynamics and VIX term structure formula are outlined in appendix A.6. This model is calibrated using only steps two and three of the calibration procedure.

When calibrating any of the nested models (Heston (1993), floating θ or $\kappa = \kappa_V = \kappa_\theta$ models) step one is not necessary. When calibrating the Heston (1993) model we estimate κ and $\bar{\theta}$ in step two and only estimate the daily V_t in step three. We calibrate κ in step two and the daily V_t and θ_t in step three for the floating θ model. For the $\kappa = \kappa_V = \kappa_\theta$ nested model κ and $\bar{\theta}$ are estimated in step two and the daily V_t and θ_t are estimated in step three.

4.2 Results

We first verify our calibration procedure by estimating the parameters of the Heston (1993) and floating θ models over a sub sample, which is equivalent to the sample used by Zhang, Zhen, Sun, and Zhao (2017). This allows us to verify our calibration method. Table 2 shows the results of their calibration, panel A, and ours, panel B. From the table we can see that our estimation is virtually identical to theirs. The slight difference could be explained by a difference in software or optimization algorithms used.

The main calibration results are presented in table 3. In panel A we present the results of calibrating the floating θ, $\kappa = \kappa_V = \kappa_\theta$ and full models using the full VIX term structure. The results using only the first two and first three maturities of the VIX term structure are presented in panels B and C, respectively. We can see in the table that the full model is able to fit the VIX term structure the best for any of the samples, as it has the lowest RMSE.

Table 3 also shows that when we are using either of the short VIX term-structure samples, κ_θ, for the full model, is essentially zero. However, when using the full VIX term structure κ_θ is 0.2406. From this evidence we can conclude that the mean reversion of θ_t is less important when modeling only the short VIX term structure, but is more important when we want to model the full VIX term structure.

Interestingly, no matter which model is used the mean daily V_t and θ_t are quite close, apart from the mean θ_t for the $\kappa = \kappa_V = \kappa_\theta$ using the full VIX term structure. In figure 2 we present the daily time series of V_t and θ_t for the full model using the two short and the full VIX term structure. We can see in the charts that most of the time θ_t is above V_t; this is because the VIX term structure is usually in contango (upward sloping). However, there are days in the sample where V_t spikes and is above θ_t; these are the days where the VIX term structure is in backwardation (downward sloping). On the days where the VIX term structure is in backwardation there was likely some event/news causing short-term implied volatility to increase drastically, while the longer-term implied volatility does not change as much. When volatility spikes, the market seems to expect it to decrease again at some point in the future.

The Heston (1993) model used by Gehricke and Zhang (2018) to model the VXX, fits the VIX term structure the worst compared with the other models. This is because the long-term mean level of the instantaneous variance, θ, is constant making it less flexible in modeling anomalies, such as days where the VIX term structure is in backwardation. Our model fits the VIX term structure well whether it is in contango or backwardation because of the flexibility of θ_t being time varying.

Figure 3 shows the daily mean squared error (MSE) of the full model calibration using the full and two short VIX term structure samples. We can see that the MSE for the shorter VIX term structures is usually lower, but especially so when volatility (V_t) goes very high. This is likely due to the model only needing to fit two or three-points of the VIX term structure by optimizing the two daily parameters. However, even with this flexibility the model did not fit the VIX spike, which was caused by the credit rating downgrade of the U.S. government in August 2011, very well. The calibration using the shorter VIX term structure samples is also better on and around this event. We can see in the figure that on most days the model is able to fit the VIX term structure well. It can be expected to have larger mean errors when trying to fit more points of the term structure using the same model.

5 Model fit

In this section we focus on how well our VIX term structure calibrated model can fit the VXX time series. We compare the fit of the calibrated model using either the continuous or discrete VXX model and calibrating to either the full, two-point or three-point short VIX term structure samples. We find that using the realistic model and calibrating to the three-point VIX term structure is best for modeling the VXX. Finally, we show that our model is also good for fitting the time series of other short-term VIX futures ETPs.

5.1 VXX model fit

We now estimate the model implied VXX time series using the VIX term structure calibrated parameters $\{\kappa_V, \kappa_\theta, \bar{\theta}, V_t, \theta_t\}$, which are estimated as reported in section 4.2. To measure the model fit we estimate the RMSE in the levels and returns between the implied and market VXX time series. We also examine the fit of the first two moments of the returns of the VXX, the correlation between model implied and market VXX returns and by graphing the implied and market time series.

From table 4 and figure 4 we can see that the realistic model outperforms the idealistic model in fitting the VXX time series. The table shows that the VXX level RMSE for the realistic model is 416.93, 150.68 and 99.11 compared with 739.69, 722.88 and 664.65 for the idealistic model using the full, two-point short and three-point short VIX term structure samples, respectively. We can observe a similar pattern looking at the VXX return RMSE, where the realistic models is 2.42%, 5.16% and 2.72% compared with 2.52%, 5.77% and 2.72% for the idealistic model, using the full, two-point short and three-point short VIX term structure samples, respectively. We also show that the mean VXX return implied by the model is a lot closer to the market value when using the realistic model, whether we use the full or short VIX term structure calibration. The standard deviation of the implied VXX returns is barely different between the two models for the full VIX term structure calibration, but when we use the shorter VIX term structure calibrations the realistic model's value is much closer. Turning to figure 4 we can see that clearly the realistic model fits the VXX price time series far better than the idealistic model, whether we use the shorter or full VIX term structure for calibrations.

Both table 4 and figure 4 also show that the three-point short VIX term structure calibration is superior for fitting the VXX time series. We can see in figure 4 that the implied VXX fits the market VXX best when using the realistic model calibrated to the three-point VIX term structure. Turning to the table we can see that the level RMSE for the realistic model is much lower when we use the three-point short (99.11) rather than the two-point short (150.68) or full (416.93) VIX term structure calibration. The correlation of market and model implied returns is also higher for the realistic model, when calibrating to the shorter term structures, and almost identical when calibrating to the full VIX term structure.

The return RMSE is lowest for the full VIX term structure calibration (2.42%), but is almost as good for the three-point VIX term structure calibration (2.72%), while the two-point VIX term structure calibration (5.16%) is much worse. Turning to the return correlations we can see that the three-point calibration outperforms the other two. Calibrating to the first three-points of the VIX term structure seems best, which can also be observed in figure 4.

Overall, the best fit for our model to the VXX time series is achieved by using the realistic VXX model and calibrating the parameters to the three-point short VIX term structure.

5.2 Short-term VIX futures ETPs model fit

We now briefly analyze the ability of our model to fit other short-term VIX futures ETPs. We examine the fit of the realistic model using either the shorter or full VIX term structure calibrations. The other short-term VIX futures ETPs we try to model are the five most liquid, after the VXX, namely the XIV, SVXY, UVXY, TVIX and VIXY (in order of market size).

Figures 5, 6 and 7 show the model fit to the other ETPs using the two-point, three-point and full VIX term structure calibration, respectively. From the figures we can see that the two-point VIX term structure calibration works fairly well for fitting the long-exposure ETPs, namely UVXY, TVIX and VIXY. However, neither the two-point nor full VIX term structure calibration allows our realistic model to fit the short-exposure ETPs, namely XIV and SVXY, very well. When the model is calibrated using the three-point VIX term structure it fits the short-term ETPs far better, as can see in figure 7.

The more volatile the underlying index the less likely a leveraged ETF is to achieve its target leverage that day, and it will often be over or under exposed. So, if the SVXY fails to achieve its target leverage of negative one it is to be expected that the model can not fit it perfectly. Surprisingly, the model seems to fit the UVXY, double-leverage, ETF quite well optically. The model fits the leveraged ETNs, XIV and TVIX, well as they do not have this replication problem. ETNs should track their target exposure much closer, since they do not have to replicate a leveraged version of the underlying VIX futures position of the SPVXSTR. Gehricke and Zhang (2017) show that the VIX futures ETNs also do not track their indicative values perfectly and are often priced inconsistently to each other, so some error in modeling these is to be expected.

Overall we can say that our realistic VIX futures ETP model is useful in modeling any short-term VIX futures ETPs, which make up most of the market capitalization and trading of all VIX futures ETPs.

6 Conclusions

In this paper we have created a new model for the VXX, which accounts for the relationship between the S&P 500, VIX index, VIX futures and the VXX. Our model builds on the work of Gehricke and Zhang (2018) by including jumps in the instantaneous variance (V_t) and making the long-run mean level of V_t a stochastic mean-reverting process. We derive simple analytical formula for the VIX term structure, VIX futures prices, short-term (less than 60 days) VIX futures prices, the VXX and any other short-term VIX futures ETPs (which make up most of the market size and activity). Our derived VXX model confirms the finding of Gehricke and Zhang (2018) that, theoretically, the roll yield is the main driver of the under-performance of the VXX relative to the VIX index.

We calibrate our theoretical model to the full VIX term structure as well as the first two and first three-points. We find that our model fits the VIX term structure better than the nested models, no matter which calibration sample we use. Namely, the nested models are the model of Gehricke and Zhang (2018), its floating θ equivalent and our model with $\kappa_V = \kappa_\theta = \kappa$. We find that the mean reversion feature of the models contributes very little when fitting the shorter VIX term structures, but is more important when fitting the full VIX term structure.

We provide a realistic and idealistic model for the VXX and show that the realistic model outperforms in fitting the VXX time series, no matter whether we calibrate to the shorter or full VIX term structure. We find that our model performs best for fitting the levels and returns of the VXX when calibrating only to the first three-points of the VIX term structure. This is likely because longer term market expectations of volatility are irrelevant to the short-term ETP prices and using the first two-points misses some relevant market information.

Lastly, we show that our model fits other short-term VIX futures ETP time series well, but fails to fit the SVXY ETF time series. This may be explained by the difficulty for ETFs to achieve the target leverage over a holding period of more than one day and/or by market frictions in the VIX futures ETP market, which has become a topic of interest for other researchers (e.g. Fernandez-Perez et al. 2018)

Our model could be extended to the mid-term VIX futures ETPs, in which case we recommend not removing the convexity adjustment in the VIX futures price formula. Our model for the VXX could be used to price options written on the VXX and other short-term VIX futures ETPs. It may also be useful for short-term forecasts of VXX and other short-term VIX futures ETP prices.

Appendix

A.1 Central moments of θ_s

In this section we derive the first and second central moments of θ_t. Let us start with the first moment, to do this we define a function $f(\theta_t) = e^{\kappa_\theta t}\theta_t$, then we get:

$$\begin{aligned} df(\theta_t) = d(e^{\kappa_\theta t}\theta_t) &= \kappa_\theta e^{\kappa_\theta t}\theta_t dt + \kappa_\theta e^{\kappa_\theta t}(\bar{\theta} - \theta_t)dt + e^{\kappa_\theta t}\sigma_\theta\sqrt{\theta_t}dB_{3,t} \\ &= \kappa_\theta\bar{\theta}e^{\kappa_\theta t}dt + e^{\kappa_\theta t}\sigma_\theta\sqrt{\theta_t}dB_{3,t}. \end{aligned} \tag{28}$$

Taking the integral of both sides of equation (28) from t to s we get:

$$\begin{aligned} e^{\kappa_\theta s}\theta_s - e^{\kappa_\theta t}\theta_t &= \kappa_\theta\bar{\theta}\int_t^s e^{\kappa_\theta u}du + \sigma_\theta\int_t^s e^{\kappa_\theta u}\sqrt{\theta_u}dB_{3,u} \\ e^{\kappa_\theta s}\theta_s - e^{\kappa_\theta t}\theta_t &= \bar{\theta}(e^{\kappa_\theta s} - e^{\kappa_\theta t}) + \sigma_\theta\int_t^s e^{\kappa_\theta u}\sqrt{\theta_u}dB_{3,u} \\ \theta_s &= \bar{\theta} + (\theta_t - \bar{\theta})e^{-\kappa_\theta(s-t)} + \sigma_\theta\int_t^s e^{-\kappa_\theta(s-u)}\sqrt{\theta_u}dB_{3,u}. \end{aligned} \tag{29}$$

Then taking the expectation of equation (29), conditional on the information at t, we get the first central moment of θ_s as:

$$E_t^Q[\theta_s] = \bar{\theta} + (\theta_t - \bar{\theta})e^{-\kappa_\theta(s-t)}. \tag{30}$$

Using the definition of the second central moment of θ_s and plugging in equations (29) and (30) we get:

$$\begin{aligned} E_t^Q[(\theta_s - E_t^Q[\theta_s])^2] &= E_t^Q\Bigg[\bigg(\bar{\theta} + (\theta_t - \bar{\theta})e^{-\kappa_\theta(s-t)} + \sigma_\theta\int_t^s e^{-\kappa_\theta(s-u)}\sqrt{\theta_u}dB_{3,u} \\ &\qquad -\bar{\theta} - (\theta_t - \bar{\theta})e^{-\kappa_\theta(s-t)}\bigg)^2\Bigg] \\ &= E_t^Q\Bigg[\bigg(\sigma_\theta\int_t^s e^{-\kappa_\theta(s-u)}\sqrt{\theta_u}dB_{3,u}\bigg)^2\Bigg], \end{aligned} \tag{31}$$

then using Ito's isometry and bringing the expectation inside the integral we get:

$$E_t^Q[(\theta_s - E_t^Q[\theta_s])^2] \;=\; \sigma_\theta^2 \int_t^s e^{-2\kappa_\theta(s-u)} E_t^Q[\theta_u]du \tag{32}$$

and substituting in $E_t^Q[\theta_u]$, from equation (6) with $s = u$, we get:

$$\begin{aligned} E_t^Q[(\theta_s - E_t^Q[\theta_s])^2] &= \sigma_\theta^2 \left(\int_t^s e^{-2\kappa_\theta(s-u)}[\bar{\theta} + (\theta_t - \bar{\theta})e^{-\kappa_\theta(u-t)}]du \right) \\ &= \sigma_\theta^2 \left(\int_t^s e^{-2\kappa_\theta(s-u)}\bar{\theta}du + \int_t^s e^{\kappa_\theta u - 2\kappa_\theta s + \kappa_\theta t}\theta_t du - \int_t^s e^{\kappa_\theta u - 2\kappa_\theta s + \kappa_\theta t}\bar{\theta}du \right) \\ &= \frac{\sigma_\theta^2 \theta_t}{\kappa_\theta}(e^{-\kappa_\theta(s-t)} - e^{-2\kappa_\theta(s-t)}) + \frac{\sigma_\theta^2 \bar{\theta}}{2\kappa_\theta}(1 - 2e^{-\kappa_\theta(s-t)} + e^{-2\kappa_\theta(s-t)}) \end{aligned} \tag{33}$$

A.2 Central moments of V_s

To find the risk-neutral expectation of instantaneous variance ($E_t^Q[V_S]$) we need to first derive a function for the future instantaneous variance V_s. The dynamics of V_t can be rewritten as:

$$dV_t \;=\; \kappa_V(\theta_t - V_t)dt + dM_t, \tag{34}$$

where

$$dM_t = \sigma_V \sqrt{V_t} dB_{2,t} + y dN_t - \lambda E^Q[y]dt, \tag{35}$$

which is a martingale in the risk-neutral measure.

Let $f(V_t) = e^{\kappa_V t} V_t$ then we get:

$$\begin{aligned} df(V_t) = d(e^{\kappa_V t}V_t) &= \kappa_V e^{\kappa_V t} V_t dt + e^{\kappa_V t}\kappa_V(\theta_t - V_t)dt + e^{\kappa_V t}dM_t \\ &= e^{\kappa_V t}\kappa_V \theta_t dt + e^{\kappa_V t} dM_t. \end{aligned} \tag{36}$$

Integrating both sides from t to s gives and solving for s we get:

$$\begin{aligned} e^{\kappa_V s}V_s &- e^{\kappa_V t}V_t = \int_t^s e^{\kappa_V u}\kappa_V \theta_u du + \int_t^s e^{\kappa_V u} dM_u \\ V_s &= V_t e^{-\kappa_V (s-t)} + \int_t^s e^{-\kappa_V (s-u)}\kappa_V \theta_u du + \int_t^s e^{-\kappa_V (s-u)} dM_u. \end{aligned} \tag{37}$$

We can then take the conditional risk-neutral expectation of V_s, which yields:

$$E_t^Q[V_s] = V_t e^{-\kappa_V (s-t)} + \int_t^s e^{-\kappa_V (s-u)}\kappa_V E_t^Q[\theta_u] du \tag{38}$$

We then substitute $E_t^Q[\theta_u]$, from equation (6) with $s = u$, into the expected future instantaneous variance, equation (38), which results in:

$$\begin{aligned} E_t^Q[V_s] &= V_t e^{-\kappa_V (s-t)} + \int_t^s \kappa_V e^{-\kappa (s-u)}[\bar{\theta} + (\theta_t - \bar{\theta})e^{-\kappa_\theta (u-t)}] du \\ &= V_t e^{-\kappa_V (s-t)} + \bar{\theta}(1 - e^{-\kappa_V (s-t)}) + \frac{\kappa_V}{\kappa_V - \kappa_\theta}(\theta_t - \bar{\theta})(e^{-\kappa_\theta (s-t)} - e^{-\kappa_V (s-t)}) \\ &= e^{-\kappa_V (s-t)} V_t + \frac{\kappa_V}{\kappa_V - \kappa_\theta}\big(e^{-\kappa_\theta (s-t)} - e^{-\kappa_V (s-t)}\big)\theta_t \\ &\quad + \left(1 - e^{-\kappa_V (s-t)} - \frac{\kappa_V}{\kappa_V - \kappa_\theta}\big(e^{-\kappa_\theta (s-t)} - e^{-\kappa_V (s-t)}\big)\right)\bar{\theta} \end{aligned} \tag{39}$$

We can get the second central moment of V_s by ... , resulting in:

$$
\begin{aligned}
& E_t^Q[(V_s - E_t^Q[V_s])^2] \\
= \; & E_t\left[\left(\int_t^s e^{-\kappa_V(s-y)}\kappa_V(\theta_y - E_t[\theta_y])dy + \int_t^s e^{-\kappa_V(s-u)}dM_u\right)^2\right] \\
= \; & E_t\left[\left(\kappa_V\sigma_\theta\int_t^s e^{-\kappa_V(s-y)}\int_t^y e^{-\kappa_\theta(y-u)}\sqrt{\theta_u}dB_{3,u}dy + \int_t^s e^{-\kappa_V(s-u)}dM_u\right)^2\right] \\
= \; & E_t\left[\left(\kappa_V\sigma_\theta\int_t^s \int_u^s e^{-\kappa_V(s-y)}e^{-\kappa_\theta(y-u)}dy\sqrt{\theta_u}dB_{3,u} + \int_t^s e^{-\kappa_V(s-u)}dM_u\right)^2\right] \\
= \; & E_t\left[\left(\frac{\kappa_V\sigma_\theta}{\kappa_V - \kappa_\theta}\int_t^s (e^{-\kappa_\theta(s-u)} - e^{-\kappa_V(s-u)})\sqrt{\theta_u}dB_{3,u} + \int_t^s e^{-\kappa_V(s-u)}dM_u\right)^2\right] \\
= \; & E_t\left[\left(\frac{\kappa_V\sigma_\theta}{(\kappa_V - \kappa_\theta)}\int_t^s (e^{-\kappa_\theta(s-u)} - e^{-\kappa_V(s-u)})\sqrt{\theta_u}dB_{3,u}\right)^2\right] \\
& + E_t\left[\frac{\kappa_V\sigma_\theta}{\kappa_V - \kappa_\theta}\int_t^s (e^{-\kappa_\theta(s-u)} - e^{-\kappa_V(s-u)})\sqrt{\theta_u}dB_{3,u} \times \int_t^s e^{-\kappa_V(s-u)}dM_u\right] \\
& + E_t\left[\left(\int_t^s e^{-\kappa_V(s-u)}dM_u\right)^2\right] \\
= \; & X + Y,
\end{aligned}
\tag{40}
$$

where

$$
\begin{aligned}
X &= E_t\left[\left(\frac{\kappa_V\sigma_\theta}{(\kappa_V-\kappa_\theta)}\int_t^s (e^{-\kappa_\theta(s-u)}-e^{-\kappa_V(s-u)})\sqrt{\theta_u}dB_{3,u}\right)^2\right] \\
&= \frac{\kappa_V^2\sigma_\theta^2}{(\kappa_V-\kappa_\theta)^2}\int_t^s (e^{-\kappa_\theta(s-u)}-e^{-\kappa_V(s-u)})^2 E_t[\theta_u]du \\
&= \frac{\kappa_V^2\sigma_\theta^2}{(\kappa_V-\kappa_\theta)^2}\int_t^s (e^{-\kappa_\theta(s-u)}-e^{-\kappa_V(s-u)})^2 (\bar{\theta}+(\theta_t-\bar{\theta})e^{-\kappa_\theta(u-t)})du \\
&= \frac{\kappa_V^2\sigma_\theta^2}{(\kappa_V-\kappa_\theta)^2}\left[\int_t^s e^{-2\kappa_\theta(s-u)}(\bar{\theta}+(\theta_t-\bar{\theta})e^{-\kappa_\theta(u-t)})du\right. \\
&\qquad -2\int_t^s e^{-\kappa_\theta(s-u)}e^{-\kappa_V(s-u)}(\bar{\theta}+(\theta_t-\bar{\theta})e^{-\kappa_\theta(u-t)})du \\
&\qquad \left.+\int_t^s e^{-2\kappa_V(s-u)}(\bar{\theta}+(\theta_t-\bar{\theta})e^{-\kappa_\theta(u-t)})du\right] \\
&= \frac{\kappa_V^2\sigma_\theta^2}{(\kappa_V-\kappa_\theta)^2}\left[\left(\frac{e^{-\kappa_\theta(s-t)}-e^{-2\kappa_\theta(s-t)}}{\kappa_\theta}-\frac{2e^{-\kappa_\theta(s-t)}(1-e^{-\kappa_V(s-t)})}{\kappa_V}\right.\right. \\
&\qquad \left.+\frac{e^{-\kappa_\theta(s-t)}-e^{-\kappa_V(s-t)}}{2\kappa_V-\kappa_\theta}\right)\theta_t+\left(\frac{(1-e^{-\kappa_\theta(s-t)})^2}{2\kappa_\theta}\right. \\
&\qquad -\frac{2(1-e^{-(\kappa_\theta-\kappa_V)(s-t)})}{\kappa_\theta+\kappa_V}+\frac{2(e^{-\kappa_\theta(s-t)}-e^{-(\kappa_\theta-\kappa_V)(s-t)})}{\kappa_V} \\
&\qquad \left.\left.+\frac{1-e^{-2\kappa_V(s-t)}}{2\kappa_V}-\frac{e^{-\kappa_\theta(s-t)}e^{-2\kappa_V(s-t)}}{2\kappa_V-\kappa_\theta}\right)\bar{\theta}\right] \qquad (41)
\end{aligned}
$$

and

$$\begin{aligned}
Y &= E_t\left[\left(\int_t^s e^{-\kappa_V(s-u)}dM_u\right)^2\right] \\
&= \int_t^s e^{-2\kappa_V(s-u)}\sigma_V^2 E_t^Q[V_u](dB_{2,u})^2 + \int_t^s e^{-2\kappa_V(s-u)}E^Q[y^2]\lambda du \\
&= \int_t^s e^{-2\kappa_V(s-u)}\sigma_V^2 e^{-2\kappa_V(u-t)}V_t du \\
&\quad + \int_t^s e^{-2\kappa_V(s-u)}\sigma_V^2 \frac{\kappa_V\theta_t}{\kappa_V-\kappa_\theta}\left(e^{-\kappa_\theta(u-t)} - e^{-\kappa_V(u-t)}\right)du \\
&\quad + \int_t^s e^{-2\kappa_V(s-u)}\sigma_V^2\bar{\theta}\left(1 - e^{-\kappa_V(u-t)} - \frac{\kappa_V}{\kappa_V-\kappa_\theta}(e^{-\kappa_\theta(u-t)} - e^{-\kappa_V(u-t)})\right)du \\
&\quad + \frac{\lambda}{2\kappa_V}\left(1-e^{-2\kappa_V(s-t)}\right)E_t[y^2] \\
&= \frac{\sigma_V^2}{\kappa_V}\left(e^{-\kappa_V(s-t)} - e^{-2\kappa_V(s-t)}\right)V_t \\
&\quad + \frac{\kappa_V\sigma_V^2}{\kappa_V-\kappa_\theta}\left(\frac{e^{-\kappa_\theta(s-t)} - e^{-\kappa_V(s-t)}}{2\kappa_V-\kappa_\theta} - \frac{e^{-\kappa_V(s-t)} - e^{-2\kappa_V(s-t)}}{\kappa_V}\right)\theta_t \\
&\quad + \sigma_V^2\left(\frac{(1-e^{-\kappa_V(s-t)})^2}{2\kappa_V} - \frac{\kappa_V}{\kappa_V-\kappa_\theta}\left[\frac{e^{-\kappa_\theta(s-t)} - e^{-\kappa_V(s-t)}}{2\kappa_V-\kappa_\theta} - \frac{e^{-\kappa_V(s-t)} - e^{-2\kappa_V(s-t)}}{\kappa_V}\right]\right)\bar{\theta} \\
&\quad + \frac{\lambda}{2\kappa_V}\left(1-e^{-2\kappa_V(s-t)}\right)E_t[y^2] \qquad (42)
\end{aligned}$$

A.3 VIX index proof

The VIX squared index is equal to the risk-neutral conditional expectation of the variance over the next 30 days, using this and lemma 1 we get:

$$\begin{aligned}
\left(\frac{VIX_t}{100}\right)^2 &= \frac{1}{\tau_0}\int_t^{t+\tau_0} E_t^Q[V_s]ds \\
&= \frac{1}{\tau_0}\Bigg[V_t\int_t^{t+\tau_0} e^{-\kappa_V(u-t)}du \\
&+ \frac{\kappa_V}{\kappa_V-\kappa_\theta}\theta_t\Bigg(\int_t^{t+\tau_0} e^{-\kappa_\theta(u-t)}du - \int_t^{t+\tau_0} e^{-\kappa_V(u-t)}du\Bigg) \\
&+ \bar{\theta}\Bigg(\int_t^{t+\tau_0} du - \int_t^{t+\tau_0} e^{-\kappa_V(u-t)}du \\
&- \frac{\kappa_V}{\kappa_V-\kappa_\theta}\Bigg(\int_t^{t+\tau_0} e^{-\kappa_\theta(u-t)}du - \int_t^{t+\tau_0} e^{-\kappa_V(u-t)}du\Bigg)\Bigg)\Bigg] \\
&= A\,V_t + B\theta_t + (1-A-B)\bar{\theta},
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
A &= \frac{1-e^{-\kappa_V\tau_0}}{\kappa_V\tau_0}, \\
B &= \frac{\kappa_V(1-e^{-\kappa_\theta\tau_0})}{\kappa_\theta\tau_0(\kappa_V-\kappa_\theta)} - \frac{(1-e^{-\kappa_V\tau_0})}{\tau_0(\kappa_V-\kappa_\theta)}.
\end{aligned}$$

Then taking the square root gives us the VIX formula presented in proposition 1.

A.4 VIX futures approximate formula

We expand the the square root form equation (15) (VIX formula) using the two variable Taylor expansion near the points $E_t^Q[V_T]$ and $E_t^Q[\theta_T]$, which results in:

$$\begin{aligned}
f(V_T,\theta_T) &= a_0 + a_1(V_T - E_t^Q[V_T]) + a_2(\theta_T - E_t^Q[\theta_T]) \\
&+\frac{1}{2}a_3(V_T - E_t^Q[V_T])^2 + \frac{1}{2}a_4(\theta_T - E_t^Q[\theta_T])^2 \\
&+a_5(V_T - E_t^Q[V_T])(\theta_T - E_t^Q[\theta_T]),
\end{aligned} \tag{44}$$

where

$$
\begin{aligned}
a_0 &= f(V,\theta)\Big|_{V=E_t^Q[V_T],\ \theta=E_t^Q[\theta_T]}, \qquad a_1 = \frac{\partial f}{\partial V}\Big|_{V=E_t^Q[V_T],\ \theta=E_t^Q[\theta_T]},\\
a_2 &= \frac{\partial f}{\partial \theta}\Big|_{V=E_t^Q[V_T],\ \theta=E_t^Q[\theta_T]}, \qquad a_3 = \frac{\partial^2 f}{\partial V^2}\Big|_{V=E_t^Q[V_T],\ \theta=E_t^Q[\theta_T]},\\
a_4 &= \frac{\partial^2 f}{\partial \theta^2}\Big|_{V=E_t^Q[V_T],\ \theta=E_t^Q[\theta_T]}, \qquad a_5 = \frac{\partial^2 f}{\partial V \partial \theta}\Big|_{V=E_t^Q[V_T],\ \theta=E_t^Q[\theta_T]},\\
f(V,\theta) &= \Big(A\ V + B\theta + (1-A-B)\bar{\theta}\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Plugging the partial differentials into equation (44) we get:

$$
\begin{aligned}
f(V_T,\theta_T) \;=\;& \left(E_t^Q\left[\left(\frac{VIX}{100}\right)^2\right]\right)^{\frac{1}{2}} + \frac{1}{2}\left(E_t^Q\left[\left(\frac{VIX}{100}\right)^2\right]\right)^{-\frac{1}{2}} A(V_T - E_t^Q[V_T])\\
&+\frac{1}{2}\left(E_t^Q\left[\left(\frac{VIX}{100}\right)^2\right]\right)^{-\frac{1}{2}} B(\theta_T - E_t^Q[\theta_T])\\
&-\frac{1}{8}\left(E_t^Q\left[\left(\frac{VIX}{100}\right)^2\right]\right)^{-\frac{3}{2}} A^2(V_T - E_t^Q[V_T])^2\\
&-\frac{1}{8}\left(E_t^Q\left[\left(\frac{VIX}{100}\right)^2\right]\right)^{-\frac{3}{2}} B^2(\theta_T - E_t^Q[\theta_T])^2\\
&-\frac{1}{8}\left(E_t^Q\left[\left(\frac{VIX}{100}\right)^2\right]\right)^{-\frac{3}{2}} AB(V_T - E_t^Q[V_T])(\theta_T - E_t^Q[\theta_T]),
\end{aligned}
\tag{45}
$$

We then take the expectation of $f(V_T,\theta_T)$ and substitute in the second central moments of V_T and θ_T, from lemma 1 where $s = T$, to get:

$$
\begin{aligned}
E_T^Q[f(V_T,\theta_T)] \;=\; & \left(E_t^Q\left[\left(\frac{VIX}{100}\right)^2\right]\right)^{\frac{1}{2}} \\
& -\frac{1}{8}\left(E_t^Q\left[\left(\frac{VIX}{100}\right)^2\right]\right)^{-\frac{3}{2}} A^2\left[X\big|_{s=T} + Y\big|_{s=T}\right] \\
& -\frac{1}{8}\left(E_t^Q\left[\left(\frac{VIX}{100}\right)^2\right]\right)^{-\frac{3}{2}} B^2 \\
& \times\left[\frac{\sigma_\theta^2\theta_t}{\kappa_\theta}\left(e^{-\kappa_\theta(T-t)} - e^{-2\kappa_\theta(T-t)}\right) + \frac{\sigma_\theta^2\bar{\theta}}{2\kappa_\theta}\left(1 - 2e^{-\kappa_\theta(T-t)} + e^{-2\kappa_\theta(T-t)}\right)\right],
\end{aligned}
\tag{46}
$$

then substituting in $E_t^Q\left[\left(\frac{VIX}{100}\right)^2\right]$ from lemma 16 gives us proposition 2.

A.5 Partial derivatives for change in log short-term VIX futures

The partial derivative of the short-term VIX futures price with respect to t in equation (20) is derived as:

$$
\frac{\partial \ln F_t^T}{\partial t} = \frac{1}{2}\frac{\frac{\partial C}{\partial t}V_t + \frac{\partial D}{\partial t}\theta_t - \left(\frac{\partial C}{\partial t} + \frac{\partial D}{\partial t}\right)\bar{\theta}}{CV_t + D\theta_t + (1 - C - D)\bar{\theta}}, \tag{47}
$$

where

$$
\frac{\partial C}{\partial t} = \kappa_V C \tag{48}
$$

$$
\frac{\partial D}{\partial t} = \frac{\kappa_V}{\kappa_V - \kappa_\theta}\left(A\kappa_\theta e^{-\kappa_\theta(T-t)} - \kappa_V C\right) + \kappa_\theta B e^{-\kappa_\theta(T-t)}. \tag{49}
$$

The other partial derivatives in equation (20) are derived as:

$$\frac{\partial \ln F_t^T}{\partial V_t} = \frac{1}{2}\frac{C}{CV_t + D\theta_t + (1-C-D)\bar{\bar{\theta}}} \tag{50}$$

$$\frac{\partial^2 \ln F_t^T}{\partial V_t^2} = -\frac{1}{2}\frac{C^2}{\left(CV_t + D\theta_t + (1-C-D)\bar{\theta}\right)^2} \tag{51}$$

$$\frac{\partial \ln F_t^T}{\partial \theta_t} = \frac{1}{2}\frac{D}{CV_t + D\theta_t + (1-C-D)\bar{\bar{\theta}}} \tag{52}$$

$$\frac{\partial^2 \ln F_t^T}{\partial \theta_t^2} = -\frac{1}{2}\frac{D^2}{\left(CV_t + D\theta_t + (1-C-D)\bar{\theta}\right)^2} \tag{53}$$

$$\frac{\partial^2 \ln F_t^T}{\partial V_t \theta_t} = -\frac{1}{2}\frac{CD}{\left(CV_t + D\theta_t + (1-C-D)\bar{\theta}\right)^2} \tag{54}$$

A.6 Floating θ model dynamics and VIX formula

Under the floating θ model the risk-neutral dynamics are given by:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dB_{1,t}, \tag{55}$$

$$dV_t = \kappa(\theta_t - V_t)dt + \sigma_V \sqrt{V_t} dB_{2,t}, \tag{56}$$

$$d\theta_t = dB_{3,t}, \tag{57}$$

which results in the following VIX term structure formula:

$$\frac{VIX_t}{100} = \sqrt{E_t^Q\left[\frac{1}{\tau_i}\int_t^{t+\tau_i} V_s ds\right]} = \sqrt{(1-G)\theta_t + GV_t}, \tag{58}$$

where:

$$G = \frac{(1 - e^{-\kappa\tau_i})}{\kappa\tau_i} \tag{59}$$

where $\tau_i = T_i/365$ and T_i is the days to maturity of any maturity VIX index in the term structure.

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Table 1: **Summary statistics of the daily returns for the SPX, VIX, VXX and XIV.** This table is an extended and updated version of the table found in Gehricke and Zhang (2018), with a longer sample size and a short exposure VIX futures ETP, the XIV, included. The table shows the summary statistics and correlations of the SPX, VIX, VXX and XIV returns from the 30 January 2009 (XIV was not launched until 29 November 2010) to the 24 April 2017. Here, r_D represents estimates using discrete daily returns and r_C represents estimates using continuously compounded daily returns. The annualised standard deviation is calculated by multiplying the standard deviation by $\sqrt{252}$. The Holding Period Return (HPR) is the return from the first day to the last day of the sample. The Compound Annual Growth Rate (CAGR) is the constant yearly growth rate that would lead to the corresponding HPR, it is calculated by $CAGR = (HPR+1)^{\frac{1}{T}} - 1$, where T is the length of the sample in years.

	SPX		VIX		VXX		XIV	
	r_D	r_C	r_D	r_C	r_D	r_C	r_D	r_C
Mean	0.06%	0.05%	0.20%	−0.07%	−0.29%	−0.36%	0.21%	0.13%
σ	1.06%	1.06%	7.49%	7.25%	3.89%	3.85%	4.02%	4.11%
Annualized σ	16.82%	16.83%	118.96%	115.05%	61.80%	61.14%	63.79%	65.18%
Skewness	−0.14	−0.25	1.28	0.69	0.89	0.64	−0.94	−1.26
Excess Kurtosis	4.83	4.83	5.74	3.65	3.36	2.64	3.64	5.17
HPR	187.47%		−75.83%		−99.94%		666.56%	
CAGR	13.70%		−15.86%		−59.65%		37.55%	

Correlations	r_D				r_C			
	SPX	VIX	VXX	XIV	SPX	VIX	VXX	XIV
SPX	1				1			
VIX	−0.7655	1			−0.7700	1		
VXX	−0.8015	0.8847	1		−0.8039	0.8833	1	
XIV	0.8267	−0.8799	−0.9967	1	0.8263	−0.8764	−0.9924	1

Table 2: **Calibrated parameters over the sample of Zhang, Zhen, Sun, and Zhao (2017)** This table shows the calibrated parameters to the Heston (1993) and the floating θ models over the sample period of Zhang, Zhen, Sun, and Zhao (2017). Panel A shows the parameters as estimated by Zhang, Zhen, Sun, and Zhao (2017) and Panel B shows our estimation of the parameters over the same sample.

	κ	$\bar{\theta}$	$\bar{V}_t$	$Std(V_t)$	$\bar{\theta}_t$	Std (θ_t)	VIX TS RMSE
Panel A: Zhang, Zhen, Sun, and Zhao (2017) calibration							
Heston	0.28	0.1651	0.0342	0.0264	–	–	1.3531
Floating θ	1.91	–	0.0299	0.0289	0.0729	0.0253	0.5473
Panel B: Our calibration							
Heston	0.28	0.1653	0.0345	0.0266	–	–	1.3616
Floating θ	1.91	–	0.0301	0.0291	0.0732	0.0254	0.5499

Table 3: **Calibrated parameters.** This table shows the estimated parameters using either the full VIX term structure, in Panel A, using only the first two points of the VIX term structure, in Panel B or using the first three points of the VIX term structure.

	κ_V	κ_θ	κ	$\bar{\theta}$	$\bar{V}_t$	$Std(V_t)$	$\bar{\theta}_t$	Std (θ_t)	VIX TS RMSE
Panel A: Full sample and full VIX term structure									
Heston			0.3814	0.1287	0.0322	0.0244			1.3193
Floating θ			2.0341		0.0281	0.0263	0.068	0.0225	0.5279
$\kappa = \kappa_V = \kappa_\theta$ model			0.5908	0.0176	0.0289	0.0248	0.1419	0.0500	0.5951
Full model	2.0969	0.2406		0.0680	0.0280	0.0260	0.0678	0.0262	0.5276
Panel B: Full sample and first two VIX term structure									
Floating θ			10.0000		0.0269	0.0314	0.0406	0.0240	0.1766
$\kappa = \kappa_V = \kappa_\theta$ model			4.2238	0.0567	0.0269	0.0289	0.0495	0.0399	0.3193
Full model	12.0459	0.000013		0.0406	0.0266	0.0321	0.0395	0.0236	0.1566
Panel C: Full sample and first three VIX term structure									
Floating θ			7.8589		0.0266	0.0307	0.0446	0.0248	0.2731
$\kappa = \kappa_V = \kappa_\theta$ model			3.1278	0.0644	0.0277	0.0278	0.0574	0.0398	0.3829
Full model	7.8863	0.000013		0.0446	0.0265	0.0306	0.0446	0.0234	0.2729

Table 4: **Model implied VXX and market VXX** This table shows some summary statistics of the performance of our calibrated model in fitting the VXX time series. The model is calibrated using either the full, first two points or first three points of the VIX term structure. Then either the idealistic or realistic model is used to imply a VXX time series, which can be compared to the market VXX time series, presented in the first column. RMSE is the root mean squared error, which can be computed for the errors between the model implied and market VXX prices.

		Full VIX term structure		First two VIX term structure		First three VIX term structure	
	Market	Idealistic	Realistic	Idealistic	Realistic	Idealistic	Realistic
κ_V		2.097		12.046		7.886	
κ_θ		0.241		0.000		0.000	
$\bar{\theta}$		0.068		0.041		0.045	
VIX TS Level RMSE		0.515		0.157		0.273	
VXX Level RMSE		739.69	416.93	722.88	150.68	664.65	99.11
VXX Return RMSE		2.52%	2.42%	5.77%	5.16%	3.12%	2.72%
VXX mean Return	-0.32%	-0.94%	-0.42%	-1.02%	-0.25%	-0.81%	-0.39%
VXX Return std. dev.	3.97%	4.96%	5.01%	6.91%	6.33%	5.05%	4.87%
Return correlation	1.00	0.89	0.88	0.56	0.58	0.79	0.83

Figure 1: **Market share by maturity target** This figure has four panels. The top left panel shows the total market capitalization of all the VIX futures ETPs grouped by their target maturity. The bottom left panel shows the daily proportion of total market capitalization for each group. The top right Panel shows the five day moving average of the daily dollar trading volume for each group. The last panel (bottom right) shows the five day moving average of the daily proportion of dollar trading volume for each group. ST represents the ETPs that track the short-term VIX futures indices, MT represents the ETPs that track the mid-term VIX futures indices, WEEK represents ETPs that provide exposure to shorter term VIX futures (weekly futures) and dynamic/hybrid represents those ETPs that are not linearly tracking one of the indices.

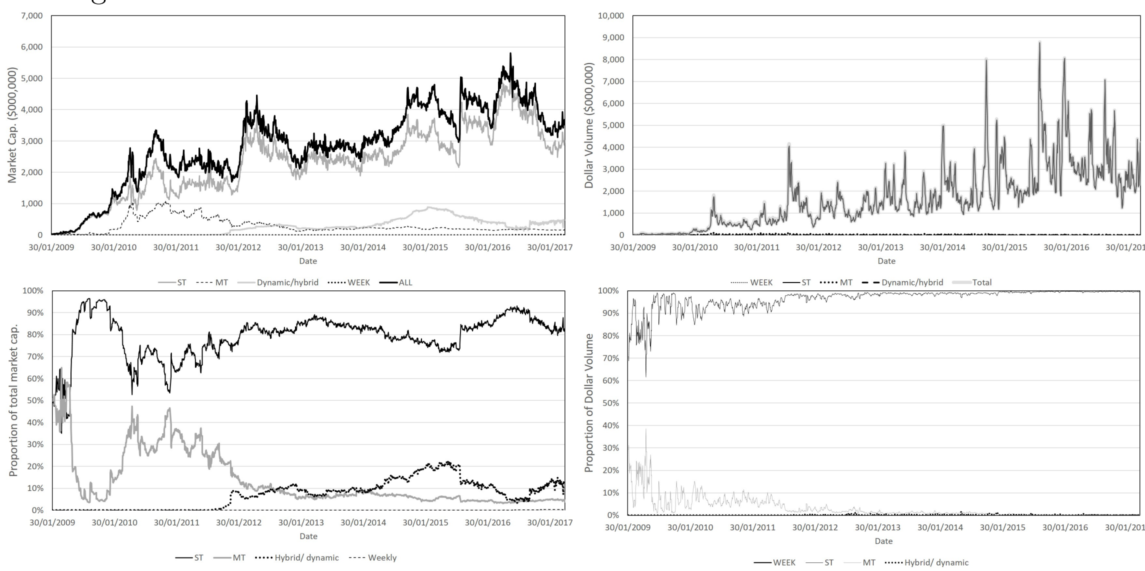

Figure 2: **Daily V_t and θ_t - full model** This figure shows the daily estimates of V_t and θ_t from calibrating the full model using the full, first two points or first three points of the VIX term structure, in order.

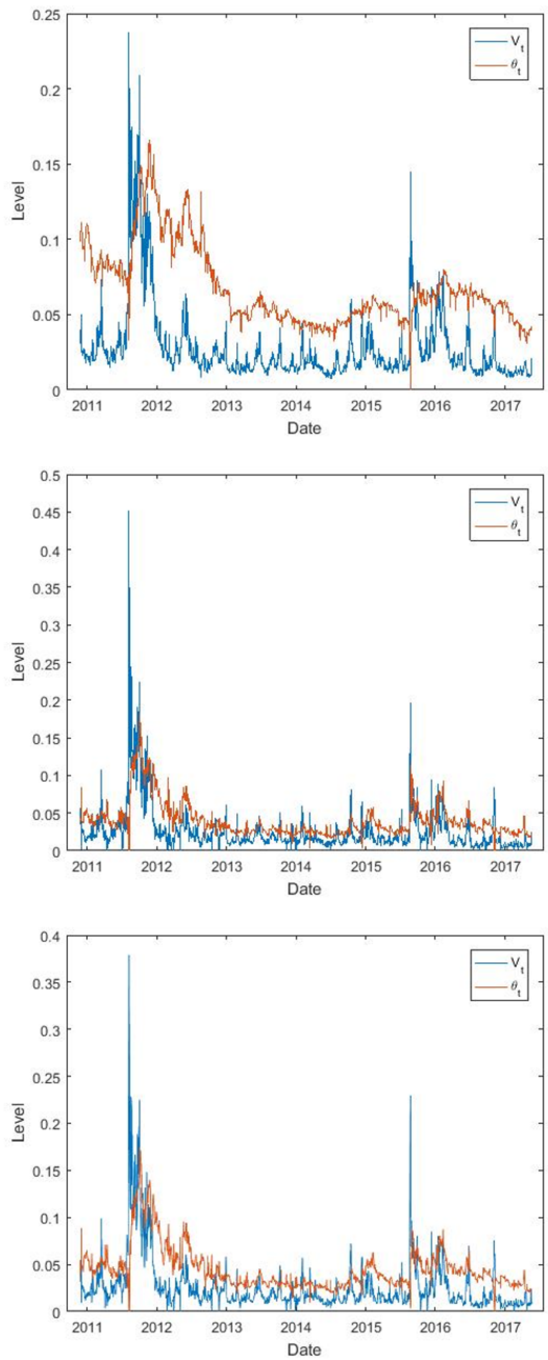

Figure 3: **Daily MSE** This figure shows the daily estimates of the MSE from calibrating the full model using the full, first two and first three-points of the VIX term structure, in order.

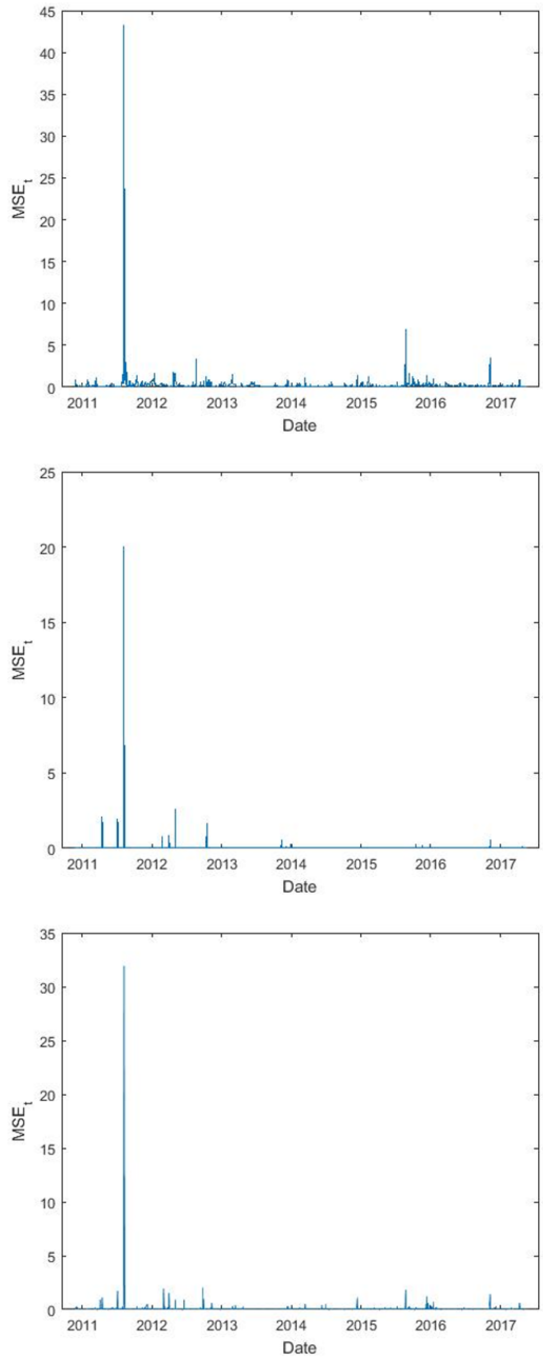

Figure 4: **Model implied vs market VXX price** This figure shows 6 graphs, each depicting the model implied and market VXX prices. The top three graphs display the model implied series using the idealistic model calibrated to the full (left), two-point (middle) and three-point (right) VIX term structure data. The bottom figures display the model implied series using realistic model. In the lower three graphs we also plot the replicated VXX time series using market VIX futures prices.

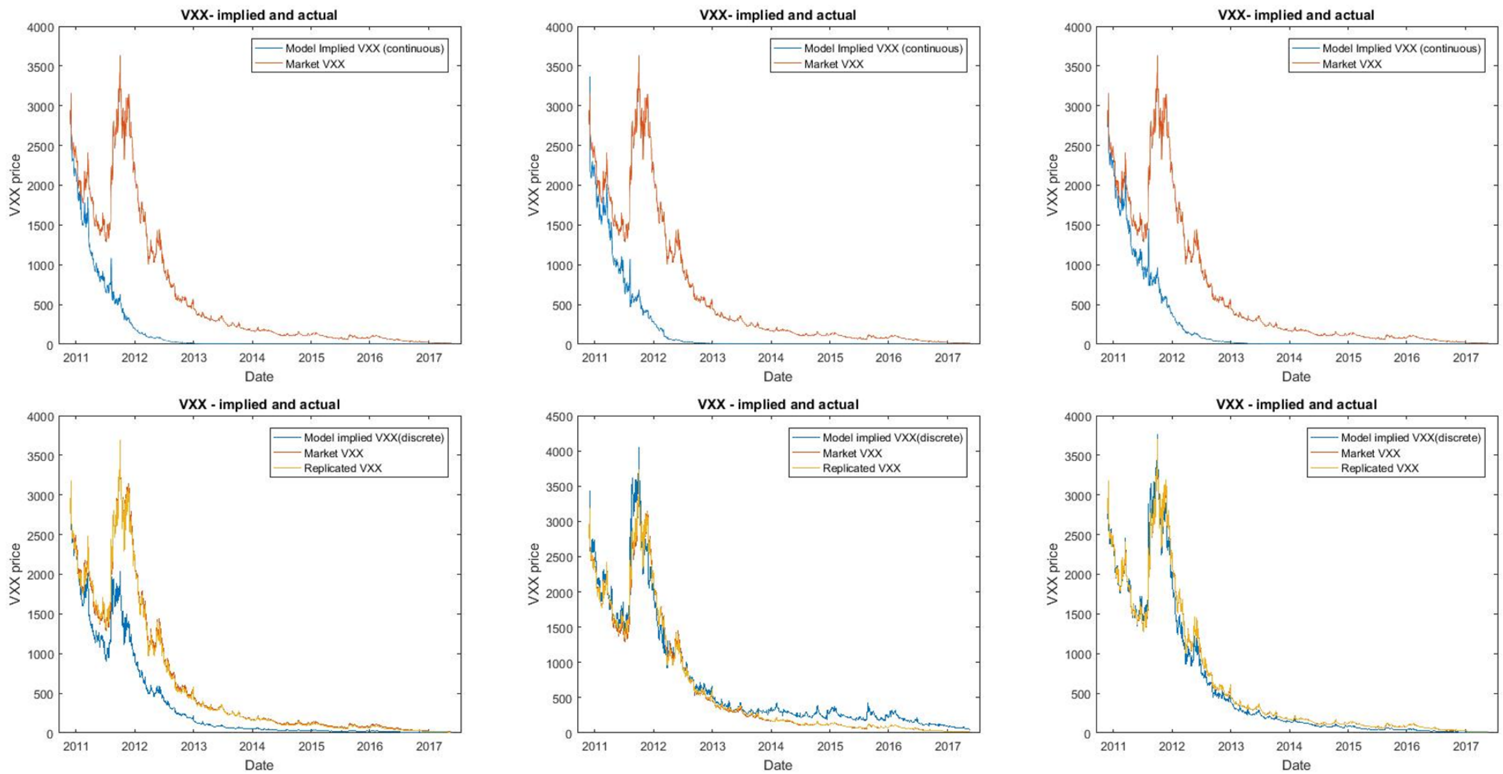

Figure 5: **Realistic model fit for short-term ETPs - full VIX term structure** This figure shows the model implied, market and replicated prices of the 5 most liquid, after the VXX, short-term VIX futures ETPs. The model implied prices are calculated using the realistic model and the parameters from calibrating to the full VIX term structure.

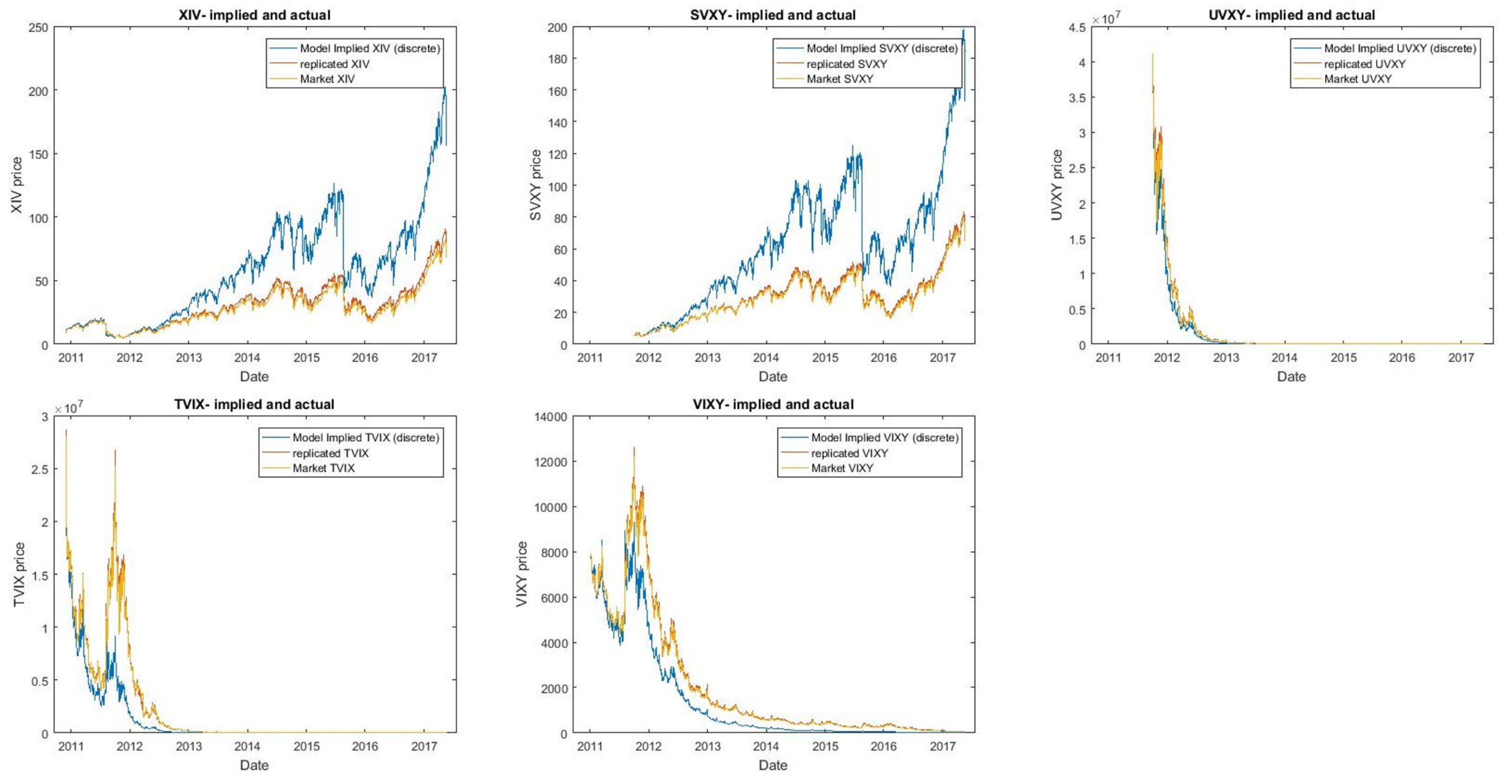

Figure 6: **Realistic model fit for short-term ETPs - first 2 points VIX term structure** This figure shows the model implied, market and replicated prices of the 5 most liquid, after the VXX, short-term VIX futures ETPs. The model implied prices are calculated using the realistic model and the parameters from calibrating to the first two points of the VIX term structure.

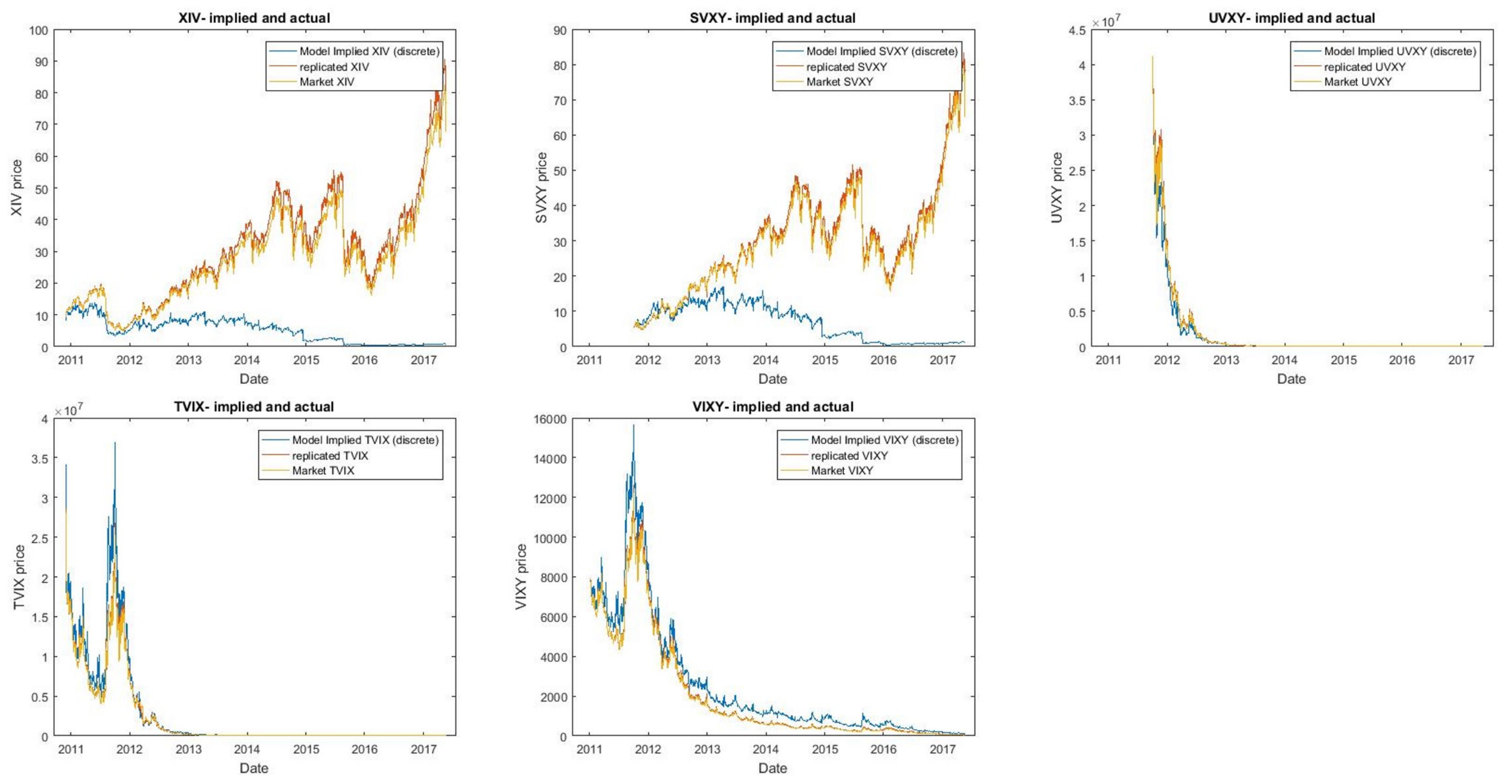

Figure 7: **realistic model fit for short-term ETPs - first 3 points VIX term structure** This figure shows the model implied, market and replicated prices of the 5 most liquid, after the VXX, short-term VIX futures ETPs. The model implied prices are calculated using the realistic model and the parameters from calibrating to the first three points of the VIX term structure.