Option Pricing under Gram-Charlier Density*

Ruizi Hu†
Department of Accountancy and Finance
Otago Business School, University of Otago
Dunedin 9054, New Zealand
ruizi.hu@postgrad.otago.ac.nz

Jin E. Zhang
Department of Accountancy and Finance
Otago Business School, University of Otago
Dunedin 9054, New Zealand
jin.zhang@otago.ac.nz

Pakorn Aschakulporn
Department of Accountancy and Finance
Otago Business School, University of Otago
Dunedin 9054, New Zealand
pakorn.aschakulporn@otago.ac.nz

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† Corresponding author. Tel: +64 27 392 8786.

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Abstract

This paper is the first to present an option pricing formula under Gram-Charlier density of arbitrary order with the analytic and explicit expression for the probabilities that options will be exercised under share measure and risk-neutral measure (namely, Π_1 and Π_2, respectively). We demonstrate the hedging ratio (i.e., Δ) of the option pricing formula under Gram-Charlier density, is equal to Π_1, as well as the equivalence between our formula and the previous derivations. Furthermore, we investigate the relationship between the cumulants with Gram-Charlier density under share measure and risk-neutral measure.

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1 Introduction

This paper is the first to present an option pricing formula under Gram-Charlier density of arbitrary order with the analytic and explicit expression for the probabilities that options will be exercised under share measure and risk-neutral measure. The Gram-Charlier density applied in option pricing is able to extend the normally distributed return in the Black and Scholes (1973) model to include higher order cumulants, which are related to the parameters implied by the Gram-Charlier density. The majority of existing literature is restricted to truncating the series expansion until the fourth moment or cumulant, and following the Longstaff's (1995) martingale restriction, Kochard (1999), Schlögl (2013), Aschakulporn (2022), and Aschakulporn and Zhang (2022) propose an option pricing formula with the untruncated Gram-Charlier series expansion; however, the probabilities that options will be exercised under share measure and risk-neutral measure are hidden in their representations rather than intuitively given as in Bakshi and Madan's (2000) formula, while the latter is not analytic due to the existence of integral and complex number. By investigating the properties of Hermite polynomials, we develop an analytic and explicit expression, and thereby derive the hedging ratio under Gram-Charlier density and discover the relationship between the cumulants with Gram-Charlier density under different measures.

The Gram-Charlier series expansion was established at the beginning of the 19th century, and since then, has been commonly employed in a variety of fields, such as statistics, mathematics, physics, and finance. The applications of Gram-Charlier densities in finance mainly focus on asset or option pricing by modelling risk-neutral asset price distributions. The dominant reasons that the Gram-Charlier density has been widely used in option pricing are because: it is an extension of the normal density where the skewness and excess kurtosis that directly appear as parametrical coefficients; non-normal skewness and kurtosis of the underlying asset significantly contribute to the phenomenon of the volatility smile, and the impact of any order cumulant on implied volatility curve can be directly observed; compared with other more advanced option pricing models (such as affine jump-diffusion model), the Gram-Charlier density can give us a parsimonious option pricing formula with arbitrary level of truncation. The option pricing framework pioneered in Black and Scholes (1973) and Merton (1973) is a pillar of modern finance theory. The Gram-Charlier-type expansions in early studies of empirical finance have been used as a semi-nonparametric device to overcome the restriction imposed by the usual normality assumption. Jarrow and Rudd (1982) are the first to develop a density expansion approach to option pricing by nesting the Black and Scholes (1973) model within an integrated Gram-Charlier series expansion of a log-normal density, and they show how a given stochastic process of an underlying security can be approximated by an arbitrary distribution in terms of a series expansion involving second and higher moments. Similar to Jarrow and Rudd (1982), Corrado and Su (1996) derive an option pricing formula that nests the Black and Scholes (1973) model within an integrated Gram–Charlier series expansion of a normal density, which is used in empirical specification tests of the Black and Scholes (1973) model. Due to the non-standard definition of the Hermite polynomial, there exists an error in Corrado and Su (1996) and is later corrected by Brown and Robinson (2002). Madan and Milne (1994) develop an expansion to approximate a risk-neutral density function, and Abken, Madan, and Ramamurtie (1996) implement this expansion to estimate risk factors embedded in observed option prices. Under the Gram–Charlier option pricing framework, Ané (1999) and Jondeau and Rockinger (2001) adapt the methodology of Madan and Milne (1994) to empirical tests involving S&P 500 index options and Franc/Mark exchange rate options, respectively. Longstaff (1995) finds martingale violations in S&P 100 index option prices using a pricing model based on a Gram–Charlier expansion of a normal density function. Kochard (1999), Knight and Satchell (2001), and Backus, Foresi, and Wu (2004) propose a method to impose a martingale restriction on the formula of Corrado and Su (1996). Jurczenko, Maillet, and Negréa (2004) provide a broad overview of the Gram–Charlier density expansion approach to option valuation along with several key extensions. Ki et al. (2005) construct a flexible option pricing model based on a mixture of Gram–Charlier density expansions. Corrado (2007) focuses on a method to impose a martingale restriction in an option pricing model developed from the Gram–Charlier density expansion. Tamaki and Taniguchi (2007) link the Gram–Charlier density expansion approach to time series moment estimates. Tanaka, Yamada, and Watanabe (2010) use the Gram-Charlier expansion under one forward measure rather than many forward measures and obtain approximate solutions for a larger class of interest-rate- and credit-risk-contingent claims. Schlögl (2013) introduces the techniques used for option pricing where the standardised distribution of the logarithmic price of the underlying asset is given by a full Gram-Charlier Type A series expansion. Lin, Huang, and Li (2015) develop a truncated Gram–Charlier expansion option pricing model and find it overall performs the best when compared to four prevalent option pricing models. Chateau and Dufresne (2017) show how Gram-Charlier distributions can be simulated, opening up the use of Monte-Carlo methods for pricing vanilla and exotic options. Aschakulporn and Zhang (2022) examine the Bakshi, Kapadia, and Madan (2003) (BKM) risk-neutral estimators with explicit skewness and kurtosis as both the input and benchmark and finds the condition of strike prices required to bound the errors of skewness.

Nevertheless, there is a considerable body of literature restricted to truncating the series expansion until the fourth order. As a polynomial approximation, once it is truncated, Gram-Charlier density may not always be valid (i.e., strictly nonnegative) for all values of its parameters in the parametric space. Without keeping the positivity of a truncating Gram-Charlier density, there might be inconsistencies among option prices. To overcome the shortcoming, Barton and Dennis (1952) obtain the conditions on parameters to guarantee positive definiteness of the underlying densities through a numerical method. Jondeau and Rockinger (2001) build on the work of Barton and Dennis (1952) and indicate how it is numerically possible to restrict parameters. Kwon (2022) is the first to establish a direct relationship between the skewness and kurtosis, and gives an analytical expression on the polynomial equations to determine the positive definite and unimodal regions for Gram-Charlier density. Lin and Zhang (2022) focus on Gram–Charlier densities to show how the valid region of higher moments or cumulants can be numerically implemented by semidefinite programming, which ensures that a series truncated at a cumulant of an arbitrary even order represents a valid probability density. Lin, Shen, and Zhang (2023) further explore into the same problem.

In this paper, the primary contribution is that we develop a new form of option pricing formula under Gram-Charlier density of arbitrary order, where the probabilities that options will be exercised under share measure and risk-neutral measure (namely, Π_1 and Π_2, respectively) can be expressed both analytically and explicitly. Compared with other option pricing formulas using the untruncated Gram-Charlier series expansion presented in previous literature, such as Kochard (1999), Schlögl (2013), Aschakulporn (2022), and Aschakulporn and Zhang (2022), Π_1 and Π_2 can be intuitively given in our formula instead of being hidden; compared with Bakshi and Madan's (2000) formula, we specify the characteristic function under Gram-Charlier density, and the advantage of our formula is its analytic expression, which is derived by investigating the properties of Hermite polynomials. The second contribution in this paper is we find that the hedging ratio, Δ^{GC}, of the option pricing formula under Gram-Charlier density, is equal to Π_1. In addition, we demonstrate the equivalence between the Bakshi and Madan's (2000) formula and our formula (as well as the other formulas with arbitrary truncation level mentioned above). Finally, to help users have a deeper understanding of the share measure, we further investigate the relationship between the cumulants with Gram-Charlier density under share measure and risk-neutral measure.

The remainder of this paper is organized as follows. Section 2 provides some necessary mathematical foundations of Gram-Charlier density and introduces some useful properties of Hermite polynomials. In Section 3, we present three different forms of option pricing formulas under Gram-Charlier density of arbitrary order and demonstrate their equivalence, and also derive the hedging ratio of the option pricing formula under Gram-Charlier density. In Section 4, we further investigate the cumulants with Gram-Charlier density under share measure and risk-neutral measure. Conclusions are offered in Section 5. The appendix gives the details of key derivations.

2 Mathematical Background

2.1 The Gram-Charlier Density

The Gram-Charlier series can be applied in option pricing by using the densities to extend the normally distributed return in the Black and Scholes (1973) model to include higher-order cumulants. The full (i.e. untruncated) Gram-Charlier series expansion allows the user to choose an arbitrary level of truncation. We start by introducing the mathematical definitions and notations of the Gram-Charlier density.

Definition 1. *$f(y)$ is defined to be the probability density function of the random variable Y. Under sufficient conditions (given by Cramér (1926)): (i)$f'(y)$ exists and is continuous, (ii)$\lim_{|y|\to+\infty} f(y) = 0$ and (iii)$\int_{-\infty}^{+\infty} [f'(y)]^2 \exp[(1/2)\cdot y^2]\, dy$ is finite, the probability density of Y can be expressed as the following full Gram-Charlier series expansion*

$$\begin{aligned} f(y) &= \sum_{k=0}^{+\infty} \frac{\rho_k}{k!} He_k(y)\phi(y) \\ &= \sum_{k=0}^{+\infty} \frac{\rho_k}{k!} \frac{(-1)^k}{\phi(y)} \frac{d^k[\phi(y)]}{dy^k}\phi(y) \\ &= \sum_{k=0}^{+\infty} \frac{\rho_k}{k!}(-1)^k \phi^{(k)}(y), \end{aligned} \tag{1}$$

where $k!$ is the factorial of k, $\phi(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)$ is the standard normal density function, $He_k(y)$ is the k-th order Hermite polynomial defined by $He_k(y) \equiv (-1)^k \frac{\phi^{(k)}(y)}{\phi(y)}$, and ρ_k is the coefficient.

Higher order Hermite polynomials can be obtained recursively using

$$He_k(y) = yHe_{k-1}(y) - (k-1)He_{k-2}(y).$$

Expressions of Hermite polynomials up to the order of 6 are given as examples here

$$\begin{aligned}
He_0(y) &= 1, \\
He_1(y) &= y, \\
He_2(y) &= y^2 - 1, \\
He_3(y) &= y^3 - 3y, \\
He_4(y) &= y^4 - 6y^2 + 3, \\
He_5(y) &= y^5 - 10y^3 + 15y, \\
He_6(y) &= y^6 - 15y^4 + 45y^2 - 15.
\end{aligned}$$

The coefficients, ρ_k, in Equation (1) are derived from the following orthogonality property of Hermite polynomials

$$\int_{-\infty}^{+\infty} He_k(y)He_j(y)\phi(y)dy = \begin{cases} k!, \ k = j \\ 0, \ k \neq j \end{cases} .$$

Multiplying both sides of the first line in Equation (1) by $He_k(y)$ and integrating over all y yields

$$\begin{aligned}
\int_{-\infty}^{+\infty} He_k(y)f(y)dy &= \int_{-\infty}^{+\infty} He_k(y) \left[\sum_{j=0}^{+\infty} \frac{\rho_j}{j!} He_j(y)\phi(y) \right] dy \\
&= \int_{-\infty}^{+\infty} \frac{\rho_k}{k!} He_k(y)He_k(y)\phi(y)dy \\
&= \rho_k.
\end{aligned} \tag{2}$$

The coefficients ρ_k through $k = 6$ for some standardized random variable are

$$\rho_0 = 1,$$

$$\rho_1 = 0,$$

$$\rho_2 = 0,$$

$$\rho_3 = \mu_3 = \kappa_3,$$

$$\rho_4 = \mu_4 - 3 = \kappa_4,$$

$$\rho_5 = \mu_5 - 10\mu_3 = \kappa_5,$$

$$\rho_6 = \mu_6 - 15\mu_4 + 30 = \kappa_6 + 10\kappa_3^2,$$

where μ_k denotes the k-th moment about the mean (i.e., the k-th central moment), and κ_k denotes the k-th cumulant. [1]

Hence, the resulting n-th order approximation to a density function is

$$f(y) = \sum_{k=0}^{n} \frac{\rho_k}{k!} He_k(y)\phi(y) = \left[1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} He_k(y)\right] \phi(y).$$

2.2 Hermite Polynomials

Before constructing the models of stock price and option price under Gram-Charlier density, it is necessary to introduce some important properties of Hermite polynomials.

Lemma 1. *The summation form of Hermite polynomial is given by*

$$He_k(y) = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} C_k^{2m} (-1)^m (2m-1)!! y^{k-2m} = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{2^k m!(k-2m)!} (-1)^m y^{k-2m}, \tag{3}$$

where $He_k(y)$ is the k-th order Hermite polynomial, $C_k^{2m} = \frac{k!}{(2m)!(k-2m)!}$ is a combinatorial number, $\lfloor \cdot \rfloor$ means the floor function of "$\cdot$" (i.e., the largest integer that is less than or equal to "$\cdot$"), $(\cdot)!$ and $(\cdot)!!$ are the factorial and double factorial of "$\cdot$", respectively.

[1] For the standardized random variable Y, the relationship between the coefficient ρ_k and the k-th central moment μ_k is given by Equation (2) with $\mu_k = E(Y^k)$. The relationship between ρ_k and the k-th cumulant κ_k can be obtained by using the cumulant generating function, and the detailed derivations will be provided in Section 4.

Proof. See Appendix A.

□

Lemma 2. *The integral form of Hermite polynomial is given by*

$$He_k(y) = \int_{-\infty}^{+\infty} (it + y)^k \phi(t)dt, \tag{4}$$

where $He_k(y)$ *is the* k*-th order Hermite polynomial,* i *is the imaginary unit with* $i^2 = -1$*, and* $\phi(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right)$ *is the standard normal density function.*

Proof. See Appendix B.

□

Remark 1. Equation (3) and (4) are two different forms of Hermite polynomials. Since we need the result of Lemma 1 to prove Lemma 2, and Lemma 2 will be used frequently in the later proofs of this paper, we give our own proofs here.

Lemma 3. *A useful formula to transform the increment in Hermite polynomial is given by*

$$He_k(y + \Delta y) = \sum_{m=0}^{k} (\Delta y)^m C_k^m He_{k-m}(y). \tag{5}$$

Proof. See Appendix C.

□

Remark 2. Since there exist two forms of binomial expansion

$$(a + b)^n = \sum_{m=0}^{n} C_n^m a^m b^{n-m} = \sum_{m=0}^{n} C_n^m a^{n-m} b^m,$$

Equation (5) can also be written as

$$He_k(y + \Delta y) = \sum_{m=0}^{k} (\Delta y)^{k-m} C_k^m He_m(y).$$

Remark 3. Our Lemma 3 is consistent with Lemma 5.7 in Aschakulporn (2022), but we present our own proof here.

Lemma 4. *Another useful formula to transform the negative sign in Hermite polynomial is given by*

$$He_k(-y) = (-1)^k He_k(y). \tag{6}$$

Proof. See Appendix D. □

Lemma 5. *Taking the derivative of Hermite polynomial gives*

$$\frac{d}{dy} He_k(y) = kHe_{k-1}(y). \tag{7}$$

Proof. See Appendix E. □

Remark 4. Our Lemma 5 is consistent with Lemma 5.3 in Aschakulporn (2022), but we present our own proof here. The sequence that satisfies Equation (7) is called an Appell sequence.

3 Option Price under Gram-Charlier Density

3.1 Stock Price under Gram-Charlier Density

The difference between the stock price model with the normal density and the Gram-Charlier density is whether the convexity adjustment term equals zero or not. By following the definition of the stock price under Gram-Charlier density in Zhang and Xiang (2008), we specify the underlying stock price at maturity in Definition 2.

Definition 2. *In a risk-neutral world, the underlying stock price at maturity is modelled by*

$$\begin{aligned} S_T &= S_t e^{\left(\mu_c + r - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Y} \\ &= F_t^T e^{\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Y}, \end{aligned} \tag{8}$$

where S_t is the current underlying stock price at time t, μ_c is the convexity adjustment term, r is the risk-free interest rate, σ is the volatility, $\tau = T - t$ is the time to maturity, $F_t^T = S_t e^{r\tau}$ is the forward price at time t, and Y is a standardized random variable with the density function $f(y) = \left[1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} He_k(y)\right] \phi(y)$.

In the no-arbitrage risk-neutral valuation framework, there exists the martingale restriction (i.e., the martingale condition) proposed by Longstaff (1995) and Backus, Foresi, and Wu (2004) that need to be required.

Lemma 6. *The martingale condition in risk-neutral probability measure, $E_t^Q[S_T] = F_t^T$, determines the convexity adjustment term, given by*

$$\mu_c = -\frac{1}{\tau} \ln \left[1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} (\sigma\sqrt{\tau})^k\right]. \tag{9}$$

Proof. See Appendix F.

□

Remark 5. If Y is normally distributed, then $\rho_k = 0$, for $k = 3, 4, ..., n$ and $\mu_c = 0$, Equation (8) degenerates to the Black and Scholes (1973) model.

Following the stock price under Gram-Charlier density defined by Equation (8), the price of a European call option under Gram-Charlier density can be calculated with the Harrison and Kreps (1979) and Harrison and Pliska (1981) risk-neutral valuation formula. Based on this valuation formula, it is possible to generate different forms of the option pricing formula under Gram-Charlier density using different methods. One way is to apply the properties of Hermite polynomials and combinatorial numbers, while the other is to implement inverse Fourier transformations.

3.2 The Option Pricing Formula under Gram-Charlier Density

Kochard (1999) first propose an option pricing formula with the untruncated Gram-Charlier series expansion, which can be regarded as a generalization of Black and Scholes (1973) formula. Later on, Schlögl (2013), Aschakulporn (2022), and Aschakulporn and Zhang (2022) present results that consistent with Kochard's (1999) formula. Although Kochard (1999) and Schlögl (2013) do not further simplify their final expressions, and Schlögl (2013) does not clarify the link between cumulants and coefficients of the Gram-Charlier density clearly, their option pricing formulas are essentially the same as Aschakulporn's (2022) result. The option pricing formula under Gram-Charlier density in Aschakulporn (2022) is given by

$$\begin{aligned} c_t^{GC} &= F_t^T e^{-r\tau} N(d_1) - K e^{-r\tau} N(d_2) \\ &\quad - K e^{-r\tau} \phi(d_2) \sum_{k=3}^{n} \left[\frac{\rho_k}{k!} (-1)^k \sum_{m=1}^{k-1} C_{k-1}^{m-1} (-\sigma\sqrt{\tau})^m He_{k-m-1}(d_2 + \sigma\sqrt{\tau}) \right], \end{aligned} \tag{10}$$

where

$$d_2 = \frac{\ln \frac{F_t^T}{K} + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, d_1 = d_2 + \sigma\sqrt{\tau},$$

$F_t^T = S_t e^{r\tau}$ is the forward price at time t, K is the strike price, μ_c is the convexity adjustment term given in Lemma 6, r is the risk-free interest rate, σ is the volatility, $\tau = T - t$ is the time to maturity, $N(\cdot)$ is the cumulative distribution function of some standard normal random variable, and $\phi(\cdot)$ is the probability density function of some standard normal random variable.

The proof of Equation (10) is consistent with the proof of Proposition 5.4 in Aschakulporn (2022). By investigating the properties of combinatorial numbers and Hermite polynomials, we are able to simplify Equation (10).

Lemma 7. *The formula of the price of a European call option under Gram-Charlier density proposed by Aschakulporn (2022) can be simplified as below*

$$c_t^{GC} = F_t^T e^{-r\tau} N(d_1) - K e^{-r\tau} N(d_2) - K e^{-r\tau} \phi(d_2) \sum_{k=3}^{n} \left[\frac{\rho_k}{k!} (-1)^k \sum_{m=1}^{k-1} (-\sigma\sqrt{\tau})^m He_{k-m-1}(d_2) \right]. \tag{11}$$

Proof. See Appendix G. □

According to Bakshi and Madan (2000), the option pricing formula can be written as a two-term form containing "Π_1" and "Π_2". The probabilities that options will be exercised under share measure and risk-neutral measure (namely, Π_1 and Π_2, respectively) can be intuitively given in this form explains the popularity of this structure. Following the previous derivation, we are able to present a new form of option pricing formula under Gram-Charlier density of arbitrary order, where Π_1 and Π_2 can be explicitly expressed as well.

Proposition 1. *The price of a European call option under Gram-Charlier density can be computed with Harrison and Kreps (1979) and Harrison and Pliska (1981) risk-neutral valuation formula, given by*

$$c_t^{GC} = F_t^T e^{-r\tau}\Pi_1 - Ke^{-r\tau}\Pi_2, \tag{12}$$

where

$$\Pi_1 = N(d_1) + e^{\mu_c\tau}\phi(d_1)\sum_{k=3}^{n}\frac{\rho_k}{k!}\left[\sum_{h=1}^{k} C_k^h He_{h-1}(-d_1)\left(\sigma\sqrt{\tau}\right)^{k-h}\right],$$

$$\Pi_2 = N(d_2) + \phi(d_2)\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}He_{k-1}(-d_2)\right],$$

$$d_2 = \frac{\ln\frac{F_t^T}{K} + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, d_1 = d_2 + \sigma\sqrt{\tau},$$

$F_t^T = S_t e^{r\tau}$ *is the forward price at time* t*,* K *is the strike price,* μ_c *is the convexity adjustment term given in Lemma 6,* r *is the risk-free interest rate,* σ *is the volatility,* $\tau = T - t$ *is the time to maturity,* $N(\cdot)$ *is the cumulative distribution function of some standard normal random variable, and* $\phi(\cdot)$ *is the probability density function of some standard normal random variable.*

Proof. See Appendix H.

□

Remark 6. Our Proposition 1 presents a new form of option pricing formula under Gram-Charlier density of arbitrary order, which is equivalent to the previous three-term's formula proposed by Kochard (1999), Schlögl (2013), and Aschakulporn (2022). In order to implement this proof, the introduction of another form of option pricing formula under Gram-Charlier density is indispensable, and therefore we will show the equivalence later in Section 3.5.

3.3 Delta of the Option Pricing Formula under Gram-Charlier Density

Lemma 8. *A useful identity involved in option pricing under Gram-Charlier density is given by*

$$F_t^T e^{-r\tau} \frac{\partial \Pi_1}{\partial S_t} - K e^{-r\tau} \frac{\partial \Pi_2}{\partial S_t} = 0, \tag{13}$$

where

$$\Pi_1 = N(d_1) + e^{\mu_c \tau} \phi(d_1) \sum_{k=3}^{n} \frac{\rho_k}{k!} \left[\sum_{h=1}^{k} C_k^h He_{h-1}(-d_1) \left(\sigma\sqrt{\tau}\right)^{k-h} \right],$$

$$\Pi_2 = N(d_2) + \phi(d_2) \sum_{k=3}^{n} \left[\frac{\rho_k}{k!} He_{k-1}(-d_2) \right],$$

$$d_2 = \frac{\ln \frac{F_t^T}{K} + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, d_1 = d_2 + \sigma\sqrt{\tau},$$

$F_t^T = S_t e^{r\tau}$ *is the forward price at time* t, S_t *is the underlying stock price modelled by Definition 2,* K *is the strike price,* μ_c *is the convexity adjustment term given by Lemma 6,* r *is the risk-free interest rate,* σ *is the volatility,* $\tau = T - t$ *is the time to maturity,* $N(\cdot)$ *is the cumulative distribution function of some standard normal random variable, and* $\phi(\cdot)$ *is the probability density function of some standard normal random variable.*

Proof. See Appendix I. □

Remark 7. Note that

$$\frac{\partial \Pi_1}{\partial S_t} = \phi(d_1) e^{\mu_c \tau} \left[1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} He_k(-d_2) \right] \frac{\partial d_1}{\partial S_t} = e^{\mu_c \tau} \left[1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} He_k(-d_2) \right] \frac{\partial N(d_1)}{\partial S_t},$$

and

$$\frac{\partial \Pi_2}{\partial S_t} = \phi(d_2) \left[1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} He_k(-d_2) \right] \frac{\partial d_2}{\partial S_t} = \left[1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} He_k(-d_2) \right] \frac{\partial N(d_2)}{\partial S_t},$$

which have been shown in the proof of Lemma 8.

If Y is normally distributed, then $\rho_k = 0$, for $k = 3, 4, ..., n$ and $\mu_c = 0$, we have $\frac{\partial \Pi_1}{\partial S_t} = \frac{\partial N(d_1)}{\partial S_t}$ and $\frac{\partial \Pi_2}{\partial S_t} = \frac{\partial N(d_2)}{\partial S_t}$.

Proposition 2. *The delta (i.e., the hedging ratio), Δ^{GC}, of the option pricing formula under Gram-Charlier density for a European call is given by*

$$\Delta^{GC} \equiv \frac{\partial c_t^{GC}}{\partial S_t} = \Pi_1, \tag{14}$$

where c_t^{GC} denotes the price of a European call option under Gram-Charlier density given in Proposition 1, S_t means the underlying stock price under Gram-Charlier density modelled by Definition 2, and

$$\Pi_1 = N(d_1) + e^{\mu_c \tau} \phi(d_1) \sum_{k=3}^{n} \frac{\rho_k}{k!} \left[\sum_{h=1}^{k} C_k^h He_{h-1}(-d_1) \left(\sigma\sqrt{\tau} \right)^{k-h} \right].$$

Proof. See Appendix J.

□

3.4 An Alternative Option Pricing Formula under Gram-Charlier Density

The return is generally constructed in terms of the stock price S_t, and the log-return of the Black and Scholes (1973) model is normally distributed. The Gram-Charlier series can be applied in option pricing by using the densities to extend the normally distributed return in Black and Scholes (1973) model to include higher-order cumulants. For notational convenience, here and hereafter, we construct the log-return under Gram-Charlier density in terms of the forward price F_t^T.

Definition 3. *The log-return R_τ under Gram-Charlier density is defined by*

$$\begin{aligned} R_\tau &= \ln \frac{S_T}{F_t^T} \\ &= \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Y, \end{aligned} \tag{15}$$

where

$$\mu_c = -\frac{1}{\tau}\ln\left[1 + \sum_{k=3}^{n}\frac{\rho_k}{k!}(\sigma\sqrt{\tau})^k\right],$$

and Y is the standardized random variable with the probability density function $f(y)$ of the form

$$f(y) = \sum_{k=0}^{n}\frac{\rho_k}{k!}(-1)^k\phi^{(k)}(y) = \sum_{k=0}^{n}\frac{\rho_k}{k!}He_k(y)\phi(y) = \left[1 + \sum_{k=3}^{n}\frac{\rho_k}{k!}He_k(y)\right]\phi(y).$$

Given the log-return under Gram-Charlier density, we are able to derive the representation of the characteristic function under Gram-Charlier density.

Lemma 9. *The characteristic function of the random variable $R_\tau = \ln\frac{S_T}{F_t^T}$ under Gram-Charlier density is given by*

$$\begin{aligned} f(\phi) &\equiv E_t^{\mathcal{Q}}\left[e^{i\phi R_\tau}\right] \\ &= e^{i\phi\mu_c\tau - \frac{1}{2}\sigma^2\tau i\phi(1-i\phi)}\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(i\phi\sigma\sqrt{\tau}\right)^k. \end{aligned} \tag{16}$$

Proof. See Appendix K.

□

According to the new defined characteristic function with log-return as random variable, we are able to develop an alternative form of option pricing formula under Gram-Charlier density to the previous one.

Proposition 3. *An alternative formula of the price of a European call option under Gram-Charlier density can be computed with Bakshi and Madan's (2000) formula, given by*

$$c_t^{GC} = F_t^T e^{-r\tau}\Pi_1 - Ke^{-r\tau}\Pi_2, \tag{17}$$

where

$$\Pi_j = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \mathcal{R}\left[\frac{f_j(\phi)\cdot e^{-i\phi\ln\frac{K}{F_t^T}}}{i\phi}\right] d\phi,$$

$$f_j(\phi) = \begin{cases} \frac{f(\phi-i)}{f(-i)},\ j=1 \\ f(\phi),\ j=2 \end{cases},$$

$F_t^T = S_t e^{r\tau}$ *is the forward price,* K *is the strike price,* r *is the risk-free interest rate,* $\tau = T - t$ *is the time to maturity,* i *is the imaginary unit with* $i^2 = -1$*,* $f(\phi) \equiv E_t^{\mathcal{Q}}\left[e^{i\phi R_\tau}\right]$ *is the characteristic function of log-return* $R_\tau = \ln\frac{S_T}{F_t^T}$*, and* $\mathcal{R}[\cdot]$ *means the real part of some complex number.*

Remark 8. According to Bakshi and Madan (2000), the representations of Π_1 and Π_2 are derived by using inverse Fourier transformations, which can be considered as a well-known result. Although $\Pi_j, j = 1, 2$ in Bakshi and Madan (2000) and in our Proposition 3 are defined in terms of the stock price S_t and the forward price F_t^T, respectively, they still have the same structure of representations. The proof is trivial, and thus it is omitted.

Remark 9. In Equation (17), we use the same notations "Π_1" and "Π_2" as what we have defined in Equation (12), because we will give a proof later (in Proposition 4) showing that this form of option pricing formula under Gram-Charlier density is equivalent to the formula presented in Proposition 1. Compared with the option pricing formula developed by Bakshi and Madan (2000), we specify the characteristic function under Gram-Charlier density in our formula, which can be regarded as an alternative option pricing formula under Gram-Charlier density of arbitrary order.

3.5 Equivalence among Option Pricing Formulas under Gram-Charlier Density

Now we are interested in whether these different forms of option pricing formula under Gram-Charlier density that presented in Lemma 7, Proposition 1 and Proposition 3 are equivalent, as well as the relationship between different terms in different formulas. We first focus on two option pricing formulas given in Proposition 1 and Proposition 3.

Proposition 4. *The two forms of option pricing formula under Gram-Charlier density (given in our Proposition 1 and Proposition 3) are equivalent, that is,*

$$\begin{aligned}\Pi_1 &= N(d_1) + e^{\mu_c \tau}\phi(d_1)\sum_{k=3}^{n}\frac{\rho_k}{k!}\left[\sum_{h=1}^{k} C_k^h He_{h-1}(-d_1)\left(\sigma\sqrt{\tau}\right)^{k-h}\right] \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty}\mathcal{R}\left[\frac{f_1(\phi)\cdot e^{-i\phi \ln \frac{K}{F_t^T}}}{i\phi}\right]d\phi,\end{aligned} \tag{18}$$

$$\begin{aligned}\Pi_2 &= N(d_2) + \phi(d_2)\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}He_{k-1}(-d_2)\right] \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty}\mathcal{R}\left[\frac{f_2(\phi)\cdot e^{-i\phi \ln \frac{K}{F_t^T}}}{i\phi}\right]d\phi,\end{aligned} \tag{19}$$

where

$$f_1(\phi) = e^{\mu_c \tau} e^{i\phi\left(\mu_c + \frac{1}{2}\sigma^2\right)\tau - \frac{1}{2}\phi^2\sigma^2\tau}\sum_{k=0}^{n}\frac{\rho_k}{k!}\left[(i\phi + 1)\,\sigma\sqrt{\tau}\right]^k,$$

$$f_2(\phi) = e^{\mu_c \tau} e^{i\phi\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau - \frac{1}{2}\phi^2\sigma^2\tau}\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(i\phi\sigma\sqrt{\tau}\right)^k.$$

Proof. See Appendix L.

□

Furthermore, considering that a simplified version of the three-term option pricing formula under Gram-Charlier density proposed by Aschakulporn (2022) has been obtained in Section 3.2, we can examine whether this form is also equivalent to the former two formulas.

Corollary 1. *The simplified version of option pricing formula under Gram-Charlier density (given in Lemma 7), which consists of three terms, is equivalent to both of the two option pricing formulas under Gram-Charlier density (given in our Proposition 1 and Proposition 3).*

The relationship between terms in these three forms is given by

$$\begin{aligned}
c_t^{GC} &= \underbrace{F_t^T e^{-r\tau} N(d_1)}_{Term\,1} - \underbrace{Ke^{-r\tau} N(d_2)}_{Term\,2} - \underbrace{Ke^{-r\tau}\phi(d_2)\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}(-1)^k \sum_{m=1}^{k-1}(-\sigma\sqrt{\tau})^m He_{k-m-1}(d_2)\right]}_{Term\,3} \\
&= \underbrace{F_t^T e^{-r\tau} A_1}_{Term\,1} - \underbrace{Ke^{-r\tau} A_2}_{Term\,2} - \underbrace{\left(Ke^{-r\tau} B_2 - F_t^T e^{-r\tau} B_1\right)}_{Term\,3} \\
&= F_t^T e^{-r\tau} \underbrace{(A_1 + B_1)}_{\Pi_1} - Ke^{-r\tau} \underbrace{(A_2 + B_2)}_{\Pi_2},
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
A_1 &:= \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{i\phi d_1 \sigma\sqrt{\tau} - \frac{1}{2}\phi^2\sigma^2\tau}}{i\phi} d\phi = N(d_1), \\
B_1 &:= \frac{e^{\mu_c \tau}}{2\pi}\int_{-\infty}^{+\infty} e^{i\phi d_1 \sigma\sqrt{\tau} - \frac{1}{2}\phi^2\sigma^2\tau} \sum_{k=3}^{n}\left[\frac{\rho_k}{k!}\sum_{h=1}^{k} C_k^h \,(i\phi)^{h-1}\left(\sigma\sqrt{\tau}\right)^k\right] d\phi, \\
A_2 &:= \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{i\phi d_2 \sigma\sqrt{\tau} - \frac{1}{2}\phi^2\sigma^2\tau}}{i\phi} d\phi = N(d_2), \\
B_2 &:= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{i\phi d_2 \sigma\sqrt{\tau} - \frac{1}{2}\phi^2\sigma^2\tau}}{i\phi} \sum_{k=3}^{n}\left[\frac{\rho_k}{k!}\left(i\phi\sigma\sqrt{\tau}\right)^k\right] d\phi, \\
F_t^T B_1 - KB_2 &= -K\phi(d_2)\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}(-1)^k \sum_{m=1}^{k-1}(-\sigma\sqrt{\tau})^m He_{k-m-1}(d_2)\right].
\end{aligned}$$

Remark 10. Based on the proof of Proposition 4, it is not too difficult to obtain Corollary 1, and thus its proof is omitted.

If the random variable Y is normally distributed, then the higher order coefficients $\rho_k, k = 3, 4, ..., n$ defined in Definition 1 are all equal to zero, and therefore $\mu_c = 0$ and Equation (8) degenerates to the Black and Scholes (1973) model. In this case, by comparing the corresponding option pricing formulas (i.e., the Black and Scholes (1973) formula and the Bakshi and Madan's (2000) formula), we can find that $\Pi_1 = N(d_1)$ and $\Pi_2 = N(d_2)$ describe the probabilities that options will be exercised under share measure and risk-neutral measure, respectively.

When the higher order coefficients $\rho_k \neq 0, k = 3, 4, ..., n$, even though Π_1 and Π_2 can still describe the probabilities that options will be exercised under the corresponding measures, $N(d_1)$ and $N(d_2)$ can no longer describe them. This is because $\Pi_1 \neq N(d_1)$ and $\Pi_2 \neq N(d_2)$ in this case.

4 Cumulants under Risk-neutral Measure and Share Measure

Given the characteristic functions $f_1(\phi)$ and $f_2(\phi)$ with log-return R_τ as random variable, we are able to compute the corresponding moment generating functions and cumulant generating functions under share measure and risk-neutral measure, and therefore determining the representations for the cumulants with Gram-Charlier density under different measures. In order to have a deeper understanding of the share measure, it is meaningful to investigate the relationship between the cumulants with Gram-Charlier density under share measure and risk-neutral measure.

4.1 Cumulant Generating Functions with Gram-Charlier Density under $\mathcal{Q}$-measure and $\mathcal{S}$-measure

Lemma 10. *The cumulant generating function (i.e., CGF) of the random variable* $R_\tau = \ln \frac{S_T}{F_t^T} = \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Y$ *with Gram-Charlier density under risk-neutral measure (i.e., $\mathcal{Q}$-measure) is given by*

$$K^{\mathcal{Q}}(\phi) = \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau\phi + \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\phi\sigma\sqrt{\tau}\right)^k. \tag{21}$$

Proof. See Appendix M.

□

Lemma 11. *The cumulant generating function (i.e., CGF) of the random variable* $R_\tau = \ln \frac{S_T}{F_t^T} = \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Y$ *with Gram-Charlier density under share measure (i.e.,*

$\mathcal{S}$-measure) is given by

$$K^{\mathcal{S}}(\phi) = \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c + \frac{1}{2}\sigma^2\right)\tau\phi + \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left[(\phi+1)\,\sigma\sqrt{\tau}\right]^k - \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k. \tag{22}$$

Proof. See Appendix N.

□

In order to deal with the term $\ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\phi\sigma\sqrt{\tau}\right)^k = \ln\left[1+\sum_{k=3}^{n}\frac{\rho_k}{k!}\left(\phi\sigma\sqrt{\tau}\right)^k\right]$ in Equation (21) and the term $\ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left[(\phi+1)\,\sigma\sqrt{\tau}\right]^k = \ln\left\{1+\sum_{k=3}^{n}\frac{\rho_k}{k!}\left[(\phi+1)\,\sigma\sqrt{\tau}\right]^k\right\}$ in Equation (22), we consider applying Taylor expansion to these two logarithm terms, respectively.

Thus, Equation (21) can be rewritten as follows

$$\begin{aligned} K^{\mathcal{Q}}(\phi) = &\frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau\phi \\ &+ \sum_{q=3}^{+\infty}\frac{1}{q!}\left(\sigma\sqrt{\tau}\right)^q\sum_{h=1}^{q}(-1)^{h-1}(h-1)!B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right)\phi^q, \end{aligned} \tag{23}$$

where

$$B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right) = \sum_{\substack{3h_3+4h_4+\cdots+nh_n=q\\ h_3+h_4+\cdots+h_n=h\\ h_3,h_4,\cdots,h_n\geq 0}}\frac{q!}{h_3!h_4!\cdots h_n!}\left(\frac{\rho_3}{3!}\right)^{h_3}\left(\frac{\rho_4}{4!}\right)^{h_4}\cdots\left(\frac{\rho_n}{n!}\right)^{h_n}$$

refers to the partial or incomplete exponential Bell polynomials. [2]

Based on Equation (23), we are able to calculate the m-th order partial derivatives of the cumulant generating function under $\mathcal{Q}$-measure with respect to ϕ, ($m = 0,1,2,3,\cdots$).

When $m = 0,1,2$, we have

$$K^{\mathcal{Q}}(\phi) = \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau\phi + \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\phi\sigma\sqrt{\tau}\right)^k, \tag{24}$$

$$\begin{aligned} \frac{\partial}{\partial\phi}K^{\mathcal{Q}}(\phi) = &\sigma^2\tau\phi + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau \\ &+ \sum_{q=3}^{+\infty}\frac{1}{(q-1)!}\left(\sigma\sqrt{\tau}\right)^q\sum_{h=1}^{q}(-1)^{h-1}(h-1)!B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right)\phi^{q-1}, \end{aligned} \tag{25}$$

[2] See Appendix O for the detailed derivation of Equation (23).

$$\frac{\partial^2}{\partial\phi^2}K^{\mathcal{Q}}(\phi) = \sigma^2\tau + \sum_{q=3}^{+\infty}\frac{1}{(q-2)!}\left(\sigma\sqrt{\tau}\right)^q\sum_{h=1}^{q}(-1)^{h-1}(h-1)!B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right)\phi^{q-2}. \tag{26}$$

When $m \geq 3$, we have

$$\frac{\partial^m}{\partial\phi^m}K^{\mathcal{Q}}(\phi) = \sum_{q=m}^{+\infty}\frac{1}{(q-m)!}\left(\sigma\sqrt{\tau}\right)^q\sum_{h=1}^{q}(-1)^{h-1}(h-1)!B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right)\phi^{q-m}. \tag{27}$$

Similarly, Equation (22) can be rewritten as follows

$$\begin{aligned} K^{\mathcal{S}}(\phi) &= \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c + \frac{1}{2}\sigma^2\right)\tau\phi - \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k \\ &\quad + \sum_{q=3}^{+\infty}\frac{1}{q!}\left(\sigma\sqrt{\tau}\right)^q\sum_{h=1}^{q}(-1)^{h-1}(h-1)!B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right)(\phi+1)^q, \end{aligned} \tag{28}$$

where

$$B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right) = \sum_{\substack{3h_3+4h_4+\cdots+nh_n=q \\ h_3+h_4+\cdots+h_n=h \\ h_3,h_4,\cdots,h_n\geq 0}}\frac{q!}{h_3!h_4!\cdots h_n!}\left(\frac{\rho_3}{3!}\right)^{h_3}\left(\frac{\rho_4}{4!}\right)^{h_4}\cdots\left(\frac{\rho_n}{n!}\right)^{h_n}$$

refers to the partial or incomplete exponential Bell polynomials. [3]

Based on Equation (28), we are able to calculate the m-th order partial derivatives of the cumulant generating function under $\mathcal{S}$-measure with respect to ϕ, $(m = 0, 1, 2, 3, \cdots)$.

When $m = 0, 1, 2$, we have

$$K^{\mathcal{S}}(\phi) = \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c + \frac{1}{2}\sigma^2\right)\tau\phi + \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left[(\phi+1)\,\sigma\sqrt{\tau}\right]^k - \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k, \tag{29}$$

$$\begin{aligned} \frac{\partial}{\partial\phi}K^{\mathcal{S}}(\phi) &= \sigma^2\tau\phi + \left(\mu_c + \frac{1}{2}\sigma^2\right)\tau \\ &\quad + \sum_{q=3}^{+\infty}\frac{1}{(q-1)!}\left(\sigma\sqrt{\tau}\right)^q\sum_{h=1}^{q}(-1)^{h-1}(h-1)!B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right)(\phi+1)^{q-1}, \end{aligned} \tag{30}$$

$$\frac{\partial^2}{\partial\phi^2}K^{\mathcal{S}}(\phi) = \sigma^2\tau + \sum_{q=3}^{+\infty}\frac{1}{(q-2)!}\left(\sigma\sqrt{\tau}\right)^q\sum_{h=1}^{q}(-1)^{h-1}(h-1)!B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right)(\phi+1)^{q-2}. \tag{31}$$

[3] See Appendix P for the detailed derivation of Equation (28).

When $m \geq 3$, we have

$$\frac{\partial^m}{\partial \phi^m} K^{\mathcal{S}}(\phi) = \sum_{q=m}^{+\infty} \frac{1}{(q-m)!} \left(\sigma\sqrt{\tau}\right)^q \sum_{h=1}^{q} (-1)^{h-1}(h-1)! B_{q,h}\left(0, 0, \rho_3, \cdots, \rho_n\right)(\phi+1)^{q-m}. \tag{32}$$

4.2 Cumulants with Gram-Charlier Density under $\mathcal{Q}$-measure and $\mathcal{S}$-measure

The corresponding cumulants under $\mathcal{Q}$-measure can be obtained by substituting $\phi = 0$ into Equation (24), (25), (26) and (27), given by

$$\kappa_0^{\mathcal{Q}} = K^{\mathcal{Q}}(\phi)\Big|_{\phi=0} = 0, \tag{33}$$

$$\kappa_1^{\mathcal{Q}} = \frac{\partial}{\partial \phi} K^{\mathcal{Q}}(\phi)\Big|_{\phi=0} = \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau, \tag{34}$$

$$\kappa_2^{\mathcal{Q}} = \frac{\partial^2}{\partial \phi^2} K^{\mathcal{Q}}(\phi)\Big|_{\phi=0} = \sigma^2\tau. \tag{35}$$

When $m \geq 3$, we have

$$\kappa_m^{\mathcal{Q}} = \frac{\partial^m}{\partial \phi^m} K^{\mathcal{Q}}(\phi)\Big|_{\phi=0} = \left(\sigma\sqrt{\tau}\right)^m \sum_{h=1}^{m} (-1)^{h-1}(h-1)! B_{m,h}\left(0, 0, \rho_3, \cdots, \rho_n\right). \tag{36}$$

Here we give the examples for cumulants under $\mathcal{Q}$-measure from $m = 3$ to $m = 6$ of the random variable $R_\tau = \ln \frac{S_T}{F_t^T} = \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Y$ with Gram-Charlier density

$$\begin{aligned}
\kappa_3^{\mathcal{Q}} &= \left(\sigma\sqrt{\tau}\right)^3 \rho_3, \\
\kappa_4^{\mathcal{Q}} &= \left(\sigma\sqrt{\tau}\right)^4 \rho_4, \\
\kappa_5^{\mathcal{Q}} &= \left(\sigma\sqrt{\tau}\right)^5 \rho_5, \\
\kappa_6^{\mathcal{Q}} &= \left(\sigma\sqrt{\tau}\right)^6 \left(\rho_6 - 10\rho_3^2\right).
\end{aligned}$$

Since the log-return R_τ can be regarded as the linear combination of normally distributed random variable Y, we can standardize R_τ to become Y by applying $\frac{R_\tau - E_t^{\mathcal{Q}}(R_\tau)}{\sigma\sqrt{\tau}} = Y$. Then the relationships between coefficients and cumulants above are completely consistent with the examples given in Section 2.1.

Similarly, the corresponding cumulants under $\mathcal{S}$-measure can be obtained by substituting $\phi = 0$ into Equation (29), (30), (31) and (32), given by

$$\kappa_0^{\mathcal{S}} = K^{\mathcal{S}}(\phi)\Big|_{\phi=0} = 0, \tag{37}$$

$$\begin{aligned} \kappa_1^{\mathcal{S}} &= \frac{\partial}{\partial \phi} K^{\mathcal{S}}(\phi)\Big|_{\phi=0} \\ &= \left(\mu_c + \frac{1}{2}\sigma^2\right)\tau + \sum_{q=3}^{+\infty} \frac{1}{(q-1)!} \left(\sigma\sqrt{\tau}\right)^q \sum_{h=1}^{q} (-1)^{h-1}(h-1)! B_{q,h}\left(0, 0, \rho_3, \cdots, \rho_n\right), \end{aligned} \tag{38}$$

$$\begin{aligned} \kappa_2^{\mathcal{S}} &= \frac{\partial^2}{\partial \phi^2} K^{\mathcal{S}}(\phi)\Big|_{\phi=0} \\ &= \sigma^2\tau + \sum_{q=3}^{+\infty} \frac{1}{(q-2)!} \left(\sigma\sqrt{\tau}\right)^q \sum_{h=1}^{q} (-1)^{h-1}(h-1)! B_{q,h}\left(0, 0, \rho_3, \cdots, \rho_n\right). \end{aligned} \tag{39}$$

When $m \geq 3$, we have

$$\begin{aligned} \kappa_m^{\mathcal{S}} &= \frac{\partial^m}{\partial \phi^m} K^{\mathcal{S}}(\phi)\Big|_{\phi=0} \\ &= \sum_{q=m}^{+\infty} \frac{1}{(q-m)!} \left(\sigma\sqrt{\tau}\right)^q \sum_{h=1}^{q} (-1)^{h-1}(h-1)! B_{q,h}\left(0, 0, \rho_3, \cdots, \rho_n\right). \end{aligned} \tag{40}$$

Proposition 5. *The relationships between cumulants with Gram-Charlier density under $\mathcal{Q}$-measure and $\mathcal{S}$-measure are given by*

when $m = 0$,

$$\kappa_0^{\mathcal{S}} = \kappa_0^{\mathcal{Q}}, \tag{41}$$

when $m \geq 1$,

$$\kappa_m^{\mathcal{S}} = \sum_{q=0}^{+\infty} \frac{\kappa_{m+q}^{\mathcal{Q}}}{q!}. \tag{42}$$

Proof. See Appendix Q.

□

5 Conclusion

As a generalization of the normal density, the Gram-Charlier series expansion can be used to approximate densities that deviate from the normal one, where the third- and fourth-order cumulants can directly appear as the parameters. The Gram-Charlier density applied in option pricing is able to extend the normally distributed return in the Black and Scholes (1973) model to include higher order cumulants, which are related to the parameters implied by the Gram-Charlier density as well. The majority of existing literature is restricted to truncating the series expansion until the fourth moment or cumulant, since for the standardized random variable, its third- and fourth-order cumulants are exactly the skewness and excess kurtosis, which are commonly used in financial applications.

In this paper, based on some indispensable properties of Hermite polynomials, we derive a new analytic form of option pricing formula under Gram-Charlier density of arbitrary order, where the probabilities that options will be exercised under share measure and risk-neutral measure (namely, Π_1 and Π_2, respectively) can be explicitly expressed. Compared with the Black and Scholes (1973) formula, we find that the hedging ratio, Δ^{GC}, of the option pricing formula under Gram-Charlier density, is equal to Π_1 instead of $N(d_1)$.

Following the derivations in previous literature, we are able to simplify the option pricing formula under Gram-Charlier density proposed by Aschakulporn (2022), which consists of three terms. Moreover, we specify the characteristic function under Gram-Charlier density, and make the Bakshi and Madan's (2000) formula become an alternative option pricing formula under Gram-Charlier density. Subsequently, the equivalence between our new formula and these two forms has been demonstrated, and the relationship between terms in different formulas is also discussed.

Additionally, we further investigate the relationship between the cumulants with Gram-Charlier density under share measure and risk-neutral measure to help users have a deeper understanding of the share measure.

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Appendix

A Proof of Lemma 1

Proof. (Note that $(-1)!! = 0!! = 1$.)

When $k = 0$ and $k = 1$, we have

$$He_0(y) = C_0^0 \cdot (-1)^0 \cdot 1 \cdot y^0 = 1$$

and

$$He_1(y) = C_1^0 \cdot (-1)^0 \cdot 1 \cdot y^{1-0} = y$$

hold, respectively.

Assume that for an arbitrary natural number $w < k$, we have

$$He_w(y) = \sum_{m=0}^{\lfloor \frac{w}{2} \rfloor} C_w^{2m} (-1)^m (2m-1)!! y^{k-2m}$$

holds, now we need to show that this summation formula also holds for the natural number k.

Since the recursive form of Hermite polynomial

$$He_k(y) = yHe_{k-1}(y) - (k-1)He_{k-2}(y)$$

is known, according to our assumption, we have

$$He_{k-1}(y) = \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} C_{k-1}^{2m} (-1)^m (2m-1)!! y^{k-1-2m}$$

and

$$He_{k-2}(y) = \sum_{m=0}^{\lfloor \frac{k-2}{2} \rfloor} C_{k-2}^{2m} (-1)^m (2m-1)!! y^{k-2-2m}$$

hold, respectively.

When k is odd, we set $k = 2u + 1$, then

$$\begin{aligned}
& He_{k-1}(y) \\
&= He_{2u}(y) \\
&= \sum_{m=0}^{u} C_{2u}^{2m} (-1)^m (2m-1)!! y^{2u-2m} \\
&= C_{2u}^0 \cdot (-1)^0 \cdot y^{2u} + C_{2u}^2 \cdot (-1)^1 \cdot y^{2u-2} + C_{2u}^4 \cdot (-1)^2 \cdot 3!! \cdot y^{2u-4} \\
&\quad + \cdots + C_{2u}^{2m} \cdot (-1)^m \cdot (2m-1)!! \cdot y^{2u-2m} + \cdots + C_{2u}^{2u} \cdot (-1)^u \cdot (2u-1)!! \cdot y^0,
\end{aligned} \tag{A.1}$$

$$
\begin{aligned}
& He_{k-2}(y) \\
&= He_{2u-1}(y) \\
&= \sum_{m=0}^{u-1} C_{2u-1}^{2m}(-1)^m(2m-1)!!y^{2u-2m-1} \\
&= C_{2u-1}^{0} \cdot (-1)^0 \cdot y^{2u-1} + C_{2u-1}^{2} \cdot (-1)^1 \cdot y^{2u-3} + C_{2u-1}^{4} \cdot (-1)^2 \cdot 3!! \cdot y^{2u-5} \\
&\quad + \cdots + C_{2u-1}^{2m-2} \cdot (-1)^{m-1} \cdot (2m-3)!! \cdot y^{2u-2m+1} + C_{2u-1}^{2m} \cdot (-1)^m \cdot (2m-1)!! \cdot y^{2u-2m-1} \\
&\quad + \cdots + C_{2u-1}^{2u-2} \cdot (-1)^{u-1} \cdot (2u-3)!! \cdot y,
\end{aligned}
\tag{A.2}
$$

and thus

$$
\begin{aligned}
& He_k(y) \\
&= He_{2u+1}(y) \\
&= yHe_{2u}(y) - 2uHe_{2u-1}(y) \\
&= \Big[C_{2u}^{0} \cdot (-1)^0 \cdot y^{2u+1} + C_{2u}^{2} \cdot (-1)^1 \cdot y^{2u-1} + C_{2u}^{4} \cdot (-1)^2 \cdot 3!! \cdot y^{2u-3} \\
&\quad + \cdots + C_{2u}^{2m} \cdot (-1)^m \cdot (2m-1)!! \cdot y^{2u-2m+1} + \cdots + C_{2u}^{2u} \cdot (-1)^u \cdot (2u-1)!! \cdot y\Big] \\
&\quad - \Big[2u \cdot C_{2u-1}^{0} \cdot (-1)^0 \cdot y^{2u-1} + 2u \cdot C_{2u-1}^{2} \cdot (-1)^1 \cdot y^{2u-3} + 2u \cdot C_{2u-1}^{4} \cdot (-1)^2 \cdot 3!! \cdot y^{2u-5} \\
&\quad + 2u \cdot C_{2u-1}^{2m-2} \cdot (-1)^{m-1} \cdot (2m-3)!! \cdot y^{2u-2m+1} \\
&\quad + 2u \cdot C_{2u-1}^{2m} \cdot (-1)^m \cdot (2m-1)!! \cdot y^{2u-2m-1} + \cdots + 2u \cdot C_{2u-1}^{2u-2} \cdot (-1)^{u-1} \cdot (2u-3)!! \cdot y\Big] \\
&= C_{2u}^{0} \cdot (-1)^0 \cdot y^{2u+1} + \Big[C_{2u}^{2} \cdot (-1)^1 - 2u \cdot C_{2u-1}^{0} \cdot (-1)^0\Big] \cdot y^{2u-1} \\
&\quad + \Big[C_{2u}^{4} \cdot (-1)^2 \cdot 3!! - 2u \cdot C_{2u-1}^{2} \cdot (-1)^1\Big] \cdot y^{2u-3} \\
&\quad + \cdots + \underbrace{\Big[C_{2u}^{2m} \cdot (-1)^m \cdot (2m-1)!! - 2u \cdot C_{2u-1}^{2m-2} \cdot (-1)^{m-1} \cdot (2m-3)!!\Big]}_{\Omega} \cdot y^{2u-2m+1} \\
&\quad + \cdots + \Big[C_{2u}^{2u} \cdot (-1)^u \cdot (2u-1)!! - 2u \cdot C_{2u-1}^{2u-2} \cdot (-1)^{u-1} \cdot (2u-3)!!\Big] \cdot y.
\end{aligned}
\tag{A.3}
$$

Since

$$
\begin{aligned}
&\Omega\\
&= \frac{(2u)!}{(2m)!(2u-2m)!}\cdot(-1)^m\cdot(2m-1)!! - 2u\cdot\frac{(2u-1)!}{(2m-2)!(2u-2m+1)!}\cdot(-1)^{m-1}\cdot(2m-3)!!\\
&= \frac{(2u-1)!\cdot(-1)^m\cdot(2m-3)!!}{(2m-2)!(2u-2m)!}\cdot\left[\frac{2u\cdot(2m-1)}{2m\cdot(2m-1)}+\frac{2u}{2u-2m+1}\right]\\
&= \frac{(2u-1)!\cdot(-1)^m\cdot(2m-3)!!}{(2m-2)!(2u-2m)!}\cdot\frac{2u\cdot[(2m-1)\cdot(2u-2m+1)+2m\cdot(2m-1)]}{2m\cdot(2m-1)\cdot(2u-2m+1)}\\
&= \frac{(2u-1)!\cdot(-1)^m\cdot(2m-3)!!}{(2m-2)!(2u-2m)!}\cdot\frac{2u\cdot(2m-1)\cdot(2u+1)}{2m\cdot(2m-1)\cdot(2u-2m+1)}\\
&= \frac{(2u+1)!}{(2m)!(2u-2m+1)!}\cdot(-1)^m\cdot(2m-1)!!\\
&= C_{2u+1}^{2m}\cdot(-1)^m\cdot(2m-1)!!,
\end{aligned}
\tag{A.4}
$$

where $m = 1, 2, \cdots, u$, and then we have

$$
\begin{aligned}
&He_k(y)\\
&= He_{2u+1}(y)\\
&= C_{2u+1}^{0}\cdot(-1)^0\cdot y^{2u+1} + C_{2u+1}^{2}\cdot(-1)^1\cdot y^{2u-1} + C_{2u+1}^{4}\cdot(-1)^2\cdot 3!!\cdot y^{2u-3}\\
&\quad + \cdots + C_{2u+1}^{2m}\cdot(-1)^m\cdot(2m-1)!!\cdot y^{2u-2m+1} + \cdots + C_{2u+1}^{2u}\cdot(-1)^u\cdot(2u-1)!!\cdot y\\
&= \sum_{m=0}^{u} C_{2u+1}^{2m}(-1)^m(2m-1)!!y^{2u-2m+1}\\
&= \sum_{m=0}^{\lfloor \frac{k}{2}\rfloor} C_k^{2m}(-1)^m(2m-1)!!y^{k-2m}.
\end{aligned}
\tag{A.5}
$$

When k is even, we set $k = 2u$, then

$$
\begin{aligned}
&He_{k-1}(y)\\
&= He_{2u-1}(y)\\
&= \sum_{m=0}^{u-1} C_{2u-1}^{2m}(-1)^m(2m-1)!!y^{2u-2m-1}\\
&= C_{2u-1}^{0}\cdot(-1)^0\cdot y^{2u-1} + C_{2u-1}^{2}\cdot(-1)^1\cdot y^{2u-3} + C_{2u-1}^{4}\cdot(-1)^2\cdot 3!!\cdot y^{2u-5}\\
&\quad + \cdots + C_{2u-1}^{2m}\cdot(-1)^m\cdot(2m-1)!!\cdot y^{2u-2m-1} + \cdots + C_{2u-1}^{2u-2}\cdot(-1)^{u-1}\cdot(2u-3)!!\cdot y,
\end{aligned}
\tag{A.6}
$$

$$\begin{aligned}
&He_{k-2}(y)\\
&= He_{2u-2}(y)\\
&= \sum_{m=0}^{u-1} C_{2u-2}^{2m}(-1)^m(2m-1)!!y^{2u-2m-2}\\
&= C_{2u-2}^{0}\cdot(-1)^0\cdot y^{2u-2} + C_{2u-2}^{2}\cdot(-1)^1\cdot y^{2u-4} + C_{2u-2}^{4}\cdot(-1)^2\cdot 3!!\cdot y^{2u-6}\\
&\quad + \cdots + C_{2u-2}^{2m-2}\cdot(-1)^{m-1}\cdot(2m-3)!!\cdot y^{2u-2m} + C_{2u-2}^{2m}\cdot(-1)^m\cdot(2m-1)!!\cdot y^{2u-2m-2}\\
&\quad + \cdots + C_{2u-2}^{2u-4}\cdot(-1)^{u-2}\cdot(2u-5)!!\cdot y^2 + C_{2u-2}^{2u-2}\cdot(-1)^{u-1}\cdot(2u-3)!!\cdot y^0,
\end{aligned} \tag{A.7}$$

and thus

$$\begin{aligned}
&He_k(y)\\
&= He_{2u}(y)\\
&= yHe_{2u-1}(y) - (2u-1)He_{2u-2}(y)\\
&= \Big[C_{2u-1}^{0}\cdot(-1)^0\cdot y^{2u} + C_{2u-1}^{2}\cdot(-1)^1\cdot y^{2u-2} + C_{2u-1}^{4}\cdot(-1)^2\cdot 3!!\cdot y^{2u-4}\\
&\quad + \cdots + C_{2u-1}^{2m}\cdot(-1)^m\cdot(2m-1)!!\cdot y^{2u-2m} + \cdots + C_{2u-1}^{2u-2}\cdot(-1)^{u-1}\cdot(2u-3)!!\cdot y^2\Big]\\
&\quad - \Big[(2u-1)\cdot C_{2u-2}^{0}\cdot(-1)^0\cdot y^{2u-2} + (2u-1)\cdot C_{2u-2}^{2}\cdot(-1)^1\cdot y^{2u-4}\\
&\quad + (2u-1)\cdot C_{2u-2}^{4}\cdot(-1)^2\cdot 3!!\cdot y^{2u-6} + \cdots + (2u-1)\cdot C_{2u-2}^{2m-2}\cdot(-1)^{m-1}\cdot(2m-3)!!\cdot y^{2u-2m}\\
&\quad + (2u-1)\cdot C_{2u-2}^{2m}\cdot(-1)^m\cdot(2m-1)!!\cdot y^{2u-2m-2} + \cdots\\
&\quad + (2u-1)\cdot C_{2u-2}^{2u-4}\cdot(-1)^{u-2}\cdot(2u-5)!!\cdot y^2 + (2u-1)\cdot C_{2u-2}^{2u-2}\cdot(-1)^{u-1}\cdot(2u-3)!!\cdot y^0\Big]\\
&= C_{2u-1}^{0}\cdot(-1)^0\cdot y^{2u} + \Big[C_{2u-1}^{2}\cdot(-1)^1 - (2u-1)\cdot C_{2u-2}^{0}\cdot(-1)^0\Big]\cdot y^{2u-2}\\
&\quad + \Big[C_{2u-1}^{4}\cdot(-1)^2\cdot 3!! - (2u-1)\cdot C_{2u-2}^{2}\cdot(-1)^1\Big]\cdot y^{2u-4}\\
&\quad + \cdots + \underbrace{\Big[C_{2u-1}^{2m}\cdot(-1)^m\cdot(2m-1)!! - (2u-1)\cdot C_{2u-2}^{2m-2}\cdot(-1)^{m-1}\cdot(2m-3)!!\Big]}_{\Omega_1}\cdot y^{2u-2m}\\
&\quad + \cdots + \Big[C_{2u-1}^{2u-2}\cdot(-1)^{u-1}\cdot(2u-3)!! - (2u-1)\cdot C_{2u-2}^{2u-4}\cdot(-1)^{u-2}\cdot(2u-5)!!\Big]\cdot y^2\\
&\quad \underbrace{-\Big[(2u-1)\cdot C_{2u-2}^{2u-2}\cdot(-1)^{u-1}\cdot(2u-3)!!\Big]}_{\Omega_2}\cdot y^0.
\end{aligned} \tag{A.8}$$

Since

$$C_{2u-1}^{0}\cdot(-1)^0\cdot y^{2u} = C_{2u}^{0}\cdot(-1)^0\cdot y^{2u}, \tag{A.9}$$

$$\begin{aligned}
\Omega_2 &= (2u-1)\cdot C_{2u-2}^{2u-2}\cdot(-1)^u\cdot(2u-3)!!\\
&= C_{2u}^{2u}\cdot(-1)^u\cdot(2u-1)!!,
\end{aligned} \tag{A.10}$$

$$
\begin{aligned}
&\Omega_1 \\
&= \frac{(2u-1)!}{(2m)!(2u-2m-1)!} \cdot (-1)^m \cdot (2m-1)!! - (2u-1) \cdot \frac{(2u-2)!}{(2m-2)!(2u-2m)!} \cdot (-1)^{m-1} \cdot (2m-3)!! \\
&= \frac{(2u-1)!}{(2m-2)!(2u-2m-1)!} \cdot (-1)^m \cdot (2m-3)!! \cdot \left[\frac{2m-1}{2m \cdot (2m-1)} + \frac{1}{2u-2m}\right] \\
&= \frac{(2u-1)!}{(2m-2)!(2u-2m-1)!} \cdot (-1)^m \cdot (2m-3)!! \cdot \frac{(2m-1)\cdot(2u-2m) + 2m \cdot (2m-1)}{2m \cdot (2m-1) \cdot (2u-2m)} \\
&= \frac{(2u)!}{(2m)!(2u-2m)!} \cdot (-1)^m \cdot (2m-1)!! \\
&= C_{2u}^{2m} \cdot (-1)^m \cdot (2m-1)!!,
\end{aligned}
\tag{A.11}
$$

where $m = 1, 2, \cdots, u-1$, and then we have

$$
\begin{aligned}
&He_k(y) \\
&= He_{2u}(y) \\
&= C_{2u}^0 \cdot (-1)^0 \cdot y^{2u} + C_{2u}^2 \cdot (-1)^1 \cdot y^{2u-2} + C_{2u}^4 \cdot (-1)^2 \cdot 3!! \cdot y^{2u-4} \\
&\quad + \cdots + C_{2u}^{2m} \cdot (-1)^m \cdot (2m-1)!! \cdot y^{2u-2m} + \cdots + C_{2u}^{2u} \cdot (-1)^u \cdot (2u-1)!! \cdot y^0 \\
&= \sum_{m=0}^{u} C_{2u}^{2m} (-1)^m (2m-1)!! y^{2u-2m} \\
&= \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} C_k^{2m} (-1)^m (2m-1)!! y^{k-2m}.
\end{aligned}
\tag{A.12}
$$

By induction, we have $He_k(y) = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} C_k^{2m} (-1)^m (2m-1)!! y^{k-2m}$ holds for any natural number k.

Hence, the summation form of Hermite polynomial is given by

$$
\begin{aligned}
He_k(y) &= \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} C_k^{2m} (-1)^m (2m-1)!! y^{k-2m} \\
&= \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!(2m-1)!!}{(2m)!(k-2m)!} (-1)^m y^{k-2m} \\
&= \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{(2m)!!(k-2m)!} (-1)^m y^{k-2m} \\
&= \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{2^k m!(k-2m)!} (-1)^m y^{k-2m},
\end{aligned}
\tag{A.13}
$$

where $He_k(y)$ is the k-th order Hermite polynomial, $C_k^{2m} = \frac{k!}{(2m)!(k-2m)!}$ is a combinatorial number, $\lfloor \cdot \rfloor$ means the floor function of "$\cdot$" (i.e., the largest integer that is less than or equal to "$\cdot$"), $(\cdot)!$ and $(\cdot)!!$ are the factorial and double factorial of "$\cdot$", respectively.

□

B Proof of Lemma 2

Proof. By Lemma 1 and binomial theorem, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty}(it+y)^k\phi(t)dt\\
&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{t^2}{2}}\left[C_k^0y^k+C_k^1(it)y^{k-1}+C_k^2(it)^2y^{k-2}+\cdots+C_k^k(it)^k\right]dt\\
&=C_k^0\cdot y^k+C_k^2\cdot i^2\cdot y^{k-2}+C_k^4\cdot i^4\cdot 3!!\cdot y^{k-4}+\cdots+C_k^{2m}\cdot i^{2m}\cdot(2m-1)!!\cdot y^{k-2m}\\
&\quad+\cdots+C_k^{2\lfloor\frac{k}{2}\rfloor}\cdot i^{2\lfloor\frac{k}{2}\rfloor}\cdot\left(2\left\lfloor\frac{k}{2}\right\rfloor-1\right)!!\cdot y^{k-2\lfloor\frac{k}{2}\rfloor}\\
&=\sum_{m=0}^{\lfloor\frac{k}{2}\rfloor}C_k^{2m}(-1)^m(2m-1)!!y^{k-2m}\\
&=He_k(y).
\end{aligned}\tag{B.1}$$

□

C Proof of Lemma 3

Proof. By Lemma 2 and binomial theorem, we have

$$\begin{aligned}
&He_k(y+\Delta y)\\
&=\int_{-\infty}^{+\infty}\left[(it+y)+\Delta y\right]^k\phi(t)dt\\
&=\int_{-\infty}^{+\infty}\left[C_k^0(it+y)^k+C_k^1(\Delta y)^1(it+y)^{k-1}+\cdots+C_k^k(\Delta y)^k(it+y)^0\right]\phi(t)dt\\
&=\sum_{m=0}^{k}(\Delta y)^mC_k^mHe_{k-m}(y).
\end{aligned}\tag{C.1}$$

□

D Proof of Lemma 4

Proof. By Lemma 2 and substitution method, we have

$$\begin{aligned}
He_k(-y)&=\int_{-\infty}^{+\infty}(it-y)^k\phi(t)dt\\
(\text{Let } t=-z)\quad&=\int_{+\infty}^{-\infty}(-iz-y)^k\phi(-z)d(-z)\\
&=(-1)^{k+1}\int_{+\infty}^{-\infty}(iz+y)^k\phi(-z)dz\\
&=(-1)^k\int_{-\infty}^{+\infty}(iz+y)^k\phi(z)dz\\
&=(-1)^kHe_k(y).
\end{aligned}\tag{D.1}$$

□

E Proof of Lemma 5

Proof. By Lemma 2, we have

$$\begin{aligned} \frac{d}{dy}He_k(y) &= \int_{-\infty}^{+\infty} \frac{d}{dy}(it+y)^k \phi(t)dt \\ &= \int_{-\infty}^{+\infty} k(it+y)^{k-1}\phi(t)dt \\ &= kHe_{k-1}(y). \end{aligned} \tag{E.1}$$

□

F Proof of Lemma 6

Proof.

$$E_t^{\mathcal{Q}}[S_T] = F_t^T e^{\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau} E_t^{\mathcal{Q}}\left[e^{\sigma\sqrt{\tau}Y}\right] = F_t^T,$$

$$\Longrightarrow e^{\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau} E_t^{\mathcal{Q}}\left[e^{\sigma\sqrt{\tau}Y}\right] = 1.$$

We first focus on $E_t^{\mathcal{Q}}\left[e^{\sigma\sqrt{\tau}Y}\right]$

$$\begin{aligned} E_t^{\mathcal{Q}}\left[e^{\sigma\sqrt{\tau}Y}\right] &= \int_{-\infty}^{+\infty} e^{\sigma\sqrt{\tau}y} f(y)dy \\ &= \int_{-\infty}^{+\infty} e^{\sigma\sqrt{\tau}y}\phi(y)dy + \sum_{k=3}^{n}(-1)^k \frac{\rho_k}{k!}\int_{-\infty}^{+\infty} e^{\sigma\sqrt{\tau}y}\phi^{(k)}(y)dy, \end{aligned} \tag{F.1}$$

When $k = 3$, we have

$$
\begin{aligned}
& \int_{-\infty}^{+\infty} e^{\sigma\sqrt{\tau}y}\phi^{(3)}(y)dy \\
&= \int_{-\infty}^{+\infty} e^{\sigma\sqrt{\tau}y}d\phi^{(2)}(y) \\
&= e^{\sigma\sqrt{\tau}y}\phi^{(2)}(y)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \phi^{(2)}(y)\sigma\sqrt{\tau}e^{\sigma\sqrt{\tau}y}dy \\
&= 0 - \sigma\sqrt{\tau}\int_{-\infty}^{+\infty} e^{\sigma\sqrt{\tau}y}d\phi^{(1)}(y) \\
&= -\sigma\sqrt{\tau}\left[e^{\sigma\sqrt{\tau}y}\phi^{(1)}(y)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \phi^{(1)}(y)\sigma\sqrt{\tau}e^{\sigma\sqrt{\tau}y}dy\right] \\
&= \sigma^2\tau\int_{-\infty}^{+\infty} e^{\sigma\sqrt{\tau}y}d\phi(y) \\
&= \sigma^2\tau\left[e^{\sigma\sqrt{\tau}y}\phi(y)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \phi(y)\sigma\sqrt{\tau}e^{\sigma\sqrt{\tau}y}dy\right] \\
&= -\left(\sigma\sqrt{\tau}\right)^3\int_{-\infty}^{+\infty} e^{\sigma\sqrt{\tau}y}\phi(y)dy \\
&= -\left(\sigma\sqrt{\tau}\right)^3\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2+\sigma\sqrt{\tau}y}dy \\
&= -\left(\sigma\sqrt{\tau}\right)^3\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(y-\sigma\sqrt{\tau}\right)^2+\frac{1}{2}\sigma^2\tau}dy \\
\left(\text{Let } z = y - \sigma\sqrt{\tau}\right) \quad &= -\left(\sigma\sqrt{\tau}\right)^3 e^{\frac{1}{2}\sigma^2\tau}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= -\left(\sigma\sqrt{\tau}\right)^3 e^{\frac{1}{2}\sigma^2\tau}.
\end{aligned}
\tag{F.2}
$$

Similarly, for $k > 3$, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{\sigma\sqrt{\tau}y}\phi^{(k)}(y)dy \\
&= \int_{-\infty}^{+\infty} e^{\sigma\sqrt{\tau}y}d\phi^{(k-1)}(y) \\
&= e^{\sigma\sqrt{\tau}y}\phi^{(k-1)}(y)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \phi^{(k-1)}(y)\sigma\sqrt{\tau}e^{\sigma\sqrt{\tau}y}dy \\
&= 0 - \sigma\sqrt{\tau}\int_{-\infty}^{+\infty} e^{\sigma\sqrt{\tau}y}d\phi^{(k-2)}(y) \\
&= -\sigma\sqrt{\tau}\left[e^{\sigma\sqrt{\tau}y}\phi^{(k-2)}(y)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \phi^{(k-2)}(y)\sigma\sqrt{\tau}e^{\sigma\sqrt{\tau}y}dy\right] \\
&= \sigma^2\tau\int_{-\infty}^{+\infty} e^{\sigma\sqrt{\tau}y}d\phi^{(k-3)}(y) \\
&= \sigma^2\tau\left[e^{\sigma\sqrt{\tau}y}\phi^{(k-3)}(y)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \phi^{(k-3)}(y)\sigma\sqrt{\tau}e^{\sigma\sqrt{\tau}y}dy\right] \\
&= -\left(\sigma\sqrt{\tau}\right)^3\int_{-\infty}^{+\infty} e^{\sigma\sqrt{\tau}y}d\phi^{(k-4)}(y) \\
&= \cdots \\
&= (-1)^{k-1}\left(\sigma\sqrt{\tau}\right)^{k-1}\int_{-\infty}^{+\infty} e^{\sigma\sqrt{\tau}y}d\phi^{(k-k)}(y) \\
&= (-1)^{k-1}\left(\sigma\sqrt{\tau}\right)^{k-1}\left[e^{\sigma\sqrt{\tau}y}\phi^{(k-k)}(y)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \phi^{(k-k)}(y)\sigma\sqrt{\tau}e^{\sigma\sqrt{\tau}y}dy\right] \\
&= (-1)^{k}\left(\sigma\sqrt{\tau}\right)^{k}\int_{-\infty}^{+\infty} e^{\sigma\sqrt{\tau}y}\phi(y)dy \\
&= (-1)^{k}\left(\sigma\sqrt{\tau}\right)^{k}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2+\sigma\sqrt{\tau}y}dy \\
&= (-1)^{k}\left(\sigma\sqrt{\tau}\right)^{k}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(y-\sigma\sqrt{\tau}\right)^2+\frac{1}{2}\sigma^2\tau}dy \\
\left(\text{Let } z = y-\sigma\sqrt{\tau}\right) \quad &= (-1)^{k}\left(\sigma\sqrt{\tau}\right)^{k}e^{\frac{1}{2}\sigma^2\tau}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= (-1)^{k}\left(\sigma\sqrt{\tau}\right)^{k}e^{\frac{1}{2}\sigma^2\tau}.
\end{aligned} \tag{F.3}$$

Thus,

$$E_t^{\mathcal{Q}}\left[e^{\sigma\sqrt{\tau}Y}\right] = \left[1+\sum_{k=3}^{n}\frac{\rho_k}{k!}(\sigma\sqrt{\tau})^k\right]e^{\frac{1}{2}\sigma^2\tau},$$

$$\Longrightarrow\ e^{\left(\mu_c-\frac{1}{2}\sigma^2+\frac{1}{2}\sigma^2\right)\tau}\left[1+\sum_{k=3}^{n}\frac{\rho_k}{k!}(\sigma\sqrt{\tau})^k\right] = 1,$$

$$\Longrightarrow\ -\mu_c\tau = \ln\left[1+\sum_{k=3}^{n}\frac{\rho_k}{k!}(\sigma\sqrt{\tau})^k\right],$$

$$\Longrightarrow\ \mu_c = -\frac{1}{\tau}\ln\left[1+\sum_{k=3}^{n}\frac{\rho_k}{k!}(\sigma\sqrt{\tau})^k\right].$$

□

G Proof of Lemma 7

Proof. The price of a European call option under Gram-Charlier density can be calculated with the Harrison and Kreps (1979) and Harrison and Pliska (1981) risk-neutral valuation formula, given by

$$\begin{aligned}
c_t^{GC} &= e^{-r\tau} E_t^{\mathcal{Q}} \left[\max\left(S_T - K, 0\right)\right] \\
&= e^{-r\tau} E_t^{\mathcal{Q}} \left[(S_T - K) \cdot 1_{S_T > K}\right] \\
&= e^{-r\tau} E_t^{\mathcal{Q}} \left[S_T \cdot 1_{S_T > K}\right] - K e^{-r\tau} E_t^{\mathcal{Q}} \left[1_{S_T > K}\right]
\end{aligned}$$

$$\left(S_T > K \implies y > -\frac{\ln \frac{F_t^T}{K} + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} := -d_2 \right)$$

$$\begin{aligned}
&= e^{-r\tau} \int_{-d_2}^{+\infty} F_t^T e^{\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}y} f(y) dy - K e^{-r\tau} \int_{-d_2}^{+\infty} f(y) dy \\
&= e^{-r\tau} F_t^T e^{\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau} \int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y} f(y) dy - K e^{-r\tau} \int_{-d_2}^{+\infty} f(y) dy \\
&= e^{-r\tau} F_t^T e^{\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau} \sum_{k=0}^{n} \frac{\rho_k}{k!} \int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y} He_k(y)\phi(y) dy - K e^{-r\tau} \sum_{k=0}^{n} \frac{\rho_k}{k!} \int_{-d_2}^{+\infty} He_k(y)\phi(y) dy \\
&= e^{-r\tau} F_t^T e^{\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau} \sum_{k=0}^{n} \frac{\rho_k}{k!} \int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y} He_k(y)\phi(y) dy - K e^{-r\tau} \sum_{k=0}^{n} \frac{\rho_k}{k!} \int_{-d_2}^{+\infty} He_k(y)\phi(y) dy \\
&= e^{-r\tau} F_t^T e^{\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau} \left[\underbrace{\int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y} \phi(y) dy}_{I_1} + \sum_{k=3}^{n} \frac{\rho_k}{k!} \underbrace{\int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y} He_k(y)\phi(y) dy}_{I_2} \right] \\
&\quad - K e^{-r\tau} \left[\underbrace{\int_{-d_2}^{+\infty} \phi(y) dy}_{I_3} + \sum_{k=3}^{n} \frac{\rho_k}{k!} \underbrace{\int_{-d_2}^{+\infty} He_k(y)\phi(y) dy}_{I_4} \right],
\end{aligned} \tag{G.1}$$

where

$$d_2 = \frac{\ln \frac{F_t^T}{K} + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, d_1 = d_2 + \sigma\sqrt{\tau},$$

$$\mu_c = -\frac{1}{\tau} \ln \left[1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} (\sigma\sqrt{\tau})^k \right],$$

$$e^{\mu_c \tau} = \frac{1}{1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} (\sigma\sqrt{\tau})^k},$$

$$f(y) = \sum_{k=0}^{n} \frac{\rho_k}{k!} (-1)^k \phi^{(k)}(y) = \sum_{k=0}^{n} \frac{\rho_k}{k!} He_k(y)\phi(y) = \left[1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} He_k(y) \right] \phi(y),$$

$k!$ is the factorial of k, $\phi(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)$ is the standard normal density function, $He_k(y)$ is the k-th order Hermite polynomial defined by $He_k(y) \equiv (-1)^k \frac{\phi^{(k)}(y)}{\phi(y)}$, ρ_k is the coefficient derived from the Hermite polynomial, $F_t^T = S_t e^{r\tau}$ is the forward price at time t, K is the strike price, r is the risk-free interest rate, σ is the volatility, and $\tau = T - t$ is the time to maturity.

We first focus on the integral parts I_1, I_2, I_3 and I_4, respectively

$$\begin{aligned}
I_1 &= \int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y}\phi(y)dy \\
&= \int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}} dy \\
&= \int_{-d_2}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2 - 2\sigma\sqrt{\tau}y + \sigma^2\tau - \sigma^2\tau}{2}} dy \\
&= e^{\frac{1}{2}\sigma^2\tau} \int_{-d_2}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(y-\sigma\sqrt{\tau})^2}{2}} dy \\
\left(\text{Let } z = -(y - \sigma\sqrt{\tau})\right) \quad &= e^{\frac{1}{2}\sigma^2\tau} \int_{-\infty}^{d_2+\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}} dz \\
&= e^{\frac{1}{2}\sigma^2\tau} N(d_2 + \sigma\sqrt{\tau}) \\
&= e^{\frac{1}{2}\sigma^2\tau} N(d_1),
\end{aligned} \tag{G.2}$$

$$\begin{aligned}
I_2 &= \int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y} He_k(y)\phi(y)dy \\
&= \int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y} (-1)^k \phi^{(k)}(y)dy \\
&= (-1)^k \int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y} d\phi^{(k-1)}(y) \\
&= (-1)^k \left[e^{\sigma\sqrt{\tau}y}\phi^{(k-1)}(y)\Big|_{-d_2}^{+\infty} - \sigma\sqrt{\tau} \int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y}\phi^{(k-1)}(y)dy \right] \\
&= (-1)^k \left[-e^{-\sigma\sqrt{\tau}d_2}\phi^{(k-1)}(-d_2) - \sigma\sqrt{\tau} \int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y} d\phi^{(k-2)}(y) \right] \\
&= (-1)^k \Big[-e^{-d_2\sigma\sqrt{\tau}}(-1)^{k-1}He_{k-1}(-d_2)\phi(-d_2) \\
&\quad - \sigma\sqrt{\tau}\left(-e^{-d_2\sigma\sqrt{\tau}}\phi^{(k-2)}(-d_2) - \sigma\sqrt{\tau} \int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y} d\phi^{(k-3)}(y) \right) \Big] \\
&= (-1)^k \Big[(-1)^k e^{-d_2\sigma\sqrt{\tau}} He_{k-1}(-d_2)\phi(-d_2) \\
&\quad + (-1)^k \sigma\sqrt{\tau} e^{-d_2\sigma\sqrt{\tau}} He_{k-2}(-d_2)\phi(-d_2) + (\sigma\sqrt{\tau})^2 \int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y}\phi^{(k-2)}(y)dy \Big] \\
&= e^{-d_2\sigma\sqrt{\tau}}\phi(-d_2) \Big[He_{k-1}(-d_2) + \sigma\sqrt{\tau}He_{k-2}(-d_2) + (\sigma\sqrt{\tau})^2 He_{k-3}(-d_2) \\
&\quad + \cdots + (\sigma\sqrt{\tau})^{k-1}He_0(-d_2) \Big] + (-1)^k(-\sigma\sqrt{\tau})^k \int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y}\phi(y)dy \\
&= e^{-d_2\sigma\sqrt{\tau}}\phi(-d_2) \sum_{m=0}^{k-1} (\sigma\sqrt{\tau})^m He_{k-m-1}(-d_2) + (\sigma\sqrt{\tau})^k e^{\frac{1}{2}\sigma^2\tau} N(d_1),
\end{aligned} \tag{G.3}$$

$$I_3 = \int_{-d_2}^{+\infty} \phi(y)dy = \int_{-\infty}^{d_2} \phi(y)dy = N(d_2), \tag{G.4}$$

$$\begin{aligned} I_4 &= \int_{-d_2}^{+\infty} He_k(y)\phi(y)dy \\ &= \int_{-d_2}^{+\infty} (-1)^k \phi^{(k)}(y)dy \\ &= (-1)^k \phi^{(k-1)}(y)\Big|_{-d_2}^{+\infty} \\ &= (-1)^{k+1}\phi^{(k-1)}(-d_2) \\ &= (-1)^{k+1}(-1)^{k-1}He_{k-1}(-d_2)\phi(-d_2) \\ &= He_{k-1}(-d_2)\phi(-d_2). \end{aligned} \tag{G.5}$$

Thus, Equation (G.1) can be written as

$$\begin{aligned} & c_t^{GC} \\ =& F_t^T e^{-r\tau} e^{\mu_c \tau} N(d_1) \\ &+ F_t^T e^{-r\tau} e^{\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau} \sum_{k=3}^{n} \frac{\rho_k}{k!} \left[e^{-d_2\sigma\sqrt{\tau}} \phi(-d_2) \sum_{m=0}^{k-1} (\sigma\sqrt{\tau})^m He_{k-m-1}(-d_2) + (\sigma\sqrt{\tau})^k e^{\frac{1}{2}\sigma^2\tau} N(d_1) \right] \\ &- Ke^{-r\tau} N(d_2) - Ke^{-r\tau} \sum_{k=3}^{n} \frac{\rho_k}{k!} He_{k-1}(-d_2)\phi(-d_2) \\ =& F_t^T e^{-r\tau} e^{\mu_c\tau} N(d_1) \left[1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} (\sigma\sqrt{\tau})^k \right] - Ke^{-r\tau} N(d_2) \\ &+ F_t^T e^{-r\tau} e^{\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau} \sum_{k=3}^{n} \frac{\rho_k}{k!} e^{-d_2\sigma\sqrt{\tau}} \phi(-d_2) \sum_{m=0}^{k-1} (\sigma\sqrt{\tau})^m He_{k-m-1}(-d_2) \\ &- Ke^{-r\tau} \sum_{k=3}^{n} \frac{\rho_k}{k!} He_{k-1}(-d_2)\phi(-d_2) \\ & \left(d_2 = \frac{\ln \frac{F_t^T}{K} + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \implies F_t^T e^{\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau} = Ke^{d_2\sigma\sqrt{\tau}} \right) \\ =& F_t^T e^{-r\tau} N(d_1) - Ke^{-r\tau} N(d_2) + Ke^{-r\tau}\phi(-d_2) \sum_{k=3}^{n} \frac{\rho_k}{k!} \left[\sum_{m=0}^{k-1} (\sigma\sqrt{\tau})^m He_{k-m-1}(-d_2) - He_{k-1}(-d_2) \right] \\ =& F_t^T e^{-r\tau} N(d_1) - Ke^{-r\tau} N(d_2) + Ke^{-r\tau}\phi(d_2) \sum_{k=3}^{n} \frac{\rho_k}{k!} \sum_{m=1}^{k-1} (\sigma\sqrt{\tau})^m He_{k-m-1}(-d_2). \end{aligned} \tag{G.6}$$

By applying Lemma 4, Equation (G.6) can be rewritten as

$$
\begin{aligned}
c_t^{GC} &= F_t^T e^{-r\tau} N(d_1) - K e^{-r\tau} N(d_2) + K e^{-r\tau}\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\sum_{m=1}^{k-1}(\sigma\sqrt{\tau})^m(-1)^{k-m-1}He_{k-m-1}(d_2)\\
&= F_t^T e^{-r\tau} N(d_1) - K e^{-r\tau} N(d_2) - K e^{-r\tau}\phi(d_2)\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}(-1)^k\sum_{m=1}^{k-1}(-\sigma\sqrt{\tau})^m He_{k-m-1}(d_2)\right],
\end{aligned}
\tag{G.7}
$$

which is the simplified version of option pricing formula under Gram-Charlier density proposed by Aschakulporn (2022).

□

H Proof of Proposition 1

Proof. Similar to the proof of Lemma 7, the price of a European call option under Gram-Charlier density can be calculated with the Harrison and Kreps (1979) and Harrison and Pliska (1981) risk-neutral valuation formula. Following the steps in Equation (G.1), we have

$$
\begin{aligned}
& c_t^{GC}\\
&= e^{-r\tau} E_t^{\mathbb{Q}}\left[\max\left(S_T - K, 0\right)\right]\\
&= e^{-r\tau} F_t^T e^{\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau}\left[\underbrace{\int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y}\phi(y)dy}_{I_1} + \sum_{k=3}^{n}\frac{\rho_k}{k!}\underbrace{\int_{-d_2}^{+\infty} e^{\sigma\sqrt{\tau}y} He_k(y)\phi(y)dy}_{I_2}\right]\\
&\quad - K e^{-r\tau}\left[\underbrace{\int_{-d_2}^{+\infty}\phi(y)dy}_{I_3} + \sum_{k=3}^{n}\frac{\rho_k}{k!}\underbrace{\int_{-d_2}^{+\infty} He_k(y)\phi(y)dy}_{I_4}\right]\\
&= F_t^T e^{-r\tau} e^{\mu_c\tau} N(d_1)\\
&\quad + F_t^T e^{-r\tau} e^{\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau}\sum_{k=3}^{n}\frac{\rho_k}{k!}\left[e^{-d_2\sigma\sqrt{\tau}}\phi(-d_2)\sum_{m=0}^{k-1}(\sigma\sqrt{\tau})^m He_{k-m-1}(-d_2) + (\sigma\sqrt{\tau})^k e^{\frac{1}{2}\sigma^2\tau}N(d_1)\right]\\
&\quad - K e^{-r\tau} N(d_2) - K e^{-r\tau}\sum_{k=3}^{n}\frac{\rho_k}{k!} He_{k-1}(-d_2)\phi(-d_2)\\
&= F_t^T e^{-r\tau} e^{\mu_c\tau} N(d_1)\left[1 + \sum_{k=3}^{n}\frac{\rho_k}{k!}(\sigma\sqrt{\tau})^k\right]\\
&\quad + F_t^T e^{-r\tau} e^{\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau}\sum_{k=3}^{n}\frac{\rho_k}{k!} e^{-d_2\sigma\sqrt{\tau}}\phi(-d_2)\sum_{m=0}^{k-1}(\sigma\sqrt{\tau})^m He_{k-m-1}(-d_2) - K e^{-r\tau}\Pi_2
\end{aligned}
$$

$$
\left(d_2 = \frac{\ln\frac{F_t^T}{K} + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \implies F_t^T e^{\left(\mu_c - \frac{1}{2}\sigma^2\right)\tau} = K e^{d_2\sigma\sqrt{\tau}}\right)
$$

$$= F_t^T e^{-r\tau} N(d_1) + K e^{-r\tau} e^{d_2\sigma\sqrt{\tau}} \sum_{k=3}^{n} \frac{\rho_k}{k!} e^{-d_2\sigma\sqrt{\tau}} \phi(d_2) \sum_{m=0}^{k-1} (\sigma\sqrt{\tau})^m He_{k-m-1}(-d_2) - K e^{-r\tau} \Pi_2$$

$$= F_t^T e^{-r\tau} N(d_1) + K e^{-r\tau} \phi(d_2) \sum_{k=3}^{n} \frac{\rho_k}{k!} \sum_{m=0}^{k-1} (\sigma\sqrt{\tau})^m He_{k-m-1}(-d_2) - K e^{-r\tau} \Pi_2. \tag{H.1}$$

where

$$d_2 = \frac{\ln \frac{F_t^T}{K} + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, d_1 = d_2 + \sigma\sqrt{\tau},$$

$$\mu_c = -\frac{1}{\tau} \ln \left[1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} (\sigma\sqrt{\tau})^k\right],$$

$$e^{\mu_c \tau} = \frac{1}{1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} (\sigma\sqrt{\tau})^k},$$

$$f(y) = \sum_{k=0}^{n} \frac{\rho_k}{k!} (-1)^k \phi^{(k)}(y) = \sum_{k=0}^{n} \frac{\rho_k}{k!} He_k(y)\phi(y) = \left[1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} He_k(y)\right] \phi(y),$$

$k!$ is the factorial of k, $\phi(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)$ is the standard normal density function, $He_k(y)$ is the k-th order Hermite polynomial defined by $He_k(y) \equiv (-1)^k \frac{\phi^{(k)}(y)}{\phi(y)}$, ρ_k is the coefficient derived from the Hermite polynomial, $F_t^T = S_t e^{r\tau}$ is the forward price at time t, K is the strike price, r is the risk-free interest rate, σ is the volatility, and $\tau = T - t$ is the time to maturity.

Now we only need to show that

$$\sum_{m=0}^{k-1} (\sigma\sqrt{\tau})^m He_{k-m-1}(-d_2) = \sum_{h=1}^{k} C_k^h He_{h-1}(-d_1)(\sigma\sqrt{\tau})^{k-h} \tag{H.2}$$

holds.

Since for any integer k and v with $k \geq v \geq 1$, we have

$$(1+x)^{k-v} = (1+x)^k \frac{1}{(1+x)^v} = \sum_{l=0}^{k-v} C_{k-v}^l x^l,$$

$$(1+x)^k = \sum_{l=0}^{k} C_k^l x^l,$$

$$\frac{1}{(1+x)^v} = \sum_{l=0}^{+\infty} (-1)^l C_{v+l-1}^l x^l,$$

$$\Longrightarrow \sum_{l=0}^{k-v} C_{k-v}^l x^l = \left(\sum_{l=0}^{k} C_k^l x^l\right) \left(\sum_{l=0}^{+\infty} (-1)^l C_{v+l-1}^l x^l\right),$$

and by comparing the coefficients of x^{k-v} on both sides, we have

$$\sum_{l=0}^{k-v}(-1)^l C_k^{k-v-l} C_{v+l-1}^{l} = 1. \tag{H.3}$$

Equation (H.3) can be transformed into

$$\sum_{l=0}^{k-v}(-1)^l C_k^{v+l} C_{v+l-1}^{v-1} = 1, \tag{H.4}$$

when taking $v = 1, 2, \cdots, k-1, k$, respectively, we can get the following equalities

$$\begin{gathered}
C_k^1 C_0^0 - C_k^2 C_1^0 + C_k^3 C_2^0 + \cdots + (-1)^{k-1} C_k^k C_{k-1}^0 = 1, \\
C_k^2 C_1^1 - C_k^3 C_2^1 + C_k^4 C_3^1 + \cdots + (-1)^{k-2} C_k^k C_{k-1}^1 = 1, \\
\cdots, \\
C_k^{k-1} C_{k-2}^{k-2} - C_k^k C_{k-1}^{k-2} = 1, \\
C_k^k C_{k-1}^{k-1} = 1.
\end{gathered}$$

By applying Lemma 3 and Equation (H.4), we have

$$
\begin{aligned}
&\sum_{h=1}^{k} C_k^h He_{h-1}(-d_1)(\sigma\sqrt{\tau})^{k-h} \\
&= \sum_{h=1}^{k} C_k^h \left[\sum_{m=0}^{h-1} (-\sigma\sqrt{\tau})^m C_{h-1}^m He_{h-1-m}(-d_2)\right] (\sigma\sqrt{\tau})^{k-h} \\
&= C_k^1 \left[C_0^0 He_0(-d_2)\right] (\sigma\sqrt{\tau})^{k-1} \\
&\quad + C_k^2 \left[(-\sigma\sqrt{\tau}) C_1^0 He_0(-d_2) + C_1^1 He_1(-d_2)\right] (\sigma\sqrt{\tau})^{k-2} \\
&\quad + C_k^3 \left[(-\sigma\sqrt{\tau})^2 C_2^0 He_0(-d_2) + (-\sigma\sqrt{\tau}) C_2^1 He_1(-d_2) + C_2^2 He_2(-d_2)\right] (\sigma\sqrt{\tau})^{k-3} \\
&\quad + \cdots \\
&\quad + C_k^h \left[(-\sigma\sqrt{\tau})^{h-1} C_{h-1}^0 He_0(-d_2) + (-\sigma\sqrt{\tau})^{h-2} C_{h-1}^1 He_1(-d_2) + \cdots + C_{h-1}^{h-1} He_{h-1}(-d_2)\right] (\sigma\sqrt{\tau})^{k-h} \\
&\quad + \cdots \\
&\quad + C_k^k \left[(-\sigma\sqrt{\tau})^{k-1} C_{k-1}^0 He_0(-d_2) + (-\sigma\sqrt{\tau})^{k-2} C_{k-1}^1 He_1(-d_2) + \cdots + C_{k-1}^{k-1} He_{k-1}(-d_2)\right] (\sigma\sqrt{\tau})^{k-k} \\
&= \left[C_k^1 C_0^0 - C_k^2 C_1^0 + C_k^3 C_2^0 + \cdots + (-1)^{k-1} C_k^k C_{k-1}^0\right] He_0(-d_2)(\sigma\sqrt{\tau})^{k-1} \\
&\quad + \left[C_k^2 C_1^1 - C_k^3 C_2^1 + C_k^4 C_3^1 + \cdots + (-1)^{k-2} C_k^k C_{k-1}^1\right] He_1(-d_2)(\sigma\sqrt{\tau})^{k-2} \\
&\quad + \cdots \\
&\quad + \left(C_k^{k-1} C_{k-2}^{k-2} - C_k^k C_{k-1}^{k-2}\right) He_{k-2}(-d_2)(\sigma\sqrt{\tau}) \\
&\quad + C_k^k C_{k-1}^{k-1} He_{k-1}(-d_2) \\
&= He_0(-d_2)(\sigma\sqrt{\tau})^{k-1} + He_1(-d_2)(\sigma\sqrt{\tau})^{k-2} + \cdots + He_{k-1}(-d_2)(\sigma\sqrt{\tau})^{k-k} \\
&= \sum_{m=0}^{k-1} (\sigma\sqrt{\tau})^m He_{k-m-1}(-d_2).
\end{aligned} \tag{H.5}
$$

Hence, Equation (H.1) can be rewritten as

$$
\begin{aligned}
c_t^{GC} &= F_t^T e^{-r\tau} N(d_1) + K e^{-r\tau} \phi(d_2) \sum_{k=3}^{n} \frac{\rho_k}{k!} \sum_{m=0}^{k-1} (\sigma\sqrt{\tau})^m He_{k-m-1}(-d_2) - K e^{-r\tau}\Pi_2 \\
&= F_t^T e^{-r\tau} \left[N(d_1) + \frac{K\phi(d_2)}{F_t^T} \sum_{k=3}^{n} \frac{\rho_k}{k!} \sum_{h=1}^{k} C_k^h He_{h-1}(-d_1) \left(\sigma\sqrt{\tau}\right)^{k-h}\right] - K e^{-r\tau}\Pi_2
\end{aligned}
$$

$$
\left(\frac{\phi(d_2)}{\phi(d_1)} = e^{d_2\sigma\sqrt{\tau} + \frac{1}{2}\sigma^2\tau} \implies \frac{K\phi(d_2)}{F_t^T} = e^{\mu_c\tau}\phi(d_1)\right)
$$

$$
\begin{aligned}
&= F_t^T e^{-r\tau} \left[N(d_1) + e^{\mu_c\tau}\phi(d_1) \sum_{k=3}^{n} \frac{\rho_k}{k!} \sum_{h=1}^{k} C_k^h He_{h-1}(-d_1) \left(\sigma\sqrt{\tau}\right)^{k-h}\right] - K e^{-r\tau}\Pi_2 \\
&= F_t^T e^{-r\tau}\Pi_1 - K e^{-r\tau}\Pi_2.
\end{aligned} \tag{H.6}
$$

□

I Proof of Lemma 8

Proof. First we focus on $\frac{\partial \Pi_1}{\partial S_t}$

$$\begin{aligned}\frac{\partial \Pi_1}{\partial S_t} &= \frac{\partial}{\partial S_t}\left[N(d_1) + e^{\mu_c \tau}\phi(d_1)\sum_{k=3}^{n}\frac{\rho_k}{k!}\sum_{h=1}^{k}C_k^h He_{h-1}(-d_1)\left(\sigma\sqrt{\tau}\right)^{k-h}\right] \\ &= \phi(d_1)\frac{\partial d_1}{\partial S_t} + e^{\mu_c \tau}\sum_{k=3}^{n}\frac{\rho_k}{k!}\sum_{h=1}^{k}C_k^h\frac{\partial}{\partial S_t}\left[\phi(d_1)He_{h-1}(-d_1)\right]\left(\sigma\sqrt{\tau}\right)^{k-h}.\end{aligned} \tag{I.1}$$

By Lemma 5, we have

$$\begin{aligned}&\frac{\partial}{\partial S_t}\left[\phi(d_1)He_{h-1}(-d_1)\right] \\ &= \frac{\partial \phi(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S_t}He_{h-1}(-d_1) + \phi(d_1)(h-1)He_{h-2}(-d_1)\frac{\partial(-d_1)}{\partial S_t} \\ &= \phi(d_1)\left[(-d_1)He_{h-1}(-d_1) - (h-1)He_{h-2}(-d_1)\right]\frac{\partial d_1}{\partial S_t} \\ &= \phi(d_1)He_h(-d_1)\frac{\partial d_1}{\partial S_t}.\end{aligned} \tag{I.2}$$

Note that

$$\begin{aligned}&\sum_{h=1}^{k}C_k^h He_h(-d_1)\left(\sigma\sqrt{\tau}\right)^{k-h} \\ &= \sum_{h=1}^{k}C_k^h\int_{-\infty}^{+\infty}(it-d_1)^h\phi(t)dt\left(\sigma\sqrt{\tau}\right)^{k-h} \\ &= \int_{-\infty}^{+\infty}\sum_{h=1}^{k}C_k^h(it-d_1)^h\left(\sigma\sqrt{\tau}\right)^{k-h}\phi(t)dt \\ &= \int_{-\infty}^{+\infty}\sum_{h=0}^{k}C_k^h(it-d_1)^h\left(\sigma\sqrt{\tau}\right)^{k-h}\phi(t)dt - \int_{-\infty}^{+\infty}C_k^0(it-d_1)^0\left(\sigma\sqrt{\tau}\right)^{k}\phi(t)dt \\ &= \int_{-\infty}^{+\infty}(it-d_1+\sigma\sqrt{\tau})^k\phi(t)dt - \left(\sigma\sqrt{\tau}\right)^k \\ &= He_k(-d_2) - \left(\sigma\sqrt{\tau}\right)^k,\end{aligned} \tag{I.3}$$

Equation (I.1) can be continued as

$$\begin{aligned}
&\frac{\partial \Pi_1}{\partial S_t} \\
&= \phi(d_1)\frac{\partial d_1}{\partial S_t} + e^{\mu_c \tau}\phi(d_1)\sum_{k=3}^{n}\frac{\rho_k}{k!}\sum_{h=1}^{k} C_k^h He_h(-d_1)\frac{\partial d_1}{\partial S_t}\left(\sigma\sqrt{\tau}\right)^{k-h} \\
&= \phi(d_1)\frac{\partial d_1}{\partial S_t} + e^{\mu_c \tau}\phi(d_1)\sum_{k=3}^{n}\frac{\rho_k}{k!}\left[He_k(-d_2) - \left(\sigma\sqrt{\tau}\right)^k\right]\frac{\partial d_1}{\partial S_t} \\
&= \phi(d_1)\left\{1 + e^{\mu_c \tau}\left[\sum_{k=3}^{n}\frac{\rho_k}{k!}He_k(-d_2) - \sum_{k=3}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k\right]\right\}\frac{\partial d_1}{\partial S_t} \\
\left(\because \sum_{k=3}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k = e^{-\mu_c \tau} - 1\right) \quad &= \phi(d_1)\left\{1 + e^{\mu_c \tau}\left[\sum_{k=3}^{n}\frac{\rho_k}{k!}He_k(-d_2) - e^{-\mu_c \tau} + 1\right]\right\}\frac{\partial d_1}{\partial S_t} \\
&= \phi(d_1)e^{\mu_c \tau}\left[1 + \sum_{k=3}^{n}\frac{\rho_k}{k!}He_k(-d_2)\right]\frac{\partial d_1}{\partial S_t}.
\end{aligned} \tag{I.4}$$

Then we consider $\frac{\partial \Pi_2}{\partial S_t}$

$$\begin{aligned}
\frac{\partial \Pi_2}{\partial S_t} &= \frac{\partial}{\partial S_t}\left[N(d_2) + \phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}He_{k-1}(-d_2)\right] \\
&= \phi(d_2)\frac{\partial d_2}{\partial S_t} + \sum_{k=3}^{n}\frac{\rho_k}{k!}\frac{\partial}{\partial S_t}\left[\phi(d_2)He_{k-1}(-d_2)\right].
\end{aligned} \tag{I.5}$$

Applying Lemma 5 again, we have

$$\begin{aligned}
&\frac{\partial}{\partial S_t}\left[\phi(d_2)He_{k-1}(-d_2)\right] \\
&= \frac{\partial \phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S_t}He_{k-1}(-d_2) + \phi(d_2)(k-1)He_{k-2}(-d_2)\frac{\partial(-d_2)}{\partial S_t} \\
&= \phi(d_2)\left[(-d_2)He_{k-1}(-d_2) - (k-1)He_{k-2}(-d_2)\right]\frac{\partial d_2}{\partial S_t} \\
&= \phi(d_2)He_k(-d_2)\frac{\partial d_2}{\partial S_t}.
\end{aligned} \tag{I.6}$$

Equation (I.5) can be continued as

$$\begin{aligned}
\frac{\partial \Pi_2}{\partial S_t} &= \phi(d_2)\frac{\partial d_2}{\partial S_t} + \phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}He_k(-d_2)\frac{\partial d_2}{\partial S_t} \\
&= \phi(d_2)\left[1 + \sum_{k=3}^{n}\frac{\rho_k}{k!}He_k(-d_2)\right]\frac{\partial d_2}{\partial S_t}.
\end{aligned} \tag{I.7}$$

Combine Equation (I.4) and (I.7) and multiply by the corresponding coefficients, we have

$$\begin{aligned} & F_t^T e^{-r\tau} \frac{\partial \Pi_1}{\partial S_t} - K e^{-r\tau} \frac{\partial \Pi_2}{\partial S_t} \\ = & F_t^T e^{-r\tau} \phi(d_1) e^{\mu_c \tau} \left[1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} He_k(-d_2) \right] \frac{\partial d_1}{\partial S_t} - K e^{-r\tau} \phi(d_2) \left[1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} He_k(-d_2) \right] \frac{\partial d_2}{\partial S_t}. \end{aligned} \tag{I.8}$$

Since

$$d_1 + d_2 = 2 \frac{\ln \frac{F_t^T}{K} + \mu_c \tau}{\sigma \sqrt{\tau}}, \quad d_1 - d_2 = \sigma \sqrt{\tau},$$

we have

$$\frac{\phi(d_2)}{\phi(d_1)} = e^{\frac{1}{2}\left(d_1^2 - d_2^2\right)} = e^{\frac{1}{2}(d_1 + d_2)(d_1 - d_2)} = e^{\ln \frac{F_t^T}{K} + \mu_c \tau} = \frac{F_t^T}{K} e^{\mu_c \tau}, \tag{I.9}$$

then

$$\phi(d_2) = \frac{F_t^T}{K} e^{\mu_c \tau} \phi(d_1). \tag{I.10}$$

Note that $\frac{\partial d_1}{\partial S_t} = \frac{\partial d_2}{\partial S_t}$, and hence Equation (I.8) can be continued as

$$\begin{aligned} & F_t^T e^{-r\tau} \frac{\partial \Pi_1}{\partial S_t} - K e^{-r\tau} \frac{\partial \Pi_2}{\partial S_t} \\ = & \underbrace{\left[F_t^T \phi(d_1) e^{\mu_c \tau} - K \phi(d_2) \right]}_{0} e^{-r\tau} \left[1 + \sum_{k=3}^{n} \frac{\rho_k}{k!} He_k(-d_2) \right] \frac{\partial d_1}{\partial S_t} \\ = & 0. \end{aligned} \tag{I.11}$$

□

J Proof of Proposition 2

Proof. Here we use the option pricing formula presented in Proposition 1 to derive Δ under Gram-Charlier density. According to Lemma 8, we have

$$\begin{aligned} \Delta^{GC} & \equiv \frac{\partial c_t^{GC}}{\partial S_t} \\ & = \frac{\partial}{\partial S_t} \left(F_t^T e^{-r\tau} \Pi_1 - K e^{-r\tau} \Pi_2 \right) \\ & = \Pi_1 + \underbrace{F_t^T e^{-r\tau} \frac{\partial \Pi_1}{\partial S_t} - K e^{-r\tau} \frac{\partial \Pi_2}{\partial S_t}}_{0} \\ & = \Pi_1. \end{aligned} \tag{J.1}$$

□

K Proof of Lemma 9

Proof. According to Definition 3, the characteristic function of the random variable $R_\tau = \ln \frac{S_T}{F_t^T} = \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Y$ under Gram-Charlier density can be derived by

$$
\begin{aligned}
& f(\phi) \\
\equiv\ & E_t^{\mathcal{Q}}\left[e^{i\phi R_\tau}\right] \\
=\ & e^{i\phi\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau} \cdot E_t^{\mathcal{Q}}\left[e^{i\phi\sigma\sqrt{\tau}Y}\right] \\
=\ & e^{i\phi\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau} \cdot \int_{-\infty}^{+\infty} e^{i\phi\sigma\sqrt{\tau}y} \cdot f\left(y\right) dy \\
=\ & e^{i\phi\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau} \cdot \int_{-\infty}^{+\infty} e^{i\phi\sigma\sqrt{\tau}y} \cdot \sum_{k=0}^{n} \frac{\rho_k}{k!} \cdot He_k(y) \cdot \phi(y)dy \\
=\ & e^{i\phi\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau} \cdot \sum_{k=0}^{n} \frac{\rho_k}{k!} \cdot \int_{-\infty}^{+\infty} e^{i\phi\sigma\sqrt{\tau}y} \cdot He_k(y) \cdot \phi(y)dy \\
=\ & e^{i\phi\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau} \cdot \sum_{k=0}^{n} \frac{\rho_k}{k!} \cdot \int_{-\infty}^{+\infty} e^{i\phi\sigma\sqrt{\tau}y} \cdot (-1)^k \cdot \phi^{(k)}(y)dy \\
=\ & e^{i\phi\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau} \cdot \sum_{k=0}^{n} \frac{\rho_k}{k!} \cdot (-1)^k \cdot \int_{-\infty}^{+\infty} e^{i\phi\sigma\sqrt{\tau}y} d\phi^{(k-1)}(y) \\
=\ & e^{i\phi\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau} \cdot \sum_{k=0}^{n} \frac{\rho_k}{k!} \cdot (-1)^k \cdot \left[0 - \int_{-\infty}^{+\infty} e^{i\phi\sigma\sqrt{\tau}y} \cdot \phi^{(k-1)}(y) \cdot \left(i\phi\sigma\sqrt{\tau}\right) dy\right] \\
=\ & e^{i\phi\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau} \cdot \sum_{k=0}^{n} \frac{\rho_k}{k!} \cdot (-1)^k \cdot \left[(-1) \cdot \left(i\phi\sigma\sqrt{\tau}\right) \cdot \int_{-\infty}^{+\infty} e^{i\phi\sigma\sqrt{\tau}y} d\phi^{(k-2)}(y)\right] \\
=\ & e^{i\phi\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau} \cdot \sum_{k=0}^{n} \frac{\rho_k}{k!} \cdot (-1)^k \cdot (-1) \cdot \left(i\phi\sigma\sqrt{\tau}\right) \cdot \left[0 - \int_{-\infty}^{+\infty} e^{i\phi\sigma\sqrt{\tau}y} \cdot \phi^{(k-2)}(y) \cdot \left(i\phi\sigma\sqrt{\tau}\right) dy\right] \\
=\ & e^{i\phi\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau} \cdot \sum_{k=0}^{n} \frac{\rho_k}{k!} \cdot (-1)^k \cdot \left[(-1)^2 \cdot \left(i\phi\sigma\sqrt{\tau}\right)^2 \cdot \int_{-\infty}^{+\infty} e^{i\phi\sigma\sqrt{\tau}y} \cdot \phi^{(k-2)}(y)dy\right] \\
=\ & \cdots \\
=\ & e^{i\phi\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau} \cdot \sum_{k=0}^{n} \frac{\rho_k}{k!} \cdot (-1)^k \cdot \left[(-1)^k \cdot \left(i\phi\sigma\sqrt{\tau}\right)^k \cdot \int_{-\infty}^{+\infty} e^{i\phi\sigma\sqrt{\tau}y} \cdot \phi(y)dy\right] \\
=\ & e^{i\phi\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau} \cdot \sum_{k=0}^{n} \frac{\rho_k}{k!} \cdot \left(i\phi\sigma\sqrt{\tau}\right)^k \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(y-i\phi\sigma\sqrt{\tau}\right)^2} \cdot e^{\frac{1}{2}\left(i\phi\sigma\sqrt{\tau}\right)^2} dy \\
=\ & e^{i\phi\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau} \cdot \sum_{k=0}^{n} \frac{\rho_k}{k!} \cdot \left(i\phi\sigma\sqrt{\tau}\right)^k \cdot e^{\frac{1}{2}\left(i\phi\sigma\sqrt{\tau}\right)^2} \\
=\ & e^{i\phi\mu_c\tau-\frac{1}{2}\sigma^2\tau i\phi(1-i\phi)} \sum_{k=0}^{n} \frac{\rho_k}{k!} \left(i\phi\sigma\sqrt{\tau}\right)^k.
\end{aligned}
\tag{K.1}
$$

□

L Proof of Proposition 4

Proof. According to Lemma 9, we have

$$
\begin{aligned}
f(\phi) &\equiv E_t^{\mathcal{Q}}\left[e^{i\phi R_\tau}\right] \\
&= e^{i\phi\mu_c\tau - \frac{1}{2}\sigma^2\tau i\phi(1-i\phi)} \sum_{k=0}^{n} \frac{\rho_k}{k!}\left(i\phi\sigma\sqrt{\tau}\right)^k,
\end{aligned}
\tag{L.1}
$$

$$
\begin{aligned}
\Longrightarrow\ f_1(\phi) &= e^{\mu_c\tau} e^{i\phi\left(\mu_c+\frac{1}{2}\sigma^2\right)\tau-\frac{1}{2}\phi^2\sigma^2\tau} \sum_{k=0}^{n} \frac{\rho_k}{k!}\left[\left(i\phi+1\right)\sigma\sqrt{\tau}\right]^k, \\
f_2(\phi) &= e^{i\phi\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau-\frac{1}{2}\phi^2\sigma^2\tau} \sum_{k=0}^{n} \frac{\rho_k}{k!}\left(i\phi\sigma\sqrt{\tau}\right)^k.
\end{aligned}
$$

By applying substitution method, Euler's formula and the property of conjugate complex number, we have

$$
\begin{aligned}
\Pi_2 &= \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \mathcal{R}\left[\frac{f_2(\phi)\cdot e^{-i\phi\ln\frac{K}{F_t^T}}}{i\phi}\right] d\phi \\
&= \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f(\phi)\cdot e^{-i\phi\ln\frac{K}{F_t^T}}}{i\phi} d\phi \\
&= \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\phi\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau-\frac{1}{2}\phi^2\sigma^2\tau}\left[\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(i\phi\sigma\sqrt{\tau}\right)^k\right]\frac{e^{-i\phi\ln\frac{K}{F_t^T}}}{i\phi} d\phi \\
&= \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\phi\left[\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau+\ln\frac{F_t^T}{K}\right]-\frac{1}{2}\phi^2\sigma^2\tau}\left(\frac{1}{i\phi}\right)\left[1+\sum_{k=3}^{n}\frac{\rho_k}{k!}\left(i\phi\sigma\sqrt{\tau}\right)^k\right] d\phi \\
&= \underbrace{\frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\phi d_2\sigma\sqrt{\tau}-\frac{1}{2}\phi^2\sigma^2\tau}\left(\frac{1}{i\phi}\right) d\phi}_{A_2} \\
&\quad + \underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\phi d_2\sigma\sqrt{\tau}-\frac{1}{2}\phi^2\sigma^2\tau}\left(\frac{1}{i\phi}\right)\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}\left(i\phi\sigma\sqrt{\tau}\right)^k\right] d\phi}_{B_2},
\end{aligned}
\tag{L.2}
$$

$$
\begin{aligned}
\Pi_1 &= \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \mathcal{R}\left[\frac{f_1(\phi)\cdot e^{-i\phi \ln \frac{K}{F_t^T}}}{i\phi}\right] d\phi \\
&= \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f_1(\phi)\cdot e^{-i\phi \ln \frac{K}{F_t^T}}}{i\phi} d\phi \\
&= \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\mu_c\tau} e^{i\phi\left(\mu_c+\frac{1}{2}\sigma^2\right)\tau - \frac{1}{2}\phi^2\sigma^2\tau} \left\{\sum_{k=0}^{n} \frac{\rho_k}{k!}\left[(i\phi+1)\,\sigma\sqrt{\tau}\right]^k\right\} \frac{e^{-i\phi \ln \frac{K}{F_t^T}}}{i\phi} d\phi \\
&= \frac{1}{2} + \frac{e^{\mu_c\tau}}{2\pi}\int_{-\infty}^{+\infty} e^{i\phi\left[\left(\mu_c+\frac{1}{2}\sigma^2\right)\tau + \ln\frac{F_t^T}{K}\right] - \frac{1}{2}\phi^2\sigma^2\tau} \left(\frac{1}{i\phi}\right)\left\{1 + \sum_{k=3}^{n}\frac{\rho_k}{k!}\left[(i\phi+1)\,\sigma\sqrt{\tau}\right]^k\right\} d\phi \\
&= \frac{1}{2} + \frac{e^{\mu_c\tau}}{2\pi}\int_{-\infty}^{+\infty} e^{i\phi d_1\sigma\sqrt{\tau} - \frac{1}{2}\phi^2\sigma^2\tau} \left(\frac{1}{i\phi}\right)\left\{1 + \sum_{k=3}^{n}\frac{\rho_k}{k!}\left[1 + \sum_{h=1}^{k} C_k^h\,(i\phi)^h\right]\left(\sigma\sqrt{\tau}\right)^k\right\} d\phi \\
&= \frac{1}{2} + \frac{e^{\mu_c\tau}}{2\pi}\int_{-\infty}^{+\infty} e^{i\phi d_1\sigma\sqrt{\tau} - \frac{1}{2}\phi^2\sigma^2\tau} \left(\frac{1}{i\phi}\right)\left\{\left[1 + \sum_{k=3}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k\right]\right. \\
&\quad \left. + \sum_{k=3}^{n}\frac{\rho_k}{k!}\sum_{h=1}^{k} C_k^h\,(i\phi)^h\left(\sigma\sqrt{\tau}\right)^k\right\} d\phi \\
&= \frac{1}{2} + \frac{e^{\mu_c\tau}}{2\pi}\int_{-\infty}^{+\infty} e^{i\phi d_1\sigma\sqrt{\tau} - \frac{1}{2}\phi^2\sigma^2\tau} \left(\frac{1}{i\phi}\right)\underbrace{\left[1 + \sum_{k=3}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k\right]}_{e^{-\mu_c\tau}} d\phi \\
&\quad + \frac{e^{\mu_c\tau}}{2\pi}\int_{-\infty}^{+\infty} e^{i\phi d_1\sigma\sqrt{\tau} - \frac{1}{2}\phi^2\sigma^2\tau} \left(\frac{1}{i\phi}\right)\sum_{k=3}^{n}\frac{\rho_k}{k!}\sum_{h=1}^{k} C_k^h\,(i\phi)^h\left(\sigma\sqrt{\tau}\right)^k d\phi \\
&= \underbrace{\frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\phi d_1\sigma\sqrt{\tau} - \frac{1}{2}\phi^2\sigma^2\tau}\left(\frac{1}{i\phi}\right) d\phi}_{A_1} \\
&\quad + \underbrace{\frac{e^{\mu_c\tau}}{2\pi}\int_{-\infty}^{+\infty} e^{i\phi d_1\sigma\sqrt{\tau} - \frac{1}{2}\phi^2\sigma^2\tau}\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}\sum_{h=1}^{k} C_k^h\,(i\phi)^{h-1}\left(\sigma\sqrt{\tau}\right)^k\right] d\phi}_{B_1}.
\end{aligned}
\tag{L.3}
$$

To show two forms of option pricing formula under Gram-Charlier density (given in our Proposition 1 and Proposition 3) are equivalent, we only need to show both of the following two conditions hold

$$\text{(i)} \quad A_j = N(d_j), j = 1, 2;$$

$$\text{(ii)} \quad B_1 = e^{\mu_c\tau}\phi(d_1)\sum_{k=3}^{n}\frac{\rho_k}{k!}\left[\sum_{h=1}^{k} C_k^h He_{h-1}(-d_1)\left(\sigma\sqrt{\tau}\right)^{k-h}\right] \text{ and } B_2 = \phi(d_2)\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}He_{k-1}(-d_2)\right].$$

We show (i) first

$$
\begin{aligned}
\frac{\partial A_j}{\partial K} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\phi d_j\sigma\sqrt{\tau}-\frac{1}{2}\phi^2\sigma^2\tau} i\phi\sigma\sqrt{\tau}\left(-\frac{1}{K\sigma\sqrt{\tau}}\right)\frac{1}{i\phi}d\phi \\
&= -\frac{1}{2\pi K}\int_{-\infty}^{+\infty} e^{i\phi d_j\sigma\sqrt{\tau}-\frac{1}{2}\phi^2\sigma^2\tau}d\phi \\
&= -\frac{1}{2\pi K}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\phi\sigma\sqrt{\tau}-d_j i\right)^2-\frac{1}{2}d_j^2}d\phi \\
\left(\text{Let } z=\phi\sigma\sqrt{\tau}-d_j i\right) \quad &= -\frac{1}{2\pi K}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2}e^{-\frac{1}{2}d_j^2}\frac{1}{\sigma\sqrt{\tau}}dz \\
&= -\frac{1}{2\pi K\sigma\sqrt{\tau}}e^{-\frac{1}{2}d_j^2}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2}dz \\
&= -\frac{1}{2\pi K\sigma\sqrt{\tau}}e^{-\frac{1}{2}d_j^2}\sqrt{2\pi} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_j^2}\left(-\frac{1}{K\sigma\sqrt{\tau}}\right) \\
&= \frac{\partial N(d_j)}{\partial K},
\end{aligned}
\tag{L.4}
$$

where $j=1,2$.

$$\therefore A_j = N(d_j)+L_j, j=1,2,$$

where L_j are two constants for $j=1,2$.

Let $d_j=0, j=1,2$, then we have $K=F_t^T e^{\left(\mu_c+\frac{1}{2}\sigma^2\right)\tau}$ for $j=1$ and $K=F_t^T e^{\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau}$ for $j=2$.

$$\therefore A_j\big|_{d_j=0} = \frac{1}{2}+\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\phi^2\sigma^2\tau}\left(\frac{1}{i\phi}\right)d\phi = \frac{1}{2} = N(0) = N(d_j)\big|_{d_j=0},$$

and thus $L_1=L_2=0$, which means (i) holds.

Then we show (ii)

$$
\begin{aligned}
B_1 &= \frac{e^{\mu_c\tau}}{2\pi}\int_{-\infty}^{+\infty} e^{i\phi d_1\sigma\sqrt{\tau}-\frac{1}{2}\phi^2\sigma^2\tau}\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}\sum_{h=1}^{k}C_k^h\left(i\phi\right)^{h-1}\left(\sigma\sqrt{\tau}\right)^k\right]d\phi \\
&\left(d_1=\frac{\ln\frac{F_t^T}{K}+\left(\mu_c+\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \implies e^{\mu_c\tau}=\frac{K}{F_t^T}e^{d_1\sigma\sqrt{\tau}-\frac{1}{2}\sigma^2\tau}\right) \\
&= \frac{K}{F_t^T}e^{d_1\sigma\sqrt{\tau}-\frac{1}{2}\sigma^2\tau}\frac{1}{2\pi}e^{-\frac{1}{2}d_1^2}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\phi\sigma\sqrt{\tau}-d_1 i\right)^2}\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}\sum_{h=1}^{k}C_k^h\left(i\phi\right)^{h-1}\left(\sigma\sqrt{\tau}\right)^k\right]d\phi \\
&= \frac{K}{F_t^T}\frac{1}{2\pi}e^{-\frac{1}{2}d_2^2}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\phi\sigma\sqrt{\tau}-d_1 i\right)^2}\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}\sum_{h=1}^{k}C_k^h\left(i\phi\right)^{h-1}\left(\sigma\sqrt{\tau}\right)^k\right]d\phi,
\end{aligned}
\tag{L.5}
$$

and

$$B_2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\phi d_2\sigma\sqrt{\tau}-\frac{1}{2}\phi^2\sigma^2\tau}\left(\frac{1}{i\phi}\right)\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}\left(i\phi\sigma\sqrt{\tau}\right)^k\right]d\phi$$
$$= \frac{1}{2\pi}e^{-\frac{1}{2}d_2^2}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\phi\sigma\sqrt{\tau}-d_2 i\right)^2}\left(\frac{1}{i\phi}\right)\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}\left(i\phi\sigma\sqrt{\tau}\right)^k\right]d\phi \tag{L.6}$$
$$= \phi(d_2)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\phi\sigma\sqrt{\tau}-d_2 i\right)^2}\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}\left(i\phi\sigma\sqrt{\tau}\right)^{k-1}\sigma\sqrt{\tau}\right]d\phi.$$

Substitute $z = \phi\sigma\sqrt{\tau} - d_1 i$ and $z = \phi\sigma\sqrt{\tau} - d_2 i$ into Equation (L.5) and (L.6), respectively, then we have

$$B_1 = \frac{K}{F_t^T}\phi(d_2)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2}\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}\sum_{h=1}^{k}C_k^h\left(\frac{iz-d_1}{\sigma\sqrt{\tau}}\right)^{h-1}\left(\sigma\sqrt{\tau}\right)^{k-1}\right]dz$$
$$= \frac{K}{F_t^T}\phi(d_2)\int_{-\infty}^{+\infty}\phi(z)\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}\sum_{h=1}^{k}C_k^h\left(iz-d_1\right)^{h-1}\left(\sigma\sqrt{\tau}\right)^{k-h}\right]dz$$
$$= \frac{K}{F_t^T}\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\sum_{h=1}^{k}C_k^h\int_{-\infty}^{+\infty}\phi(z)\left(iz-d_1\right)^{h-1}\left(\sigma\sqrt{\tau}\right)^{k-h}dz \tag{L.7}$$
$$= e^{\mu_c\tau}\phi(d_1)\sum_{k=3}^{n}\frac{\rho_k}{k!}\sum_{h=1}^{k}C_k^h He_{h-1}(-d_1)\left(\sigma\sqrt{\tau}\right)^{k-h},$$

and

$$B_2 = \phi(d_2)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2}\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}\left(iz-d_2\right)^{k-1}\right]dz$$
$$= \phi(d_2)\int_{-\infty}^{+\infty}\phi(z)\sum_{k=3}^{n}\frac{\rho_k}{k!}\left(iz-d_2\right)^{k-1}dz$$
$$= \phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\int_{-\infty}^{+\infty}\phi(z)\left(iz-d_2\right)^{k-1}dz \tag{L.8}$$
$$= \phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}He_{k-1}(-d_2).$$

Hence, the two forms of option pricing formula under Gram-Charlier density (given in our Proposition 1 and Proposition 3) are equivalent.

Furthermore, we are also able to show that the simplified version of the three-term option pricing formula under Gram-Charlier density (given in Lemma 7) is equivalent to those presented in Proposition 1 and Proposition 3. This requires both of (i) and another new condition (iii) hold, which is given by

$$\text{(iii)}\quad F_t^T B_1 - KB_2 = -K\phi(d_2)\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}(-1)^k\sum_{m=1}^{k-1}(-\sigma\sqrt{\tau})^m He_{k-m-1}(d_2)\right].$$

Based on the proof of (ii), it is not too difficult to show (iii).

Applying Lemma 2 to the both sides of (iii), we have

$$\begin{aligned} LHS &= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\sum_{h=1}^{k}C_k^h He_{h-1}(-d_1)\left(\sigma\sqrt{\tau}\right)^{k-h} - K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}He_{k-1}(-d_2) \\ &= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\sum_{h=1}^{k}C_k^h\int_{-\infty}^{+\infty}\phi(z)\left(iz-d_1\right)^{h-1}\left(\sigma\sqrt{\tau}\right)^{k-h}dz \\ &\quad - K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\int_{-\infty}^{+\infty}\phi(z)\left(iz-d_2\right)^{k-1}dz, \end{aligned} \tag{L.9}$$

and

$$\begin{aligned} RHS &= -K\phi(d_2)\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}(-1)^k\sum_{m=1}^{k-1}(-\sigma\sqrt{\tau})^m He_{k-m-1}(d_2)\right] \\ &= (-1)^{k+1}K\phi(d_2)\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}\sum_{m=1}^{k-1}(-\sigma\sqrt{\tau})^m\int_{-\infty}^{+\infty}\left(iz+d_2\right)^{k-m-1}\phi(z)dz\right]. \end{aligned} \tag{L.10}$$

Thus, (iii) can be rewritten as below

$$
\begin{aligned}
&LHS \\
&= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\sum_{h=1}^{k}C_k^h\int_{-\infty}^{+\infty}\phi(z)\,(iz-d_1)^{h-1}\left(\sigma\sqrt{\tau}\right)^{k-h}dz \\
&\quad - K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\int_{-\infty}^{+\infty}\phi(z)\,(iz-d_2)^{k-1}dz \\
&= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\left[\int_{-\infty}^{+\infty}\sum_{h=1}^{k}C_k^h\left(\sigma\sqrt{\tau}\right)^{k-h}(iz-d_1)^{h-1}\phi(z)dz-\int_{-\infty}^{+\infty}(iz-d_2)^{k-1}\phi(z)dz\right] \\
&= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\int_{-\infty}^{+\infty}\left[\frac{\sum_{h=0}^{k}C_k^h\,(\sigma\sqrt{\tau})^{k-h}\,(iz-d_1)^h-(\sigma\sqrt{\tau})^k}{iz-d_1}-(iz-d_2)^{k-1}\right]\phi(z)dz \\
&= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\int_{-\infty}^{+\infty}\left[\frac{(iz-d_1+\sigma\sqrt{\tau})^k-(\sigma\sqrt{\tau})^k}{iz-d_1}-(iz-d_2)^{k-1}\right]\phi(z)dz \\
&= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\int_{-\infty}^{+\infty}\left[\frac{(iz-d_2)^{k-1}(iz-d_2-iz+d_1)-(\sigma\sqrt{\tau})^k}{iz-d_1}\right]\phi(z)dz \\
&= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\int_{-\infty}^{+\infty}\left[\frac{(iz-d_2)^{k-1}\sigma\sqrt{\tau}-(\sigma\sqrt{\tau})^k}{iz-d_2-\sigma\sqrt{\tau}}\right]\phi(z)dz \\
&= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\int_{+\infty}^{-\infty}\left[\frac{(-iz-d_2)^{k-1}\sigma\sqrt{\tau}-(\sigma\sqrt{\tau})^k}{-iz-d_2-\sigma\sqrt{\tau}}\right]\phi(-z)d(-z) \\
&= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\int_{-\infty}^{+\infty}\left[\frac{(-1)^k(iz+d_2)^{k-1}\sigma\sqrt{\tau}+(\sigma\sqrt{\tau})^k}{iz+d_2+\sigma\sqrt{\tau}}\right]\phi(z)dz \\
&= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\int_{-\infty}^{+\infty}\left[\frac{(-1)^k(iz+d_2)^{k-1}\sigma\sqrt{\tau}+(\sigma\sqrt{\tau})^k}{(iz+d_2)\left(1+\frac{\sigma\sqrt{\tau}}{iz+d_2}\right)}\right]\phi(z)dz \\
&= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\int_{-\infty}^{+\infty}\Bigg[(-1)^k\sigma\sqrt{\tau}\,(iz+d_2)^{k-2}\sum_{m=1}^{\infty}\left(\frac{-\sigma\sqrt{\tau}}{iz+d_2}\right)^{m-1} \\
&\quad+\left(\sigma\sqrt{\tau}\right)^k(iz+d_2)^{-1}\sum_{m=0}^{\infty}\left(\frac{-\sigma\sqrt{\tau}}{iz+d_2}\right)^{m}\Bigg]\phi(z)dz
\end{aligned}
$$

$$
\begin{aligned}
&= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}\int_{-\infty}^{+\infty}\left[(-1)^k\sigma\sqrt{\tau}\sum_{m=1}^{\infty}\left(-\sigma\sqrt{\tau}\right)^{m-1}(iz+d_2)^{k-m-1}\right.\\
&\quad\left.+\left(\sigma\sqrt{\tau}\right)^k\sum_{m=0}^{\infty}\left(-\sigma\sqrt{\tau}\right)^{m}(iz+d_2)^{-m-1}\right]\phi(z)dz\\
&= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}(-1)^{k-1}\int_{-\infty}^{+\infty}\left[\sum_{m=1}^{\infty}\left(-\sigma\sqrt{\tau}\right)^{m}(iz+d_2)^{k-m-1}\right.\\
&\quad\left.-\left(-\sigma\sqrt{\tau}\right)^k\sum_{m=0}^{\infty}\left(-\sigma\sqrt{\tau}\right)^{m}(iz+d_2)^{-m-1}\right]\phi(z)dz\\
&= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}(-1)^{k-1}\int_{-\infty}^{+\infty}\left[\sum_{m=1}^{k-1}\left(-\sigma\sqrt{\tau}\right)^{m}(iz+d_2)^{k-m-1}\right.\\
&\quad\left.+\sum_{m=k}^{\infty}\left(-\sigma\sqrt{\tau}\right)^{m}(iz+d_2)^{k-m-1}-\sum_{m=0}^{\infty}\left(-\sigma\sqrt{\tau}\right)^{k+m}(iz+d_2)^{-m-1}\right]\phi(z)dz\\
&= K\phi(d_2)\sum_{k=3}^{n}\frac{\rho_k}{k!}(-1)^{k-1}\int_{-\infty}^{+\infty}\left[\sum_{m=1}^{k-1}\left(-\sigma\sqrt{\tau}\right)^{m}(iz+d_2)^{k-m-1}\right]\phi(z)dz\\
&= -K\phi(d_2)\sum_{k=3}^{n}\left[\frac{\rho_k}{k!}(-1)^k\sum_{m=1}^{k-1}(-\sigma\sqrt{\tau})^m He_{k-m-1}(d_2)\right]\\
&= RHS,
\end{aligned}
\tag{L.11}
$$

which ends the proof of (iii).

□

M Proof of Lemma 10

Proof. Recall the characteristic function of the random variable $R_\tau = \ln\frac{S_T}{F_t^T} = \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Y$ in Lemma 9 and $f_2(\phi) = f_2^{\mathcal{Q}}(\phi)$ defined in Proposition 3, we have

$$
\begin{aligned}
f^{\mathcal{Q}}(\phi) &\equiv E_t^{\mathcal{Q}}\left[e^{i\phi R_\tau}\right]\\
&= f_2^{\mathcal{Q}}(\phi)\\
&= e^{i\phi\mu_c\tau-\frac{1}{2}\sigma^2\tau i\phi(1-i\phi)}\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(i\phi\sigma\sqrt{\tau}\right)^k.
\end{aligned}
\tag{M.1}
$$

Then we can obtain the moment generating function (i.e., MGF) with Gram-Charlier density under $\mathcal{Q}$-measure by substitution method

$$
\begin{aligned}
M^{\mathcal{Q}}(\phi) &\equiv E_t^{\mathcal{Q}}\left[e^{\phi R_\tau}\right]\\
&= e^{\phi\mu_c\tau-\frac{1}{2}\sigma^2\tau\phi(1-\phi)}\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\phi\sigma\sqrt{\tau}\right)^k\\
&= e^{\frac{1}{2}\sigma^2\tau\phi^2+\left(\mu_c-\frac{1}{2}\sigma^2\right)\tau\phi}\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\phi\sigma\sqrt{\tau}\right)^k.
\end{aligned}
\tag{M.2}
$$

Thus, the cumulant generating function (i.e., CGF) with Gram-Charlier density under

$\mathcal{Q}$-measure is given by the logarithm of the corresponding moment generating function

$$\begin{aligned} K^{\mathcal{Q}}(\phi) &\equiv \ln M^{\mathcal{Q}}(\phi) \\ &= \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau\phi + \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\phi\sigma\sqrt{\tau}\right)^k. \end{aligned} \tag{M.3}$$

□

N Proof of Lemma 11

Proof. Recall the characteristic function of the random variable $R_\tau = \ln\frac{S_T}{F_t^T} = \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Y$ in Lemma 9 and $f_1(\phi) = f_1^{\mathcal{Q}}(\phi)$ defined in Proposition 3, we have

$$\begin{aligned} f^{\mathcal{S}}(\phi) &\equiv E_t^{\mathcal{S}}\left[e^{i\phi R_\tau}\right] \\ &= f_1^{\mathcal{Q}}(\phi) \\ &= \frac{f^{\mathcal{Q}}(\phi - i)}{f^{\mathcal{Q}}(-i)} \\ &= \frac{E_t^{\mathcal{Q}}\left[e^{i(\phi-i)R_\tau}\right]}{E_t^{\mathcal{Q}}\left[e^{i(-i)R_\tau}\right]} \\ &= e^{i\phi\mu_c\tau + \frac{1}{2}\sigma^2\tau i\phi(1+i\phi)}\frac{\sum_{k=0}^{n}\frac{\rho_k}{k!}\left[\left(1+i\phi\right)\sigma\sqrt{\tau}\right]^k}{\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k}. \end{aligned} \tag{N.1}$$

Then we can obtain the moment generating function (i.e., MGF) with Gram-Charlier density under $\mathcal{S}$-measure by substitution method

$$\begin{aligned} M^{\mathcal{S}}(\phi) &\equiv E_t^{\mathcal{S}}\left[e^{\phi R_\tau}\right] \\ &= e^{\phi\mu_c\tau + \frac{1}{2}\sigma^2\tau\phi(1+\phi)}\frac{\sum_{k=0}^{n}\frac{\rho_k}{k!}\left[\left(1+\phi\right)\sigma\sqrt{\tau}\right]^k}{\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k} \\ &= e^{\frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c + \frac{1}{2}\sigma^2\right)\tau\phi}\frac{\sum_{k=0}^{n}\frac{\rho_k}{k!}\left[\left(1+\phi\right)\sigma\sqrt{\tau}\right]^k}{\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k}. \end{aligned} \tag{N.2}$$

Thus, the cumulant generating function (i.e., CGF) with Gram-Charlier density under $\mathcal{S}$-measure is given by the logarithm of the corresponding moment generating function

$$\begin{aligned} K^{\mathcal{S}}(\phi) &\equiv \ln M^{\mathcal{S}}(\phi) \\ &= \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c + \frac{1}{2}\sigma^2\right)\tau\phi + \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left[\left(\phi+1\right)\sigma\sqrt{\tau}\right]^k - \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k. \end{aligned} \tag{N.3}$$

□

O Proof of Equation (23)

Proof.

$$
\begin{aligned}
& K^{\mathbb{Q}}(\phi) \\
= & \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau\phi + \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\phi\sigma\sqrt{\tau}\right)^k \\
= & \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau\phi + \ln\left[1 + \sum_{k=3}^{n}\frac{\rho_k}{k!}\left(\phi\sigma\sqrt{\tau}\right)^k\right] \\
= & \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau\phi \\
& + \left[\frac{\rho_3}{3!}\left(\sigma\sqrt{\tau}\right)^3\phi^3 + \frac{\rho_4}{4!}\left(\sigma\sqrt{\tau}\right)^4\phi^4 + \cdots\right] \\
& - \frac{1}{2}\left[\frac{\rho_3}{3!}\left(\sigma\sqrt{\tau}\right)^3\phi^3 + \frac{\rho_4}{4!}\left(\sigma\sqrt{\tau}\right)^4\phi^4 + \cdots\right]^2 \\
& + \frac{1}{3}\left[\frac{\rho_3}{3!}\left(\sigma\sqrt{\tau}\right)^3\phi^3 + \frac{\rho_4}{4!}\left(\sigma\sqrt{\tau}\right)^4\phi^4 + \cdots\right]^3 \\
& - \frac{1}{4}\left[\frac{\rho_3}{3!}\left(\sigma\sqrt{\tau}\right)^3\phi^3 + \frac{\rho_4}{4!}\left(\sigma\sqrt{\tau}\right)^4\phi^4 + \cdots\right]^4 + \cdots \\
= & \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau\phi \\
& + \frac{\rho_3}{3!}\left(\sigma\sqrt{\tau}\right)^3\phi^3 + \frac{\rho_4}{4!}\left(\sigma\sqrt{\tau}\right)^4\phi^4 + \frac{\rho_5}{5!}\left(\sigma\sqrt{\tau}\right)^5\phi^5 \\
& + \frac{1}{6!}\left(\sigma\sqrt{\tau}\right)^6\left[\rho_6 - \frac{1}{2}\cdot 6!\cdot\frac{2!}{2!0!}\cdot\frac{(\rho_3)^2}{(3!)^2}\right]\phi^6 \\
& + \frac{1}{7!}\left(\sigma\sqrt{\tau}\right)^7\left[\rho_7 - \frac{1}{2}\cdot 7!\cdot\frac{2!}{1!1!}\cdot\frac{\rho_3}{3!}\cdot\frac{\rho_4}{4!}\right]\phi^7 \\
& + \frac{1}{8!}\left(\sigma\sqrt{\tau}\right)^8\left[\rho_8 - \frac{1}{2}\cdot 8!\cdot\frac{2!}{1!1!}\cdot\frac{\rho_3}{3!}\cdot\frac{\rho_5}{5!} - \frac{1}{2}\cdot 8!\cdot\frac{2!}{2!0!}\cdot\frac{(\rho_4)^2}{(4!)^2}\right]\phi^8 \\
& + \frac{1}{9!}\left(\sigma\sqrt{\tau}\right)^9\left[\rho_9 - \frac{1}{2}\cdot 9!\cdot\frac{2!}{1!1!}\cdot\frac{\rho_3}{3!}\cdot\frac{\rho_6}{6!} - \frac{1}{2}\cdot 9!\cdot\frac{2!}{1!1!}\cdot\frac{\rho_4}{4!}\cdot\frac{\rho_5}{5!} + \frac{1}{3}\cdot 9!\cdot\frac{3!}{3!0!0!}\cdot\frac{(\rho_3)^3}{(3!)^3}\right]\phi^9 \\
& + \cdots \\
& + \frac{1}{q!}\left(\sigma\sqrt{\tau}\right)^q\left[\sum_{h=1}^{q}\sum_{\substack{3h_3+4h_4+\cdots+nh_n=q\\ h_3+h_4+\cdots+h_n=h\\ h_3,h_4,\cdots,h_n\geq 0}}\frac{(-1)^{h-1}}{h}\cdot q!\cdot\frac{h!}{h_3!h_4!\cdots h_n!}\cdot\left(\frac{\rho_3}{3!}\right)^{h_3}\cdot\left(\frac{\rho_4}{4!}\right)^{h_4}\cdots\left(\frac{\rho_n}{n!}\right)^{h_n}\right]\phi^q \\
& + \cdots
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau\phi \\
&\quad + \sum_{q=3}^{+\infty}\frac{1}{q!}\left(\sigma\sqrt{\tau}\right)^q \sum_{h=1}^{q} \sum_{\substack{3h_3+4h_4+\cdots+nh_n=q \\ h_3+h_4+\cdots+h_n=h \\ h_3,h_4,\cdots,h_n\geq 0}} \frac{(-1)^{h-1}}{h} q! \frac{h!}{h_3!h_4!\cdots h_n!}\left(\frac{\rho_3}{3!}\right)^{h_3}\left(\frac{\rho_4}{4!}\right)^{h_4}\cdots\left(\frac{\rho_n}{n!}\right)^{h_n}\phi^q \\
&= \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c - \frac{1}{2}\sigma^2\right)\tau\phi + \sum_{q=3}^{+\infty}\frac{1}{q!}\left(\sigma\sqrt{\tau}\right)^q \sum_{h=1}^{q}(-1)^{h-1}(h-1)!B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right)\phi^q,
\end{aligned}
\tag{O.1}
$$

where

$$
B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right) = \sum_{\substack{3h_3+4h_4+\cdots+nh_n=q \\ h_3+h_4+\cdots+h_n=h \\ h_3,h_4,\cdots,h_n\geq 0}} \frac{q!}{h_3!h_4!\cdots h_n!}\left(\frac{\rho_3}{3!}\right)^{h_3}\left(\frac{\rho_4}{4!}\right)^{h_4}\cdots\left(\frac{\rho_n}{n!}\right)^{h_n}
$$

refers to the partial or incomplete exponential Bell polynomials.

□

P Proof of Equation (28)

Proof.

$$
\begin{aligned}
&K^{\mathcal{S}}(\phi) \\
&= \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c + \frac{1}{2}\sigma^2\right)\tau\phi + \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left[(\phi+1)\,\sigma\sqrt{\tau}\right]^k - \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k \\
&= \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c + \frac{1}{2}\sigma^2\right)\tau\phi - \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k + \ln\left\{1 + \sum_{k=3}^{n}\frac{\rho_k}{k!}\left[(\phi+1)\,\sigma\sqrt{\tau}\right]^k\right\} \\
&= \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c + \frac{1}{2}\sigma^2\right)\tau\phi - \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k \\
&\quad + \left[\frac{\rho_3}{3!}\left(\sigma\sqrt{\tau}\right)^3(\phi+1)^3 + \frac{\rho_4}{4!}\left(\sigma\sqrt{\tau}\right)^4(\phi+1)^4 + \cdots\right] \\
&\quad - \frac{1}{2}\left[\frac{\rho_3}{3!}\left(\sigma\sqrt{\tau}\right)^3(\phi+1)^3 + \frac{\rho_4}{4!}\left(\sigma\sqrt{\tau}\right)^4(\phi+1)^4 + \cdots\right]^2 \\
&\quad + \frac{1}{3}\left[\frac{\rho_3}{3!}\left(\sigma\sqrt{\tau}\right)^3(\phi+1)^3 + \frac{\rho_4}{4!}\left(\sigma\sqrt{\tau}\right)^4(\phi+1)^4 + \cdots\right]^3 \\
&\quad - \frac{1}{4}\left[\frac{\rho_3}{3!}\left(\sigma\sqrt{\tau}\right)^3(\phi+1)^3 + \frac{\rho_4}{4!}\left(\sigma\sqrt{\tau}\right)^4(\phi+1)^4 + \cdots\right]^4 + \cdots \\
&= \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c + \frac{1}{2}\sigma^2\right)\tau\phi - \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k \\
&\quad + \frac{\rho_3}{3!}\left(\sigma\sqrt{\tau}\right)^3(\phi+1)^3 + \frac{\rho_4}{4!}\left(\sigma\sqrt{\tau}\right)^4(\phi+1)^4 + \frac{\rho_5}{5!}\left(\sigma\sqrt{\tau}\right)^5(\phi+1)^5 \\
&\quad + \frac{1}{6!}\left(\sigma\sqrt{\tau}\right)^6\left[\rho_6 - \frac{1}{2}\cdot 6!\cdot\frac{2!}{2!0!}\cdot\frac{(\rho_3)^2}{(3!)^2}\right](\phi+1)^6 \\
&\quad + \frac{1}{7!}\left(\sigma\sqrt{\tau}\right)^7\left[\rho_7 - \frac{1}{2}\cdot 7!\cdot\frac{2!}{1!1!}\cdot\frac{\rho_3}{3!}\cdot\frac{\rho_4}{4!}\right](\phi+1)^7 \\
&\quad + \frac{1}{8!}\left(\sigma\sqrt{\tau}\right)^8\left[\rho_8 - \frac{1}{2}\cdot 8!\cdot\frac{2!}{1!1!}\cdot\frac{\rho_3}{3!}\cdot\frac{\rho_5}{5!} - \frac{1}{2}\cdot 8!\cdot\frac{2!}{2!0!}\cdot\frac{(\rho_4)^2}{(4!)^2}\right](\phi+1)^8 \\
&\quad + \frac{1}{9!}\left(\sigma\sqrt{\tau}\right)^9\left[\rho_9 - \frac{1}{2}\cdot 9!\cdot\frac{2!}{1!1!}\cdot\frac{\rho_3}{3!}\frac{\rho_6}{6!} - \frac{1}{2}\cdot 9!\cdot\frac{2!}{1!1!}\cdot\frac{\rho_4}{4!}\frac{\rho_5}{5!} + \frac{1}{3}\cdot 9!\cdot\frac{3!}{3!0!0!}\cdot\frac{(\rho_3)^3}{(3!)^3}\right](\phi+1)^9 \\
&\quad + \cdots \\
&\quad + \frac{1}{q!}\left(\sigma\sqrt{\tau}\right)^q\left[\sum_{h=1}^{q}\sum_{\substack{3h_3+4h_4+\cdots+nh_n=q\\ h_3+h_4+\cdots+h_n=h\\ h_3,h_4,\cdots,h_n\geq 0}}\frac{(-1)^{h-1}}{h}q!\frac{h!}{h_3!h_4!\cdots h_n!}\left(\frac{\rho_3}{3!}\right)^{h_3}\left(\frac{\rho_4}{4!}\right)^{h_4}\cdots\left(\frac{\rho_n}{n!}\right)^{h_n}\right](\phi+1)^q \\
&\quad + \cdots
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c + \frac{1}{2}\sigma^2\right)\tau\phi - \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k \\
&\quad + \sum_{q=3}^{+\infty}\frac{1}{q!}\left(\sigma\sqrt{\tau}\right)^q \sum_{h=1}^{q}\sum_{\substack{3h_3+4h_4+\cdots+nh_n=q\\ h_3+h_4+\cdots+h_n=h\\ h_3,h_4,\cdots,h_n\geq 0}} \frac{(-1)^{h-1}}{h} q! \frac{h!}{h_3!h_4!\cdots h_n!}\left(\frac{\rho_3}{3!}\right)^{h_3}\left(\frac{\rho_4}{4!}\right)^{h_4}\cdots\left(\frac{\rho_n}{n!}\right)^{h_n}(\phi+1)^q \\
&= \frac{1}{2}\sigma^2\tau\phi^2 + \left(\mu_c + \frac{1}{2}\sigma^2\right)\tau\phi - \ln\sum_{k=0}^{n}\frac{\rho_k}{k!}\left(\sigma\sqrt{\tau}\right)^k \\
&\quad + \sum_{q=3}^{+\infty}\frac{1}{q!}\left(\sigma\sqrt{\tau}\right)^q \sum_{h=1}^{q}(-1)^{h-1}(h-1)!B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right)(\phi+1)^q,
\end{aligned}
\tag{P.1}
$$

where

$$
B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right) = \sum_{\substack{3h_3+4h_4+\cdots+nh_n=q\\ h_3+h_4+\cdots+h_n=h\\ h_3,h_4,\cdots,h_n\geq 0}} \frac{q!}{h_3!h_4!\cdots h_n!}\left(\frac{\rho_3}{3!}\right)^{h_3}\left(\frac{\rho_4}{4!}\right)^{h_4}\cdots\left(\frac{\rho_n}{n!}\right)^{h_n}
$$

refers to the partial or incomplete exponential Bell polynomials.

□

Q Proof of Proposition 5

Proof. By comparing Equation (33), (34), (35) and (36) with Equation (37), (38), (39) and (40), we are able to investigate the relationships between cumulants with Gram-Charlier density under $\mathcal{Q}$-measure and $\mathcal{S}$-measure.

When $m = 0, 1, 2$, we have

$$\kappa_0^{\mathcal{S}} = \kappa_0^{\mathcal{Q}} = 0, \tag{Q.1}$$

$$\kappa_1^{\mathcal{S}} = \kappa_1^{\mathcal{Q}} + \sigma^2\tau + \sum_{q=3}^{+\infty}\frac{1}{(q-1)!}\left(\sigma\sqrt{\tau}\right)^q\sum_{h=1}^{q}(-1)^{h-1}(h-1)!B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right), \tag{Q.2}$$

$$\kappa_2^{\mathcal{S}} = \kappa_2^{\mathcal{Q}} + \sum_{q=3}^{+\infty}\frac{1}{(q-2)!}\left(\sigma\sqrt{\tau}\right)^q\sum_{h=1}^{q}(-1)^{h-1}(h-1)!B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right). \tag{Q.3}$$

When $m \geq 3$,

$$
\begin{aligned}
\kappa_m^{\mathcal{S}} &= \kappa_m^{\mathcal{Q}} + \sum_{q=m+1}^{+\infty}\frac{1}{(q-m)!}\left(\sigma\sqrt{\tau}\right)^q\sum_{h=1}^{q}(-1)^{h-1}(h-1)!B_{q,h}\left(0,0,\rho_3,\cdots,\rho_n\right) \\
&= \kappa_m^{\mathcal{Q}} + \sum_{q=m+1}^{+\infty}\frac{1}{(q-m)!}\kappa_q^{\mathcal{Q}} \\
&= \kappa_m^{\mathcal{Q}} + \frac{\kappa_{m+1}^{\mathcal{Q}}}{1!} + \frac{\kappa_{m+2}^{\mathcal{Q}}}{2!} + \frac{\kappa_{m+3}^{\mathcal{Q}}}{3!} + \cdots \\
&= \sum_{q=0}^{+\infty}\frac{\kappa_{m+q}^{\mathcal{Q}}}{q!}.
\end{aligned}
\tag{Q.4}
$$

It is obvious that the cases $m = 1, 2$ are also consistent with Equation (Q.4), then we can combine them together.

Hence, the relationships between cumulants with Gram-Charlier density under $\mathcal{Q}$-measure and $\mathcal{S}$-measure are given by

when $m = 0$,

$$\kappa_0^{\mathcal{S}} = \kappa_0^{\mathcal{Q}}, \tag{Q.5}$$

when $m \geq 1$,

$$\kappa_m^{\mathcal{S}} = \sum_{q=0}^{+\infty} \frac{\kappa_{m+q}^{\mathcal{Q}}}{q!}. \tag{Q.6}$$

□