How Heterogeneity Drives the Yield, Correlation, and Volatility of Oil and Stock

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Abstract

We explore the effects of households' heterogeneity on the crude oil, stock, and bond markets in an equilibrium model. The asset prices, volatilities, and correlations admit closed-form solutions. The key parts of all the asset yields can be explained by different weighted averages of the expected growth rates of goods or consumption portions among households, and the time-varying movements of all these voatilities and correlations can be well captured by differences between two certain sorts of weight averages, and all these weights depend on households' heterogeneous beliefs and preferences. The model estimation shows excellent performance in fitting three markets, including term structures of interest rates and crude oil futures, volatilities, and the correlation between crude oil futures and stock. Our model can explain many empirical regularities, including the time-varying correlation-volatility of oil and stock, decreasing and convex volatility term structure of crude oil futures, and the V-shaped relationship between the futures volatility and the slope of the futures curve.

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1 Introduction

The crude oil market had experienced a massive boom and bust from 2004 to 2009. Since 2008 the correlation between crude oil futures and S&P 500 stock index jumped up sharply, while before that, the correlation mostly fluctuated around zero (see Figure 3). These phenomena, along with similar patterns in other commodity futures, have raised considerable interest in understanding the joint dynamics of commodity futures and stock markets and the determinant factors for such dynamics.

Some empirical studies attribute the increased comovements of commodity futures and stock markets to the financialization of commodities, referring to the influx of institutional investors into the commodity futures markets, and others emphasize the role of speculation.[1] However, both interpretations may be driven by more primitive economic factors, such as preferences and beliefs. This paper investigates how an equilibrium model with heterogeneity in preferences and beliefs can generate the observed dynamics of the crude oil futures and stock prices and other critical empirical regularities of term structures of interest rates and crude oil futures prices and volatilities.

We adopt a pure-exchange equilibrium model with a commodity, crude oil, and a generic consumption good. Households are heterogeneous in preferences (time, crude oil, and consumption good) and beliefs about the growth rates of crude oil and the consumption good supplies. All households observe the supplies with incomplete but symmetric information; they learn different expected growth rates of the supplies based on their own beliefs about how the expected growth rates evolve. When the households' models of the expected growth rates are different enough, they will never agree. Such households' estimated growth rates of the fundamentals are generally different.

Households have both time and good additive logarithm utilities. Taking the generic consumption good as the numeraire, the prices of stock, the claim to the stream of the consumption good, crude oil, and bonds are the consumption-weighted households' evalu-

[1] Tang and Xiong (2012) find that the growing index investment in commodity markets promotes the comovement between indexed commodities and stocks. Prices of non-energy commodity futures have shown to be increasingly correlated with oil prices, especially for those within the index group. Other empirical studies, such as Silvennoinen and Thorp (2013) and Singleton (2013), also support this view. However, several studies (Hamilton and Wu, 2015; Fattouh, Kilian and Mahadeva, 2013; Irwin and Sanders, 2011) test the influence of speculation and weaken the explanatory power of financialization.

ations. Households' heterogeneity in beliefs makes the consumption distributions among households stochastic. Thus, our equilibrium model generates stochastic volatilities and correlation of stock and crude oil, which are functions of the differences in households' estimated growth rates of the fundamentals, together with the consumption distributions.

To estimate, we try to fit asset prices, volatilities, and correlations with data in stock, bond, and oil futures markets over time. In detail, the time-series include the dividend yield of S&P 500 index, volatilities and correlation between the prices of 1-month oil futures and S&P 500 index, the term structures of Treasury yields, convenience yields of oil futures,[2] and volatilities of oil futures, all covering six maturities, through 3 and 6 months, 1, 2, 3, 5 years. Compared with methods to match the moments between data and model, our estimation provides estimated state variables that generate the time-varying prices, volatilities, and correlation over time. The model performs very well in fitting the data. It captures the key time-series features of its empirical counterparts. Among the time-series that the model tries to match, most of the R^2s are well-above 90% with an average of 97%. In addition to capturing the time-series properties of asset prices, volatilities, and correlations, our model also sheds light on the term structures of interest rates, oil convenience yields and their volatilities, and the relationship between oil volatility and the slopes of the oil futures curve.

In our model, heterogeneity in beliefs and preferences generates fluctuation in wealth and thus in relative consumption, amplifying the volatilities and correlations among stock, bond, and oil futures and drives them variating over time. Without disagreement, both volatilities and correlations are constant, and volatilities are much smaller than those observed in the data. Disagreement leads to different investments and consumption, and different preferences lead to different evaluations of investment opportunities. Since asset prices are consumption-weighted individual evaluations, heterogeneous beliefs and relative consumptions generate stochastic volatilities and correlations. For example, if a household with higher evaluation gets wealthier and thus consumes more, it would cause extra asset price variation due to the relative consumption changes.

[2]Because the model yields a closed-form solution of the forward price, we ignore the difference between the futures and forward prices (see the support from Geman (2009)), and use the futures data in the estimation.

Convenience yield in our model positively depends on a weighted average of households' oil expected growth rates. Because the weight is each household's relative oil consumption, we can use this weighted average as a proxy about how the market concerns crude oil's future availability. Higher future availability would decrease the time value of the forward contract. This explanation echos that of Casassus, Liu and Tang (2013), and they find that the convenience yield of a commodity depends on its relative scarcity to other related commodities.

Moreover, we show how the slope of the term structure of convenience yields varies over time. We find that the slope of the convenience yield curve approximately negatively depends on the *oil-consumption-weighted* average of the beliefs about expected oil growth rate, adjusted by the speed of mean reversion. Suppose the weighted growth rate is relatively high, meaning that the near-future oil availability is higher than the long-run mean level. The current high expected growth rate will fall over time because of the mean-reverting property, so the convenience yield curve is decreasing with maturity in this case. On the other hand, the weighted expected oil growth rate is relatively low, then the term structure of convenience yields is upward-sloping.

Our model can also generate the Samuelson (1965) effect, that is, commodity futures price typically exhibits a declining volatility term structure. In our model, disagreement about oil growth is a major source of futures volatility. The Samuelson effect follows as the disagreement tends to decline over time. Routledge, Seppi and Spatt (2000) develop an equilibrium model and show that the Samuelson effect violation can happen in extreme conditions like high inventory. The other two papers about the volatility term structure are Hitzemann (2015) and Khan, Khokher and Simin (2017). Khan, Khokher and Simin (2017) show that the negative slope of volatility term structure comes from the mean-reverting of supply. However, we find that it is not enough to generate the time-varying slopes of the futures volatility curve. Hitzemann (2015) explains that because the uncertainty can be adjusted by inventory in the long run, the volatility is smaller in longer maturities, but they only use the model to match the futures volatility within one-year maturity. In contrast, we employ a full spectrum of futures maturities available and fit the volatility term structure over time. For example, the conditional slope of the volatility curve depends on the degree of disagreement.

The previous discussions imply our model also generates the V-shape relationship between the volatility futures prices and the slopes of the futures price curve since high disagreement tends to generate steep futures price curves. This V-shape relation are documented by Kogan, Livdan and Yaron (2009) and Carlson, Khokher and Titman (2007). Because the futures price curve contains both term structures of yields and convenience yields, we further study how the two term-structures contribute to the V-shape relation. Our empirical study shows that the information in the bond market does not play any role in explaining the V-shape relationship; meanwhile, the level and slope of the convenience yield curve significantly determine the V-shape relationship, matching the negative and positive sides of the V-shaped curve, respectively.

This paper adds to the literature on commodity price dynamics, especially about crude oil. There are two main strands in the theoretical literature, and one builds on the assumption of long-run risk (Ready, 2018; Hitzemann, 2015), the other is based on affine factor models (Casassus and Collin-Dufresne, 2005; Chiang, Hughen and Sagi, 2015; Khan, Khokher and Simin, 2017; Heath, 2019). Concerning the characteristics of the commodity market, Sockin and Xiong (2015) and Goldstein and Yang (2015) capture the information friction, and Routledge, Seppi and Spatt (2000) and Khan, Khokher and Simin (2017) build on inventory. Both Ready (2018) and Hitzemann (2015) focus on macroeconomic shock and long-run risk. Besides, several other studies pay attention to special constraints in production or trading in the commodity market, such as collateral constraint (Tang and Zhu, 2016), adjustment cost (Carlson, Khokher and Titman, 2007), limit to arbitrage (Acharya, Lochstoer and Ramadorai, 2013) and constraint in investment (Kogan, Livdan and Yaron, 2009; Casassus, Collin-Dufresne and Routledge, 2018; Liu, Qiu and Tang, 2011).

In recent years, the interactions among different commodity market participants have become one of the main factors to understand the commodity markets, especially the debate about financialization. This paper belongs to the literature with heterogeneous market participants. Our framework in the endowment economy is mostly close to Basak and Pavlova (2016) and Baker and Routledge (2017). However, our model is different from theirs in several aspects. First, consumption in the model of Basak and Pavlova (2016) only occurs at the end of the economy; ours includes intertemporal consumption. Second, all investors have the same beliefs in their models, leading to relatively stable asset price volatilities and correlations. As to the model implications, the correlations in the model of Basak and Pavlova (2016) are always positive, which is at odds with the data, e.g., we do see significant negative equity-commodity correlations as depicted in Figure 3. Last, we rigorously estimate the model to fit time-series data and perform empirical studies with the estimated model. In contrast, they both use numerical examples to illustrate some unconditional qualitative implications of their models.

The rest of the paper is organized as follows. Section 2 discusses the model and its solutions, and the estimation results are reported in Section 3. Section 4 discusses empirical evidence and model implications, and Section 5 concludes.

2 Model

In this section, we present our model mainly based on the framework of Li (2007) and Li and Muzere (2010).

2.1 Consumption Good and Commodity

We consider a continuous-time pure exchange economy as a complete market. There are 2 goods: one is a generic good, e.g., a bundle of many varieties; the other commodity is crude oil, denoted by δ and h respectively. The supplies of the 2 goods follow the form below

$$\begin{pmatrix} \mathrm{d}\delta(t)/\delta(t) \\ \mathrm{d}h(t)/h(t) \end{pmatrix} = \begin{pmatrix} \mu_\delta(t) \\ \mu_h(t) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \sigma_\delta^\top \\ \sigma_h^\top \end{pmatrix} \mathrm{d}B(t), \tag{1}$$

where $\mu_\delta(t)$, $\mu_h(t)$ are the mean growth rates or drifts and σ_δ, σ_h are constant volatility vectors, which represent the fundamental risk, and $B(t)$ is a 2-dimensional Brownian motion defined on some suitable probability space. And we denote the vector $\mu(t) = (\mu_\delta(t),\, \mu_h(t))^\top$ and square matrix $\sigma = (\sigma_\delta,\, \sigma_h)^\top$.

2.2 Preferences and Beliefs

There are N types of households in the economy. Household i has logarithmic preferences represented by

$$\int_0^\infty e^{-\rho_i t}\left[\log c_i(t) + \lambda_i \log s_i(t)\right] \mathrm{d}t,$$

where $\rho_i > 0$ is the subjective discount rate, $\lambda_i \geqslant 0$ represents[3] the heterogeneity in consumption of crude oil, $c_i(t)$ represents household consumption of the generic good and s_i represents the consumption of crude oil. So in preferences, households differ in two dimensions, subjective discount rate ρ and consumption of commodities λ.

The N households differ not only in preferences but also in their beliefs about the dynamics of the fundamentals. They all observe the supply $\delta(t)$ and $h(t)$ continuously and agree on the same dynamic pattern in equation (1), where the mean-reverting process of expected growth rate $\mu(t)$ is unobservable. The key difference lies in that they disagree on the covariance matrix in the dynamics of $\mu(t)$. So following standard filtering theory (e.g., Liptser and Shiryaev, 2013, and see details in Appendix A.1), they learn from the observed supply and update the dynamics of expected growth rates by

$$\mathrm{d}\widehat{\mu}_i(t) = \kappa_i[\alpha_i - \widehat{\mu}_i(t)]\,\mathrm{d}t + \widehat{\Sigma}_i\,\mathrm{d}\widehat{B}_i(t), \tag{2}$$

based on their beliefs about the dynamic of μ, where we denote by $\widehat{\mu}_i(t) = \mathbb{E}_t^i\left[\mu(t)\right]$ the conditional expectation based on the information set of $(\delta, h)^\top$, and α_i is a two-dimensional constant vector, and both κ_i and $\widehat{\Sigma}_i$ are two constant square matrices, and $\widehat{B}_i(t)$ is the

[3] To avoid confusion, we explain the use of superscripts and subscripts as follows. There are five dimensions, households, goods (assets), Brownian motion, maturity and time. The indexes, i and j, are only used to indicate households, and δ, h for the generic good and the commodity in goods markets respectively, and all capital letters including D, H, S, C for financial markets, respectively. Notice that there are three dimensions, households, goods (assets) and Brownian motion, so if any of them appears alone, we just write it in subscripts, such as $\widehat{\mu}_i$ (household i's expected growth rates of supply of both goods) or σ_h (the transpose of the second row in σ, which is the volatility vector of crude oil). If there are two dimensions in a variable, we prefer to set the number of household i as the subscript and the symbol of good as the superscript in case of misunderstanding i as an exponent sign, except for when t is included in expectation $\mathbb{E}_t^i[\cdot]$, or variables in financial markets such μ_S^i, μ_H^i. For example, $\widehat{\mu}_i^h$ is household i's expected growth rate of the supply of crude oil, and $\widehat{\mu}_S^i$ is household i's expected growth rate of stock price. If dimension about Brownian motion is also included, we always put it into the subscript with a comma as $\sigma_\delta = (\sigma_{\delta,1}, \sigma_{\delta,2})^\top$.

innovation process, such that the observed supply can also be interpreted as

$$\begin{pmatrix} \mathrm{d}\delta(t)/\delta(t) \\ \mathrm{d}h(t)/h(t) \end{pmatrix} = \widehat{\mu}_i(t)\,\mathrm{d}t + \sigma\,\mathrm{d}\widehat{B}_i(t). \tag{3}$$

This equation shows immediately that

$$\mathrm{d}\widehat{B}_i(t) - \mathrm{d}\widehat{B}_j(t) = -\sigma^{-1}\left(\widehat{\mu}_i(t) - \widehat{\mu}_j(t)\right)\mathrm{d}t. \tag{4}$$

Due to the fact that μ is unobservable, households make their consumption and investment decision based on their filtered probability space as defined by equations (2) and (3).

2.3 Assets and Commodity Markets

We use the generic consumption as the numeraire, or normalize the price of the generic good to equal 1. The spot price of crude oil is $R(t)$. In the financial markets, households can trade a net-zero supply of bonds, stock that claims to the stream of the generic good, and forward contracts written on crude oil.

The risk-free asset, $P(t)$, evolves according to

$$\frac{\mathrm{d}P(t)}{P(t)} = r(t)\,\mathrm{d}t,$$

where $r(t)$ is the instantaneous interest rate, and stock price $S(t)$, under household i's beliefs, evolves as[4]

$$\mathrm{d}S(t) + \delta(t)\,\mathrm{d}t = S(t)\mu_S^i(t)\,\mathrm{d}t + S(t)\sigma_S(t)\cdot\mathrm{d}\widehat{B}_i(t),$$

and similarly, the market value of a forward contract of crude oil $C(t)$ follows

$$\mathrm{d}C(t) = C(t)\mu_C^i(t)\,\mathrm{d}t + C(t)\sigma_C(t)\cdot\mathrm{d}\widehat{B}_i(t),$$

[4]Note that the volatility does not vary across different probability measures in our context.

where μ^i_S and μ^i_C are household i's forecasts of the mean growth rates, and σ_S and σ_C are the volatility vectors of stock and forward contract of crude oil, respectively.

Let $\mu^i_A(t) = (\mu^i_S(t),\, \mu^i_C(t))^\top$and $\sigma_A(t) = (\sigma_S(t),\, \sigma_C(t))^\top$, then the following equation must hold in equilibrium

$$\mathrm{d}\widehat{B}_i(t) - \mathrm{d}\widehat{B}_j(t) = -\sigma_A^{-1}(t)\left(\mu^i_A(t) - \mu^j_A(t)\right)\mathrm{d}t.$$

Then, combining this with equation (4) yields

$$\sigma_A^{-1}(t)\left(\mu^i_A(t) - \mu^j_A(t)\right) = \sigma^{-1}\left(\widehat{\mu}_i(t) - \widehat{\mu}_j(t)\right), \tag{5}$$

and the price of risk is

$$\theta_i(t) = \sigma_A^{-1}(t)\left(\mu^i_A(t) - r(t)\mathbf{1}\right),$$

therefore, equation (4) implies that

$$\beta_{ij}(t) \equiv \theta_i(t) - \theta_j(t) = \sigma^{-1}\left(\widehat{\mu}_i(t) - \widehat{\mu}_j(t)\right), \tag{6}$$

and

$$\mathrm{d}\widehat{B}_i(t) - \mathrm{d}\widehat{B}_j(t) = -\beta_{ij}(t)\,\mathrm{d}t.$$

These relations must hold in equilibrium.

Given the price of risk, the state price density for household i is then given by

$$\xi_i(t) = \exp\left[-\int_0^t \left(r(a) + \frac{1}{2}\|\theta_i(a)\|^2\right)\mathrm{d}a - \int_0^t \theta_i(a)\cdot \mathrm{d}\widehat{B}_i(a)\right].$$

Using equation (6), we have that the ratio of state price density between households is given by

$$\frac{\xi_i(t)}{\xi_j(t)} = \exp\left(-\frac{1}{2}\int_0^t \|\beta_{ij}(a)\|^2\mathrm{d}a - \int_0^t \beta_{ij}(a)\cdot \mathrm{d}\widehat{B}_i(a)\right). \tag{7}$$

Because of the complete markets in our setup, the equilibrium stochastic discount factor is unique, and $\xi_i(t)$ and $\xi_j(t)$ are merely different in two measures. Our another state variable $\eta_{ij}(t)$ is also related with this ratio. Notice that it does not depend on any endogenous variables. This observation is very useful to construct the equilibrium.

2.4 Households Optimization

Households choose consumption plans subject to budget constraints to maximize expected utility. Such a problem is well understood in the literature. Essentially, the optimization problem can be solved by the *martingale approach* as follows:

$$\max_{\{c_i(t),\, s_i(t)\}_0^\infty} \mathbb{E}^i \left[\int_0^\infty e^{-\rho_i t} \left(\log c_i(t) + \lambda_i \log s_i(t) \right) \mathrm{d}t \right],$$

subject to a static budget constraint

$$\mathbb{E}^i \left[\int_0^\infty \xi_i(t) \left(c_i(t) + R(t) s_i(t) \right) \mathrm{d}t \right] \leq W_i(0),$$

where $W_i(0)$ is the initial wealth of household i. The trading strategy that finances the optimal consumption plans can then be found by

$$W_i(t) = \frac{1}{\xi_i(t)} \mathbb{E}_t^i \left[\int_t^\infty \xi_i(a) \left(c_i^*(a) + R(a) s_i^*(a) \right) \mathrm{d}a \right]. \tag{8}$$

The first-order conditions for household i are

$$e^{-\rho_i t} \frac{1}{c_i(t)} = \psi_i \xi_i(t), \quad e^{-\rho_i t} \frac{\lambda_i}{s_i(t)} = \psi_i \xi_i(t) R(t), \tag{9}$$

where ψ_i is a constant. Substituting these conditions into wealth equation yields

$$W_i(t) = \frac{1+\lambda_i}{\rho_i} \frac{e^{-\rho_i t}}{\psi_i \xi_i(t)}. \tag{10}$$

Then, these conditions show the propensities of consumption are constant and given by

$$\frac{c_i(t)}{W_i(t)} = \frac{\rho_i}{1+\lambda_i} \quad \text{and} \quad \frac{R(t) s_i(t)}{W_i(t)} = \frac{\rho_i \lambda_i}{1+\lambda_i} \tag{11}$$

for the generic consumption good and crude oil.

Let η_{ji} be the consumption ratios of the generic good between households j and i, that is

$$\begin{aligned}\eta_{ji}(t) &\equiv \frac{c_j(t)}{c_i(t)} = e^{-(\rho_j-\rho_i)t}\frac{\psi_i\xi_i(t)}{\psi_j\xi_j(t)} \\ &= \frac{\psi_i}{\psi_j}\exp\left[-\int_0^t\left(\rho_j-\rho_i+\frac{1}{2}\|\beta_{ij}(a)\|^2\right)\mathrm{d}a-\int_0^t\beta_{ij}(a)\cdot\mathrm{d}\widehat{B}_i(a)\right]. \qquad (12)\end{aligned}$$

These consumption ratios, which only depend on exogenous variables, are the key state variables to characterize the equilibrium. This equation also shows that $e^{(\rho_j-\rho_i)t}\eta_{ji}(t)$ is a martingale under household i's beliefs, that is

$$\mathbb{E}_t^i\left[e^{(\rho_j-\rho_i)a}\eta_{ji}(a)\right] = e^{(\rho_j-\rho_i)t}\eta_{ji}(t), \quad \forall a > t.$$

This martingale property is used in deriving stock and commodity prices.

These results directly show that households' shares in both the generic consumption good and crude oil are also exogenous, which are crucial in expressing the equilibrium asset prices. Household j's' consumption weight in the generic good is

$$\frac{c_j(t)}{\sum_j c_j(t)} = \frac{\eta_{ji}(t)}{\sum_j \eta_{ji}(t)}. \qquad (13)$$

and similarly, by equation (9), household j's consumption weight in crude oil is

$$\frac{s_j(t)}{\sum_j s_j(t)} = \frac{\lambda_j\eta_{ji}(t)}{\sum_j \lambda_j\eta_{ji}(t)}. \qquad (14)$$

From now on, we refer these two weights as generic consumption-weight and oil consumption-weight, respectively.

2.5 Equilibrium Prices and Volatility

Market clearing conditions[5] are

$$\sum_{i=1}^{N} c_i(t) = \delta(t), \quad \sum_{i=1}^{N} s_i(t) = h(t),$$

and together with the first-order conditions in equation (9) and the definition of η_{ji}, we can find that the state price density for household i is

$$\xi_i(t) = \frac{e^{-\rho_i t}}{\psi_i} \times \frac{\sum_j \eta_{ji}(t)}{\delta(t)}, \tag{15}$$

and the spot price of crude oil[6]

$$R(t) = \frac{\overline{\lambda}_\eta \, \delta(t)}{h(t)}. \tag{16}$$

Also, the equilibrium interest rate (see Appendix A.2) is given by

$$r(t) = \overline{(\rho + \mu_\delta)}_\eta - \|\sigma_\delta\|^2. \tag{17}$$

Notice that

$$\frac{\lambda_j \delta(t)}{h(t)} \quad \text{and} \quad \rho_j + \widehat{\mu}_j^\delta(t) - \|\sigma_\delta\|^2$$

are the price of crude oil and interest rate, respectively, in a homogeneous economy in which only household j exists. Given the consumption weights in equation (13), the price of crude oil and interest rate in the heterogeneous economy are the consumption-weighted homogeneous counterparts. The interest rate consists of intertemporal substitution, ρ_j from the preference side and $\widehat{\mu}_j^\delta(t)$ from the supply side, and precautionary saving caused by the risk from the supply side, $\|\sigma_\delta\|^2$. However, compared with homogeneous economies,

[5]Storage of crude oil by households and firms only accounts for a small proportion. Data show that Strategic Petroleum Reserve, which is the main storage of crude oil, is mostly stable, so we ignore the storage for simplicity.

[6]We explain the symbol of weighted average for simplicity as follows. All pattern like $\overline{f(X_a, Y)}_\omega$ represents $\frac{\sum_j \omega_j f(X_j^a, Y_j)}{\sum_j \omega_j}$, and if η appears in the weight ω, it stands for η_{ji} which varies with j given a certain fixed i. For example, $\overline{\mu}_{\lambda\eta} = \frac{\sum_j \lambda_j \eta_{ji}(t) \widehat{\mu}_j(t)}{\sum_j \lambda_j \eta_{ji}(t)}$, and we always omit the t and hat symbol in any learned variable $\widehat{X}$ for simplicity.

household heterogeneity provides extra fluctuations to the oil price and interest rate because the consumption weights are stochastic, driving by heterogeneous beliefs as shown by equation (12).

The stock price, claimed to the stream of the generic consumption good, is given by (see Appendix A.3)

$$S(t) = \frac{1}{\xi_i(t)} E_t^i \left[\int_t^{\infty} \xi_i(a)\delta(a)\,\mathrm{d}a \right] = \overline{(1/\rho)}_\eta \cdot \delta(t). \tag{18}$$

The stock price is also the consumption-weighted average of the counterparts (δ/ρ_j) for homogeneous economies. Without the heterogeneity in preferences, the price-to-dividend ratio (S/δ), as well as the oil price to relative supply ratio (Rh/δ), becomes constant. Thus, the disagreements drive these ratios stochastic and fluctuate between $\frac{1}{\max_j \rho_j}$ and $\frac{1}{\min_j \rho_j}$ for the stock and $\min_j \lambda_j$ and $\max_j \lambda_j$ for crude oil. These endogenous fluctuations show that household heterogeneity provides extra volatilities for both stock and crude oil prices.

The closed-form solutions of stock price $S(t)$ and spot price $R(t)$ enable us to compute the price dynamics. Applying Itô lemma, the stock and commodity spot prices yields the volatilities (see Appendix A.3)

$$\sigma_S(t) = \sigma_\delta + \sigma^{-1}\big(\,\overline{\mu}_{\eta/\rho} - \overline{\mu}_\eta\,\big) \tag{19a}$$

$$\sigma_R(t) = \sigma_\delta - \sigma_h + \sigma^{-1}\big(\,\overline{\mu}_{\lambda\eta} - \overline{\mu}_\eta\,\big). \tag{19b}$$

From these expressions, the variances and covariance of the instantaneous returns for the stock and spot price of crude oil are then given by $\|\sigma_S(t)\|^2$, $\|\sigma_R(t)\|^2$, and $\sigma_S(t) \cdot \sigma_R(t)$, respectively. If households have homogeneous beliefs, then both the volatilities of stock and crude oil spot prices come from the fundamentals, but empirical results show that the price volatilities of stock and crude oil are much higher than that of fundamentals. The heterogeneity in preferences and beliefs can generate excessive volatilities. Based on equation (19), the extra volatile components are derived from the different weighted averages of the households' heterogeneous beliefs between the consumption-weighted and the valuation-weighted. Furthermore, the heterogeneity in preferences and beliefs also drives the correlation between stock and crude oil deviated from that implied by the fundamentals. These volatilities and the correlation are evolving stochastically because η_{ji} and μ_j are stochastic.

2.6 Bond, Forward, and Volatilities

There are two kinds of consumption goods in the economy: the generic consumption and crude oil. Naturally, there are two real bonds, and one pays 1 unit of the generic consumption good; the other pays 1 unit of crude oil. As shown later, the term structures of these two bonds fully characterize the term structure of the forward prices or convenience yields of crude oil. Because we use the generic consumption as the numeraire, the price of a zero-coupon bond that pays 1 unit of the generic consumption good or crude oil at T is given by

$$P(t,T) = \frac{1}{\xi_i(t)} \mathbb{E}_t^i \left[\xi_i(T)\right] \quad \text{or} \quad P_c(t,T) = \frac{1}{R(t)\xi_i(t)} \mathbb{E}_t^i \left[\xi_i(T) R(T)\right],$$

respectively, where the price for crude oil bond is expressed in the units of crude oil.

Due to the affine structure of each household's model (Duffie and Kan, 1996), the prices of bond and forward price are (see Appendices A.4)

$$P(t,T) = \overline{\mathbf{f}_{\eta}^{\delta,\tau}}(t) \tag{20a}$$

$$P_c(t,T) = \overline{\mathbf{f}_{\lambda\eta}^{h,\tau}}(t), \tag{20b}$$

where, $\tau = T - t$ and for $a = \delta,\ h$,

$$\mathbf{f}_j^{a,\tau}(t) := F(\tau, \widehat{\mu}_j^a(t)) = \exp\Big[- b_j^a \tau - \frac{1 - e^{-\kappa_j^a \tau}}{\kappa_j^a} \Big(\widehat{\mu}_j^a(t) - \gamma_j^a(\tau) \Big) \Big] \tag{21}$$

and

$$b_j^a = \rho_j + \gamma_j^a(0) - \|\sigma_a\|^2, \qquad \gamma_j^a(\tau) = \alpha_j^a - \frac{\widehat{\Sigma}_j^a \cdot \sigma_a}{\kappa_j^a} - \frac{\|\widehat{\Sigma}_j^a\|^2}{4\kappa_j^{a\,2}} \cdot \big(3 - e^{-\kappa_j^a \tau}\big).$$

The bond price measured in the generic consumption (crude oil) is the generic (crude oil) consumption-weighted average of that in the homogeneous economies, as $F(T-t, \widehat{\mu}_j^\delta)$ ($F(T-t, \widehat{\mu}_j^h)$) is the bond price in a homogeneous economy with household j only. The bond prices in any of the homogeneous economies form a one-factor Gaussian term structure of interest rates. However, the term structure of interest rates in the heterogeneous economy is no longer admitted into the affine structure due to stochastic consumption weights.

The forward price is determined by

$$\frac{1}{\xi_i(t)}\mathbb{E}_t^i\left\{\xi_i(T)[R(T)-H(t,T)]\right\}=0,$$

that is, the forward contract worths zero at initiation. Therefore, the forward price is given by

$$H(t,T)=\frac{R(t)}{P(t,T)}\cdot\frac{1}{R(t)\xi_i(t)}\mathbb{E}_t^i\left[\xi_i(T)R(T)\right]=R(t)\cdot\frac{P_c(t,T)}{P(t,T)}. \tag{22}$$

Thus, the forward price of crude oil is the price of a bond for receiving 1 unit of crude oil measured in the generic consumption divided by the price of a bond for receiving 1 unit the generic consumption. The impact of heterogeneity comes through the two bonds and the spot crude price.

A direct calculation show the volatility vector of Treasury yield $\sigma_y(t,T)$ as (see Appendix A.5),

$$\sigma_y(t,T)=\frac{1}{\tau}\sigma^{-1}\left(\overline{\mu}_\eta-\overline{\mu}_{\eta\mathbf{f}_{\delta,\tau}}\right)+\overline{\left(\frac{1-e^{-\kappa_\delta\tau}}{\kappa_\delta\tau}\Sigma_\delta\right)}_{\eta\mathbf{f}_{\delta,\tau}}, \tag{23}$$

and the forward price volatility vector $\sigma_H(t,T)$ as (see Appendix A.5),

$$\sigma_H(t,T)=\sigma_\delta-\sigma_h+\sigma^{-1}\left(\overline{\mu}_{\lambda\eta\mathbf{f}_{h,\tau}}-\overline{\mu}_{\eta\mathbf{f}_{\delta,\tau}}\right)+\overline{\left(\frac{1-e^{-\kappa_\delta\tau}}{\kappa_\delta}\Sigma_\delta\right)}_{\eta\mathbf{f}_{\delta,\tau}}-\overline{\left(\frac{1-e^{-\kappa_h\tau}}{\kappa_h}\Sigma_h\right)}_{\lambda\eta\mathbf{f}_{h,\tau}}, \tag{24}$$

where $\tau=T-t$. Similar to the spot price, we can then compute the volatility of crude forward price and its correlations with the stock price as $\|\sigma_H(t,T)\|$ and $\sigma_H(t,T)\cdot\sigma_S(t)/\|\sigma_H(t,T)\|/\|\sigma_S(t)\|$. By the fact that $F(0,\cdot)=1$, we have $\sigma_H(t,t)=\sigma_R(t)$. In addition to the differences in beliefs, the forward price volatility also depends on the volatilities of households' learnings about the growth rates of the aggregate supplies.

3 Model Estimation

To investigate how well the model can explain the data and what insights the model can provide, we estimate a version of the model that consists of three households.[7] We use the S&P 500 stock index as the stock that claims to the generic consumption good, and crude oil futures as the model's forward contracts.[8]

3.1 Data

In the estimation, we try to match three blocks of data with their counterparts in the model. The first block consists of four time-series from the stock and crude oil futures markets, including dividend yield of S&P 500 stock index, annual rolling volatility of returns of the S&P 500 index, front-month futures of crude oil, and the annual rolling correlation between the two series. The second block consists of twelve time-series yields, include Treasury yields and convenience yields of crude oil futures, covering six maturities, 3 and 6 months, 1, 2, 3, and 5 years for both yields. The third block consists of futures price volatility of crude oil across the same six maturities as for the yields. We get the Treasury yields from *Federal Reserve Economic Data*, and the S&P 500 index data from CRSP. The futures prices for the NYMEX light sweet crude oil contract (CL) are downloaded from *Bloomberg*. All the realized volatilities and correlation are calculated in one-year rolling. Because the oil data starts from April 1983, our time range starts from March 1984 to December 2019, and we ignore the missing data of long-maturity futures in early period. All data are annualized at a monthly frequency. To fix the parameter α and σ, we get the data of annual *oil including lease condensate production* from EIA as

[7] Our experiments seem to suggest that a model with two households is not flexible enough to capture the essential features presented in the data.

[8] The forward and futures prices should be different in our model, but futures price does not admit any analytical solution, so we use forward price instead to match the futures price in the data. Also see Geman (2009) for related empirical evidence.

the oil production, and also calculate the annual dividend based on CRSP. Details to set α and σ will be discussed in next subsection.

3.2 Parameter

The estimated model contains 3 households and 2 assets including S&P 500 stock index and crude oil futures, denoted as δ and h, respectively. We summarize the model parameters and state variables in Table 1. There are 6 types and totally 25 numbers of constant parameters,[9] α, κ_i, $\widehat{\Sigma}_i$, λ, ρ, σ, in which α and σ are fixed and the other 20 parameters are obtained by estimation. Here α, κ_i and $\widehat{\Sigma}_i$ are parameters from households' learnings the dynamics of the fundamentals and λ, ρ, σ are parameters from households' preferences. There are also two sets of state variables, the ratios of the generic consumptions $\eta_{ji}(t)$ and the updated growth rates of the fundamentals $\widehat{\mu}_i(t)$. We assume all households agree on that the long term mean level of consumption and oil supply, equal to the observed sample means, so we fix $\alpha_\delta = 0.0580$ and $\alpha_h = 0.0094$. Given the annual dividend and oil production data, we fix the volatility matrix as the Cholesky decomposition of the two time-series' covariance matrix, in which $\sigma_{\delta,1} = 0.0733$, $\sigma_{\delta,2} = 0.0107$, and $\sigma_{h,2} = 0.0618$.

3.3 Methodology

One challenge of estimating the heterogeneous belief model is that there is more than one probability measure, and estimations using different measures lead to different results. One Brownian motion under one's belief is not a Brownian motion any more under another household's beliefs. Another feature is that the household-specific state variables are highly correlated with each other. All state variables jointly determine the equilibrium prices, so the household-specific state variables must be estimated simultaneously with the implied dynamics. Given these two points, we build our estimation by minimizing the sum of the squares of the distance between data and model results and maximizing the joint transition likelihood of households' filtered growth rates. The joint transition follows

[9] $\widehat{\Sigma}_i$ and σ are assumed to be triangle matrices. The consumption portion of crude oil λ in utility is just a relative value, so λ_1 is set to be 1, and given the freedom of other two, it can still provide heterogeneity.

Table 1: List of parameters

	demension	**pattern**
constant parameters		
α_i	2	$\alpha_i = \begin{pmatrix} \alpha_i^\delta \\ \alpha_i^h \end{pmatrix}$
κ_i	2	$\kappa_i = \begin{bmatrix} \kappa_i^\delta & 0 \\ 0 & \kappa_i^h \end{bmatrix}$
$\widehat{\Sigma}_i$	2×2	$\widehat{\Sigma}_i \triangleq \begin{bmatrix} (\widehat{\Sigma}_i^\delta)^\top \\ (\widehat{\Sigma}_i^h)^\top \end{bmatrix} = \begin{bmatrix} \widehat{\Sigma}_{i,1}^\delta & \widehat{\Sigma}_{i,2}^\delta \\ 0 & \widehat{\Sigma}_{i,2}^h \end{bmatrix}$
λ	3×1	$(1, \lambda_2, \lambda_3)^\top$
ρ	3×1	$(\rho_1, \rho_2, \rho_3)^\top$
σ	2×2	$\sigma \triangleq \begin{bmatrix} \sigma_\delta^\top \\ \sigma_h^\top \end{bmatrix} = \begin{bmatrix} \sigma_{\delta,1} & \sigma_{\delta,2} \\ 0 & \sigma_{h,2} \end{bmatrix}$
time-series		
$\eta(t)$	$3 \times T$	$\begin{pmatrix} \eta_{13}(t) \\ \eta_{23}(t) \\ 1 \end{pmatrix}$
$\widehat{\mu}_i(t)$	$2 \times T$	$\begin{pmatrix} \widehat{\mu}_i^\delta(t) \\ \widehat{\mu}_i^h(t) \end{pmatrix}, \ i = 1, 2, 3$

a normal distribution, and we use the means under each household's probability measure implied by equations (3) and (4) to compute the likelihood. Note that the variance-covariance of the joint transition of the estimated growth rates is measure independent.

3.4 Estimation Results and Model Performance

Table 2 reports the estimated parameters and Figure 1 and 2 show the state variables. Table 3 shows the statistics of these state variables. In detail, the estimation indicates that household 2 is relatively pessimistic in the consumption supply, also with lower speed of the fluctuation of the growth rates (κ_δ and κ_h). While household 3 is relatively optimistic, in line with the smallest λ and the highest ρ, and it is reasonable to explain this agent as the speculator, who is most impatient and not want to hold physical crude oil because λ_3 is nearly zero.

Table 2: Estimation result of parameters

parameter	estimate	(SE)	parameter	estimate	(SE)
α_1^δ	0.0580	()	α_1^h	0.0094	()
α_2^δ	0.0580	()	α_2^h	0.0094	()
α_3^δ	0.0580	()	α_3^h	0.0094	()
κ_1^δ	0.5839	(0.0009)	κ_1^h	1.9885	(0.0317)
κ_2^δ	0.0729	(0.0004)	κ_2^h	0.4046	(0.0009)
κ_3^δ	0.0667	(0.0003)	κ_3^h	0.8122	(0.1217)
$\widehat{\Sigma}_{1,1}^\delta$	0.0459	(0.0000)	$\widehat{\Sigma}_{2,1}^\delta$	0.0151	(0.0000)
$\widehat{\Sigma}_{1,2}^\delta$	-0.0449	(0.0000)	$\widehat{\Sigma}_{2,2}^\delta$	-0.0016	(0.0000)
$\widehat{\Sigma}_{1,2}^h$	-0.2158	(0.0059)	$\widehat{\Sigma}_{2,2}^h$	-0.1121	(0.0002)
$\widehat{\Sigma}_{3,1}^\delta$	-0.0262	(0.0000)	$\sigma_{\delta,1}$	0.0733	-
$\widehat{\Sigma}_{3,2}^\delta$	0.0089	(0.0000)	$\sigma_{\delta,2}$	0.0107	-
$\widehat{\Sigma}_{3,2}^h$	0.1135	(0.0205)	$\sigma_{h,2}$	0.0618	-
λ_1	1	-	ρ_1	0.0050	(0.0000)
λ_2	12.9912	(0.0447)	ρ_2	0.0263	(0.0000)
λ_3	0.0619	(0.0018)	ρ_3	0.1581	(0.0000)

Table 3: Statistics of state variables

	$\widehat{\mu}_1^\delta$	$\widehat{\mu}_2^\delta$	$\widehat{\mu}_3^\delta$	$\widehat{\mu}_1^h$	$\widehat{\mu}_2^h$	$\widehat{\mu}_3^h$	η_{13}	η_{23}
Mean	-0.0311	-0.0404	-0.0516	0.1063	0.0399	0.0672	0.3432	1.2739
Median	-0.0402	-0.0416	-0.0465	0.0745	0.0331	0.0606	0.3745	0.9842
Std.	0.0685	0.0342	0.0336	0.1918	0.1254	0.1208	0.2120	0.9480

Figures 3, 4, and 5 depict the model-implied time-series to their counterparts in the data, and the time series in each figure match each block explained in Appendix B.1. Table 3 shows the statistics of the estimated state variables and Table 4 shows the analysis of errors between the model and the data. The three figures and the analysis of errors show

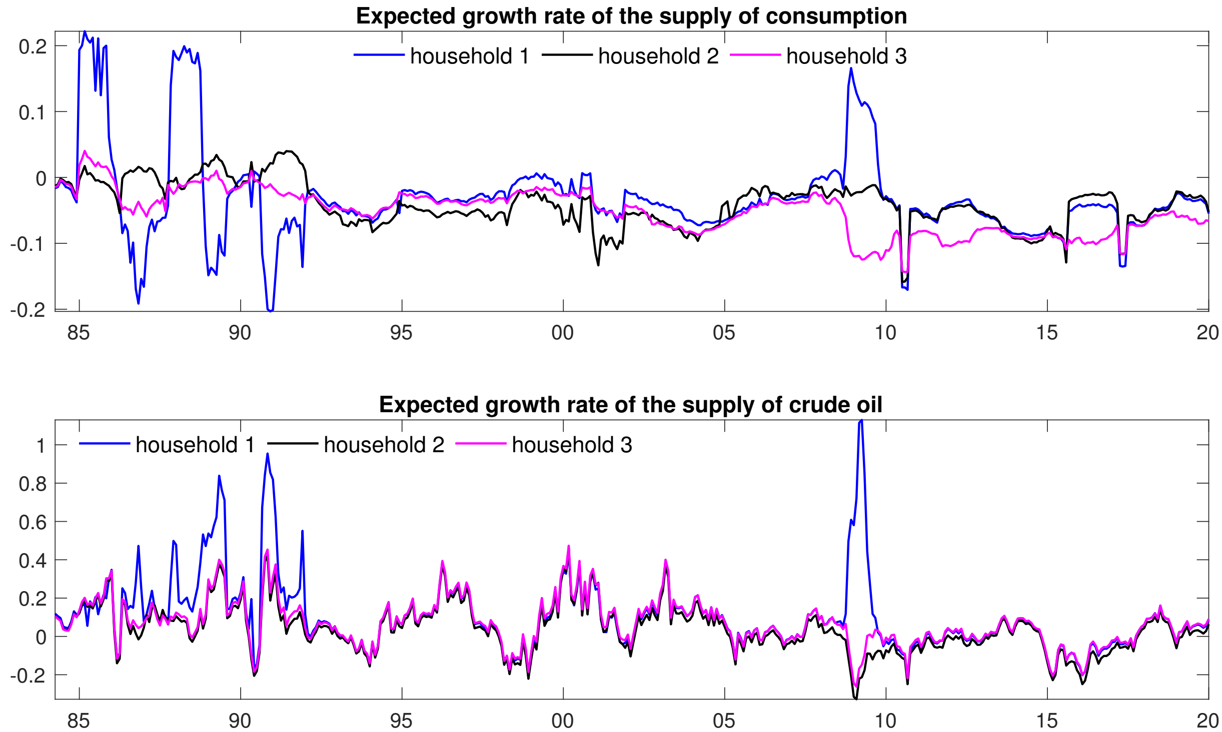

Figure 1: Households' expected growth rates $\widehat{\mu}_j$. This figure plots the three households' expected growth rates of the supply of consumption $\widehat{\mu}_j^{\delta}$ and crude oil $\widehat{\mu}_j^{h}$.

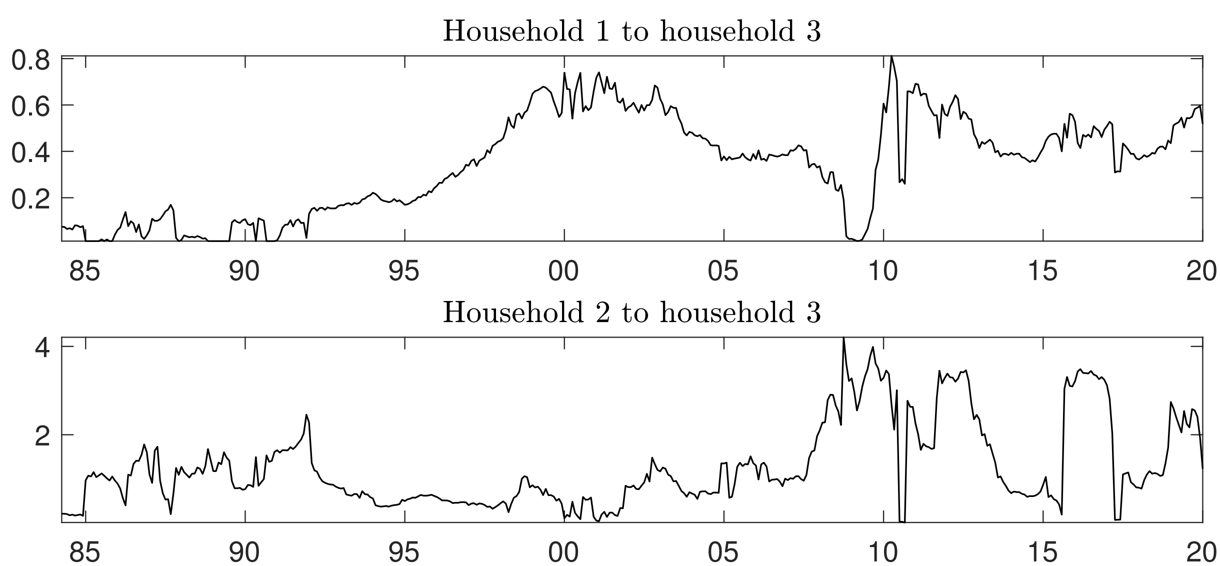

Figure 2: Consumption ratios of generic good between households. This figure plots the consumption ratios of generic good between households, η_{13} and η_{23}.

Table 4: Analysis of the pricing errors. This table shows the analysis of the error between data and model results. The time-series in Panel A (Figure 3), Panel B and C (Figure 4), and Panel D (Figure 5) correspond the three blocks explained in Appendix B.1. Notice that in Panel C, 1-month futures yield fluctuates from -50 to 50 nearly. R-square $= 1 - \sum_T (y_t - \widehat{y}_t)^2 / \sum_T y_t^2$, RMSE $= \sqrt{\sum_T (y_t - \widehat{y}_t)^2 / T}$, and here y_t, $\widehat{y}_t$, as well as RMSE are in the unit of percentage.

A: dividend, volatility and correlation

time-series	R-square	RMSE (in %)
S&P 500 dividend yield	0.9971	0.1350
volatility of S&P 500	0.9780	2.6215
volatility of 1-month oil futures	0.9825	4.8953
correlation of S&P 500 and 1-month oil futures	0.9920	2.1835
volatility of 3-month zero-coupon bond yield	0.9700	0.1539
correlation of S&P 500 and 3-month bond yield	0.9639	2.8368

B: bond yields

time-series	R-square	RMSE (in %)
3 month	0.9607	0.8902
6 month	0.9734	0.7594
1 year	0.9838	0.6116
2 year	0.9908	0.4918
3 year	0.9894	0.5428
5 year	0.9824	0.7335

C: oil convenience yields

time-series	R-square	RMSE (in %)
3 month	0.9036	4.5545
6 month	0.9418	3.4879
1 year	0.9404	3.0286
2 year	0.9250	2.5157
3 year	0.9608	1.2689
5 year	0.9204	1.3156

D: volatility of bond yields

time-series	R-square	RMSE (in %)
3 month	0.9700	0.1539
6 month	0.9916	0.0759
1 year	0.9829	0.1086
2 year	0.9864	0.1109
3 year	0.9884	0.1072
5 year	0.9821	0.1379

E: volatility of oil futures

time-series	R-square	RMSE (in %)
3 month	0.9362	0.2503
6 month	0.8716	0.3355
1 year	0.8336	0.3658
2 year	0.9932	2.5532
3 year	0.9812	3.8170
5 year	0.9703	4.3695

time-series	R-square	RMSE (in %)
total	0.9731	2.3796

that the model with heterogeneous in beliefs and preferences indeed greatly captures the time-series variations in crude oil futures, stock, and bond markets in the past thirty years. Figure 3 depicts the time-series of the dividend yields of the S&P 500 index, volatilities of S&P 500 index, and crude oil futures, the correlation between the two, for the model-implied and observed in the data. As shown by equation (18), the dividend yield depends on households' heterogeneous time preferences (subjective discount rates) and the generic consumption ratios. The consumption ratios vary over time due to households' different investment portfolios that are caused by heterogeneous beliefs. Thus, dividend yield fluctuates with the consumption ratios. Since the volatilities of these consumption ratios ηs are the differences in beliefs βs, dramatic changes of the dividend yields tend to accompany with dramatic changes of heterogeneous beliefs.

Moreover, equation (19) shows that volatilities of stock and crude futures prices directly depend on the differences in beliefs, so this is why the rapid fluctuations of dividend yield and volatilities of stock and crude futures prices happened coincidently, especially in two periods, around 1987 and 2009. The figure also indicates that the model can match the correlation between the S&P stock index and the crude oil futures. We delay further discussions about the comovement of stock and crude oil markets in Section 4.1.

Figure 4 shows that the model fits the term structures of Treasury yields and the convenience yields of the crude oil futures very well for the whole sample period, with maturities across 3 and 6 months and 1, 2, 3, and 5 years. As discussed previously, by equations (20a) and (22), bond prices are the consumption-weighted average of that in homogeneous economies for both the generic consumption good and the crude oil. The dynamics of households' expected growth rates of the fundamentals are one set of the key determinants for the bond yields and the convenience yields. Recall that the convenience yields are exactly the yields of bonds that pay 1 unit of the crude oil.

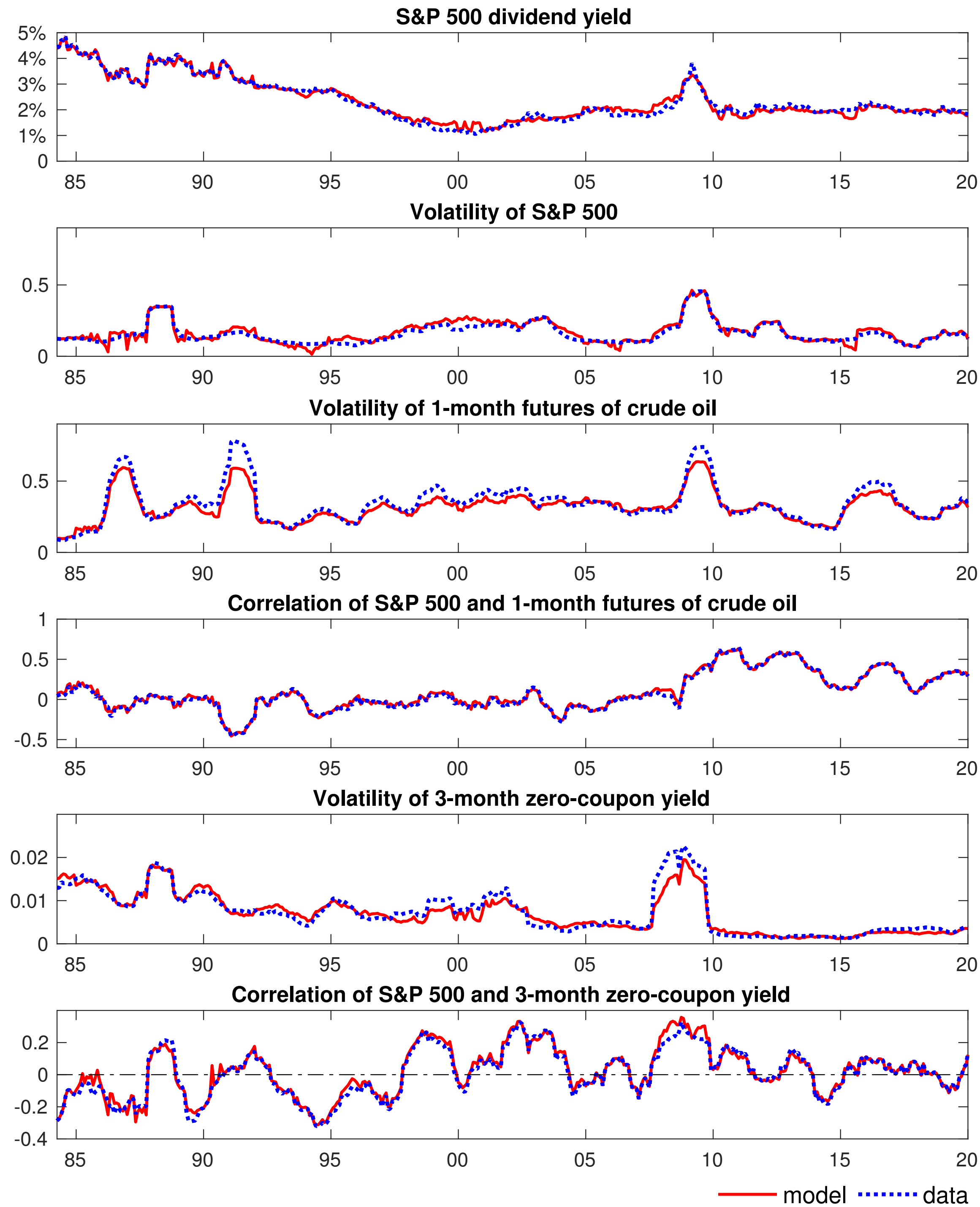

Figure 3: Equity and crude futures markets. This figure plots model-implied results and corresponding data of four time-series, including the dividend yield of the S&P 500 stock index, volatilities of the S&P 500 index, and the crude oil price respectively, and their correlation. The front-month crude oil futures is used here. The frequency is monthly, and both volatilities and correlation are annual-rolling estimates based on daily returns.

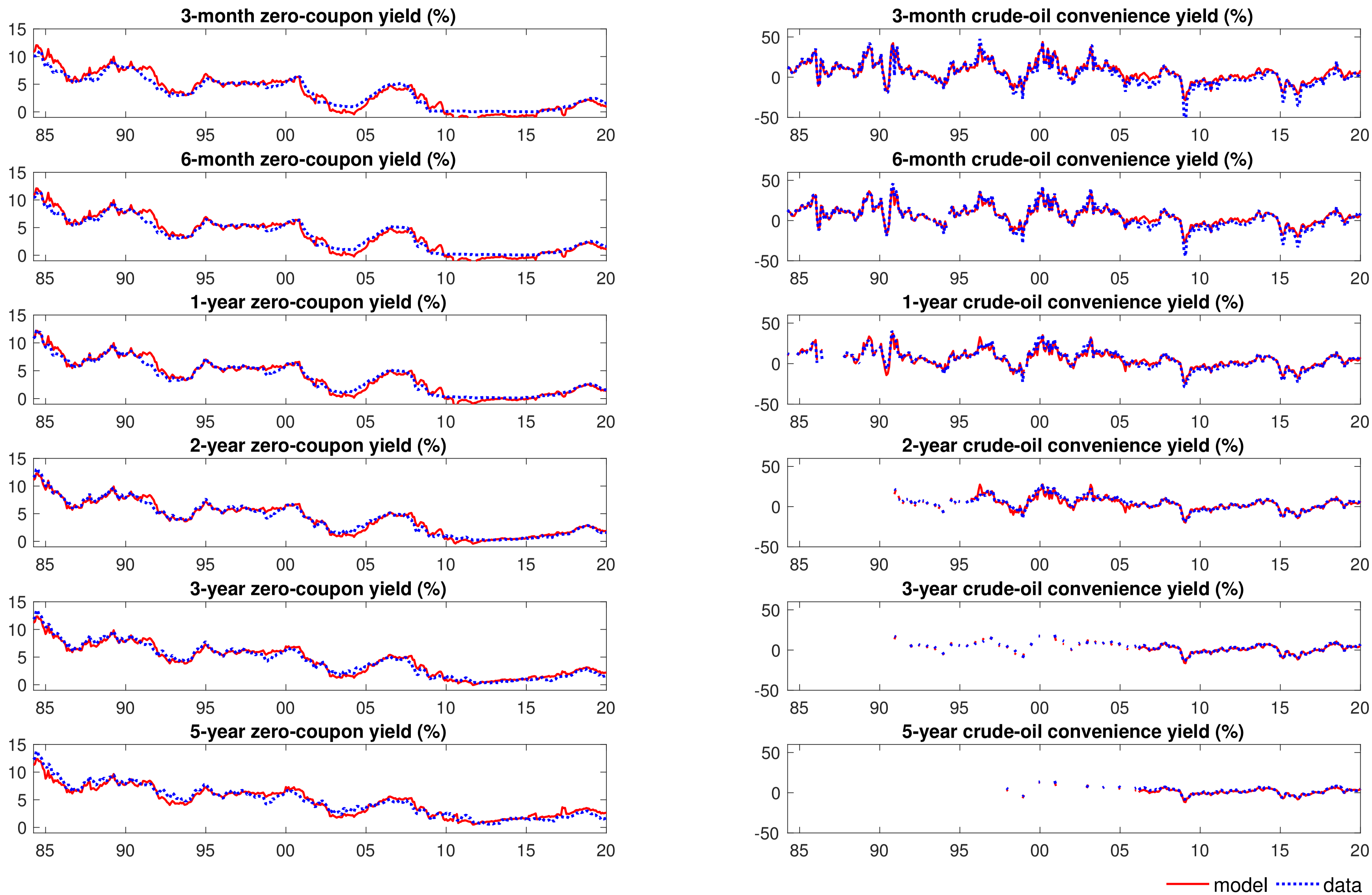

Figure 4: Bond and futures markets. This figure plots the yields of Treasury bonds and the convenience yields of the crude oil futures with different maturities.

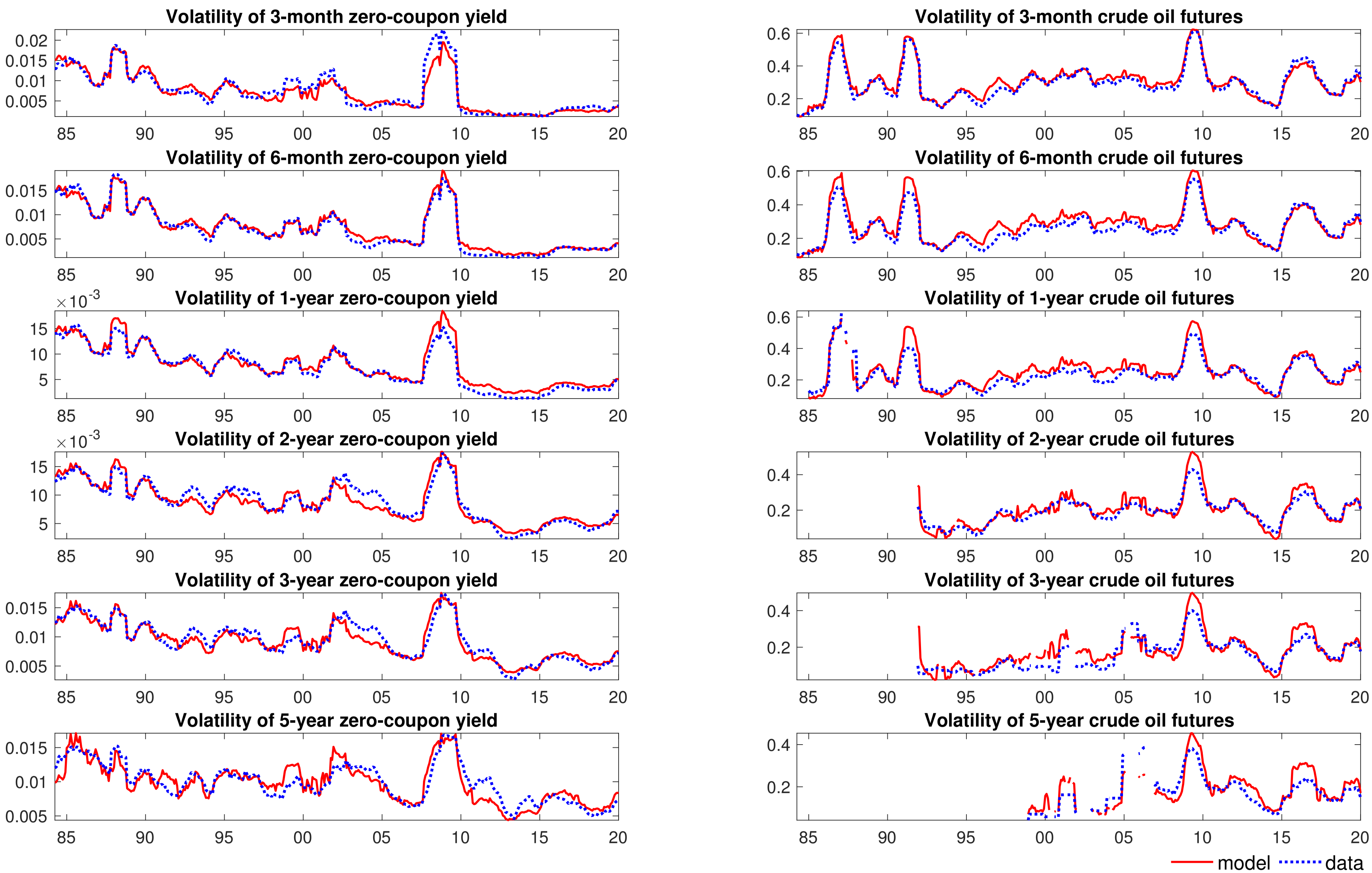

Figure 5: Volatilities of futures across different maturities. This figure plots the time-varying volatilities of futures, with different maturities. data of 3, 6 months, 1, 2, 3 and 5 year are used in estimation and others are the out-of-sample results.

4 Empirical Implications

In this section, we investigate further the empirical implications of our model and compare them with empirical evidence. First, we discuss further the volatility-correlation in oil and stock markets. Second, we present how the level and slope of the term structure of the convenience yield of crude oil vary over different periods. Finally, we investigate the property of the term structure of futures prices and show the V-shaped relationship between the volatility and the slope of futures prices.

4.1 Oil Futures and Stock Markets

Figure 3 shows that our model can generate the correlation and volatilities of oil futures and stock observed in the data. Based on the closed-form solution in equation (19), the volatilities of assets are decomposed into the constant component from the endowment process and the time-varying component driven by the disagreements, which play a crucial role in the time-varying movement of volatilities and correlation between assets.

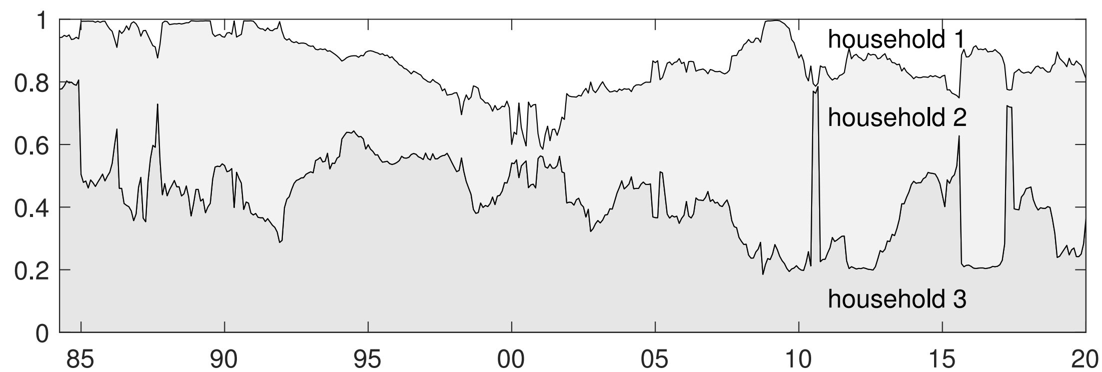

Figure 6: Households' proportion of the consumption of generic good. This figure shows the each household's proportion of the consumption of generic good.

To figure out the movement of the disagreement among households in detail, Figure 6 shows each household's proportion of the consumption of the generic good. These two together provide the heterogeneity in the system. According to equation (19), if there is no heterogeneity, the volatility in the financial market would just come from the volatility of fundamental σ, which is constant. The disagreement among households plays a crucial role in driving the time-varying volatilities of the stock and crude oil futures, much higher than the volatility of fundamental. Notice that in Figure 1, the disagreement on the expected growth rates of crude oil is relatively smaller than that of generic good; however, because the volatility of the oil supply $\|\sigma_h\|$ is smaller than the volatility of the generic good's supply $\|\sigma_\delta\|$, even a small difference between $\widehat{\mu}_j^h$ can be magnified by σ^{-1}.

Figure 3 has already shown that our model can capture the time-varying volatilities and correlation between stock and oil markets very well. Comparing Figure 3 with Figure 1, interestingly, we can also find that the heterogeneity captures several famous moments during the past decades, such as the Black Monday shock in 1987, Kuwait's Iraqi invasion around 1990, and the financial crisis in 2009.

4.2 Convenience Yields

Equation (22) indicates that the convenience yield for the crude forward is the yield-to-maturity of the crude oil bond $P_c(t,T)$, which is defined as, for $\tau = T - t$,

$$\begin{aligned} y_c(\tau) &= -\frac{\log P_c(t,T)}{\tau} \\ &\approx \overline{(\rho + \mu_h)}_{\lambda\eta} - \|\sigma_h\|^2 + \frac{\tau}{2} \cdot \overline{\left(\kappa_h(\alpha_h - \mu_h) - \Sigma_h \cdot \sigma_h\right)}_{\lambda\eta} - \frac{\tau^2}{4} \cdot \overline{(\|\Sigma_h\|^2)}_{\lambda\eta}. \end{aligned} \tag{25}$$

The last expression is the second-order approximation of the convenience yields when τ is small, showing that the slope of the convenience yield curve is the oil consumption weighted average of that in homogeneous economies (see Appendix A.6). Figure 4 shows that the convenience yield is very volatile, and the approximated convenience yield in equation (25) reveals that the *oil consumption-weighted* average of the expected growth rates of oil

$$\sum_j \frac{\lambda_j \eta_{ji}(t)}{\sum_j \lambda_j \eta_{ji}(t)} \widehat{\mu}_j^h(t) \tag{26}$$

is the determinant factor of the time-varying convenience yield. Actually this average expected growth rate represents the market's expectation of future availability of crude oil. Suppose that this market's expectation of growth is higher in a period, the future supply of oil may become more abundant and thus less valuable. In this case, we would observe high convenience yields and lower futures prices. Similarly, we would expect low convenience yields and high futures prices if the market's expectation of future growth is low.

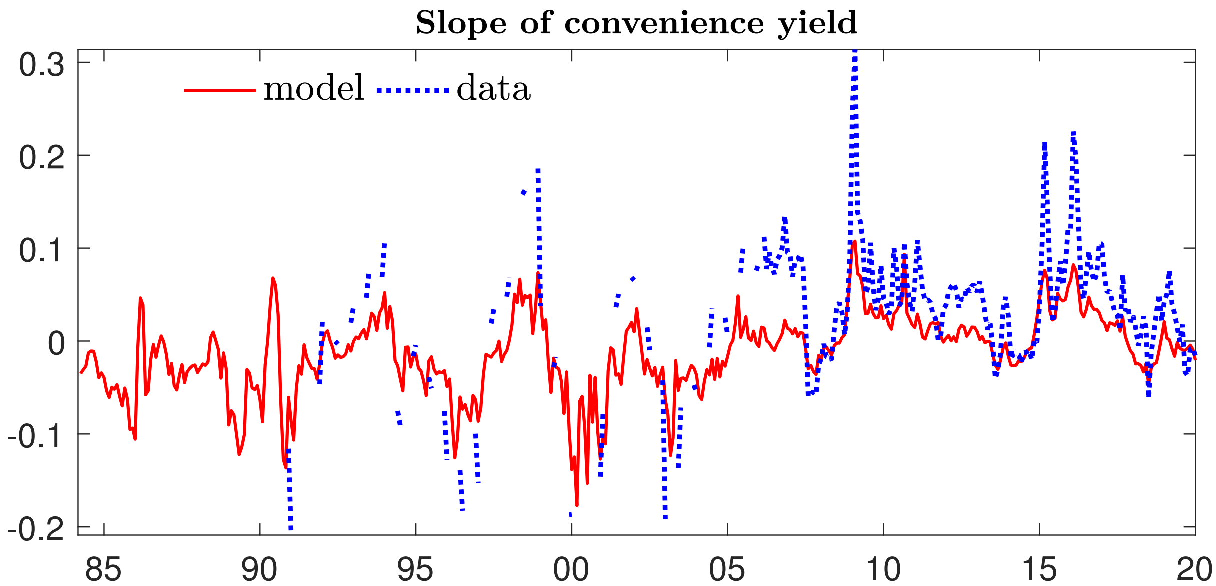

Figure 7: Slope of convenience yield curve. This figure plots the model and data slopes of convenience yield curve, both calculated as the difference between the 6-month and 3-year convenience yields, $slope_{y_c}(t) = y_c(t, t + 1/2) - y_c(t, t + 1/4)$.

Table 5: Slope of convenience yield curve. This table shows two regression results. The first regression checks the model and data slopes of convenience yield curve, both calculated as the difference between the 6-month and 3-year convenience yields, $slope_{y_c}(t) = y_c(t, t+1/2) - y_c(t, t+1/4)$. The second regression is between the slope of convenience yield curve in data and the implied indicator of the slope of convenience yield curve based on equation (25).

	Estimate	***t*-stat**	R^2
	$A.\ \ slope_{y_c}(t) = \beta_0 + \beta_1\, \widehat{slope_{y_c}}(t) + \epsilon_t$		0.753
β_0	0.0262	10.8	
β_1	1.7649	26.04	
	$B.\ \ slope_{y_c}(t) = \beta_0 - \frac{\sum_j \lambda_j \eta_{ji}(t) \kappa_j^h \widehat{\mu}_j^h(t)}{\sum_j \lambda_j \eta_{ji}(t)} \cdot \beta_1 + \epsilon_t$		0.615
β_0	0.0302	10	
β_1	1.9072	18.89	

The slope of convenience yield curve is also very volatile, and Figure 7 shows both the model and data slopes of the convenience yield curve, calculated as the difference between 3-year and 6-month convenience yields. The approximation in equation (25) shows that the time-varying component of the slope of the convenience yield curve is[10]

$$-\sum_j \frac{\lambda_j \eta_{ji}(t)}{\sum_j \lambda_j \eta_{ji}(t)}\, \kappa_j^h \widehat{\mu}_j^h(t) \tag{27}$$

This expression is the oil-consumption-weighted average of speeds of mean reversion because $-\kappa_j^h \widehat{\mu}_j^h(t)$ represents the time-varying speed of mean reversion in household j's learning dynamics. This market's speed of mean reversion is the dominant determinant of the time-varying slope of the convenience yield curve. It is highly correlated with the market's expectation of growth given by equation (26). Thus, we should expect the convenience yield curve is steep upward if the short-term convenience yields are low, and vice versa. Table 5 reports the results of two regressions. The first regression is between the data and model slopes of the convenience yield curve. The second one examines how the model's weighted average speed of mean reversion can explain the convenience yield curve slope in the data. Both regressions show very significant results, verifying that the estimated model captures the time-varying slopes of the convenience yield curve well.

4.3 Term Structures of Volatilities

First, let us focus on the term structure of the volatility of Treasury yields. As Figure 8 shows, our model also generates the hump-shaped term structure of volatility in the Treasury yields in the data. According to equation (23), if there is no heterogeneity in beliefs, the remaining component is $\frac{1-e^{-\kappa_j^\delta \tau}}{\kappa_j^\delta \tau} \widehat{\Sigma}_j^\delta$, which is always decreasing with maturity. So it is the component related to the heterogeneity that plays the key role in the hump shape of the term structure of the volatility of Treasury yields.

Based on equation (24), we also get the instantaneous forward volatility, and Figure 5 shows it matches the data very well unconditionally. Thus, our model matches the term structure of futures volatility not only conditionally but also unconditionally. Indeed, the

[10]The other time-varying component is related to $\gamma_j^h(\tau)$, driving the time-varying the consumption weights. However, this component is much less varying because $\gamma_j^h(\tau)$ does not vary much across houdeholds.

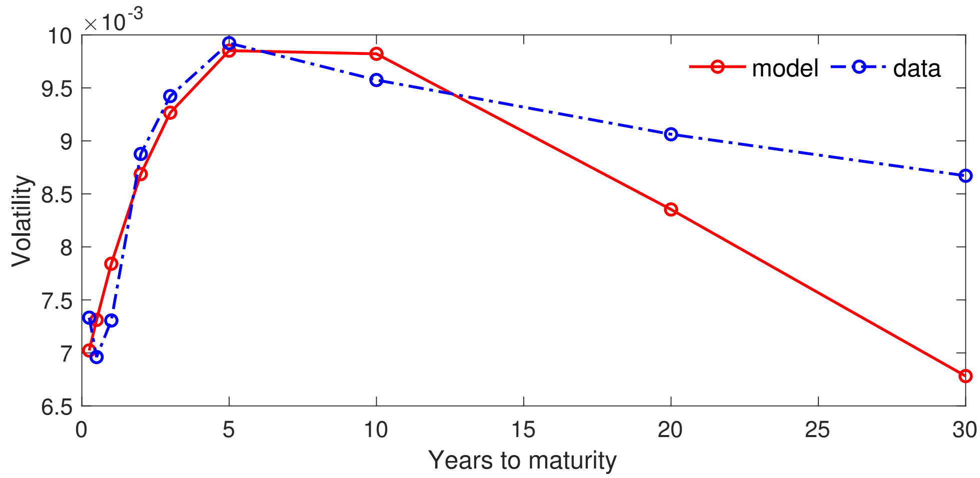

Figure 8: Term structure of the volatility of treasury yield. This figure plots the volatilities of Treasury yield across maturity (years). Notice that 10, 20, and 30-year volatilities are out-of-sample result.

model generates declining term structures of futures volatilities, known as the Samuelson (1965) effect, and also, our model matches the convex shape of this term structure. In Figure 9, (a) shows the annualized standard deviation of historical futures prices across maturities, and (b) shows the average of the instantaneous volatilities of futures prices across maturities based on the estimation.

If there is no heterogeneity in beliefs, then βs are zero; thus, the futures volatility resonates with the result in Casassus and Collin-Dufresne (2005). However, this setup does not generate enough variations in futures prices to explain the volatility term structure observed in the data. So heterogeneity in beliefs plays a pivotal role in generating the excess time-varying volatility beyond economic fundamentals. To further examine how heterogeneity in beliefs drives the time-varying slopes of the volatility term structure, we use the consumption-weighted dispersion of beliefs to measure the degree of disagreement among households as follows.

$$dispersion(\widehat{\mu}_a(t)) = \sum_j \frac{\omega_j(a)\eta_{ji}(t)\left[\widehat{\mu}_j^a(t) - mean(\widehat{\mu}_a(t))\right]^2}{\omega_j(a)\eta_{ji}(t)}, \tag{28}$$

where

$$mean(\widehat{\mu}_a(t)) = \sum_j \frac{\omega_j(a)\eta_{ji}(t)\widehat{\mu}_j^a(t)}{\omega_j(a)\eta_{ji}(t)}, \quad \omega_j(\delta) = 1 \text{ and } \omega_j(h) = \lambda_j.$$

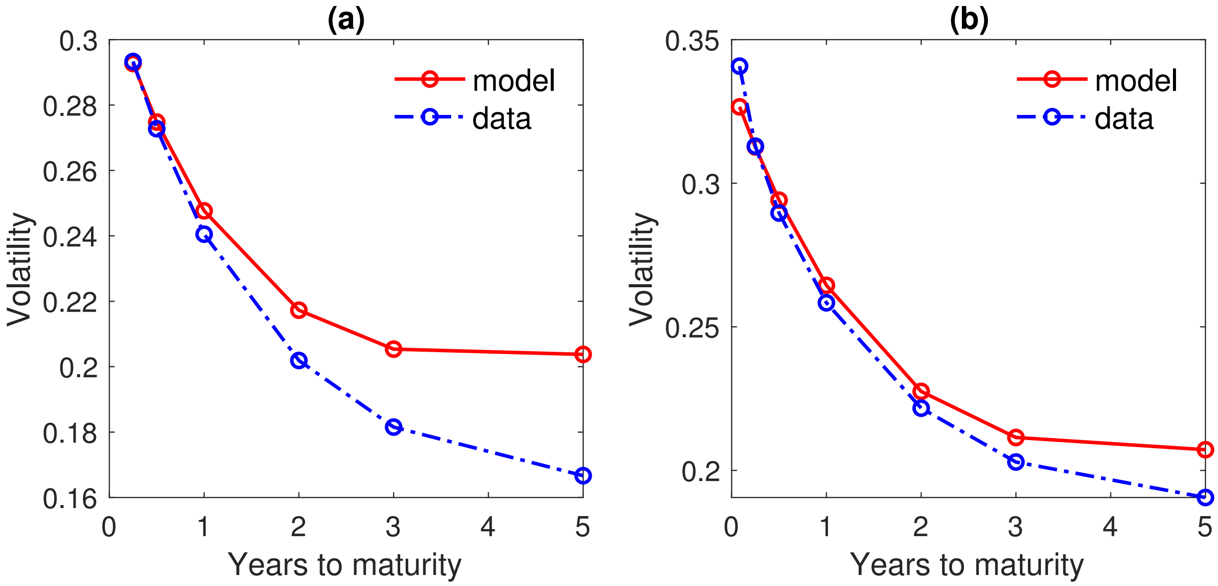

Figure 9: Term structure of crude oil futures volatility. This figure plots the crude oil futures volatility across maturity (years) calculated in two ways. (a) shows the annualized standard deviation of historical futures price across maturities, and (b) shows the mean of the instantaneous volatilities of futures price, both across maturities, 3 and 6 month, and 1, 2, 3 and 5 year.

We can also offer some intuitions on what drives the time-varying slopes of volatility term structure over time. Figure 10 shows that the slope of the volatility term structure depends on the level of the volatility, so based on the closed-form solution of volatility, higher decreasing of the slope of volatility term structure happens when the dispersion of the heterogeneous beliefs is high. And if we build a proxy of disagreement between households as the sum of the two corresponding consumption-weighted dispersions of the three households' estimated expected growth rates of the two goods in each time, Table 6 shows that, if focusing on the maturity, the curve of the oil volatility is decreasing as the effect of the disagreement about expected growth rates declines in long run, and if focusing on different periods, a steeper curve corresponds to a higher disagreement.

4.4 V-shape Relationship

Both Kogan, Livdan and Yaron (2009) and Carlson, Khokher and Titman (2007) present that there exists a V-shape relationship between the volatility of futures prices and the slope of the futures price curve. Following local weighted average regression

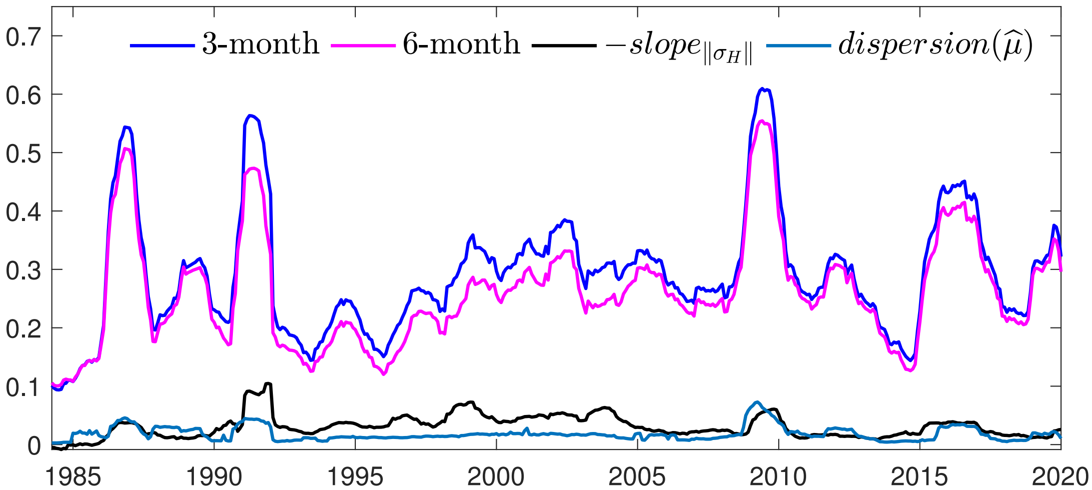

Figure 10: Slope of futures volatilities and disagreement across time. This figure plots our estimated disagreement between households and the slope of futures volatilities across time in data, together with volatilities of 3-month and 6-month futures. The calculation of dispersion follows equation (28). slope is the difference between the volatilities of 3-month and 6-month futures, $slope_{\|\sigma_H\|}(t) = \|\sigma_H(t, t+1/2)\| - \|\sigma_H(t, t+1/4)\|$. Notice that volatility term structure is mostly downward sloping, so $-slope_{\|\sigma_H\|}$ is plotted in the figure.

Table 6: Regressions between volatility term structure and the dispersion. This table shows two groups of regression. The first six columns show six regressions, in which each is between the futures volatility given a certain maturity and the estimated dispersions, $\|\sigma_{H(t,t+\tau)}\| = \alpha + \beta_1 \cdot dispersion(\widehat{\mu}_\delta(t)) + \beta_2 \cdot dispersion(\widehat{\mu}_h(t)) + \epsilon(t)$. And the last column shows the regression between the slope of futures volatility and the estimated dispersions, $slope_{\|\sigma_H\|}(t) = \alpha + \beta_1 \cdot dispersion(\widehat{\mu}_\delta(t)) + \beta_2 \cdot dispersion(\widehat{\mu}_h(t)) + \epsilon(t)$. Here the $dispersion(\widehat{\mu})$ is calculated following equation (28). Slope is the difference between the volatilities of 3-month and 6-month futures, $slope_{\|\sigma_H\|}(t) = \|\sigma_H(t, t+1/2)\| - \|\sigma_H(t, t+1/4)\|$. All estimated coefficients are followed with t-Statistic below.

	Volatility						$slope_{\|\sigma_H\|}$
	3-month	6-month	1-year	2-year	3-year	5-year	
α	0.16	0.14	0.12	0.11	0.08	0.1	-0.019
	29.05	29.27	23.38	24.76	13.04	13.74	-11.58
β_1	6.46	6.14	6.02	4.82	5.58	4.68	-0.32
	18.01	19.54	17.70	15.57	12.54	9.89	-3.01
β_2	9.03	7.51	6.05	6.26	2.49	1.97	-1.52
	9.22	8.74	6.59	7.59	2.09	1.55	-5.32

in Kogan, Livdan and Yaron (2009), Figure 11 shows that the model can also provide a consistent result in this regard. Actually, the results in Table 6 and Figure 10 have already provides the explanation. On the one hand, the effect of heterogeneity on oil volatility declines with maturity, so the futures prices are much dispersed in short term. On the other hand, the descending slope of volatility curve positively depends on the heterogeneity. So steeper price curve corresponds to higher futures volatilities, linked by the time-varying disagreement among households.

We choose four maturities in Figure 11 and all present the V-shape relationship. Given that the futures volatility is highly attributed to households' heterogeneity, this result indicates that the degree of heterogeneity is higher when futures prices are in either extreme backwardation or contango. Following the setup in Kogan, Livdan and Yaron (2009) and Carlson, Khokher and Titman (2007), the slope of the futures price curve is calculated as

$$\begin{aligned} slope_H &= \ln\left[\frac{H(t,t+\tau_2)}{H(t,t+\tau_1)}\right] \\ &= [y(\tau_2)-y_c(\tau_2)]\tau_2 - [y(\tau_1)-y_c(\tau_1)]\tau_1 \\ &= -\ln\left[\frac{P(t,t+\tau_2)}{P(t,t+\tau_1)}\right] - y_c(\tau_1)(\tau_2-\tau_1) - [y_c(\tau_2)-y_c(\tau_1)]\tau_2 \\ &= -slope_P - level_{y_c}(\tau_2-\tau_1) - slope_{y_c}\tau_2. \end{aligned} \tag{29}$$

This decomposition of the slope of futures price curve help identify what is the underlying determinant factor to generate the V-shape relationship. Table 7 provides the rusults of regressions of futures price volatility on each component in equation (29). The results in Table 7 show that the information in bond market is not consistent to explain the V-shape relationship while the level and slope of the term structure of convenience yield significantly determine the V-shape relationship, matching the negative and positive sides respectively. Given the discussion in section 4.2, we can also find that the implied indicator $\frac{\sum_j \lambda_j \eta_{ji}(t)\kappa_j^h \widehat{\mu}_j^h(t)}{\sum_j \lambda_j \eta_{ji}(t)}$ is consistent in the explanation of both the convenience yield curve and the V-shape relationship between the slope of futures price curve and the futures volatility.

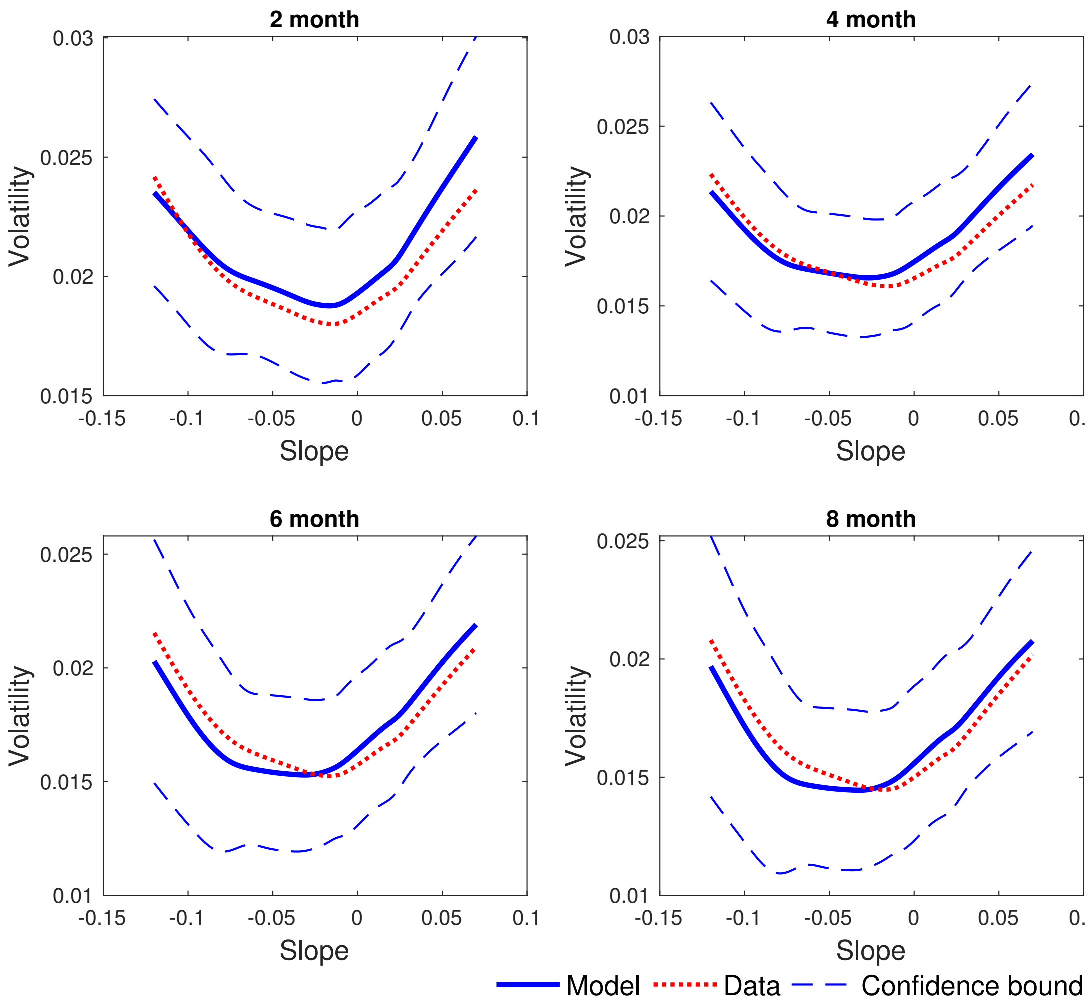

Figure 11: V-shape of volatility of futures prices to slope of futures price curve. Futures volatility is plotted as a function of the slope of the futures price curve. Four maturities are picked and all of them are out of sample. Both data and model are calculated by receptive field weighted regression following Kogan, Livdan and Yaron (2009). See the caption to Table 7 for a detailed description of the slope.

Table 7: Regressions between volatility of futures price and the slope of futures price curve. This table shows the regression between volatility of futures price and the slope of futures price curve. To test the V-shape, the second regression separates the slope into positive and negative as two parts. 4 maturities are picked, matching with Figure 11. In regression A, the $slope_H(t) = \ln\left[\frac{H(t,t+0.5)}{H(t,t+0.25)}\right]$ is calculated by picking 6-month and 3-month futures. Given equation (29), the independent variables in regressions B, C, and D are based on the three decomposed components, slope of Treasury bond price $slope_P$, level of the term structure of convenience yield $level_{y_c}$, and the slope of the term structure of convenience yield $slope_{y_c}$ respectively, followed with model output. The model output of $slope_P$ and $level_{y_c}$ is directly calculated by closed-form solution, and the model output of $slope_{y_c}$ is based on the implied indicator of the slope of convenience yield curve, discussed in section 4.2. All estimated coefficients are followed with t-Statistic below.

	2 Months		4 Months		6 Months		8 Months	
	Data	Model	Data	Model	Data	Model	Data	Model
	$A.\ \lVert\sigma_{H(t,t+\tau)}\rVert = \alpha_\tau + \beta_{1,\tau}(slope_H(t))^+ + \beta_{2,\tau}(slope_H(t))^- + \epsilon_\tau(t)$							
$\beta_{1,\tau}$	0.13	0.11	0.12	0.11	0.11	0.11	0.093	0.11
	6.3	4.5	6.4	4.4	6.4	4.3	5.9	4.3
$\beta_{2,\tau}$	-0.044	-0.03	-0.024	-0.027	-0.015	-0.024	-0.0045	-0.011
	-2.4	-1.3	-1.5	-1.2	-1	-1.1	-0.32	-0.5
	$B.\ \lVert\sigma_{H(t,t+\tau)}\rVert = \alpha_\tau + \beta_{1,\tau}(slope_P(t))^+ + \beta_{2,\tau}(slope_P(t))^- + \epsilon_\tau(t)$							
$\beta_{1,\tau}$	-0.39	-0.3	-0.28	-0.25	-0.2	-0.18	0.057	0.014
	-4.2	-3.4	-3.4	-3.1	-2.5	-2.5	0.63	0.18
$\beta_{2,\tau}$	-0.036	0.025	-0.21	-0.1	-0.27	-0.15	-0.39	-0.23
	-0.33	0.26	-2.1	-1.2	-3	-1.8	-4.3	-2.9
	$C.\ \lVert\sigma_{H(t,t+\tau)}\rVert = \alpha_\tau + \beta_{1,\tau}(level_{y_c}(t))^+ + \beta_{2,\tau}(level_{y_c}(t))^- + \epsilon_\tau(t)$							
$\beta_{1,\tau}$	0.0036	0.0074	0.00037	0.0034	-0.00011	0.0025	-0.00037	0.0017
	0.88	1.5	0.1	0.79	-0.032	0.62	-0.12	0.45
$\beta_{2,\tau}$	-0.027	-0.036	-0.026	-0.034	-0.025	-0.033	-0.023	-0.029
	-6.2	-6.6	-6.7	-7.1	-6.9	-7.2	-6.7	-6.8
	$D.\ \lVert\sigma_{H(t,t+\tau)}\rVert = \alpha_\tau + \beta_{1,\tau}(slope_{y_c}(t))^+ + \beta_{2,\tau}(slope_{y_c}(t))^- + \epsilon_\tau(t)$							
$\beta_{1,\tau}$	0.049	0.11	0.04	0.11	0.035	0.1	0.031	0.089
	3	5.5	2.7	6	2.5	6.1	2.4	5.7
$\beta_{2,\tau}$	-0.032	-0.02	-0.022	-0.0082	-0.019	-0.0045	-0.012	-0.0013
	-1.6	-1.4	-1.2	-0.63	-1.1	-0.37	-0.76	-0.11

5 Conclusion

We proposed an equilibrium model with heterogeneous households in terms of preferences and beliefs about fundamentals to understand the joint dynamics of stock and crude oil futures. The model admits closed-form solutions to the stock price, crude futures, and their volatilities and correlation, and the crucial driving forces behind these market quantities are the heterogeneous beliefs and their direct implication on the wealth distribution among the households. These explicit solutions enable us to take the model to the market data, including Treasury bonds, stock, and crude oil futures.

We estimate the model with Treasury bonds, S&P 500 stock index, crude oil futures, the correlation between the latter two, and volatilities of S&P 500 index and crude oil futures prices. The model fits these time-series very well with one set of parameters. Thus, it offers explanations and rationals for the observed empirical phenomena, including the time-varying price volatilities, the correlation between stock and crude futures, and the term structure of the convenience yields of the crude futures. Furthermore, the estimated model also generates other empirical regularities regarding the convenience yields of the crude oil futures and relations between the slope of the term structure and the volatilities of the crude oil futures prices.

Even though we focus on the stock and crude oil markets in this paper, the model can be easily extended to include other commodities. The correlation dynamics among multiple commodities and between stock and commodities can be studied and understood in an equilibrium framework.

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Appendix A Model

A.1 Learning of Expected Growth Rates

The learning of expected growth rates follows the standard filtering theory. Denote by $X(t) = \big(\log\delta(t), \log h(t)\big)^\top$ the observable vector of fundamentals, and by $\Lambda^2 = \big(\|\sigma_\delta\|^2, \|\sigma_h\|^2\big)^\top$ the vector of volatilities. Then the dynamics of the fundamentals, equation (1), can be rewritten, by Itô's Lemma, as the first equation of the followings

$$\mathrm{d}X(t) = \big(\mu(t) - \frac{1}{2}\Lambda^2\big)\,\mathrm{d}t + \sigma\,\mathrm{d}B(t), \tag{30a}$$

$$\mathrm{d}\mu(t) = \kappa_i\big(\alpha_i - \mu(t)\big)\,\mathrm{d}t + \widetilde{\sigma}_i\,\mathrm{d}\widetilde{B}_i(t). \tag{30b}$$

The last equation describes household i's beliefs about the dynamics of μ, where α is a two-dimensional constant vector, and both κ_i and $\widetilde{\sigma}_i$ are two constant square matrices, and $\widetilde{B}_i(t)$ is a two-dimensional Brownian motion, independent of $B(t)$.

We assume that the expected growth rates, μ, are not observable. Thus, given this linear system of equation (30), household i optimally estimate the growth rate $\mu(t)$ by following the standard Optimal Linear Nonstationary Filtering (Liptser and Shiryaev, 2013, Theorem 10.3 in Chapter 10). The optimal estimate of the expected growth rate $\widehat{\mu}_i(t) = \mathbb{E}^i\left[\mu(t)|\mathcal{F}_t^X\right]$ satisfies

$$\mathrm{d}\widehat{\mu}_i(t) = \kappa_i[\alpha_i - \widehat{\mu}_i(t)]\,\mathrm{d}t + \Sigma_i(t)(\sigma\sigma^\top)^{-1}\sigma\,\mathrm{d}\widehat{B}_i(t) \tag{31}$$

where

$$\begin{aligned} \mathrm{d}\widehat{B}_i(t) &= \sigma^{-1}\left[\mathrm{d}X(t) - \big(\widehat{\mu}_i(t) - \frac{1}{2}\Lambda^2\big)\,\mathrm{d}t\right] \\ &= \sigma^{-1}\left[\Big(\frac{\mathrm{d}\delta(t)}{\delta(t)}, \frac{\mathrm{d}h(t)}{h(t)}\Big)^\top - \widehat{\mu}_i(t)\mathrm{d}t\right], \end{aligned} \tag{32}$$

and $\widehat{B}_i(t)$, known as *innovation process* in filtering theory, is a two-dimensional Brownian motion, containing the same information as the process $X(t)$. The variance-covariance of the best estimate $\Sigma_i(t) = \mathbb{E}^i\left[\big(\mu(t) - \widehat{\mu}_i(t)\big)\big(\mu(t) - \widehat{\mu}_i(t)\big)^\top|\mathcal{F}_t^\Delta\right]$ is a symmetric nonnegative definite matrix, satisfying the matrix Riccati differential equation

$$\frac{\mathrm{d}\Sigma_i(t)}{\mathrm{d}t} = -\kappa_i \Sigma_i(t) - \Sigma_i(t)(\kappa_i)^\top + \widetilde{\sigma}_i \widetilde{\sigma}_i^\top - \Sigma_i(t)(\sigma\sigma^\top)^{-1}\Sigma_i(t).$$

Given any initial value of $\Sigma_i(0)$, the solution to this Riccati equation, $\Sigma_i(t)$, always converges to its steady state $\overline{\Sigma}_i$ when $\mathrm{d}\Sigma_i/\mathrm{d}t = 0$. Thus, the steady state variance $\overline{\Sigma}_i$ is the solution to the algebraic Riccati equation $-\kappa\Sigma_i - \Sigma_i(\kappa)^\top + \widetilde{\sigma}_i\widetilde{\sigma}_i^\top - \Sigma_i(\sigma\sigma^\top)^{-1}\Sigma_i^\top = 0$. At steady state, equation (31) is simplified as

$$\mathrm{d}\widehat{\mu}_i(t) = \kappa_i[\alpha_i - \widehat{\mu}_i(t)]\,\mathrm{d}t + \widehat{\Sigma}_i\,\mathrm{d}\widehat{B}_i(t), \tag{33}$$

where matrix $\widehat{\Sigma}_i = \overline{\Sigma}_i(\sigma\sigma^\top)^{-1}\sigma$ is a constant matrix. For simplicity, we assume either the system reaches its steady state or starts with the initial value of $\overline{\Sigma}_i$, thus, $\Sigma_i(t)$ does not vary over time. Also, instead of specifying $\widetilde{\sigma}_i$ and solving the algebraic Riccati equation, we directly treate $\widehat{\Sigma}_i$ as parameters in this paper. Thus, household i makes consumption and investment decisions based on her filtered probability space implied by equations (32) and (33), denoted by $(\Omega, \{\mathcal{F}_t^X\}, \mathbb{P}_i)$. Notice that these two equations correspond to equations (3) and (2), respectively, in Section 2 of the main text.

A.2 Interest Rate and Price of Risk

Based on the model, we can get

$$\begin{aligned}
\frac{\mathrm{d}\delta(t)}{\delta(t)} &= \widehat{\mu}_i^\delta(t)\mathrm{d}t + \sigma_\delta \cdot \mathrm{d}\widehat{B}_i(t) \\
\frac{\mathrm{d}\xi_i(t)}{\xi_i(t)} &= -r(t)\mathrm{d}t - \theta_i(t) \cdot \mathrm{d}\widehat{B}_i(t) \\
\frac{\mathrm{d}\eta_{ji}(t)}{\eta_{ji}(t)} &= -(\rho_j - \rho_i)\mathrm{d}t - \beta_{ij}(t) \cdot \mathrm{d}\widehat{B}_i(t).
\end{aligned}$$

Then applying Itô's Lemma to the equilibrium state price density given by equation (15) under $\mathbb{P}_i$, we can find that

$$\begin{aligned}
\frac{\mathrm{d}\xi_i(t)}{\xi_i(t)} &= -\rho_i \mathrm{d}t + \frac{\mathrm{d}\sum_j \eta_{ji}}{\sum_j \eta_{ji}} - \frac{\mathrm{d}\delta}{\delta} - \left(\frac{\mathrm{d}\sum_j \eta_{ji}}{\sum_j \eta_{ji}}\right)^\top \cdot \frac{\mathrm{d}\delta}{\delta} + \left(\frac{\mathrm{d}\delta}{\delta}\right)^\top \cdot \frac{\mathrm{d}\delta}{\delta} \\
&= -\rho_i \mathrm{d}t + \left(\rho_i \mathrm{d}t - \frac{\sum_j \eta_{ji}\rho_j}{\sum_j \eta_{ji}} \mathrm{d}t - \frac{\sum_j \eta_{ji}\beta_{ij}}{\sum_j \eta_{ji}} \mathrm{d}\widehat{B}_i\right) - \left(\widehat{\mu}_i^\delta \mathrm{d}t + \sigma_\delta \mathrm{d}\widehat{B}_i\right) \\
&\quad + \frac{\sum_j \eta_{ji}\beta_{ij}^\top \sigma_\delta}{\sum_j \eta_{ji}} \mathrm{d}t + \|\sigma_\delta\|^2 \mathrm{d}t \\
&= -\left(\frac{\sum_j \eta_{ji}\rho_j}{\sum_j \eta_{ji}} - \frac{\sum_j \eta_{ji}\beta_{ij}^\top \sigma_\delta}{\sum_j \eta_{ji}} + \widehat{\mu}_i^\delta - \|\sigma_\delta\|^2\right)\mathrm{d}t - \left(\sigma_\delta + \frac{\sum_j \eta_{ji}\beta_{ij}}{\sum_j \eta_{ji}}\right)\mathrm{d}\widehat{B}_i \\
&\equiv -r(t)\mathrm{d}t - \theta_i(t)\mathrm{d}\widehat{B}_i.
\end{aligned}$$

So matching the diffusion terms, we obtain the price of risk

$$\theta_i(t) = \sigma_\delta + \frac{\sum_j \eta_{ji}(t)\beta_{ij}(t)}{\sum_j \eta_{ji}(t)}, \quad \text{with} \quad \beta_{ii}(t) = 0.$$

Matching the drift terms, we find the equilibrium interest rate

$$\begin{aligned}
r(t) &= \frac{\sum_j \eta_{ji}\rho_j}{\sum_j \eta_{ji}} - \frac{\sum_j \eta_{ji}\beta_{ij}^\top \sigma_\delta}{\sum_j \eta_{ji}} + \widehat{\mu}_i^\delta - \|\sigma_\delta\|^2 \\
&= \frac{\sum_j \eta_{ji}\rho_j}{\sum_j \eta_{ji}} - \frac{\sum_j \eta_{ji}(\widehat{\mu}_i^\delta - \widehat{\mu}_j^\delta)}{\sum_j \eta_{ji}} + \widehat{\mu}_i^\delta - \|\sigma_\delta\|^2 \\
&= \frac{\sum_j \eta_{ji}(t)[\rho_j + \widehat{\mu}_j^\delta(t)]}{\sum_j \eta_{ji}(t)} - \|\sigma_\delta\|^2.
\end{aligned}$$

Notice that here we use the result based on

$$\sigma\beta_{ij}(t) = \widehat{\mu}_i(t) - \widehat{\mu}_j(t),$$

and the first row indicates

$$\sigma_\delta^\top \beta_{ij}(t) = \widehat{\mu}_i^\delta(t) - \widehat{\mu}_j^\delta(t).$$

A.3 Stock and Crude Oil Spot Prices and Volatilities

Given $e^{(\rho_j-\rho_i)t}\eta_{ji}(t)$ is a martingale under household i's beliefs, that is

$$\mathbb{E}_t^i\left[e^{(\rho_j-\rho_i)s}\eta_{ji}(s)\right] = e^{(\rho_j-\rho_i)t}\eta_{ji}(t), \quad \forall s > t,$$

and given equation (15), the price of stock claimed to the stream of generic consumption good is

$$\begin{aligned}
S(t) &= \frac{1}{\xi_i(t)}\mathbb{E}_t^i\left[\int_t^\infty \xi_i(s)\delta(s)\,\mathrm{d}s\right] \\
&= e^{\rho_i t}\frac{\delta(t)}{\sum_j \eta_{ji}(t)}\mathbb{E}_t^i\left[\int_t^\infty e^{-\rho_i s}\sum_j \eta_{ji}(s)\,\mathrm{d}s\right] \\
&= e^{\rho_i t}\frac{\delta(t)}{\sum_j \eta_{ji}(t)}\sum_j \mathbb{E}_t^i\left[\int_t^\infty e^{-\rho_i s-(\rho_j-\rho_i)s}\cdot e^{(\rho_j-\rho_i)s}\eta_{ji}(s)\,\mathrm{d}s\right] \\
&= e^{\rho_i t}\frac{\delta(t)}{\sum_j \eta_{ji}(t)}\sum_j e^{(\rho_j-\rho_i)t}\eta_{ji}(t)\int_t^\infty e^{-\rho_j s}\,\mathrm{d}s \\
&= \frac{\sum_j \frac{\eta_{ji}(t)}{\rho_j}}{\sum_j \eta_{ji}(t)}\delta(t).
\end{aligned}$$

Apply Itô's Lemma to $S(t)$ and put every other item with $\mathrm{d}t$ into "Drift",

$$\begin{aligned}
\frac{\mathrm{d}S(t)}{S(t)} &= \frac{\mathrm{d}\sum_j \frac{\eta_{ji}(t)}{\rho_j}}{\sum_j \frac{\eta_{ji}(t)}{\rho_j}} - \frac{\mathrm{d}\sum_j \eta_{ji}(t)}{\sum_j \eta_{ji}(t)} + \frac{\mathrm{d}\delta(t)}{\delta(t)} + \mathrm{Drift}_i\cdot\mathrm{d}t \\
&= \frac{\sum_j \frac{\eta_{ji}(t)}{\rho_j}\cdot\frac{\mathrm{d}\eta_{ji}}{\eta_{ji}}}{\sum_j \frac{\eta_{ji}(t)}{\rho_j}} - \frac{\sum_j \eta_{ji}(t)\cdot\frac{\mathrm{d}\eta_{ji}}{\eta_{ji}}}{\sum_j \eta_{ji}(t)} + \frac{\mathrm{d}\delta(t)}{\delta(t)} + \mathrm{Drift}_i\cdot\mathrm{d}t, \\
&= \left[\frac{\sum_j \frac{\eta_{ji}(t)}{\rho_j}\beta_{ij}(t)}{\sum_j \frac{\eta_{ji}(t)}{\rho_j}} - \frac{\sum_j \eta_{ji}(t)\beta_{ij}(t)}{\sum_j \eta_{ji}(t)} + \sigma_\delta\right]^\top \mathrm{d}\widehat{B}_i(t) + \mathrm{Drift}\cdot\mathrm{d}t.
\end{aligned}$$

So the volatility vector of stock price is given by equation (19a).

Similarly, given the spot price of crude oil by equation (16), the volatility vector of spot price is given by equation (19b).

A.4 Bond and Forward Prices

By Girsanov Theorem (Karatzas and Shreve, 1991), the relation $\mathrm{d}\widehat{B}_j(t) = \mathrm{d}\widehat{B}_i(t) + \beta_{ij}(t)\mathrm{d}t$ implies

$$\begin{aligned}\frac{\mathrm{d}\mathbb{P}_j}{\mathrm{d}\mathbb{P}_i} &= \exp\left(-\frac{1}{2}\int_0^t \|\beta_{ij}(a)\|^2\mathrm{d}a - \int_0^t \beta_{ij}(a)\cdot \mathrm{d}\widehat{B}_i(a)\right) \\ &= \frac{\xi_i(t)}{\xi_j(t)} = \frac{\psi_j}{\psi_i}e^{(\rho_j-\rho_i)t}\eta_{ji}(t).\end{aligned}$$

Thus, $\frac{\xi_i(t)}{\xi_j(t)}$ and $e^{(\rho_j-\rho_i)t}\eta_{ji}(t)$ serve as the Randon-Nikodym derivative process of the probability measure $\mathbb{P}_j$ with respect to $\mathbb{P}_i$. Given the closed-form solution of $\xi_i(t)$ in equation (15), and the property of $e^{(\rho_j-\rho_i)t}\eta_{ji}(t)$,

$$\begin{aligned}P(t,T) &= \frac{1}{\xi_i(t)}\mathbb{E}_t^i\left[\xi_i(T)\right] \\ &= \frac{1}{\sum_j \eta_{ji}(t)}\mathbb{E}_t^i\left[e^{-\rho_i(T-t)}\delta(t)\delta^{-1}(T)\sum_j \eta_{ji}(T)\right] \\ &= \frac{1}{\sum_j \eta_{ji}(t)}\sum_j \eta_{ji}(t)\mathbb{E}_t^i\left[e^{-\rho_j(T-t)}\delta(t)\delta^{-1}(T)\cdot e^{(\rho_j-\rho_i)(T-t)}\frac{\eta_{ji}(T)}{\eta_{ji}(t)}\right] \\ &= \frac{1}{\sum_j \eta_{ji}(t)}\sum_j \eta_{ji}(t)\mathbb{E}_t^j\left[e^{-\rho_j(T-t)}\delta(t)\delta^{-1}(T)\right] \\ &= \frac{\sum_j \eta_{ji}(t)\mathbb{E}_t^j\Big[\exp\big\{-\int_t^T\big(\rho_j+\widehat{\mu}_j^\delta(s)-\frac{1}{2}\|\sigma_\delta\|^2\big)\mathrm{d}s - \int_t^T \sigma_\delta\cdot\mathrm{d}\widehat{B}_j(s)\big\}\Big]}{\sum_j \eta_{ji}(t)}.\end{aligned}$$

Given the spot price of crude oil in equation (16), the state price in equation (15), and the dynamics of the supply of crude oil given by

$$\frac{\mathrm{d}h(t)}{h(t)} = \widehat{\mu}_i^h(t)\,\mathrm{d}t + \sigma_h\cdot\mathrm{d}\widehat{B}_i(t),$$

the price of real bond denomited by crude oil is

$$
\begin{aligned}
P_c(t,T) &= \frac{1}{R(t)\xi_i(t)}\mathbb{E}_t^i\left[\xi_i(T)R(T)\right] \\
&= \mathbb{E}_t^i\left[\frac{e^{-\rho_i T}}{e^{-\rho_i t}}\cdot\frac{h(t)}{h(T)}\cdot\frac{\sum_j\lambda_j\eta_{ji}(T)}{\sum_j\lambda_j\eta_{ji}(t)}\right] \\
&= \frac{\sum_j\lambda_j\mathbb{E}_t^i\left[e^{-\rho_i(T-t)}\,h(t)h^{-1}(T)\cdot\eta_{ji}(T)\right]}{\sum_j\lambda_j\eta_{ji}(t)} \\
&= \frac{\sum_j\lambda_j\eta_{ji}(t)\mathbb{E}_t^i\left[e^{-\rho_j(T-t)}\,h(t)h^{-1}(T)\cdot e^{(\rho_j-\rho_i)(T-t)}\frac{\eta_{ji}(T)}{\eta_{ji}(t)}\right]}{\sum_j\lambda_j\eta_{ji}(t)} \\
&= \frac{\sum_j\lambda_j\eta_{ji}(t)\mathbb{E}_t^j\left[e^{-\rho_j(T-t)}\cdot h(t)h^{-1}(T)\right]}{\sum_j\lambda_j\eta_{ji}(t)} \\
&= \frac{\sum_j\lambda_j\eta_{ji}(t)\mathbb{E}_t^j\Big[\exp\big\{-\int_t^T\big(\rho_j+\widehat{\mu}_j^h(s)-\frac{1}{2}\|\sigma_h\|^2\big)\mathrm{d}s-\int_t^T\sigma_h\cdot\mathrm{d}\widehat{B}_j(s)\big\}\Big]}{\sum_j\lambda_j\eta_{ji}(t)}.
\end{aligned}
$$

The results above show that solving the prices of both bonds comes to solve

$$
F(T-t,\widehat{\mu}_j^a(t)) = \mathbb{E}_t^j\Big[\exp\Big\{-\int_t^T\big(\rho_j+\widehat{\mu}_j^a(s)-\frac{1}{2}\|\sigma_a\|^2\big)\mathrm{d}s-\int_t^T\sigma_a\cdot\mathrm{d}\widehat{B}_j(s)\Big\}\Big]
$$

for $a=\delta,h$.

Picking out the dynamics of $\widehat{\mu}_j^a$ from vector $\widehat{\mu}_j$ in equation (2), we have

$$
\mathrm{d}\widehat{\mu}_j^a(t) = \kappa_j^a\left[\alpha_j^a-\widehat{\mu}_j^a(t)\right]\,\mathrm{d}t+\widehat{\Sigma}_j^a\cdot\mathrm{d}\widehat{B}_j(t). \tag{34}
$$

This stochastic differential equation admits an explicit solution as

$$
\widehat{\mu}_j^a(s) = \alpha_j^a - e^{-\kappa_j^a(s-t)}\Big(\alpha_j^a-\widehat{\mu}_j^a(t)\Big)+\int_t^s e^{-\kappa_j^a(s-u)}\widehat{\Sigma}_j^a\cdot\mathrm{d}\widehat{B}_j(u)
$$

for any $s\geqslant t\geqslant 0$. Therefore,

$$
\begin{aligned}
Z = \int_t^T\widehat{\mu}_j^a(s)\mathrm{d}s+\int_t^T\sigma_a\cdot\mathrm{d}\widehat{B}_j(s) &= (T-t)\,\alpha_j^a-\frac{1-e^{-\kappa_j^a(T-t)}}{\kappa_j^a}\cdot\Big(\alpha_j^a-\widehat{\mu}_j^a(t)\Big) \\
&\quad+\int_t^T\Big\{\int_t^s e^{-\kappa_j^a(s-u)}\widehat{\Sigma}_j^a\cdot\mathrm{d}\widehat{B}_j(u)\Big\}\,\mathrm{d}s+\int_t^T\sigma_a\cdot\mathrm{d}\widehat{B}_j(s).
\end{aligned}
$$

By changing the order of integration, we have

$$
\begin{aligned}
\int_t^T \left\{ \int_t^s e^{-\kappa_j^a (s-u)} \widehat{\Sigma}_j^a \cdot \mathrm{d}\widehat{B}_j(u) \right\} \mathrm{d}s + \int_t^T \sigma_a \cdot \mathrm{d}\widehat{B}_j(s) \\
= \int_t^T \left\{ \int_u^T e^{-\kappa_j^a (s-u)} \mathrm{d}s \right\} \widehat{\Sigma}_j^a \cdot \mathrm{d}\widehat{B}_j(u) + \int_t^T \sigma_a \cdot \mathrm{d}\widehat{B}_j(u) \\
= \int_t^T \left\{ \frac{1 - e^{-\kappa_j^a (T-u)}}{\kappa_j^a} \widehat{\Sigma}_j^a + \sigma_a \right\} \cdot \mathrm{d}\widehat{B}_j(u),
\end{aligned}
$$

and thus, Z is normally distributed with mean

$$
\mu_Z = (T - t)\, \alpha_j^a - \frac{1 - e^{-\kappa_j^a (T-t)}}{\kappa_j^a} \cdot \Big(\alpha_j^a - \widehat{\mu}_j^a(t) \Big)
$$

and variance

$$
\begin{aligned}
V_Z &= \int_t^T \left\| \frac{1 - e^{-\kappa_j^a (T-u)}}{\kappa_j^a} \widehat{\Sigma}_j^a + \sigma_a \right\|^2 \mathrm{d}u \\
&= \frac{\|\widehat{\Sigma}_j^a\|^2}{(\kappa_j^a)^2} \left[(T - t) - 2 \cdot \frac{1 - e^{-\kappa_j^a (T-t)}}{\kappa_j^a} + \frac{1 - e^{-2\kappa_j^a (T-t)}}{2\kappa_j^a} \right] \\
&\quad + \frac{2\widehat{\Sigma}_j^a \cdot \sigma_a}{\kappa_j^a} \left[T - t - \frac{1 - e^{-\kappa_j^a (T-t)}}{\kappa_j^a} \right] + \|\sigma_a\|^2 (T - t).
\end{aligned}
$$

Therefore, for $\tau = T - t$,

$$
\begin{aligned}
F(\tau, \widehat{\mu}_j^a(t)) &= \mathbb{E}_t^j \Big[\exp\Big\{ - \big(\rho_j + \frac{1}{2}\|\sigma_a\|^2\big)\tau - Z \Big\} \Big] \\
&= \exp\left[-\big(\rho_j + \frac{1}{2}\|\sigma_a\|^2\big)\tau - \mu_Z + \frac{1}{2} V_Z \right].
\end{aligned}
$$

Substituting μ_Z and V_Z yields equation (21), and hence bond prices given in (20).

A.5 Volatilities of Treasury Yield and Forward Price

For simplicity, denote

$$
\mathbf{f}_j^{a,\tau}(t) = F(\tau, \widehat{\mu}_j^a(t)),
$$

in which $\tau = T - t$, asset $a = \delta, h$, and

$$\overline{\mathbf{f}_{\omega\eta}^{a,\tau}}(t) = \frac{\sum \omega_j(a)\eta_{ji}(t)\mathbf{f}_j^{a,\tau}(t)}{\sum \omega_j(a)\eta_{ji}(t)},$$

where

$$\omega(a) = \begin{cases} (1, 1, \cdots, 1)_N^\top & \text{if} \quad a = \delta \\ (\lambda_1, \lambda_2, \cdots, \lambda_N)^\top & \text{if} \quad a = h \end{cases}.$$

Then the bond price is

$$P(t, T) = \overline{\mathbf{f}_\eta^{\delta,\tau}}(t)$$

and the forward price is

$$H(t, T) = R(t) \cdot \frac{\overline{\mathbf{f}_{\lambda\eta}^{h,\tau}}(t)}{\overline{\mathbf{f}_\eta^{\delta,\tau}}(t)}.$$

We can also take differentiation on $H(t, T)$ from both sides and put every other item with $\mathrm{d}t$ into "Drift",

$$\frac{\mathrm{d}H}{H} = \frac{\mathrm{d}R}{R} + \frac{\mathrm{d}\overline{\mathbf{f}_{\lambda\eta}^{h,\tau}}}{\overline{\mathbf{f}_{\lambda\eta}^{h,\tau}}} - \frac{\mathrm{d}\overline{\mathbf{f}_\eta^{\delta,\tau}}}{\overline{\mathbf{f}_\eta^{\delta,\tau}}} + \mathrm{Drift} \cdot \mathrm{d}t.$$

The volatility of $H(t, T)$ only comes from the first three items $\mathrm{d}R/R$, $\mathrm{d}\overline{\mathbf{f}_{\lambda\eta}^{h,\tau}}/\overline{\mathbf{f}_{\lambda\eta}^{h,\tau}}$ and $\mathrm{d}\overline{\mathbf{f}_\eta^{\delta,\tau}}/\overline{\mathbf{f}_\eta^{\delta,\tau}}$. Notice that $\mathrm{d}R/R$ has been solved in Appendix A.3 and equation (19b), then without loss of generality, to take deviation on $\mathrm{d}\overline{\mathbf{f}_{\omega\eta}^{a,\tau}}/\overline{\mathbf{f}_{\omega\eta}^{a,\tau}}$ and put every item with $\mathrm{d}t$ into

"Drift" again, we can find that

$$
\begin{aligned}
\frac{\mathrm{d}\overline{\mathbf{f}^{a,\tau}_{\omega\eta}}}{\overline{\mathbf{f}^{a,\tau}_{\omega\eta}}} &= \frac{\mathrm{d}(\sum_j \omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j)}{\sum_j \omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j} - \frac{\mathrm{d}(\sum_j \omega_j \eta_{ji})}{\sum_j \omega_j \eta_{ji}} + \mathrm{Drift} \cdot \mathrm{d}t \\
&= \sum_j \left[\frac{\omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j}{\sum_j \omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j} \cdot \frac{\mathrm{d}(\eta_{ji} \mathbf{f}^{a,\tau}_j)}{\eta_{ji} \mathbf{f}^{a,\tau}_j} - \frac{\omega_j \eta_{ji}}{\sum_j \omega_j \eta_{ji}} \cdot \frac{\mathrm{d}\eta_{ji}(t)}{\eta_{ji}(t)} \right] + \mathrm{Drift} \cdot \mathrm{d}t \\
&= \sum_j \left[\left(\frac{\omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j}{\sum_j \omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j} - \frac{\omega_j \eta_{ji}}{\sum_j \omega_j \eta_{ji}} \right) \cdot \frac{\mathrm{d}\eta_{ji}(t)}{\eta_{ji}(t)} + \frac{\omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j}{\sum_j \omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j} \cdot \frac{\mathrm{d}\mathbf{f}^{a,\tau}_j}{\mathbf{f}^{a,\tau}_j} \right] \\
&\qquad\qquad + \mathrm{Drift} \cdot \mathrm{d}t \\
&= \sum_j \left[\left(\frac{\omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j}{\sum_j \omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j} - \frac{\omega_j \eta_{ji}}{\sum_j \omega_j \eta_{ji}} \right) \cdot \frac{\mathrm{d}\eta_{ji}(t)}{\eta_{ji}(t)} \right. \\
&\qquad\qquad \left. + \frac{\omega_j \eta_{ji}}{\sum_j \omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j} \cdot \frac{\partial \mathbf{f}^{a,\tau}_j}{\partial \mu} \mathrm{d}\widehat{\mu}^a_j(t) \right] + \mathrm{Drift} \cdot \mathrm{d}t \\
&= \sum_j \left[\left(\frac{\omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j}{\sum_j \omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j} - \frac{\omega_j \eta_{ji}}{\sum_j \omega_j \eta_{ji}} \right) \cdot \frac{\mathrm{d}\eta_{ji}(t)}{\eta_{ji}(t)} \right. \\
&\qquad\qquad \left. - \frac{\omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j}{\sum_j \omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j} \cdot \frac{1 - e^{-\kappa^a_j \tau}}{\kappa^a_j} \mathrm{d}\widehat{\mu}^a_j(t) \right] + \mathrm{Drift} \cdot \mathrm{d}t \\
&= \sum_j \left[\left(\frac{\omega_j \eta_{ji}}{\sum_j \omega_j \eta_{ji}} - \frac{\omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j}{\sum_j \omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j} \right) \cdot \beta_{ij}(t) \right. \\
&\qquad\qquad \left. - \frac{\omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j}{\sum_j \omega_j \eta_{ji} \mathbf{f}^{a,\tau}_j} \cdot \frac{1 - e^{-\kappa^a_j \tau}}{\kappa^a_j} \widehat{\Sigma}^a_j \right]^\top \mathrm{d}\widehat{B}_i(t) + \mathrm{Drift} \cdot \mathrm{d}t.
\end{aligned}
$$

So by combining all the results, we can get the forward volatility vector given by equation (24). Moreover, we can also directly get the volatility vector of Treasury yield based on the diffusion of $-\frac{1}{\tau} \cdot \mathrm{d}\overline{\mathbf{f}^{\delta,\tau}_\eta} / \overline{\mathbf{f}^{\delta,\tau}_\eta}$.

A.6 Convenience Yield

Denote by $\tau = T - t$ the time interval, we can directly get the formulas of bond and convenience yieldy from $P(t,T) = e^{-y(\tau)\tau}$ and $H(t,T) = R(t) e^{[y(\tau) - y_c(\tau)]\tau}$. Comparing this with the forward price given by (22) shows

$$y_c(\tau) = -\frac{\log P_c(t,T)}{\tau},$$

which is the same as the yield for a bond denominated by crude oil.

For a small τ, we can obtain an approximation as follows. First notice that, by equation (21),

$$\begin{aligned}
F(\tau, \widehat{\mu}_j^h(t)) &\approx 1 - b_j^h\tau - \tau\Big(1 - \frac{1}{2}\kappa_j^h\tau\Big)\Big(\widehat{\mu}_j^h(t) - \gamma_j^h(\tau)\Big) \\
&= 1 - \tau\bigg(\rho_j + \widehat{\mu}_j^h(t) - \|\sigma_h\|^2 + \gamma_j^h(0) - \gamma_j^h(\tau) + \frac{1}{2}\kappa_j^h\big(\gamma_j^h(\tau) - \widehat{\mu}_j^h(t)\big)\tau\bigg) \\
&\approx 1 - \tau\bigg(\rho_j + \widehat{\mu}_j^h(t) - \|\sigma_h\|^2 + \frac{\tau}{2}\Big[\kappa_j^h\big(\alpha_j^h - \widehat{\mu}_j^h(t)\big) - \widehat{\Sigma}_j^h \cdot \sigma_h\Big] - \frac{\tau^2}{4}\|\widehat{\Sigma}_j^h\|^2\bigg)
\end{aligned}$$

thus,

$$\begin{aligned}
y_c(\tau) &= -\frac{1}{\tau}\log\Big(\frac{\sum_j \lambda_j\eta_{ji}(t)F(\tau, \widehat{\mu}_j^h(t))}{\sum_j \lambda_j\eta_{ji}(t)}\Big) \\
&\approx -\frac{1}{\tau}\log\Bigg(1 - \frac{\tau\sum_j \lambda_j\eta_{ji}(t)\bigg(\rho_j + \widehat{\mu}_j^h(t) - \|\sigma_h\|^2 + \frac{\tau}{2}\Big[\kappa_j^h\big(\alpha_j^h - \widehat{\mu}_j^h(t)\big) - \widehat{\Sigma}_j^h \cdot \sigma_h\Big] - \frac{\tau^2}{4}\|\widehat{\Sigma}_j^h\|^2\bigg)}{\sum_j \lambda_j\eta_{ji}(t)}\Bigg) \\
&\approx \frac{\sum_j \lambda_j\eta_{ji}(t)\Big(\rho_j + \widehat{\mu}_j^h(t)\Big)}{\sum_j \lambda_j\eta_{ji}(t)} - \|\sigma_h\|^2 + \frac{\tau}{2}\cdot\frac{\sum_j \lambda_j\eta_{ji}(t)\Big[\kappa_j^h\big(\alpha_j^h - \widehat{\mu}_j^h(t)\big) - \widehat{\Sigma}_j^h \cdot \sigma_h\Big]}{\sum_j \lambda_j\eta_{ji}(t)} \\
&\quad - \frac{\tau^2}{4}\cdot\frac{\sum_j \lambda_j\eta_{ji}(t)\|\widehat{\Sigma}_j^h\|^2}{\sum_j \lambda_j\eta_{ji}(t)}.
\end{aligned}$$

Appendix B Estimation

B.1 Time-series to Match

There are 3 blocks of the data, and three households are used in estimation. The 6 observed time series to match in first block are dividend yield of S&P 500 index, volatility of S&P 500 return, volatility of the 3-month Treasury yield, correlation between the S&P 500 return and 3-month Treasury yield, volatility of the 1-month futures of crude oil, and correlation between the S&P 500 return and the 1-month futures return of crude oil,

$$
\begin{aligned}
\frac{\delta(t)}{S(t)} &= \frac{\eta_{13}(t)+\eta_{23}(t)+1}{\frac{\eta_{13}(t)}{\rho_1}+\frac{\eta_{23}(t)}{\rho_2}+\frac{1}{\rho_3}} \\
\mathrm{Vol}(S(t)) &= \|\sigma_S(t)\| \\
\mathrm{Vol}(y(t,t+1/4)) &= \|\sigma_y((t,t+1/4)\| \\
\mathrm{Corr}(S(t),y(t,t+1/4)) &= \frac{\sigma_S(t)\cdot\sigma_y(t,t+1/4)}{\|\sigma_S(t)\|\cdot\|\sigma_y(t,t+1/4)\|} \\
\mathrm{Vol}(H(t,t+1/12)) &= \|\sigma_H(t,t+1/12)\| \\
\mathrm{Corr}(S(t),H(t,t+1/12)) &= \frac{\sigma_S(t)\cdot\sigma_H(t,t+1/12)}{\|\sigma_S(t)\|\cdot\|\sigma_H(t,t+1/12)\|},
\end{aligned}
$$

in which $\sigma_S(t)$, $\sigma_y(t,t+1/4)$, and $\sigma_H(t,t+1/12)$ have been represented in equations (19a), (23), and (24) respectively. For simplicity, we use $\beta_{ij} = \left(\beta_{ij,1},\, \beta_{ij,2}\right)^\top = \sigma^{-1}\left(\widehat{\mu}_i - \widehat{\mu}_j\right)$, and without loss of generality, we pick $i=3$ so β_{3j} and η_{j3} are in these equations.

The second block is the term structures of Treasure yield and convenience yield, both covering 6 maturities,

$$
\begin{aligned}
y(t,T) &= -\ln\frac{\sum_j \eta_{j3}(t)\cdot F(T-t,\widehat{\mu}_j^\delta(t))}{\sum_j \eta_{j3}(t)} \\
y_c(t,T) &= -\ln\frac{\sum_j \lambda_j\eta_{j3}(t)\cdot F(T-t,\widehat{\mu}_j^h(t))}{\sum_j \lambda_j\eta_{j3}(t)},
\end{aligned}
$$

where, for $a=\delta,\, h$,

$$
F(\tau,\widehat{\mu}_j^a(t)) = \exp\Big[-b_j^a\tau - \frac{1-e^{-\kappa_a\tau}}{\kappa_a}\Big(\widehat{\mu}_j^a(t)-\gamma_j^a(\tau)\Big)\Big]
$$

and

$$
b_j^a = \rho_j + \gamma_j^a(0) - \|\sigma_a\|^2, \qquad \gamma_j^a(\tau) = \alpha_j^a - \frac{\widehat{\Sigma}_j^a\cdot\sigma_a}{\kappa_j^a} - \frac{\|\widehat{\Sigma}_j^a\|^2}{4\kappa_j^{a2}}\cdot\big(3-e^{-\kappa_j^a\tau}\big).
$$

The third block is the term structures of the volatilities of zero coupon yield $\|\sigma_y(t,T)\|$ and volatilities of oil futures $\|\sigma_H(t,T)\|$, also covering the 6 maturities, based on equations (23) and (24).