Economic Uncertainty, Aggregate Debt, and the Real Effects of Corporate Finance

Timothy C. Johnson*

October, 2016

Abstract

To explore the real effects of corporate leverage on aggregate risk and welfare, I develop a tractable general equilibrium model with endogenous capital structure driven by time-varying economic uncertainty. Fitting the model leads to interesting insights as to the shortcomings of the standard paradigm. Contrary to the trade-off version, the empirical relation between uncertainty and aggregate debt is *positive*. This association is not attributable to supply-side constraints or adjustment costs. An alternative formulation in which debt incentives rise with uncertainty can account for the observed dynamics of uncertainty, credit spreads and leverage. This version, unlike the trade-off model, implies that the real effects of debt on the equilibrium can become severely negative.

KEYWORDS: DEBT DYNAMICS, UNCERTAINTY SHOCKS, CREDIT SPREADS, REAL EFFECTS OF FINANCE

JEL CLASSIFICATIONS: E21, E32, G12, G32

1. Introduction

Do corporate debt markets have real effects on aggregate growth and risk? That question is one facet of the broader topic of the real effects of financial contracting generally, which has been the subject of intense study since the events of 2008-2009. Yet, with

*University of Illinois, Urbana-Champaign. tcj@illinois.edu I am grateful to Heitor Almeida, Dirk Hackbarth, and George Pennacchi for helpful comments. I also thank seminar participants at UIUC and at Texas A&M University.

some important exceptions noted below, the literature lacks a baseline model that embeds endogenous capital structure choice by firms within a general equilibrium dynamic economy. Analyzing potential real effects requires a general equilibrium treatment in order to trace the effects of credit risk forward to aggregate consumption, and then back again into credit decisions via the discount rates that reflect credit's contribution to aggregate risk. This paper offers a tractable model of debt in equilibrium that permits both quantitative and qualitative analysis of real effects.

While the ultimate goal is to explore the potential real effects of defaultable corporate debt, a first step is to ask whether the theory can deliver a realistic description of the facts of credit markets. This is open question. There are many reasons why it might not. Indeed, since the financial crisis, many have viewed standard macro-finance models as inherently incapable of capturing crucial features of debt in the real world. However, lacking a benchmark equilibrium model that includes debt, there is little basis on which to judge the most important failures of standard paradigms. Hence an initial objective of the analysis is to shed light on the descriptive strengths and weaknesses of the model.

To this end, the model is intentionally sparse in the sense that it incorporates no supply-side or intermediation frictions in the capital markets. There are also no capital structure adjustment costs. The simplicity of the model allows the key mechanisms to be clearly illuminated. The solution delivers an explicit characterization (up to two algebraic equations) of optimal capital structure and default policies. Explicit expressions are also derived for aggregate default losses, default probabilities, credit spreads and credit risk premia. Consumption, marginal utility, firm value, and welfare, are all also directly obtainable as solutions to ordinary differential and algebraic equations. All of these can be quickly solved, so that estimation of the model is also feasible.

The corporate finance literature has made important progress in modeling fluctuations in debt in partial equilibrium. See especially Hackbarth, Miao, and Morellec (2006), Bhamra, Kuehn, and Strebulaev (2010), Chen (2010). To date, the consensus framework for analysis has been the classic trade-off model of capital structure choice. The starting point for the current paper is, therefore, also a trade-off model. I present a continuous-time economy with recursive preferences, whose leverage varies endogenously because of fluctuations in exogenous uncertainty. Uncertainty (the risk of costly default) is one of the two prongs of the trade-off decision (the other being tax benefits), and is thus a key driver of debt quantities. In addition to embedding the standard trade-off framework, the model can also incorporate other distortions giving rise to a demand for debt.

Despite its lack of frictions, the benchmark model seems to provide a reasonable description of financial crises: periodic increases in uncertainty lead to both higher default probabilities and higher discount rates, causing falls in asset prices and rises in volatility and credit spreads. Moreover, debt quantities contract substantially, and the credit contractions can be followed by persistent recessions. When uncertainty spikes are followed by negative output realizations, the result is a wave of default and inefficient liquidation.

However the trade-off model has a fundamental problem in matching the time-series of aggregate credit spreads and leverage. In the data, these two quantities are strongly positively correlated, whereas the model implies that they should be almost perfectly negatively correlated. On further investigation, the problem is with leverage. When proxies for aggregate uncertainty are fed into the estimated model, the implied time-series of credit spreads closely tracks its counterpart in the data. However, the implied leverage series moves opposite to the realized path of leverage. This is because, surprisingly, the empirical association between uncertainty and leverage is *positive*. I verify this result using a variety of proxies and regression specifications, and find that the effect is robust and strongly economically significant.

This finding is puzzling for the model and for economic intuition. The inherent logic of the trade-off set up implies that debt becomes less attractive as economic risk increases. Intuitively, and with recent experience in mind, we associate recessions (when risk is high) with credit contractions. To gain some insight into the causes of the leverage-uncertainty relationship, I examine the experience of recent recessions.

It turns out that most significant debt build-ups occur at the onset of recessions, as output flattens (e.g., starting in 2007). This is also when risk measure are typically spiking. By the time debt starts to contract, recessions are well under way, and uncertainty is declining (e.g., after the fourth quarter of 2008). From the point of view of theory, the feature of the data that demands explanation is not the credit contractions during economic downturns, but rather the expansions of credit that precede them.

Moreover, there is evidence that the debt increases at the onsets of recessions are voluntary policy decisions of firms, and not a consequence of unmodeled frictions. Debt increases in absolute as well as relative terms, so restructuring constraints are obviously not binding. Precautionary borrowing (liquidity hoarding) is also not the full story: debt increases even after netting out increases in corporate cash holdings. Finally, debt does not increase because of capital market supply issues: net equity issuance is *negative* during these episodes. In short, the model's first-order problem is not the absence of capital market frictions. If anything, improving its depiction of aggregate corporate debt dynamics would seem to call for some kind of negative "financial shocks."

What could lead firms to intentionally build up debt even as its risk and cost increase? I present an alternative version of the model in which the motivation for debt is a subsidy that increases in value (to the firm) with default risk. Under this model, creditors receive a degree of default protection from the government in the event of a systematic jump. (This can be interpreted as a reduced form depiction of deposit insurance in a model with a competitive banking sector.) The subsidy means that, in effect, the market misprices default risk, and firms respond to this incentive. This formulation, which is readily incorporated into the original setting, can account for the positive relation between leverage and uncertainty.

I then return to the topic of real effects. The models allow direct computation of the contribution of debt to aggregate risk and welfare. Using fitted parameters for both versions, I show that the unconditional welfare loss, under both formulations of debt, is on the order of 4-5 percent of permanent consumption. However , unlike the trade-off case, the deposit-insurance version of the model implies that the welfare losses can become very large – 10-20 percent of permanent consumption – when uncertainty is high.

To summarize, the paper offers a tractable framework that allows us to embed the canonical trade-off model into a general equilibrium with time varying economic uncertainty. It helps us identify an important shortcoming in the model's depiction of debt dynamics, and points us in a possible direction for reconciling its predictions to the empirical evidence. The alternative formulation implies that debt can play a major role in amplifying real risks. Besides having important implications for asset pricing, the results suggest that our understanding of debt incentives may be missing a crucial element.

1.1 Related literature

There is long history in macroeconomics of investigating the role of capital supply frictions (usually modeled via a one-period debt contract) in general equilibrium.[1] Contemporary work, including Arellano, Bai, Kehoe, et al. (2012), Gilchrist, Sim, and Zakrajšek (2014) and Christiano, Motto, and Rostagno (2014), has highlighted the interaction of uncertainty shocks with financial frictions and shown that the combination can provide a quantitatively good description of the 2007-2009 experience.

For the most part, these models do not speak to the chief issues of finance: the pricing of debt and equity, and the endogenous choice of capital structure. Financial research has seen only a few attempts to date to tackle capital structure determination in a general equilibrium context. Two related works are Miao and Wang (2010) and Gomes and Schmid (2012). Both papers solve business cycle models with trade-off formulations of capital structure choice, subject to adjustment frictions. The former authors use habit

[1] See the large literature following Kiyotaki, Moore, et al. (1997), Carlstrom and Fuerst (1997) and Bernanke, Gertler, and Gilchrist (1999). Recent contributions include Brunnermeier and Sannikov (2014), Jermann and Quadrini (2012) and Bigio (2015).

preferences with endogenous labor supply and idiosyncratic liquidity shocks. The latter use recursive preferences and allow for cross-section heterogeneity in productivity and leverage. Both settings are richer than the one used here, although each presents substantial computational challenges.[2], Both papers present calibrations that do a reasonable job on a broad set of real and financial moments. Neither work aims to critique the implications of the trade-off framework, or specifically examines the ability of the models to match observed aggregate debt dynamics. And neither paper is concerned with the role of time-varying uncertainty.

Closest to the current paper is the work of Gourio (2013) who endogenizes debt and equity in an economy with single-period firms and time-varying disaster risk. His model generates large fluctuations in employment and investment through the risk that the capital stock will be destroyed in a crisis. In contrast to my model, Gourio assumes that default losses do not entail real loss of resources (they are rebated to consumers). The real consequences of debt then stem from the overinvestment of firms in response to tax shields. Like the baseline model here, his model implies that leverage contracts strongly with increases in uncertainty.

The outline of the paper is as follows. Section 2 describes the model, shows how to solve for its equilibrium, and characterizes the key real and financial dynamics. Section 3 fits the trade-off version to the U.S. data, and highlights the surprising positive empirical relation between uncertainty and leverage. The alternative formulation of debt is also fitted to the data and is shown to resolve the puzzle. Section 4 assesses the real effects of debt in both versions of the model on aggregate risk and welfare. A final section summarizes and concludes.

[2]In Miao and Wang (2010) there are six aggregate state variables. In Gomes and Schmid (2012) the state space is infinite dimensional.

2. Model

This section describes an economy in which corporate financial policies are driven by exogenous changes in uncertainty. Debt policy affects aggregate risk through costly default. The default risk in turn affects marginal utility, and hence feeds back into discount rates, the cost of credit, and hence debt decisions.

2.1 Firms, Shocks, and Aggregate Output

The model is set in continuous time on an infinite horizon. There is a single consumption good, and a single class of agents. The economy is endowed with a continuum of productive projects, whose measure is denoted M, each of which produces a non-negative stream of goods until termination. Following Gomes and Schmid (2012), a project is the model's depiction of a firm: each firm owns a single project.

The projects are all stochastically identical. Let $Y^{(i)}$ denote the instantaneous output flow of project i (net of wages). I assume $Y^{(i)}$ follows the pure-jump stochastic process

$$\frac{dY_t^{(i)}}{Y_t^{(i)}} = \mu\ dt + d\left[\sum_{j=1}^{\mathcal{J}_t}\left(e^{\varphi_j^{(i)}} - 1\right)\right].$$

Here $\mathcal{J}_t$ is a regular Poisson process with intensity λ, and the percentage jump sizes $\varphi_j^{(i)}$ are assumed drawn from a distribution, $\mathcal{F}^{\varphi}(t)$, that will depend on the aggregate state. The jump process itself, $\mathcal{J}_t$, is common across firms. Thus a jump is a systematic event. If a jump occurs at time t, the sign of the jump is a Bernoulli random variable (with both outcomes having equal probability) that is also common across firms. However, conditional on the sign, the individual jump incidences are assumed to be *i.i.d.* across firms. Specifically, I will take the $\varphi^{(i)}$ to be drawn from particular gamma distributions defined over the positive or negative real line, depending on the sign of the jump. (I will also impose that the density functions are monotonic.) The distribution $\mathcal{F}^{\varphi}(t)$ is thus a gamma-binomial convolution. Intuitively, firms differ in their exposure to a systematic event, although this is only revealed *ex post* and does not carry over from one event to the next.

The common scale of the jumps, denoted σ_t is assumed to vary exogenously as

$$d\sigma_t = m(\sigma_t)\, dt + s(\sigma_t)\, dW_t.$$

Here W is a standard Brownian motion. The drift and diffusion functions are assumed to be such that σ_t is stationary on a finite interval, $[\sigma_l, \sigma_u]$. Technically, given the sign of the jump, σ will determine the mean of the jump size distribution (hence $\mathcal{F}^{\varphi}(t) = \mathcal{F}^{\varphi}(\sigma_t)$). Intuitively, σ represents aggregate economic uncertainty.

Integrating over firms, let Y denote aggregate output. Ignoring entry and exit for the moment, aggregate dynamics are

$$dY_t = \mu\ Y_t\ dt + d\ \int_i^{M_t}\ Y_t^{(i)} \left[\sum_{j=1}^{\mathcal{J}_t} \left(e^{\varphi_j^{(i)}} - 1\right)\right]\ di$$

$$= \mu\ Y_t\ dt + Y_t\ d\left[\sum_{j=1}^{\mathcal{J}_t} \left(\mathrm{E_t}\left[e^{\varphi_j^{(i)}} | \varphi_j > 0\right] 1_{\{j,+\}} + \mathrm{E_t}\left[e^{\varphi_j^{(i)}} | \varphi_j < 0\right] 1_{\{j,-\}} - 1\right)\right]. \tag{1}$$

where $1_{\{j,+\}}$ and $1_{\{j,-\}}$ are indicators for the sign of the jth jump. Applying a law of large numbers, the stochastic term is

$$Y_t\ d\left[\sum_{j=1}^{\mathcal{J}_t} \left(\Phi^+(t) 1_{\{+\}} + \Phi^-(t) 1_{\{-\}} - 1\right)\right]$$

where $\Phi^{\pm}(t)$ are the moment generating functions of the positive and negative jump size distributions, evaluated at one. Thus aggregate output follows a binomial process. Conditional on σ_t and the sign of the jump, the size of aggregate shocks is not random.

The stochastic specification of the economy is obviously stylized. Aggregate output, and that of each project, is constant except on the occurrence of a jump. It is not hard to generalize the model to include (i) firm-specific jump events, and (ii) a diffusion component to aggregate output. However, the simplicity of the set-up here still admits rich behavior while maintaining parsimony.

2.2 Debt

Each firm has access to debt financing in the form of a floating-rate line of credit, or, equivalently via issuing perpetual notes with a floating coupon rate. The firm faces no restrictions or transactions costs in altering its quantity of debt: it may freely draw down or repay any amount at any time. Equity finance is also assumed costless. The firm will thus re-optimize its capital structure continuously. Increases in debt are paid to equity holders; decreases are funded by equity holders. The firm retains no resources.[3] Debt is not constrained to be positive: the firm could hold net cash balances.

Denoting the asset value of the ith firm $V^{(i)}$ and its optimal debt $B^{(i)}$, it is intuitively clear (and will be verified below) that, since firms are stochastically identical, these values are both linear in firm output. So write $V^{(i)} = v(\sigma)Y^{(i)}$ and $B^{(i)} = b(\sigma)Y^{(i)}$.

The terms of the debt contract stipulate that the coupon rate is paid continuously at a rate that re-sets instantaneously in order to ensure that (outside of default) the market value of the debt is always equal to its face value. This is a convenient form of the contract, that is not unrealistic for a firm that is bank-financed or must roll over a significant proportion of its liabilities periodically.

Following the usual assumptions in trade-off models, I assume the firm receives a tax deduction for coupon interest paid, and that this deduction is realized continuously as long as the firm is alive. On a pre-tax basis, this is equivalent to a subsidy paid continuously to the firm. I make one non-standard simplifying assumption regarding the

[3] Technically, there is no physical capital in the model. As described below, household investment increases the mass of firms. It is straightforward to reformulate the economy so that this mass is literally the capital stock.

tax shield on profits. I assume that it is proportional to the face value of debt times a fixed statutory interest rate, $\bar{r}$, rather than to the interest rate on the firm's debt. It is also straightforward to allow the rate to depend on the debt's credit spread. But what I preclude is dependence on the economy's real interest rate. This is one of the steps that permits the capital structure decision to be decoupled from the full equilibrium solution.[4]

While tax shields are the primary motivation for debt in the structural corporate finance literature, it is straightforward to incorporate some alternative formulations of the debt subsidy. In particular, I will also consider a reduced-form depiction of deposit insurance in which creditors of a firm that has defaulted receive a payment (a transfer from the government) of ΘB_{t-} where B_{t-} is the face value of debt prior to default. Formally, the model could be augmented to incorporate a perfectly competitive (and minimally capitalized) banking sector that issues lines of credit to firms and finances them by selling partially insured deposits to households. The deposit insurance could also be implicit, as in too-big-to-fail guarantees.

As with all such models, the basic non-contractibility built into the set-up is that – even though households hold all the debt and equity claims of each firm – the firm management cannot commit in advance *not* to act in the interests of equity holders alone by defaulting when optimal (for them) to do so. For simplicity, I assume the project terminates upon abandonment. Default is always inefficient as it result in the destruction of strictly positive income streams. Since there is no physical capital debt recovery from the firm is zero. [5]

A familiar result in the partial-equilibrium capital structure literature is that floating rate debt is riskless when asset value follows a diffusion process. In the present context,

[4]Of course, in practice, tax shields depend on nominal interest rates, not real ones. So linking the tax shield to the real riskless rate is not necessarily more realistic, especially since that rate may not always be positive.

[5]The zero-recovery assumption is the extreme case of inefficient liquidation or "fire sales" of assets. Clearly this represents an upper bound on the default losses and resulting increase in aggregate risk attributable to debt.

this implies that diffusive changes in uncertainty σ_t will never trigger default. Instead, default will occur following sufficiently negative jumps in output. If it is optimal for a firm to exit upon a bad downward jump in $Y^{(i)}$, say of percentage $\varphi^{\star}$, then, by linearity, all firms with jumps of the same size or worse will also exit. Hence there is a largest (least negative) jump size such that default will occur with any jump below this level and not otherwise. Let $\varphi^{\star}(\sigma)$ denote this critical value. It will be derived below. However, even without knowing it, we can immediately deduce the effect of exit on the dynamics of aggregate output. In equation (1) above, we replace $\mathrm{E_t}\left[e^{\varphi_j^{(i)}}|\varphi_j^{(i)}<0\right]$ with $\mathrm{E_t}\left[e^{\varphi_j^{(i)}}1_{\left\{\varphi_j^{(i)}>\varphi_t^{\star}\right\}}|\varphi_j^{(i)}<0\right]$. That is, for downward jumps, we lose the mass of firms that experience jumps worse than the threshold. Let $d = d(\sigma)$ be this expectation, and let $u = u(\sigma)$ be the corresponding value for upward jumps. Then the effect of exit on output is simply to alter the downward aggregate jump size d.

2.3 Households, Investment, and Consumption

There is a representative household characterized by preferences of the stochastic differential utility class (Duffie and Epstein (1992), Duffie and Skiadas (1994)), the continuous-time analog of Epstein and Zin (1989) preferences. Specifically, agents maximize the lifetime value function,

$$J_t = \mathrm{E_t}\left[\int_t^{\infty} f(C_s, J_s)\, ds\right].$$

where

$$f(C,J) = \frac{\beta C^{\rho}/\rho}{((1-\gamma)\ J)^{1/\theta-1}} - \beta\theta J.$$

Here β is the rate of time preference, γ is the coefficient of relative risk aversion, $\rho = 1 - 1/\psi$, where ψ is the elasticity of intertemporal substitution, and $\theta \equiv \frac{1-\gamma}{\rho}$. (I assume $\gamma \neq 1, \rho \neq 0$.)

Households' aggregate income is assumed equal to Y_t, up to a constant proportionality factor.[6] A government sector is assumed to collect corporate taxes, net of tax shields, and rebate any surplus to households.

Real investment is determined at the household level. Households are endowed with a technology (R&D) for generating a flow of new projects.[7] Specifically, if a fraction ι of aggregate output Y is expended, this is assumed to increase the mass of projects at the proportional rate $\zeta(\iota)$, where $\zeta()$ is an increasing, concave function. The flow of new projects shows up as an additional term, $\zeta(\iota)\, dt$ in the growth rate of aggregate output, dY/Y.

When new firms are created, they are distributed uniformly across households. Each household sells its firm(s) to all the others. Each firm then sells its initially optimal quantity of debt, the value of which passes to the equity holders. These financial transactions between households and themselves result in no net flow of real goods.[8]

2.4 Solution

The model is tractable for two main reasons. First, the assumptions are sufficient to ensure that there is a single state variable (the level of uncertainty) that characterizes everything. In particular, the distribution of firm sizes does not enter into any aggregate quantities. Second, the components of the valuation problem can be effectively separated from each other and derived sequentially, as the following results demonstrate.

Proposition 1. *Let $V^{(i)}$ denote the value of the ith firm's project and $B^{(i)}$ the face value of its debt. Then, prior to default, $V^{(i)}$ and $B^{(i)}$ are linear in output $Y^{(i)}$: $V = v(\sigma)Y^{(i)}$*

[6] If we assume that firms' total output is a constant returns-to-scale function of labor input, and that households supply labor inelastically and are paid their marginal product, then total wages plus dividends will just be Y_t divided by the capital share.

[7] Having households, rather than firms, do the investing is another help in decoupling steps of the solution process. However, it shuts down some effects that may be of interest, like the distortionary impact of tax shields.

[8] There is an implicit assumption that households, or the R&D sector, do not borrow against their valuable growth options. To the extent that tax shields are the incentive for debt, this is reasonable because *future* projects have no current profits to shield.

and $B = b(\sigma)Y^{(i)}$.

The optimal default policy is for owners to abandon the firm on a jump of $V_t^{(i)}$ *below* $B_{t-}^{(i)}$ *and the optimal market leverage ratio* $B^{(i)}/V^{(i)} \equiv \ell$ *is the same for all firms and is related to the abandonment threshold* $\varphi^\star$ *via*

$$e^{\varphi^\star} = \ell(\sigma).$$

The first-order condition for the optimal quantity of debt is

$$\tfrac{1}{2}\lambda d^{-\gamma}\ f^{\varphi-}(\varphi^\star) = \bar{r}\tau \tag{2}$$

where $f^{\varphi-}$ *is the density function of the negative jumps whose distribution is denoted* $\mathcal{F}^{\varphi-}$. *The aggregate output drop on a down jump is*

$$d = \int_{\varphi^\star}^{0} e^{\varphi}\ d\mathcal{F}^{\varphi-}. \tag{3}$$

There is a unique solution of the preceding two equations for d *and* $\varphi^\star$. *That solution implies* $\ell \in (0,1)$. *The optimal leverage policy is incentive compatible for equity holders.*

Note: all proofs appear in Appendix A.

The proposition characterizes optimal leverage and default as a function of the state σ in essentially closed form. The only required inputs are the distribution function of the negative jumps, their intensity, and the tax shield and risk aversion parameters. We do not need to solve for other features of the equilibrium (such as investment or the output growth rate) in order to examine debt policy. Notice also that the debt equations make no reference to the dynamic specification of the state variable, σ.

The final statement in the proposition is also an important result in achieving tractability. Many formulations of dynamic capital structure problems in continuous time embed a commitment problem and thus require careful demonstration that policies constitute a Nash equilibrium in the game between managers and the market (which prices the firm's claims conditional on policy beliefs). See He and Milbradt (2016) for example. However, the solution here obviates this difficulty. Intuitively, this is a consequence of the stipulation that the price of the debt contract per unit face value is always one, which implies that no policy can expropriate value from existing debt holders. As a result, equity-maximizing managers cannot do better than maximizing firm value.

A second nice consequence of the assumed form of the debt contract is that it is easy to evaluate the credit spread. It is just risk-neutral default intensity, which is simply the true intensity times a risk-aversion factor.

Corollary 2.1. *The credit spread on the firm's debt is*

$$\tfrac{1}{2}\lambda\ d(\sigma)^{-\gamma}\ (1 - \mathcal{F}^{-\varphi}(\varphi^{\star})).$$

The proof in the proposition is also readily modified to handle the case of default insurance, as described above.

Corollary 2.2. *With default insurance, the optimal default policy is unchanged. The first-order condition for the optimal quantity of debt is*

$$\tfrac{1}{2}\lambda\ d^{-\gamma}\ [(1-\Theta)\ f^{\varphi-}(\varphi^{\star}) - \Theta\ \mathcal{F}^{\varphi-}(\varphi^{\star})] = \bar{r}\tau. \tag{4}$$

The aggregate output drop on a down-jump is again given by equation (3).

Having solved for the optimal debt policies and the contribution of default to aggregate risk, the full aggregate dynamics are now determined.

Proposition 2. *The household's value function is $J = j(\sigma)\ Y^{1-\gamma}/(1-\gamma)$, and optimal consumption is $C = c(\sigma)Y$, where $j(\sigma)$ and $c(\sigma)$ are the solutions to (respectively) an ordinary differential equation and an algebraic equation given in the appendix.*

With the consumption process determined, the stochastic discount factor, the riskless rate, and the market price of risk are all immediately obtainable, and are also given in the appendix. Also the appendix shows that the differential equation defining j is guaranteed to have a unique solution, and is easy to solve numerically.[9] The same is true of the equation for firm value in the following result.

Proposition 3. *The firm's price-output ratio $v(\sigma)$ is the unique solution to an ordinary differential equation given in the appendix. The optimal debt-output ratio is $b(\sigma) = v(\sigma)\ell(\sigma)$, where ℓ was determined above.*

Given the prices of the firm's claims, it is straightforward to derive their risk premia, expressions for which are also given in the appendix.

2.5 Parametric Assumptions

To take the model to data, and to illustrate properties numerically, requires specification of the jump distribution and the uncertainty process.

For the firm-specific output jumps, recall that the scale of the jump size distribution is driven by σ, and that the sign is the outcome of an independent Bernoulli draw. I assume the size of (log) down jumps is drawn from a two-parameter gamma distribution, whose mean is σ. The second distribution parameter, denoted $L \geq 1$, fixes the jump size standard deviation as $L\sigma$. Averaging, the expected decline conditional on a down jump is

$$d_{nd} = \mathrm{E}[e^{\varphi}|\text{down jump}] = \left[\frac{1}{1 + L^2\sigma}\right]^{L^2}.$$

where the subscript nd ("no default") indicates that this is the down jump size before taking into account the output loss due to exiting firms. Next, I assume jumps are

[9] This follows from the fact that the equation is to be solved on the closed interval $[\sigma_l, \sigma_u]$ and the coefficient on the second order term is zero at the endpoints, because the σ process is bounded). This is equivalent to providing two boundary conditions, which gives existence and uniqueness.

symmetrical in the following sense:

$$u = \mathrm{E}[e^{\varphi}|\text{up jump}] = 1/d_{nd}.$$

Given the size of the up-jumps u, I assume these jumps are distributed exponentially, i.e., gamma distributed with $L = 1$. (The distribution of up jumps plays no role in the solution for optimal debt policy.)

For the uncertainty dynamics, a convenient choice is the so-called Jacobi process, which is stationary and bounded. The specification is

$$d\sigma_t = \kappa(\bar{\sigma} - \sigma_t)\,dt \;+\; s_0\,\sqrt{(\sigma_u - \sigma_t)(\sigma_t - \sigma_l)}\,dW_t.$$

Besides the upper and lower limits, σ_u and σ_l, this process requires the choice of the unconditional mean, $\bar{\sigma}$, the mean-reversion speed, κ, and the volatility of volatility parameter, s_0. The stationary distribution is then in the beta class[10] on the open interval (σ_l, σ_u). The instantaneous variance of σ is a symmetric quadratic function, centered at the midpoint of the range. This will mean that *volatility risk* is itself increasing in σ most of the time, since, for reasonable calibrations, the mean of the distribution, $\bar{\sigma}$, will be closer to σ_l than to σ_u.

For ease of interpretation, I take the jump intensity to be $\lambda = 1$ in annualized units, meaning that, on average, there is one output jump per year. Hence σ can be interpreted as the instantaneous scale of the annual percentage output shocks.

Finally, I will assume the investment technology is given by a simple functional form that guarantees a unique solution to the investment first-order condition is

$$\zeta(\iota) = \zeta_0\;\iota^{\zeta_1} \tag{5}$$

[10] See Gouriéroux and Valéry (2004).

with $\zeta_0 > 0$ and $1 > \zeta_1 > 0$.

2.6 Solution Properties

To understand the model's implications about debt quantities (leverage) and prices (credit spreads), consider the two equations (2) and (3) in Proposition 1. Each of these can be viewed as an equation for the equilibrium output drop, d, as a function of leverage, ℓ. We can readily deduce the properties of each, and how each varies with σ.

Equation (3) is just accounting: it says output after a down-jump is zero for firms that exit. As leverage increases, the default threshold $\varphi^\star$ rises and more firms default, lowering d. Hence this line is downward sloping in ℓ.

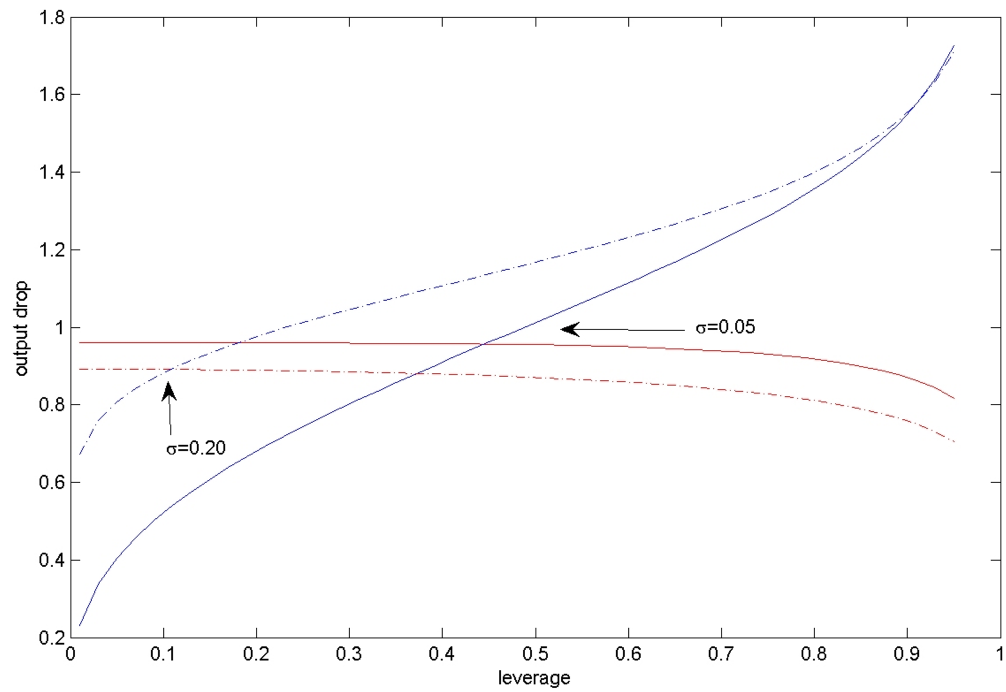
The first order condition (2) equates the marginal benefit of an additional unit of debt to its marginal increase in default probability under the risk-neutral measure. Here risk aversion contributes the factor $d^{-\gamma}$ which is equal to the increase in marginal utility conditional on an output decline. This factor is greater than one (since $d < 1$) meaning that default risk is systematic, which raises the marginal cost of debt. The true marginal default probability rises with leverage. Hence, for the left side to be constant, optimal leverage can be higher only if the marginal utility factor is lower (closer to one) or that d is higher. The equation thus describes an upward sloping relation.

Figure 1 plots the two equations for d as a function of ℓ using two values of σ.[11] The solid lines are for $\sigma = 0.05$ and they intersect at an optimal leverage of about 45 percent. The dashed lines show what happens when uncertainty rises to $\sigma = 0.20$. The accounting equation (red line) shifts down: for any given value of the default threshold, there is now more probability mass beyond it; more default means lower d. An increase in uncertainty also causes the first order condition line to shift strongly upward. For a given level of leverage, increasing risk raises the marginal default probability. Since the marginal benefit are constant, this can only happen if the marginal utility factor $d^{-\gamma}$

[11]The numerical example in this section assume $\gamma = 7$, $L = 3$, $\overline{r} = 0.05$, and $\tau = 0.35$.

falls, meaning a rise in d.

Figure 1: Optimal Debt

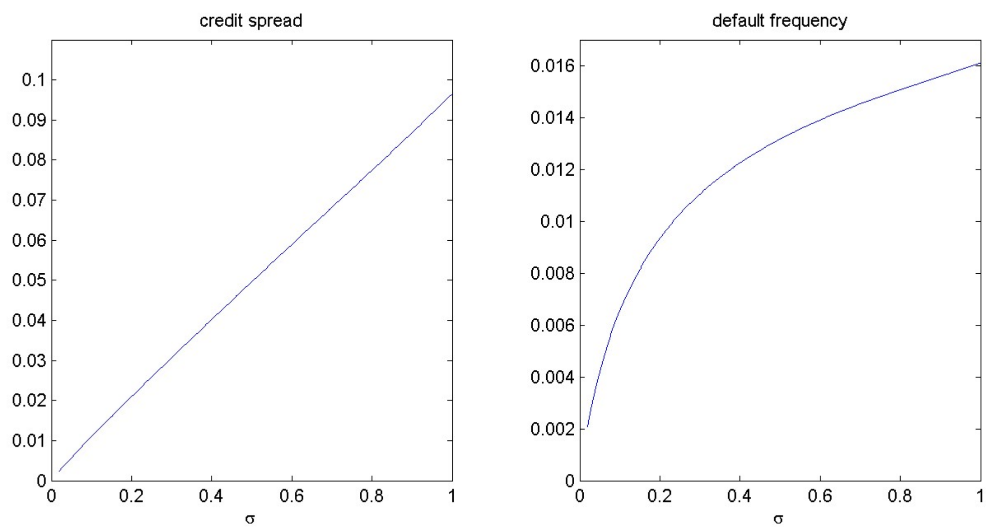


The figure plots the two equations (2) and (3) that determine optimal leverage ℓ and equilibrium output drops d. The former are the upward sloping lines; the latter are the downward sloping ones. Solid lines use $\sigma = 0.05$. Dashed lines use $\sigma = 0.20$. The parameters are $\gamma = 7$, $\lambda = 1$, $L = 3$, $\hat{r} = 0.05$, $\tau = 0.35$.

The conclusion is that increased uncertainty causes optimal leverage to contract. At $\sigma = 0.20$ the two lines cross at a little over 10 percent. Indeed, the full function $\ell(\sigma)$ is everywhere monotonically decreasing. The first order condition is the driving force behind this. Intuitively, the inherent logic of the trade-off framework requires that increases in default risk make debt less attractive. The equilibrium effect that also sees increases in default risk amplified by larger drops in marginal utility reinforces the partial-equilibrium intuition.

Next consider credit spreads. From Corollary 1, these are determined by both risk and risk aversion. The risk aversion factor $d^{-\gamma}$ clearly rises with σ since d falls. We have seen that optimal debt declines steeply as σ rises. The effect on default risk depends on the fatness of the tail of the jump distribution. The right-hand panel of Figure 2 illustrates that the equilibrium default probability actually rises with σ, despite the lower leverage. As the right-panel illustrates, the risk-aversion parameter strongly amplifies this increase. The divergence between credit spreads and default probabilities becomes extreme during high σ excursions. For example, at $\sigma = 0.5$ the true default frequency is a little more than one percent, while the credit spread is almost five times as large.

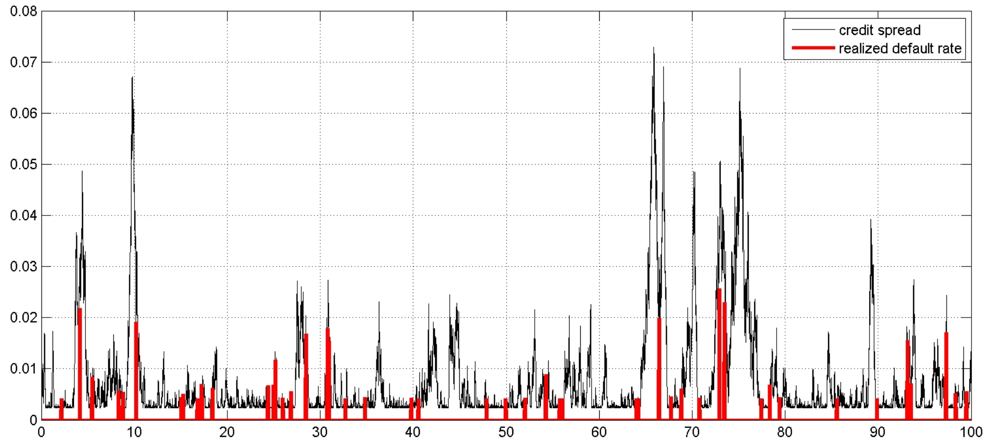
Figure 2: Credit risk



The credit spread (left panel) and default frequency (right panel) are plotted using parameters $\gamma = 7$, $\lambda = 1$, $L = 3$, $\hat{r} = 0.05$, $\tau = 0.35$.

In sum, although the model is quite stylized and omits many frictions that undoubtedly play important roles in credit markets, it does at least seem to capture important features of debt-driven crises. Rare excursions into high σ states will see strong contractions in credit and spikes in credit spreads and credit risk premia (or discount rates). If a negative output jump does occur in such a state (which will not always happen), then there will be a wave of inefficient default imposing a real cost on households. (Figure 3 shows a simulated 100-year history of credit spreads and default rates.)

Figure 3: Credit risk



The figure ploat a 100-year simulation of credit spreads and realized default rates when the state variable parameters are $\bar{\sigma} = 0.05$, $\sigma_l = 0.02$, $\sigma_u = 1.0$, $\kappa = 0.63$, and $s_0 = 0.77$. Other parameters are as given in the caption of Figure 1.

3. Empirical Evaluation

How well does the general equilibrium trade-off model do at explaining the observed behavior of debt quantities and prices? This is an open question. While it has been widely asserted that standard models have been discredited for their descriptive failures, the literature has lacked a benchmark model with endogenous debt by which to judge such assertions.

In the context of the current study, assessing the model's empirical validity is also a necessary step towards addressing the broader question of the real effects of debt. There is little point in drawing conclusions about welfare unless we have confidence in the underlying model's representation of the economy.

This section first fits the standard trade-off version of the model to U.S. data, and examines the implications of the fitted specification. In fact, the model is shown to have a fundamental difficulty. Specifically, leverage and credit spreads are positively correlated in the data, and negatively correlated in the model. Comparing the implied dynamic paths with actual experience, realized uncertainty shocks in the data do an excellent job explaining the observed history of credit spreads. The problem is with leverage: it turns out that aggregate leverage is strongly positively associated with uncertainty. This is a new finding that is shown to hold across a variety of proxies and of regression specifications.

I then attempt to diagnose the source of the problem by examining corporate cash-flows in more detail during the three most recent recessions. The question is: what friction or distortion is the model missing? I argue that data tell us that the answer is not shocks to capital supply or costly adjustment of financing.[12] Instead, it looks like firms face some positive incentive to increase leverage in the face of rising uncertainty. I then empirically fit the version of the model with deposit-insurance type debt subsidy. This version resolves the central empirical shortcoming of the trade-off model.

3.1 Estimation

The tractability of the model developed in Section 2 permits estimation of its parameters by the method of simulated moments. For any parameter values, the exact solutions to all quantities are numerically obtainable in a matter of seconds. Notably, no linearizations (or approximations of any order) are required. Moreover, the ergodic distribution of the state variable σ_t is known in closed-form, obviating issues of simulation convergence in computing population moments.

I fit the model to a collection of real and financial moments whose empirical counterparts correspond to the quantities that the model is being asked to speak to: the level and dynamic properties of credit spreads and leverage; the distribution of output shocks; the levels of savings and firm valuation; and the risk premium in credit spreads. Choosing empirical quantities to correspond to quantities in the model necessarily requires some

[12] To be clear, I am not suggesting capital supply frictions are not present and important in the real world. I am only observing that invoking them will not resolve the particular puzzle highlighted here: the increases in leverage at the on-sets of recessions.

subjective judgements. The primary measure of credit spreads is the difference between Moody's Baa yield and the yield on 20-year Treasury bonds. Debt is the sum of debt securities and loans in the U.S. nonfinancial corporate sector from the Federal Reserve's

Table 1: Data and Model Moments

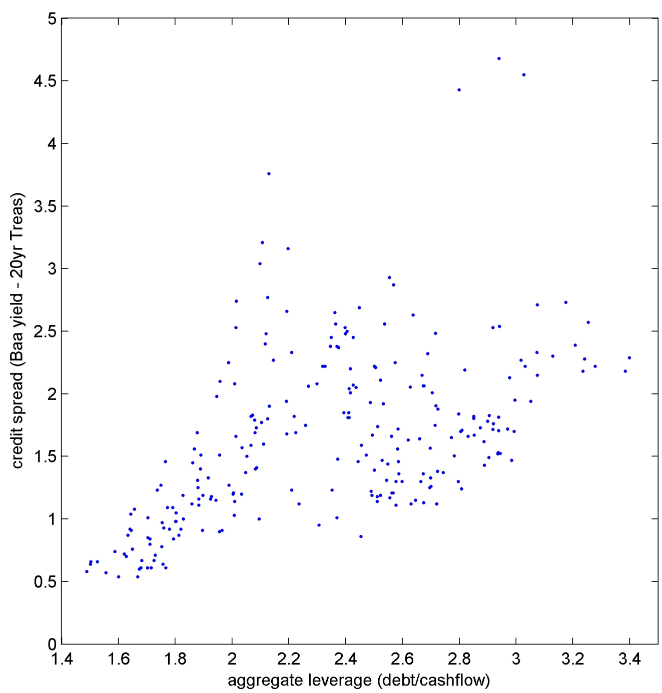
	Data	Model:T-O	Model:D-I
output growth	0.0158	0.0170	0.0226
output standard dev	0.0306	0.0300	0.0302
output skewness	-0.3973	-0.3023	-0.2743
output kurtosis	5.5771	5.8204	5.6432
investment rate	0.0780	0.0799	0.0782
default rate	0.0087	0.0157	0.0101
equity valuation	4.6342	2.7287	6.3098
leverage	2.3360	2.4338	2.5633
leverage standard dev	0.4578	0.4397	0.2513
leverage change std dev	0.0741	0.1441	0.0738
credit spread	0.0165	0.0186	0.0112
credit spread standard dev	0.0068	0.0081	0.0104
credit spread change std dev	0.0032	0.0027	0.0031
leverage-credit spread correlation	0.5229	-0.9789	0.5268

The table shows statistics from the data used to estimate two versions of the model from Section 2, along with the implied moments from the estimated models. The first four rows show standardized moments of quarterly log changes in output measured as the cash-flow of U.S. nonfinancial corporations. The investment rate is the annual household savings rate. The default rate is the annual average number of bankruptcy filings divided by the number of firms. Equity valuation and leverage are the ratios, respectively, of the market value of equity and the net total debt of U.S. nonfinancial corporations scaled by the annualized output series. The credit spread is the difference in yields-to-maturity of the Moody's Baa benchmark and 20-year U.S. Treasury bonds. Further details of the data series and estimation are given in Appendix B. The columns labeled T-O and D-I correspond to the trade-off and default-insurance versions of the model.

Flow of Funds accounts. Output is the operating cash-flow of this sector measured as net operating surplus plus consumption of fixed capital, also from the Flow of Funds. Details of the other data choices are described in Appendix B.

Table 1 shows the resulting model fit in the column labeled T-O.[13] In terms of unconditional moments, the trade-off model can well explain many important features of the data. However, the final line in the table shows the problem described above: namely, the trade-off model implies an almost perfectly negative correlation between leverage and credit spreads, while the correlation in the data is reliably positive. The positive correlation is a robust finding across choices of proxies and sample period, as discussed in the appendix. The relation is shown visually in Figure 4.

Figure 4: Leverage and Credit Spreads



The figure plots credit spreads against aggregate leverage in the U.S. data. Leverage is measured as the ratio of total debt to quarterly cashflow of the nonfinancial corporate sector, from the Flow of Funds accounts. The credit spread is the difference between the Moody's benchmark Baa yield-to-maturity and the interpolated yield-to-maturity on 20-year Treasury bonds.

[13]Parameter point estimates are given in the appendix.

The takeaway from the estimation exercise is that the trade-off model is misspecified in an important way. However, the results do not reveal whether the problem lies with the description of prices of debt (credit spreads) or quantities (leverage), or both. To investigate further, I examine the model-implied histories of these series.

3.2 Uncertainty and Leverage

The model posits that uncertainty shocks drive capital structure decisions and bond prices. Recent advances in the empirical literature provide a number of potential measures of fundamental uncertainty that can be used to assess the model predictions. Here I employ the index constructed in Jurado, Ludvigson, and Ng (2015) (JLN) which averages the forecast standard deviations from time-series models of a large and diverse panel of economic and financial statistics. The series is exogenous with respect to credit market outcomes in the sense that, at each point in time, it is based on specifications fitted to rolling windows of backward-looking data.

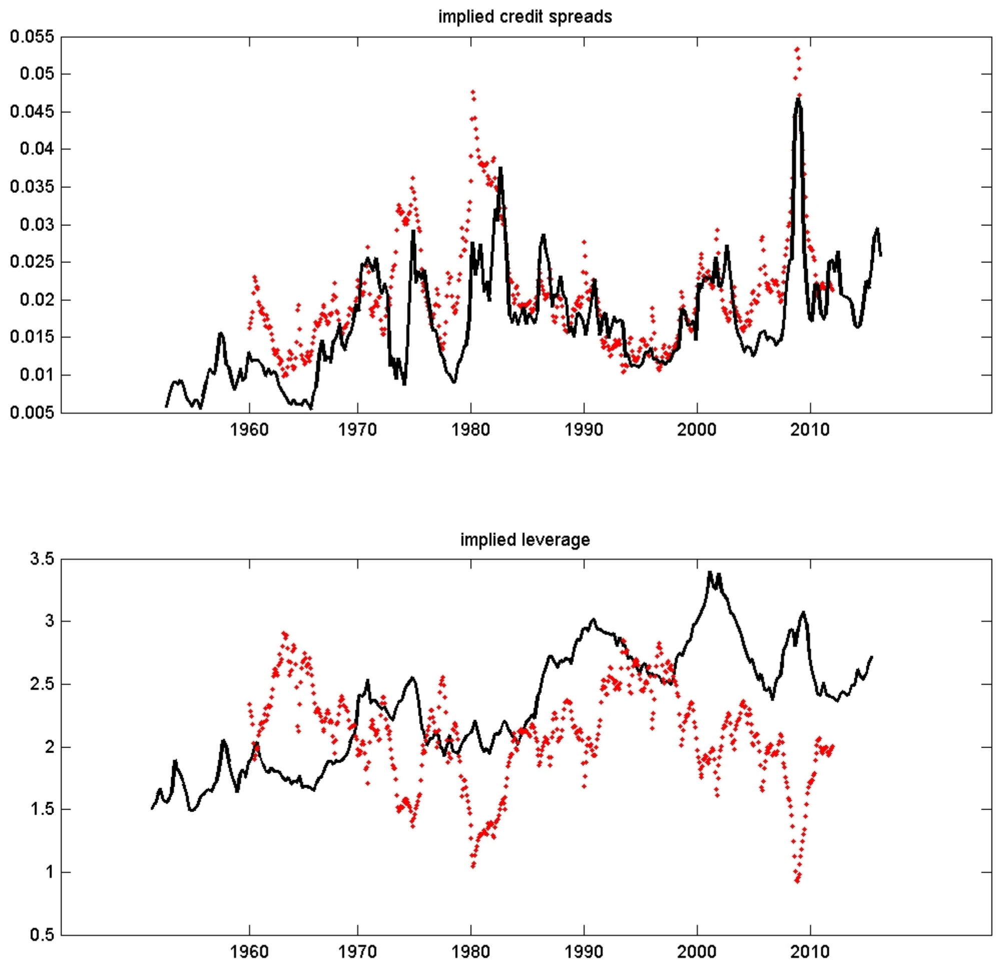
In the upper panel of Figure 5, I plot the model-implied history of credit spreads when the JLN series is taken as the realization of the model's uncertainty state and fed into the estimated model.[14] The fit to the empirical counterpart series is actually remarkably good. This provides strong support for the basic premise of the model that time-varying uncertainty explains credit spread fluctuations.

By contrast, the figure's lower panel shows that the fitted model – coupled with the time-series of uncertainty realizations – completely fails to describe the dynamics of debt. In fact, the plot suggest a new empirical fact: that debt responds *positively* to uncertainty shocks. This finding is sufficiently unexpected and diverges so strongly from the theoretical prediction that it deserves further investigation. To this end, Table 2 shows the results of a number of regression specifications using different proxies for debt,

[14] The units of the JLN measure do not map directly to a corresponding model quantity. For this exercise, the series is rescaled to match the scale of the fitted σ process.

output, and uncertainty.

Figure 5: Model-implied Histories of Leverage and Credit Spreads



Uncertainty shocks are fed in to the fitted trade-off model as described in the text. The uncertainty series is from Jurado, Ludvigson, and Ng (2015). The top panel plots model-implied credit spread series along with the actual realization from the data. The bottom panel does likewise for the model implied leverage.Leverage is measured as the ratio of total debt to quarterly cashflow of the nonfinancial corporate sector, from the Flow of Funds accounts. The credit spread is the difference between the Moody's benchmark Baa yield to maturity and the interpolated yield to maturity on 20-year Treasury bonds.

Table 2: Aggregate Debt Dynamics

Panel A: Nonfinancial Corporate Debt ; Cash-flow						
Dependent var		$(b_t - y_t) - (b_{t-4} - y_{t-4})$			$(b_t - b_{t-4})$	
Sample	1984Q1-2013:Q1	1969Q4-2014:Q4	1961Q3-2011:Q4	1984Q1-2013:Q1	1969Q4-2014:Q4	1961Q3-2011:Q4
VIX	0.1249			0.0847		
	(3.26)			(2.70)		
SPF		0.0778			0.0364	
		(3.38)			(2.14)	
JLN			0.4893			0.3365
			(4.97)			(3.63)
$b_{t-4} - y_{t-4}$	-0.2349	-0.1806	-0.0743			
	(3.25)	(2.71)	(1.93)			
$y_t - y_{t-4}$				0.0535	-0.0444	0.0447
				(0.32)	(0.30)	(0.35)
N	117	181	202	117	181	202

Panel B: Nonfin. Corp. & Noncorp. Debt; Bus. GDP						
Dependent var		$(b_t - y_t) - (b_{t-4} - y_{t-4})$			$(b_t - b_{t-4})$	
Sample	1984Q1-2013:Q1	1970Q1-2014:Q4	1970Q1-2011:Q4	1984Q1-2013:Q1	1970Q1-2014:Q4	1970Q1-2011:Q4
VIX	0.0890			0.0965		
	(2.95)			(3.35)		
SPF		0.0425			0.0399	
		(2.86)			(2.64)	
JLN			0.2479			0.2751
			(4.11)			(5.30)
$b_{t-4} - y_{t-4}$	-0.2099	-0.1465	-0.1001			
	(2.44)	(2.49)	(1.56)			
$y_t - y_{t-4}$				0.9790	0.7121	0.6441
				(4.80)	(4.37)	(6.51)
N	117	180	168	117	180	168

Caption to Table 2 The table reports time-series regressions of quarterly aggregate debt on measures of economic uncertainty. If the first panel, b is total debt securities and loans, minus cash items (including checkable, time and savings deposits, money-market funds, and foreign deposit) for the U.S. nonfinancial corporate sector and y is the total cashflow of this sector measured as net operating surplus plus consumption of fixed capital. All data are from the Federal Reserve's Flow of Funds accounts. In the second panel, b is the combined total debt (net of cash items) for corporate and noncorporate nonfinancial businesses and y is gross domestic product of nonfarm U.S. businesses from NIPA Table 1.3.5. Debt and output variables are in logarithms. VIX, SPF, and JLN are respectively the CBOE VIX index (extend backward from 1990 by the author), the dispersion in current-quarter forecasts of real and nominal GDP as tabulated by the Survey of Professional Forecasters (SPF), and the 3-month-ahead average forecast dispersion of macroeconomic statistics constructed by Jurado, Ludvigson, and Ng (2015). The uncertainty series are year-on-year log differences contemporaneous with the dependent variable. Numbers in parentheses are Newey and West (1987) t-statistics using 8 lags.

In addition to the JLN measure, the table also uses the dispersion in economists' forecast of GDP, as tabulated from the Survey of Professional Forecasters (SPF) data.[15] The table shows regressions of year-on-year changes in leverage on year-on-year log changes in the two uncertainty series. (Standard errors correct for the serial correlation in overlapping residuals.) Also shown are specifications using differences in the debt series alone as the dependent variable and controlling for contemporaneous cashflow changes. Finally, the tests are repeated using a broader measure of debt that includes the noncorporate private sector, and a measure of output (rather than cashflow) for this sector.[16] The regressions establish that the positive relationship is statistically strong and robust to measurement choices for both dependent and independent variables.

3.3 Leverage Dynamics in Recessions

From a business-cycle point of view, the positive relation between leverage and uncertainty is surprising because it is well known that uncertainty is strongly countercyclical, while debt issuance is procyclical (Covas and den Haan (2011), Jermann and Quadrini (2012)). Intuitively, especially with recent experience in mind, one expects that leverage contracts in downturns when uncertainty is high. To get a better understanding of what intuition – and the trade-off model – are missing, this section takes a closer look at financing behavior around recessions.

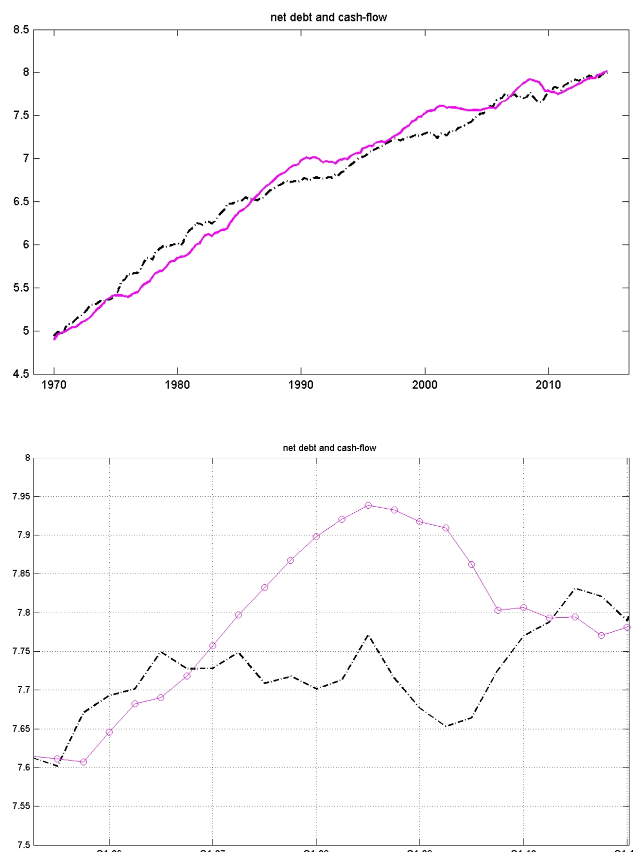
Figure 6 plots the time series of aggregate debt and output separately. The series are in logs and the means have been aligned. These quantities correspond to $\log Y$ and $\log B = \log b + \log Y$ in the model. From the top panel, the data show essentially three episodes in the last 40 years during which the two quantities diverge significantly. These

[15] The series here averages the dispersion in current-quarter forecasts of real and nominal GDP. Using other forecast horizons and extracting a principal component from the dispersion series produce similar results.

[16] Debt is measured as before (net of cash items) using the combined Flow of Funds accounts for corporate and noncorporate nonfinancial businesses. Output is Gross Domestic Product of nonfarm U.S. businesses from NIPA Table 1.3.5.

correspond approximately to the last three U.S. recessions. Interestingly, in each episode, the recession starts with a *positive* excursion of debt from its stochastic trend (i.e., the output series). The lower panel magnifies the two series to focus on the Great Recession. For the non-financial corporate sector, the "credit crunch" starting in 2009 actually saw debt merely revert to its steady state level, essentially unwinding an anomalous build-up of debt at the end of the expansion in 2007 as output stagnated.

Figure 6: Debt and Cashflow – U.S. Nonfinancial Corporations



Both panels plot the log of total debt (net of cash items) for nonfinancial U.S. corporations as a solid line, and the log total operating cashflow of this sector as a dashed line. The series have been aligned by subtracting a constant from log debt. The lower panel shows the same data for the 2007-2009 recession.

The positive leverage-uncertainty relation in the data arises from the fact that uncertainty spikes also happen early in recessions or around expansion peaks. The fact that debt rises much faster than output as output stalls at peaks also explains how countercyclical leverage can be consistent with procyclical debt *issuance*.

The data thus focus our attention in an unexpected direction. From the perspective of the model, there is no puzzle at all in credit contractions during recessions. Rather the puzzle is leverage build-ups that precede them. What is behind the increases in leverage at the onsets of recessions? What frictions or distortions could resolve the puzzle?

In considering these questions, the data present some directly relevant features. First, the debt increases at the onset of recessions are not due to precautionary borrowing in the face of uncertainty. The measures used in the regressions for debt are *net* of cash holdings. Second, the leverage increases are not due to debt adjustment frictions on the downside, as postulated by many structural models. Leverage is increasing because debt is *increasing* in these periods, not merely staying fixed while output goes down. Third, debt is not rising because of the operating losses: operating cashflows remain positive in all the downturns. Finally, and most tellingly, leverage is not reflecting capital supply frictions on the equity side. As Table 3 shows, in the periods of debt build-ups going into the last three recessions, firms actively *reduced* their equity financing by buying back stock and continuing to pay dividends, while operating cashflows were sufficient to fund investment.

The numbers in Table 3 appear to establish that increases in leverage at the onsets of recessions are, in fact, the result of voluntary policy decisions by firms. They could have been entirely avoided by *not* borrowing *more*, and cutting discretionary payouts to equity holders. Within the trade-off model, there is no mechanism that can account for this behavior.

3.4 Alternative Model

Why would firms borrow more precisely when their operating cashflows are stalling and the economic environment poses increased risk? One possibility is that debt markets

Table 3: U.S. nonfinancial corporate sector: sources and uses of funds (billions)

	1988:Q4-1990:Q2	1999:Q3-2001:Q1	2007:Q1-2008:Q3
cash flow from operations	1481.85	2529.69	3971.49
change in net debt	309.11	508.85	1037.37
physical investment	915.57	1945.43	2434.03
interest expense	266.57	312.26	437.57
income tax	177.31	287.42	482.88
change in net receivables	26.50	-47.24	209.89
dividends + equity repurchases	387.43	573.31	1766.27

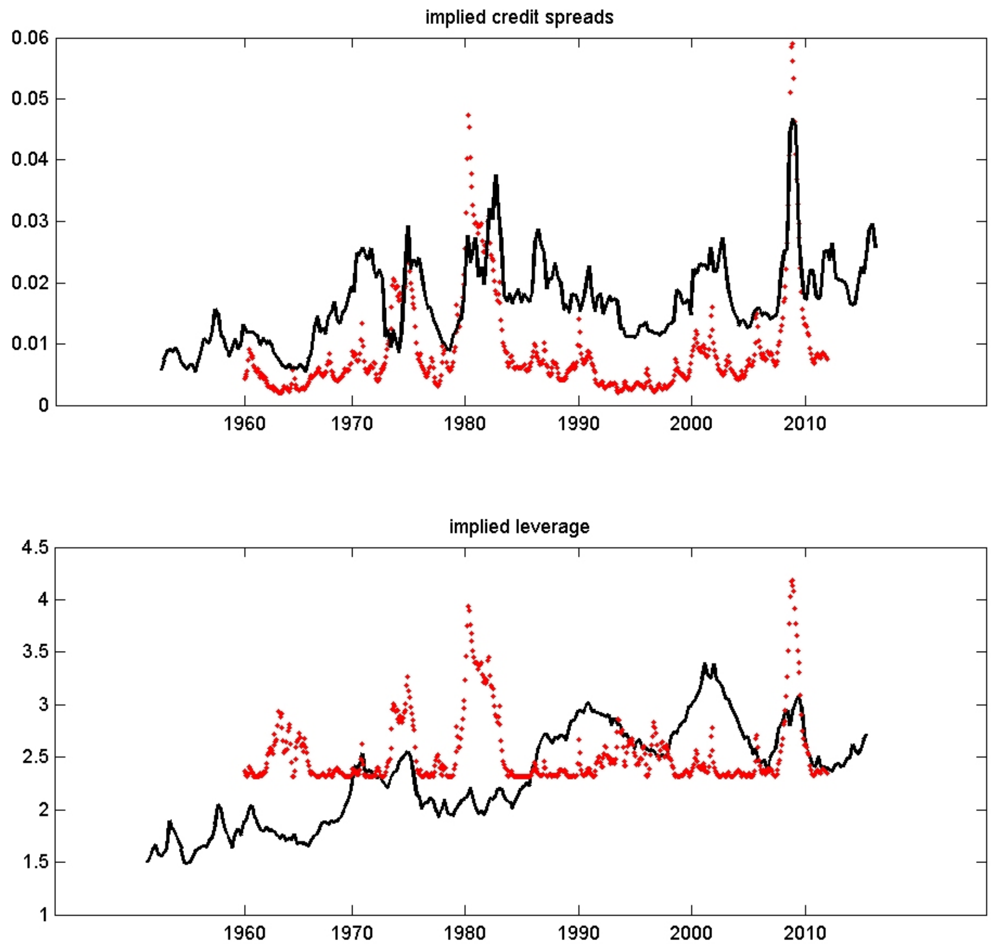
are sending firms the wrong signal by not raising borrowing costs *enough*, and managers are responding rationally by replacing equity with overpriced debt. In the context of the model, this sort of mispricing could be captured by a subsidy to debtholders that becomes more valuable with risk.[17] The model can capture such a subsidy in the version with reduced-form deposit insurance described in Section 2.

To illustrate, I fit this version of the model using the same target moments as used for fitting the trade-off model. The fit is shown in the column labeled D-I in Table 1. This version matches the observed positive correlation of leverge and credit-spreads while also achieving a reasonable match on the other properties of the data it is being asked to explain. Feeding in the uncertainty shocks from Jurado, Ludvigson, and Ng (2015) to the estimated model now results in the time series for leverage and credit spreads shown in Figure 7. As with the trade-off model, the deposit-insurance version performs well on the dynamic of credit spreads. (Although its level is too low in low volatility states, it spikes to values even higher than observed in high volatility states.) Its fit to

[17] Alternative mechanisms might include agency-induced commitment by managers to equity repurchases and dividends. Huang (2016) presents empirical evidence for such commitment in the period after 2007. In Levy and Hennessy (2007) countercyclical leverage arises from the need to maintain high managerial ownership in bad times to solve agency problems.

the observed leverage series, while having room for improvement, does appear to provide an accurate account of the 2007-2009 period.

Figure 7: Alternative Model-implied Histories



Uncertainty shocks are fed in to the fitted deposit-insurance model as described in the text. The uncertainty series is from Jurado, Ludvigson, and Ng (2015). The top panel plots model-implied credit spread series along with the actual realization from the data. The bottom panel does likewise for the model implied leverage.

Summarizing, the trade-off model is unable to account for the joint dynamics of credit spreads and leverage, and its principal shortcomings will not be solved by including financing frictions or capital-structure adjustment costs. The deposit-insurance version of the model offers one potential resolution of the empirical problems. In particular, it implies a positive relation between uncertainty and leverage. We will see that this dynamic has important implications for the real effects of debt.

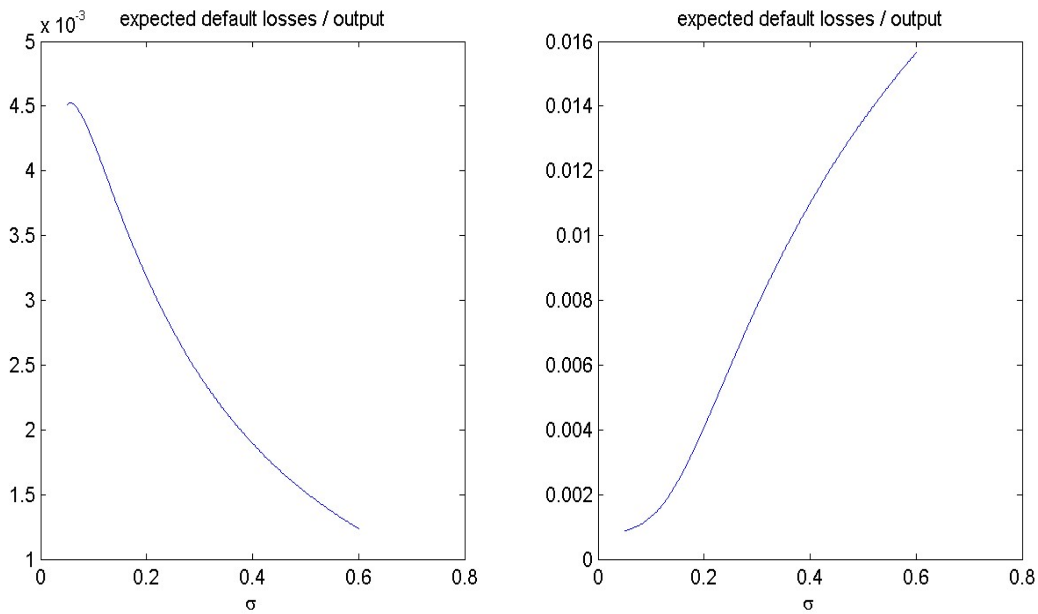
4. The Real Effects of Debt

This work was motivated by the topic of quantifying the real distortions induced by corporate credit decisions. Here I exhibit the conclusions implied by the model versions estimated in the previous section. I do so in three ways. First, I quantify how much default increases aggregate risk. Second, I show the effect of this increase in output risk on investment, and hence growth. Third, I compute the total effect on the welfare via the representative agent's value function. For each computation, the estimated economies are compared with an otherwise equal one in which the debt benefit functions have been set to zero and hence there is no leverage.

As we have seen, the key difference between the two versions of the model is that the trade-off formulation implies that leverage contracts with uncertainty whereas the default-insurance formulation implies that it expands. The left-hand panel of Figure 8 demonstrates that the former dynamic implies that the effect of defaults on output consequently shrinks as σ rises. Even though the default rate rises with σ, the rapid contraction in credit means that it takes increasingly large output jumps to trigger default. (Recall the default threshold is directly related to the firms leverage: $\ell = \exp(\varphi^{\star})$.) Hence those firms that do default in high σ states are ones whose output would have been small anyway. By contrast, in the right-hand panel we see expected aggregate default losses rise, rather than falling, with increases in risk under the alternative debt subsidy. Hence many more firms will default on a down-jump in high-σ states under this formulation, and the loss of their output is not negligible, exceeding 1%.

The top two panels of Figure 9 show the effect of default on the second moment of

Figure 8: Aggregate Default Losses

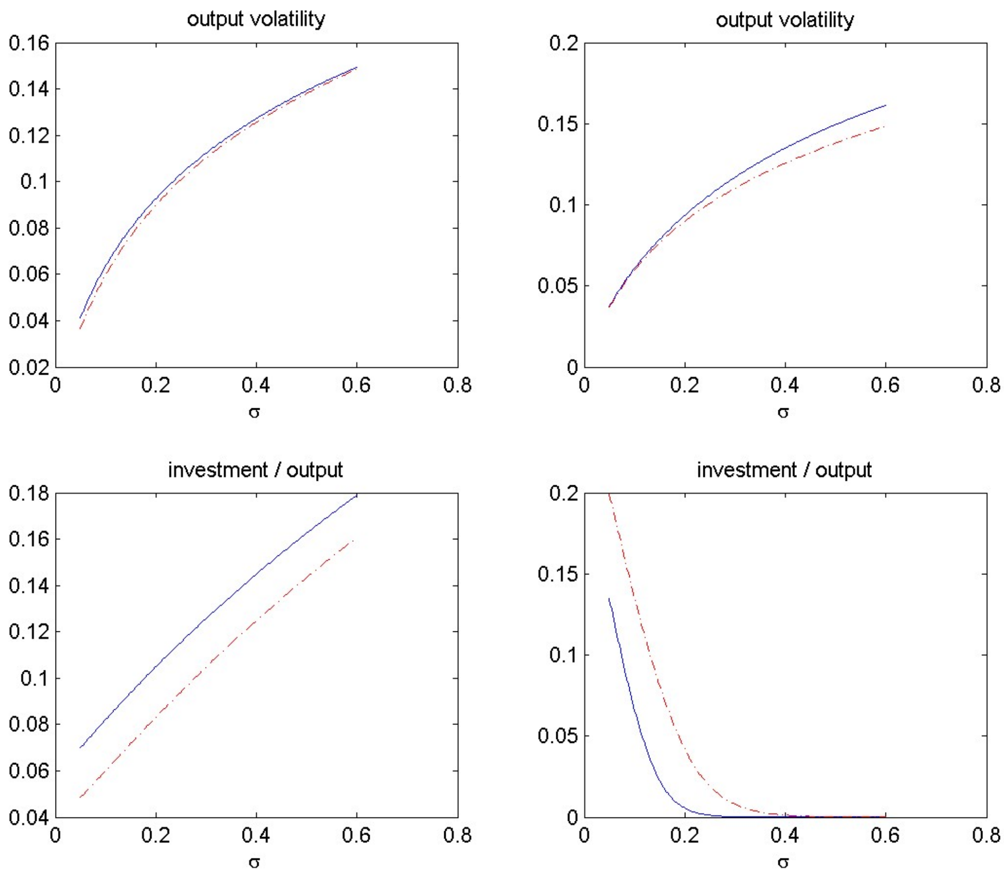


The figure shows the aggregate percentage default losses conditional on a systematic negative jump for the two versions of the model estimated in Section 3. The left panel shows the result for the trade-off model. The right panel shows the result for the default-insurance model.

the aggregate output process as compared to the no-debt equivalent economies. This is another way of quantifying the effect from the previous plots: debt substantially increases aggregate risk in the default-insurance version (top right panel) and has minor effects in the trade-off version (top left). The bottom two panels show the impact of these risk effects on aggregate investment. In the trade-off version, the effect of debt on the economy is to actually increase investment.[18] For the deposit insurance case, by contrast the extra default losses substantially inhibit investment, resulting in lower growth for the economy.

[18] This increase, and the fact that investment increases with σ (with or without debt) is a consequence of a low (< 1) estimated elasticity of intertemporal substitution for this model. All the estimated parameters are given in the appendix.

Figure 9: Real Effects of Debt

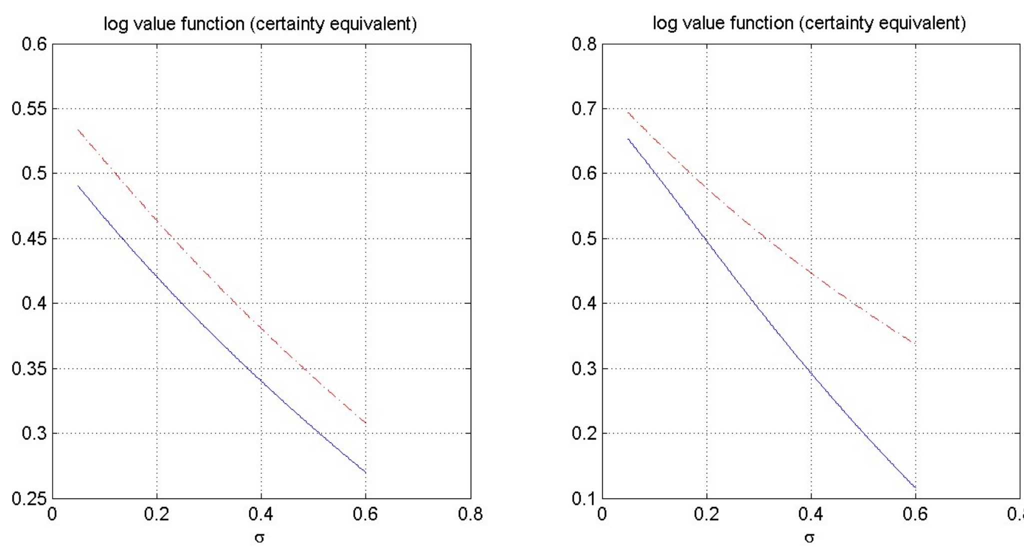


The figure shows The top panels show output volatility for the two versions of the model estimated in Section 3 plotted as solid lines. The values for equivalent economies without debt are plotted as dashed lines. The left panel corresponds to the trade-off model, and the right panel corresponds to the default-insurance model. The bottom panels show optimal investment in the respective economies.

The net effect of uncertainty and growth on the well-being of the representative agent is quantified by her value function, which is readily obtainable in terms of current income, Y, and uncertainty, σ for both versions of the model. According to Proposition 2 of Section 2, the form of the function is $j(\sigma)\, Y^{1-\gamma}/(1-\gamma)$. So the difference between two economies with the same income level can be summarized in terms of percentage income by the log certainty equivalent function $\log(j)/(1-\gamma)$.[19] Figure 10 plots the value function in these units for each version. Reading left to right, the functions steeply decline with σ under both models, indicating that times of high uncertainty are truly bad states.

The dashed line in each panel shows the function in the analogous economies with no debt. Reading along the vertical axis, the difference between the two lines is the equivalent change in permanent income between the two cases. At low levels of uncertainty, the cost of the debt distortion in similar in the trade-off and default-insurance cases. Integrating over the steady-state distribution of σ, the average welfare loss is 4.35% in the former case and 5.19% in the latter. These levels are economically significant, and indicate that even modest increases in risk induced by default losses imposes costs that would seem difficult to justify by (unmodelled) gains from subsidizing debt.

Figure 10: Welfare Effect of Debt



The figure plots the representative agent's welfare in certainty equivalent units (solid line) and in the equivalent economies without debt (dashed line). The left panel corresponds to the trade-off model, and the right panel corresponds to the default-insurance model.

The more striking conclusion from the plots is the steep rise in welfare losses under

[19] Note that $j > 0$, and in both estimated models $\gamma > 1$. Hence lower values of j imply higher (less negative) value functions.

the default-insurance version when uncertainty rises. While excursions to high values are rare, when they occur, the costs imposed by the debt subsidy rise to 10 and even 20 percent of permanent income. In this model, we see debt playing a major amplification role in terms of welfare in recessions. The leverage build up induced by debt guarantees creates a truly financial element to the stress the economy experiences. The same is not true under the trade-off model. There, leverage contracts sufficiently that the debt welfare cost is essentially the same at high and low levels of uncertainty.

Section 3 presented an empirical case – based on the positive correlation of leverage and credit spreads – for believing that the corporate sector acts as if debt incentives rise with economic uncertainty. We see here that, if this is so, these incentives embed an enormous degree of potential welfare costs in the economy.

5. Conclusion

This paper has presented a tractable general equilibrium that includes optimal capital structure decisions by firms. The primary objective was to have a way to quantify the real effects induced by corporate leverage on aggregate risk and welfare. But a preliminary step was to examine the empirical performance of a standard macro-finance model with debt in order to understand the nature of its descriptive shortcomings. In fact, a frictionless version of the trade-off model does provide a reasonable description of financial crises: uncertainty shocks lead to increases in credit spreads and discount rates, and to substantial contractions in credit.

Unexpectedly, the primary empirical difficulty of the trade-off model is that aggregate leverage of U.S. nonfinancial firms actually increases with uncertainty rather than declining. This finding is driven by increases in leverage at the peak of the last three business cycles in which corporations added debt even as output stalled and uncertainty increased. These debt build-ups cannot be explained by precautionary cash hoarding, and are not attributable to capital structure adjustment costs. Moreover, while much research has documented the real effects of financial frictions on firm outcomes, these frictions do not help resolve the puzzle at issue here. If anything, the data seem to point towards a loosening of borrowing capacity in these episodes.

To overcome the trade-off model's deficiency, I propose an alternative formulation in which the value of debt subsidies (to the firm) increases with default risk, which I view as a reduced-form depiction of deposit insurance or too-big-to-fail subsidies. Under this formulation, debt is effectively underpriced even as credit risk rises. This version explains the volatility-leverage relation, and implies a positive correlation between leverage and credit spreads.

Exploiting the tractability of the model, I estimate parameters for both debt formulations to return to the question of real effects. I find average welfare losses from the estimated models on the order of 3-5 percent of permanent income. The deposit-insurance version of the model implies, however, that things can get much worse than that – welfare losses up to 20 percent of income – in high uncertainty states. In this version, crises truly have a financial element, even without capital supply frictions.

The analysis here highlights the importance of incorporating general equilibrium effects in modeling debt dynamics, as debt decisions both affect, and are affected by, risk and marginal utility. The finding also point to the importance of understanding the true subsidies giving rise to debt, both for financial research (in asset pricing and corporate finance) and also more broadly for policy analysis that incorporates the real effects of finance on macroeconomic risk and welfare.

References

Al-Najjar, Nabil I., 2004, Aggregation and the law of large numbers in large economies, *Games and Economic Behavior* 47, 1–35.

Arellano, Cristina, Yan Bai, Patrick J Kehoe, et al., 2012, Financial frictions and fluctuations in volatility, *Federal Reserve Bank of Minneapolis Research Department Staff Report* 466.

Baxley, John V, and Sarah E Brown, 1981, Existence and uniqueness for two-point boundary value problems, *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* 88, 219–234.

Bernanke, Ben, Mark Gertler, and Simon Gilchrist, 1999, The financial accelerator in a quantitative business cycle framework, in John Taylor, and Michael Woodford, ed.: *Handbook of Macroeconomics* (North Holland: Amsterdam).

Bhamra, Harjoat Singh, Lars-Alexander Kuehn, and Ilya A. Strebulaev, 2010, The aggregate dynamics of capital structure and macroeconomic risk, *Review of Financial Studies* 23, 4187–4241.

Bigio, Saki, 2015, Endogenous liquidity and the business cycle, *The American Economic Review* 105, 1883–1927.

Brunnermeier, Markus K, and Yuliy Sannikov, 2014, A macroeconomic model with a financial sector, *American Economic Review* 104, 379–421.

Carlstrom, Charles T., and Timothy S. Fuerst, 1997, Agency costs, net worth, and business fluctuations: A computable general equilibrium analysis, *American Economic Review* 87, 893–910.

Chen, Hui, 2010, Macroeconomic conditions and the puzzles of credit spreads and capital structure, *Journal of Finance* 65, 2171–2212.

Christiano, Lawrence J, Roberto Motto, and Massimo Rostagno, 2014, Risk shocks, *American Economic Review* 104, 27–65.

Covas, Francisco, and Wouter J. den Haan, 2011, The cyclical behavior of debt and equity finance, *The American Economic Review* 101, 877–899 Papers and proceedings.

Duffie, Darrell, and Larry G. Epstein, 1992, Asset pricing with stochastic differential utility, *Review of Financial Studies* 5, 411–436.

Duffie, Darrell, and Costis Skiadas, 1994, Continuous-time security pricing: A utility gradient approach, *Journal of Mathematical Economics* 23, 107–132.

Epstein, Larry G., and Stanley E. Zin, 1989, Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* 57, 937–969.

Gilchrist, Simon, Jae W Sim, and Egon Zakrajšek, 2014, Uncertainty, financial frictions, and investment dynamics, Discussion paper National Bureau of Economic Research.

Gomes, Joao F., and Lukas Schmid, 2012, Equilibrium credit spreads and the macroeconomy, Wharton School working paper.

Gouriéroux, Christian, and Pascale Valéry, 2004, Estimation of a Jacobi process, University of Toronto working paper.

Gourio, Francois, 2013, Credit risk and disaster risk, *American Economis Journal: Macroeconomics* 5, 1–34.

Hackbarth, Dirk, Jianjun Miao, and Erwan Morellec, 2006, Capital structure, credit risk, and macroeconomic conditions, *Journal of Financial Economics* 82, 519–550.

He, Zhiguo, and Konstantin Milbradt, 2016, Dynamic debt maturity, *Review of Financial Studies* forthcoming.

Huang, Ruidi, 2016, Are open market share repurchase programs really flexible?, University of Illinois Urbana-Champaign working paper.

Jermann, Urban, and Vincenzo Quadrini, 2012, Macroeconomic effects of financial shocks, *American Economic Review* 102, 238–271.

Jurado, Kyle, Sydney C Ludvigson, and Serena Ng, 2015, Measuring uncertainty, *American Economic Review* 105, 1177–1215.

Kiyotaki, Nobuhiro, John Moore, et al., 1997, Credit chains, *Journal of Political Economy* 105, 211–248.

Levy, Amnon, and Christopher Hennessy, 2007, Why does capital structure choice vary with macroeconomic conditions?, *Journal of Monetary Economics* 54, 1545–1564.

Miao, Jianjun, and Pengfei Wang, 2010, Credit risk and business cycles, Boston University working paper.

Newey, Whitney K., and Kenneth D. West, 1987, A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix, *Econometrica* 55, 703–708.

Vazza, Diane, and Nick W. Kraemer, 2015, Default, transition, and recovery: 2014 annual global corporate default study and rating transitions, Discussion paper Standard & Poor's Ratings Services.

Appendix

A. Proofs

This appendix provides the proofs of the results in Section 2.

The first proposition solves the firm's capital structure problem before having found the pricing kernel. This is possible because we can deduce enough about aggregate dynamics in advance from the following lemmas.

The first lemma formalizes the dynamics of output deduced in the text.

Lemma 1. *Assume that firm value and optimal debt are linear linear in output – $V^{(i)} = v(\sigma)Y^{(i)}$ and $B^{(i)} = b(\sigma)Y^{(i)}$ – and that $b < v$ for all σ.*

Then aggregate output, including entry and exit effects, obeys the stochastic differential equation:

$$\frac{dY}{Y} = \mu_Y \ dt + \ d\left[\sum_{j=1}^{\mathcal{J}_t} \left((u_t - 1)1_{\{j,+\}} + (d_t - 1)1_{\{j,-\}}\right)\right]. \tag{17}$$

where $\mu_Y = (\mu + \zeta(I/Y))$ includes the growth in the mass of firms due to aggregate investment, and $1_{\{j,\pm\}}$ are indicators for the sign of the jth jump, and $u_t = u(\sigma_t)$ is

$$\mathrm{E_t}\left[e^{\varphi_j} | \varphi_j > 0\right]$$

and

$$d_t = \mathrm{E_t}\left[e^{\varphi_j} 1_{\{\varphi_j > \varphi^\star\}} | \varphi_j < 0\right] = \int_{\varphi^\star}^{0} e^{\varphi} \ d\mathcal{F}^{\varphi-}(\sigma_t)$$

where $\varphi^\star = \varphi^\star(\sigma)$ is the critical threshold for jumps below which firms default.

Proof. Formally, the probabilistic structure of the model assumes a set of *i.i.d.* firms indexed by $m \in [0, \bar{M}]$ where the interval is technically a dense limit of increasing countable subsets.[20] At time t, the set of firms that have come into existence is indexed by the subinterval

[20] Formally, the "continuum" can be described as the limit of economies with countable, increasing

$[0, M_t]$, where $M_t < \bar{M}$. For simplicity, we take M_t to be nondecreasing in t, meaning that the "mass" M_t counts firms with zero output (those that have exited). Also notice that the structure implicitly imposes that the distribution of output (or any other characteristic) of entering firms, i.e., those in $[M_t, M_t + dM_t)$, is the same as that of those that have previously entered by time t.

With this set-up, the dynamics of Y before considering entry and exit follow from a law of large numbers applied across firms at each point in time, as described in the text.

We deduce the existence of a single default-inducing jump threshold for all firms from the linearity assumption. (If no such threshold exists at t, take $\varphi^\star(t) = -\infty$.) The assumption $b < v$ implies that, absent a jump, there is no default due to changes in the aggregate state σ_t. The effect entry is to increase the mass of firms according to $dM/M = \zeta(I/Y)\ dt$ by assumption. Since the output distribution of the entering firms is the same as that of incumbents, their contribution to dY is just $Y dM$. *QED*

The next lemma characterizes the form of the economy's value function, and deduces the dynamics of consumption and marginal utility.

Lemma 2. *Given an output process described by (17), the representative agent's value function is of the form* $J = j(\sigma)\ Y^{1-\gamma}/(1-\gamma)$.

The aggregate consumption process is $C = c(\sigma)Y$. *The functions* $j(\sigma)$ *and* $c(\sigma)$ *are characterized (respectively) by an ordinary differential equation and an algebraic equation given in the proof.*

Let Λ *denote the pricing kernel. Its dynamics may be written*

$$\frac{d\Lambda}{\Lambda} = \eta_0(\sigma)\ dt + \eta_1(\sigma)\ dW + d\left[\sum_{j=1}^{\mathcal{J}_t} \big((u(\sigma)^{-\gamma} - 1)1_{\{j,+\}} + (d(\sigma)^{-\gamma} - 1)1_{\{j,-\}}\big)\right].$$

index sets, each of which is endowed with a finitely additive measure of total mass M. Al-Najjar (2004) shows that integration of random variables in the limit economy is well defined and a strong law of large numbers applies.

Proof. Given the aggregator function $f(C, J)$, the Bellman equation for J tells us that $\max_C\{\mathrm{E}[dJ] + f(C, J)\ dt\} = 0$. Under the conjectured form for $J = J(\sigma, Y)$, and using the known dynamics of σ, and Y, we have $\mathrm{E}[dJ]/J =$

$$\frac{j(\sigma)'}{j(\sigma)}\ m(\sigma) + (1-\gamma)\mu_Y(\sigma) + \tfrac{1}{2}s^2(\sigma)\ \frac{j(\sigma)''}{j(\sigma)} + \tfrac{1}{2}\lambda[(u(\sigma)^{1-\gamma} - 1) + (d(\sigma)^{1-\gamma} - 1)]$$

using the version of Itô's lemma for jumping processes. Dividing $f(C, J)$ by J and using the conjectured form of C, we get the two terms[21]

$$\beta\theta c(\sigma)^\rho j(\sigma)^{-\frac{1}{\theta}} - \beta\theta.$$

Adding these to the $\mathrm{E}[dJ]/J$ terms and multiplying by j gives the ODE:

$$\beta\theta c(\sigma)^\rho j(\sigma)^{1-\frac{1}{\theta}} - \beta\theta j(\sigma) + j(\sigma)'m(\sigma) + \tfrac{1}{2}s^2(\sigma)j(\sigma)'' + (1-\gamma)\mu_Y j(\sigma) + \tfrac{1}{2}\lambda[(u(\sigma)^{1-\gamma} - 1) + (d(\sigma)^{1-\gamma} - 1)]j = 0$$

or, more compactly,

$$\tfrac{1}{2}s^2 j'' + j'm + \beta\theta c^\rho j^{1-\frac{1}{\theta}} + \left((1-\gamma)\mu_Y + \tfrac{1}{2}\lambda[(u^{1-\gamma} - 1) + (d^{1-\gamma} - 1)] - \beta\theta\right) j = 0 \qquad (18)$$

which must hold at the optimal consumption policy. Recall that $\mu_Y = \mu + \zeta(1 - c(\sigma))$. Hence, the FOC for consumption is simply

$$\beta\theta\rho c^{\rho-1} j^{1-\frac{1}{\theta}} = \zeta'(1 - c). \qquad (19)$$

Given any smooth function $c(\sigma)$, the ODE defining j is to be solved on the closed interval $[\underline{\sigma}, \bar{\sigma}]$, and coefficient $s(\sigma)$ on the second order term is zero at the endpoints. This is equivalent to two mixed boundary conditions (i.e., a relation $g(j', j) = 0$), which suffices for existence and uniqueness of a solution. (Baxley and Brown (1981).) Then (19) is just an algebraic equation for $c(\sigma)$ given $j(\sigma)$. Formally, the implicit solution can be inserted in the coefficients of the

[21] Recall $f(C, J) = \frac{\beta C^\rho/\rho}{((1-\gamma)\ J)^{1/\theta - 1}} - \beta\theta J$.

ODE. In practice, solving the two equations iteratively rapidly yields a convergent solution for the pair of functions. The existence of these solutions verifies the conjectured functional forms.

Given these J and C functions, Duffie and Skiadas (1994) show that the pricing kernel under stochastic differential utility is

$$\Lambda_t = e^{\int_0^t \; f_J(C_u, J_u) \; du} \; f_C(C_t, J_t).$$

Here $f_C(C, J) = \beta \; c(\sigma)^{\rho-1} j(\sigma)^{1-\frac{1}{\theta}} \; Y^{-\gamma}$

The drift and diffusion coefficient of Λ can be readily evaluated (as functions of σ) by Itô's lemma and straightforward algebra, and are not of immediate interest. What is important is that $d\Lambda/\Lambda$ inherits the jump structure of $Y^{-\gamma}$, which is equivalent to the conclusion of the lemma. *QED*

We now proceed to the proof of Proposition 1. While the preceding lemma would appear to have characterized aggregate dynamics, in fact, only the *form* of the pricing kernel has been determined. What has not been pinned down are the critical jump threshold $\varphi^\star(\sigma)$ and the size of the downward jump $d(\sigma)$.

Proof of Proposition 1.

To start, assume firm value is linear in output prior to default.

The proposition first asserts that the optimal default policy for equity holders is to abandon if and only if, following a jump to $Y_t^{(i)}$, the value of the firm is below the pre-jump level of optimal debt, $B_{t-}^{(i)}$. If equity holders do not abandon, then their optimal debt policy at t is to adjust to $B_t^{(i)}$. If they do so, they repay the difference $B_{t-}^{(i)} - B_t^{(i)} > 0$ to debt holders, and their claim is now worth $V_t^{(i)} - B_t^{(i)}$. Clearly they will do this if and only if the debt repayment is less than the value they receive:

$$V_t^{(i)} - B_t^{(i)} > B_{t-}^{(i)} - B_t^{(i)} \iff V_t^{(i)} > B_{t-}^{(i)}$$

as asserted.

From this observation, it follows that we can link the optimal leverage ratio prior to a jump with the critical default threshold. Default occurs iff $V_t^{(i)} \leq B_{t-}^{(i)}$. So dividing by $B_{t-}^{(i)}$, we have

$$e^{\varphi^\star} \equiv \frac{Y_t^{(i),\star}}{Y_{t-}^{(i)}} = \frac{V_t^{(i),\star}}{V_{t-}^{(i)}} = \frac{B_{t-}^{(i)}}{V_{t-}^{(i)}}$$

where the second inequality uses the conjectured linearity. Since the left side here is only a function of the aggregate state, the linearity of $V^{(i)}$ thus implies that of $B^{(i)}$. Denote the optimal market leverage ratio ℓ. Then we have characterized the optimal bankruptcy barrier given optimal leverage as

$$e^{\varphi^\star} = b(\sigma)/v(\sigma) = \ell(\sigma). \tag{20}$$

The value of the ith firm is characterized by the condition

$$\mathrm{E}[d\Lambda V^{(i)}]/(\Lambda V^{(i)}) = -((1-\tau)Y^{(i)} + \bar{r}\tau B^{(i)})\,dt/V^{(i)}.$$

The numerator on the right is the firm's after-tax earnings when interest deduction is permitted at the statutory rate $\bar{r}$, and the tax rate is τ.

Let us conjecture that, prior to default, $V^{(i)} = v(\sigma)(1-\tau)Y^{(i)}$. Given the form of the pricing kernel, applying Itô's lemma to the left side of the above condition gives the equation

$$\tfrac{1}{2}s^2(\sigma)v'' + [m(\sigma) + \eta_1(\sigma)s(\sigma)]v' + \big(\eta_0 + \mu + \tfrac{1}{2}\lambda[(u^{1-\gamma}-1) + (d^{1-\gamma}-1) + \bar{r}\tau\ell(\sigma)]\big)\,v + 1 = 0. \tag{21}$$

As with the j equation above, existence and uniqueness of a solution to this equation will verify the linearity conjecture.

Now we consider the first order condition that maximizes v with respect to b, or, equivalently, with respect to ℓ.

Differentiating (21), there are contributions from the benefit flow term as well as from the down-jump term $d^{1-\gamma}-1$. The latter term is the expectation of the percentage jump in the product $\Lambda V^{(i)}$. The ratios $V_t^{(i)}/V_{t-}^{(i)}$ and Λ_t/Λ_{t-} are independent given a jump, and the pricing kernel term is $d^{-\gamma}$. The firm takes this component as given and not affected by its default decision. However, the jump in own-firm value is affected. Hence we differentiate

$$\int_{\varphi^\star}^{0} e^{\varphi}\ d\mathcal{F}^{\varphi-}$$

and multiply by $d^{-\gamma}$. From above, we know $\varphi^\star = \log(\ell)$. Differentiating this and using the chain rule gives the FOC as

$$\tfrac{1}{2}\lambda d^{-\gamma}\ f^{\varphi-}(\varphi^\star) = \bar{r}\tau$$

where $f^{\varphi-}$ is the density function of the negative jumps. And from Lemma 1,

$$d = \int_{\varphi^\star}^{0} e^{\varphi}\ d\mathcal{F}^{\varphi-}.$$

The preceding two equations form a system whose solutions are d and $\varphi^\star$. This closes the problem. It is easy to see that the first equation describe a locus of points d that is monotonically increasing from zero in $|\varphi^\star|$. The second describes a locus that monotonically decreases to zero as long as the density function does so, which has been assumed. Hence the system has a unique interior solution. (The fact that $\varphi^\star = 0$ is not a solution verifies the assertion that $b < v$ for all σ. That is, jumps alone can trigger default.)

So far, the derivation has assumed that leverage would be chosen to maximize the value of the firm. The proposition also asserts that resulting policy would also followed by managers who could not commit to maximizing firm value, and instead maximized the value of equity. Intuitively, this is a consequence of the stipulation that the price, p, of the debt contract per unit face value is always one, which implies that no policy can expropriate value from existing debt holders.

Formally, if the firm is at the firm-value maximizing value, policy pair V', B' then equity holders can costlessly move to any V'', B'' by paying (or receiving if negative) the difference in debt amounts $B' - B''$. Including this payment, equity holders will have achieved net value $V'' - B'$. But, by assumption, this is strictly less than the original value they had, $V' - B'$.

QED

Proof of Corollary 2.1

Let P denote the value of an arbitrary debt contract and $p = P/B$ be its price per unit face value. Let τ denote the sooner of the firm's default time and the repayment time of the contract. (The debt contract considered in the paper has no formal maturity. However, the firm has the right to alter the amount outstanding costlessly at any time. We can consider a repayment of amount ΔB as applying *pro rata* randomly across bonds. So any individual bond can be considered to have a stochastic retirement time.) Then on $[0, \tau)$, p solves the valuation equation

$$\tfrac{1}{2}s^2(\sigma)p'' + [m(\sigma) + \eta_1(\sigma)s(\sigma)]p' + \left(\eta_0 + \tfrac{1}{2}\lambda[(u^{-\gamma}\mathrm{E}_t\left[\frac{p^+}{p}\right] - 1) + (d^{-\gamma}\mathrm{E}_t\left[\frac{p^-}{p}\right] - 1)]\right)p + \Gamma = 0. \tag{22}$$

where Γ is the coupon rate and $\frac{p^+}{p}$ and $\frac{p^-}{p}$ denote the fractional changes in p conditional on an up and down jump, respectively.

We require that Γ be set such that $p = 1$ solves this equation. And we are assuming $p = 0$ on default. In that case, the equation reads

$$\eta_0 + \tfrac{1}{2}\lambda[(u^{-\gamma} - 1) + (d^{-\gamma}\mathcal{F}^{\varphi-}(\varphi^\star) - 1)] + \Gamma = 0.$$

or

$$\eta_0 + \tfrac{1}{2}\lambda[(u^{-\gamma} - 1) + (d^{-\gamma} - 1)] - \tfrac{1}{2}\lambda d^{-\gamma}(1 - \mathcal{F}^{\varphi-}(\varphi^\star))] + \Gamma = 0.$$

We then recognize that the first two term are the drift rate of the pricing kernel, Λ, which is equal to minus the instantaneous riskless rate, r. Hence,

$$\Gamma = r + \tfrac{1}{2}\lambda d^{-\gamma}(1 - \mathcal{F}^{\varphi-}(\varphi^\star)).$$

QED

Proof of Corollary 2.2

The assumption now is that, creditors of a firm that has defaulted receive a payment ΘB_{t-} where B_{t-} is the face value of debt prior to default. The government does not have the ability to create the value lost due to default, however. Those losses create the same decline in aggregate consumption as in the base case. (So implicitly a tax on all households must fund the creditors' insurance payout.)

To derive the effect on optimal capital structure, we revisit the equation (21) for firm value. Previously, the contribution from the expected change in $\Lambda\, V^{(i)}$ from down jumps was $\frac{1}{2}\lambda$ times

$$d^{-\gamma} \int_{\varphi^\star}^{0} e^{\varphi}\, d\mathcal{F}^{\varphi-}\; - 1.$$

Now there is an additional contribution to the left-hand term from the default insurance that creditors collect:

$$d^{-\gamma}\left[\int_{\varphi^\star}^{0} e^{\varphi}\, d\mathcal{F}^{\varphi-} + \Theta \frac{B^{(i)}}{V^{(i)}} \int_{-\infty}^{\varphi^\star} d\mathcal{F}^{\varphi-}\right].$$

Differentiating the new term with respect to $\ell = B^{(i)}/V^{(i)}$ adds the two terms

$$\Theta \int_{-\infty}^{\varphi^\star} d\mathcal{F}^{\varphi-} + \Theta\, f^{\varphi-}(\varphi^\star).$$

So the full FOC becomes

$$\tfrac{1}{2}\lambda\ d^{-\gamma}\ [(1-\Theta)\ f^{\varphi-}(\varphi^\star) - \Theta\ \mathcal{F}^{\varphi-}(\varphi^\star)] = \bar{r}\tau.$$

As in Proposition 1, this FOC can be solved jointly with the equation $d = \int_{\varphi^\star}^{0}\ e^{\varphi}\ d\mathcal{F}^{\varphi-}$.

Besides altering the optimal leverage, the firm value equation must be solved with the extra term given above. In addition, the solution for the credit spread picks up a factor of $(1-\Theta)$. *QED*

Proposition 2 now simply finishes the characterizations of the quantities in Lemma 2 above. Now that d and $\varphi^\star$ have been determined explicitly, the coefficients in the differential equation for $j(\sigma)$ and the algebraic equation for $c(\sigma)$ are fully specified. The proof just finishes the description of the pricing kernel.

Proof of Proposition 2

The Lemma determined that

$$\Lambda_t = e^{\int_0^t\ f_J(C_u, J_u)\ du}\ f_C(C_t, J_t),$$

and

$$f_C(C, J) = \beta\ c(\sigma)^{\rho-1} j(\sigma)^{1-\frac{1}{\theta}}\ Y^{-\gamma}.$$

Denote the product of c and j terms in this expression as $a(\sigma)$. Also, after some cancellations,

$$f_J(C, J) = \beta\theta \left[(1-\frac{1}{\theta})\ c(\sigma)^{\rho} j(\sigma)^{-\frac{1}{\theta}} - 1\right].$$

The task is to evaluate $d\Lambda/\Lambda$. The integral term just contributes an f_J term to the drift. To this we add df_C/f_C, which is

$$\left[\tfrac{1}{2}\frac{a''}{a}s^2 + \frac{a'}{a}m + \mu_Y\right]\ dt + \frac{a'}{a}s\ dW + d\left[\sum_{j=1}^{J_t}\left((u^{-\gamma}-1)1_{\{j,+\}} + (d^{-\gamma}-1)1_{\{j,-\}}\right)\right].$$

The diffusion coefficient here is $sa'/a = s[(\rho-1)c'/c + (1-1/\theta)j'/j]$, which is called η_1 in the Proposition. Likewise η_0 is the drift term plus f_J. The full expression for a''/a is omitted for brevity. The expression in the proposition for riskless rate is just minus the drift of $d\Lambda/\Lambda$. *QED*

Likewise, there is nothing formally to prove for Proposition 3, since the proof of Proposition 1 already deduced the ODE solved by $v(\sigma) = V^{(i)}/(1-\tau)Y^{(i)}$. There it was only necessary to observe its form in order to take the first order condition for optimal debt. Now that the kernel and the debt policy have been explicitly obtained, the ODE is fully specified and (as observed above) a unique solution exists. We can redefine v to be that solution times $(1-\tau)$ to obtain the solution in terms of pre-tax output $V^{(i)} = v(\sigma)Y^{(i)}$.

The following corollary computes the risk premia for the firm's claims.

Corollary A.1. *The expected excess return to the firm's assets is*

$$\pi_V = -\frac{v'}{v}s\eta_1 + \tfrac{1}{2}\lambda\left((u-1)+(d-1)+(u^{-\gamma}-1)+(d^{-\gamma}-1)-(u^{1-\gamma}-1)-(d^{1-\gamma}-1)\right).$$

The expected excess return to the firm's debt is

$$\pi_F = \tfrac{1}{2}\lambda\ (d(\sigma)^{-\gamma}-1)\ (1-\mathcal{F}^{-\varphi}(\varphi^{\star})).$$

The expected excess return to the firm's equity, π_E is given by the solution to

$$\pi_V = \frac{1}{1-\ell}\pi_E + \frac{\ell}{1-\ell}\pi_F.$$

Proof of Corollary

The valuation ODE for V equates

$$\tfrac{1}{2}\frac{v''}{v}s^2 + \frac{v'}{v}m + \mu + \frac{1-\tau}{v} + \bar{r}\tau\frac{b}{v} + \eta_0$$

to

$$-\left(\eta_1 s\frac{v'}{v} + \tfrac{1}{2}\lambda((u^{1-\gamma}-1) + (d^{1-\gamma}-1)\right).$$

If we add to each side $\frac{1}{2}\lambda((u^{-\gamma}-1)+(d^{-\gamma}-1))$ we can then substitute out the sum of these terms and η_0 for $-r$ in the top expression. Then add to each side $\frac{1}{2}\lambda((u-1)+(d-1))$ and the top expression becomes the (true) expected excess returns to dV/V. We conclude

$$\pi_V = -\frac{v'}{v}s\eta_1 + \tfrac{1}{2}\lambda\left((u-1) + (d-1) + (u^{-\gamma}-1) + (d^{-\gamma}-1) - (u^{1-\gamma}-1) - (d^{1-\gamma}-1)\right).$$

which we can also write as

$$\pi_V = -\frac{v'}{v}s\eta_1 + \tfrac{1}{2}\lambda\left[(u-1)(u^{-\gamma}-1) + (d-1)(d^{-\gamma}-1)\right].$$

The debt contract has no expected change per unit time, outside of default. So its expected excess return is the coupon rate minus the riskless rate plus the instantaneous default intensity. But this is just the difference between the credit spread, determined above, (which is also the risk neutral default intensity) and the true default intensity, giving the expression in the corollary. Finally, by no arbitrage, the risk premium on the firm is the value weighted combination of debt and equity claims. This is the last assertion in the corollary.

QED

B. Data

This appendix describes the data and estimation procedure used in Section 3.

Data

The model has several simplifying assumptions about firms and the economy that make choice of empirical counterparts somewhat subjective. The list below discusses the proxies chosen and some possible alternatives.

Leverage:

The quantity b in the model is firm's debt face vale divided by (pre-tax) cash-flow, or output. In the data, I thus need to choose pairs of (debt,output) measures that correspond to the same set of firms. The firm is supposed to be representative of the entire economy. The broadest measure, and the main proxy used, is from the Federal Reserves Z1 reports (The flow-of-funds accounts) for the U.S. nonfinancial corporate sector. Specifically debt is bank loans and bonds (long and short term), and output is net operating surplus plus consumption of fixed capital. These series are available quarterly from 1951:Q4.

For robustness, I also examine the same series netting cash and cash equivalents out of the numerator (specifically, checkable, time, and foreign deposits, holding of money market mutual funds, and Treasury securities). To check robustness to the inclusion of off-balance sheet liabilities in total debt, another version adds retirement entitlements (pensions and healthcare liabilities) to the numerator.

For further robustness, I also consider two other pairs of (debt,output) series. The first, also from the flow of funds accounts, includes the noncorporate sector, meaning primarily private firms, partnerships, and proprieterships. The debt measure is constructed from the same variables. The corresponding output measure is now taken to be the non-farm business GDP number from NIPA Table 1.3.5. This series is available from 1969:Q1.

To address concerns that aggregate data is dominated by large firms, a final alternative leverage measure is constructed from median firm values in quarterly Compustat data. Specifically, for all nonfinancial firms with a reporting quarter ending within each calendar quarter, I compute net debt as long term debt total plus debt in current liabilities minus cash and short term investment. Cash-flow is operating profits before interest, depreciation, and taxes. I then take the median value across firms of the ratio of net debt to cash-flow. Firms for which the denominator is non-positive are excluded, as are firms missing any of the numerator items. The quarterly Compustat series is available from 1976:Q1.

Credit spread:

In choosing a credit spread series, the main consideration is, again, that the model speaks to a firm representative of the entire corporate sector. The question then is how to define the average creditworthiness of firms overall.

Based on the average default rate of the entire private sector (see below) and long-term default frequencies from S&P by rating, the representative firm in the economy appears to be approximately of credit grade BB or BBB. The natural candidate to measure credit in this range is the time-series of seasoned Baa-rated yields-to-maturity from Moody's that goes back to 1916. (A Moody's Baa rating corresponds to an S&P BBB rating.) According to Moody's, this series is for debt with maturity of at least 20 years. So I subtract the constant maturity 20-year Treasury yield from the Federal Reserve Bank of St. Louis (FRED), interpolating between 10 and 30 year yields when the 20 year series in unavailable.[22] I measure

[22] Choosing a series that fixes the rating level over time does impose a measurement bias because it misses fluctuations in the population credit quality. Intuitively, the effect of this bias should be straightforward: it should mean that fluctuations in credit spreads are understated.

both yields at the end of calendar quarters. The Treasury yields are available from 1953:Q1.

For comparison with other works in the asset pricing literature, I also consider the credit spread defined as the difference between Baa yields and Aaa yields. Some authors have viewed Treasury bonds as an inappropriate benchmark because of potential liquidity premia or tax effects embedded in their prices.

To address the concern that the firm's borrowing cost in the model is for floating-rate debt and should therefor correspond to a short-maturity interest rate, I also construct a credit spread based on commercial paper yields. The main drawback with this series is that commercial paper is only issued by large high-quality borrowers, making its spread unrepresentative. Commercial paper rates are obtained from FRED. I concatenate separate pre- and post-1998 series for 90-day maturity. (The post-1998 series is for A2/P2 rated nonfinancial issuers. The earlier series does not specify the issuer type.) The CP-TB spread is defined as the difference between this rate and the current 3-month Treasury bill rate, also from FRED.

Default rate:

To assess representative borrower quality in the U.S. corporate sector, I obtain a time-series of annual total bankruptcy filings by U.S. firms for 1981-2015 from the American Bankruptcy Institute. [23] I divide total bankruptcies by the total number of firms in the U.S. from the Statistics of U.S. Businesses [24] compiled from the U.S. Census Bureau's Survey of Business Ownership. The latter series are available from 1988-2012. I average the annual ratio of the two numbers to obtain the unconditional default frequency 0.0087 used in the estimation.

For comparison to rated bond issuers, average global default rates by rating and

[23] http://www.abi.org/newsroom/bankruptcy-statistics
[24] https://www.sba.gov/advocacy/firm-size-data#susb

issuer type are available for 1981-2014 are obtained from S&P's Global Corporate Default Study (Vazza and Kraemer (2015)). For such issuers, the average one-year default rate for nonfinancial firms worldwide over this period is 0.0181 (Table 16).

Investment rate:

In the model, investment is made directly by households through their savings decisions. Therefore I measure average investment as the personal savings rate (savings as a fraction of disposable household income) from NIPA Table 2.1. The value used in the estimation is the average of annual rates from 1980-2015.

Equity valuation:

The model's equity valuation as a fraction of output is again supposed to be representative of the entire economy. The flow of funds tables include market valuation of equity (less intercompany holdings) for the nonfinancial corporate public sector. This is value is divided by the cashflow series constructed as described above. The value used in the estimation is the average of quaterly ratios from 1980:Q1-2015:Q1.

A key quantity in assessing the models examined in the text is the correlation between credit spreads and leverage. To get a sense of the robustness of the finding of a positive correlation, Table A1 shows the correlations for pairwise combinations of the measures described above.

Estimation

The two models in Section 2 are estimated by minimum-square-error criterion applied to the moments (or statistics) listed in the text's Table 1, with one exception. Instead of targeting the default rate itself, the estimation targets the ratio of the credit spread to the default rate in order to better identify the credit risk premium. In addition, since the trade-off model assumes zero recovery on debt, the estimation for that specification deflates the model's value by an average loss factor (one minus recovery rate) of 0.6 for comparison with the empirical counterpart.

Model moments at each point in the parameter space are computed by sampling from the stationary distribution of the σ process The squared error for each moment is expressed as a percentage deviation.

The estimation fixes six of the parameters to be the same in both models. The jump intensity is held constant at $\lambda = 1$ for ease of interpretation, e.g., so that jump magnitudes can be viewed as expected annual rates. The jump shape parameter L is fixed at 4.0 and the scale of the production function ζ_1 is fixed at 0.975. These parameters are poorly identified by the data moments. For comparability across specifications, the upper and lower bounds of the state variable are held fixed at $\sigma_l = 0.05, \sigma_u = 0.60$. Finally, the tax-rate is fixed at 0.30, since it is not really a free parameter.

The resulting estimates for the remaining parameters are given in Table A2 for each specification.

Table A1: Leverage Credit Spread Correlations

	Baa-20yr	Baa-Aaa	CP-TB
Nonfinancial corporate debt/cashflow	0.5229	0.2595	0.0278
minus cash	0.5641	0.3191	0.0603
plus retirement liabilities	0.4997	0.2568	0.0587
	1969:Q1-2015:Q1	1969:Q1-2015:Q1	1980:Q1-2015:Q1
Nonfinancial corporate+noncorporate net debt/			
non-farm business GDP	0.3237	0.1025	0.2120
	1969:Q1-2015:Q1	1969:Q1-2015:Q1	1980:Q1-2015:Q1
Median Compustat nonfinancial net debt/cashflow	0.1187	0.1383	0.1248
	1976:Q1-2015:Q2	1976:Q1-2015:Q2	1980:Q1-2015:Q1

Table A2: Parameter Estimates

Parameter		Model:T-O	Model:D-I
Risk aversion	γ	7.7074	8.4004
E.I.S.	ψ	0.3927	3.8295
Subjective discount rate	β	0.0595	0.1292
Interest deduction rate	$\bar{r}$	0.1246	0.0130
Debt recovery/insurance rate	θ	0.4000	0.4013
Production function curvature	ζ_0	0.1465	0.0808
Output growth constant	μ	0.0577	0.0814
Uncertainty mean	$\bar{\sigma}$	0.0925	0.0988
Uncertainty mean-reversion	κ	0.2271	0.1808
Uncertainty diffusion	s_0	0.2599	0.1805