Bakshi, Kapadia, and Madan (2003) Risk-Neutral Moment Estimators*

Pakorn Aschakulporn†
Department of Accountancy and Finance
Otago Business School, University of Otago
Dunedin 9054, New Zealand
beam.aschakulporn@otago.ac.nz

Jin E. Zhang
Department of Accountancy and Finance
Otago Business School, University of Otago
Dunedin 9054, New Zealand
jin.zhang@otago.ac.nz

First Version: 11 April 2019
This Version: 14 November 2019

Keywords: Risk-neutral moment estimators
JEL Classification Code: G13

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* Jin E. Zhang has been supported by an establishment grant from the University of Otago and the National Natural Science Foundation of China grant (Project No. 71771199).

† Corresponding author. Tel: +64 21 039 8000.

Bakshi, Kapadia, and Madan (2003) Risk-Neutral Moment Estimators

Abstract

This is the first study of the errors of the Bakshi, Kapadia, and Madan (2003) risk-neutral moment estimators with the density of the underlying explicitly specified. This was accomplished using the Gram-Charlier expansion. To obtain skewness with (absolute) errors less than 10^{-3}, the range of strikes $(K_{\min}, K_{\max})$ must contain at least 3/4 to 4/3 of the forward price and have a step size (ΔK) of no more than 0.1% of the forward price. The range of strikes and step size corresponds to truncation and discretisation errors, respectively.

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1 Introduction

Risk-neutral moment estimators are functions which convert option prices to the moments of the underlying asset. These estimators are much more complicated than the estimators for physical moments, as physical moments can be obtained from an underlying asset's return using standard statistical methods. The moments most used are the first and second moments, which correspond to the mean and variance, respectively. Higher (standardised) moments include the third and fourth moments which correspond to the skewness and kurtosis, respectively. These moments can be used to form a basis of understanding of the behaviour of the asset. For a more practical use, the third risk-neutral moment for the S&P500 is the basis for the Chicago Board Options Exchange (CBOE) skewness (SKEW) index. The SKEW[1] was designed to capture the tail-risk. The method used to calculate the risk-neutral moments in the CBOE SKEW is Bakshi, Kapadia, and Madan (2003), which will now be referred to as BKM. As with all numerical calculations the BKM calculations too have errors. This paper examines the error and convergence of BKM's risk-neutral skewness and kurtosis estimators and finds the region in which the errors of the skewness estimator is bounded by 10^{-3}.

The BKM is an extension of Demeterfi, Derman, Kamal, and Zou (1999) and Carr and Madan (2001b) which propose methods to price variance swaps via static replication using options. Volatility swaps (viz. realized volatility forward contracts) provide pure exposure to volatility in contrast to trading volatility through stock options which are impure as it is contaminated by the option's dependence on the stock price. This underpins the CBOE VIX, the volatility index of the S&P500. The BKM method now provides the foundation for recent literature with regard to risk-neutral moment estimators. Neumann and Skiadopoulos (2013), Conrad, Dittmar, and Ghysels (2013), Chang, Christoffersen, and Jacobs (2013), and Stilger, Kostakis, and Poon (2017) studied the time-series relationship between BKM risk-neutral moments and the equity market. Cheng (2018)

[1] http://www.cboe.com/products/vix-index-volatility/volatility-indicators/skew

studied the time-series relationship between BKM risk-neutral moments and the VIX. Chatrath, Miao, Ramchander, and Wang (2016) and Ruan and Zhang (2018) studied the time-series relationship between BKM risk-neutral moments and the crude oil market. Christoffersen, Jacobs, and Chang (2011) summarises various risk-neutral estimators including those for volatility, skewness, and kurtosis with respect to their underlying asset for various markets. Bakshi and Madan (2006) study the spread between risk-neutral and physical volatilities of the S&P100 index. Liu and Faff (2017) propose a forward-looking market symmetric index which is used to compete with the CBOE SKEW and therefore the BKM method.

Liu and van der Heijden (2016) and Lee and Yang (2015) study the errors of the BKM method when the true value of the risk-neutral moment is unknown. Liu and van der Heijden (2016) uses various option pricing models to create their benchmark risk-neutral moments. This was done using Monte Carlo simulations on the Black-Scholes-Merton model (Black and Scholes, 1973; Merton, 1973), Heston stochastic volatility model (Heston, 1993), Merton jump-diffusion model (Merton, 1976), and Bates stochastic volatility jump-diffusion model (combination of Merton and Heston models) (Bates, 1996). As simulations are used, the benchmark is expected to converge to the true value as more simulations are made. Lee and Yang (2015) uses historical information to calibrate their benchmark Black-Scholes model and uses the benchmark to compare with the Bates stochastic volatility jump-diffusion model. The values of the risk-neutral estimators are not calculated, instead, the errors are determined relative to the benchmark. The Black-Scholes model has also been used as a benchmark by Dennis and Mayhew (2002) and Jiang and Tian (2005). Both Liu and van der Heijden (2016) and Lee and Yang (2015) study the errors of the BKM method with little control over the inputs. They, therefore, only have approximations of the true moments and introduces additional errors into their analysis.

In this paper, the benchmark used is created following Zhang and Xiang (2008) in using the Gram-Charlier series to create virtual options. These options are created by specifying parameters such as the skewness and excess kurtosis (hereinafter, kurtosis). The Gram-Charlier series has been used in many papers, for example, Backus, Foresi, and Wu (2004), Longstaff (1995), and Jarrow and Rudd (1982). Using these options, the BKM estimators are applied and compared to the true moments used to create the options. Both the BKM methodology and Gram-Charlier series can be used to calculate the VIX. In addition to the moments, these two methods are also compared.

The remainder of this paper is organized as follows. Section 2 presents the method used to create virtual options, a brief dive into how BKM determines the risk-neutral skewness, and the method used to quantify and analyse the errors and convergence of the BKM method. Section 3 describes the data. Section 4 provides the numerical results and Section 6 concludes. The appendix gives the details of key derivations.

2 Methodology

2.1 Creating Virtual Options

To test the BKM higher-order risk-neutral moment estimators, virtual options with known properties can be created using Zhang and Xiang (2008) as a basis. The options are created based on the probability density function of the return of the underlying asset. The density used in basic models tends to be based on the normal distribution function. However, the returns of stocks are known to have both skewness and kurtosis which are not modelled by the symmetric normal distribution. The Gram-Charlier series can be used to create a new density based on the normal distribution function with specified skewness and kurtosis. This series is similar to the Taylor series; however, it also combines probability densities by using Hermite polynomials. The Gram-Charlier series used is given by

$$f(y) = n(y) - \frac{\lambda_1}{3!}\frac{d^3 n(y)}{dy^3} + \frac{\lambda_2}{4!}\frac{d^4 n(y)}{dy^4}, \tag{1}$$

where $n(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$, λ_1 is the skewness, and λ_2 is the kurtosis. The density has a mean, variance, skew, and kurtosis of 0, 1, λ_1, and λ_2, respectively.

[Insert Figure 1 about here.]

Figure 1 shows the region of skewness and kurtosis, which ensures a valid density function. The derivation of this region is shown in appendix A.

The stock price at maturity, S_T, can be modelled by

$$S_T = F_t^T e^{\left(-\frac{1}{2}\sigma^2+\mu_c\right)\tau+\sigma\sqrt{\tau}y}, \tag{2}$$

where F_t^T, σ, τ, and μ_c is the forward price, standard deviation, time to maturity $(T-t)$, and convexity adjustment term, respectively. The forward price, F_t^T, is related to the current stock price (S_t) by $F_t^T = S_t e^{(r-q)\tau}$, where q is the continuous dividend rate. The convexity adjustment term, μ_c, is required to keep this model in the risk-neutral world (by ensuring that the stock price satisfies the martingale condition). From this, the price of a European call option will be

$$c_t = e^{-r\tau} E_t^{\mathcal{Q}} \left[\max(S_T - K, 0)\right]. \tag{3}$$

In terms of $F_t^T, K, \tau, r, \sigma, \lambda_1$, and λ_2, the price is

$$c_t = F_t^T e^{-r\tau} N(d_1) - K e^{-r\tau} N(d_2) + K e^{-r\tau} \left(\frac{\lambda_1}{3!} A + \frac{\lambda_2}{4!} B\right) \sigma\sqrt{\tau} \tag{4}$$

where

$$\begin{aligned}
A &= -\left(d_2 - \sigma\sqrt{\tau}\right) n(d_2) \\
B &= -\left(1 - d_2^2 + \sigma\sqrt{\tau} d_2 - \sigma^2\tau\right) n(d_2) \\
d_2 &= \frac{\ln(F_t^T/K) + \left(-\frac{1}{2}\sigma^2 + \mu_c\right)\tau}{\sigma\sqrt{\tau}}, d_1 = d_2 + \sigma\sqrt{\tau}
\end{aligned}$$

For a given set of parameters ($F_t^T, r, \sigma, \lambda_1$, and λ_2), the prices for each virtual option can be created for a range of strikes, K, and maturities, τ. The derivation of Equation (4) is shown in appendix B. European put options can be found using the put-call parity

$$c_t - p_t = F_t^T e^{-r\tau} - K e^{-r\tau}. \tag{5}$$

With these virtual European put and call options, the BKM skewness and kurtosis estimators can be examined and compared against the true skewness (λ_1) and true kurtosis (λ_2), respectively.

2.2 Calculating BKM Skew and Kurtosis

The BKM method calculates the risk-neutral (return) skewness ($n = 3$) and kurtosis ($n = 4$) using the basic normalized nth central moments (viz. standardised moments) equation

$$n\text{th standardised moment} = \frac{E_t^{\mathcal{Q}}\left[\left(R\left(t,\tau\right) - E_t^{\mathcal{Q}}\left[R\left(t,\tau\right)\right]\right)^n\right]}{\left\{E_t^{\mathcal{Q}}\left[\left(R\left(t,\tau\right) - E_t^{\mathcal{Q}}\left[R\left(t,\tau\right)\right]\right)^2\right]\right\}^{\frac{n}{2}}} \tag{6}$$

where $R\left(t,\tau\right) \equiv \ln\left[S_T\right] - \ln\left[S_t\right]$, the log returns. However, BKM makes a small approximation to the mean and sets dividends, q, to zero:

$$E_t^{\mathcal{Q}}\left[R\left(t,\tau\right)\right] \approx \mu(t,\tau) = e^{r\tau} - 1 - \frac{1}{2}E_t^{\mathcal{Q}}\left[R\left(t,\tau\right)^2\right] - \frac{1}{3!}E_t^{\mathcal{Q}}\left[R\left(t,\tau\right)^3\right] - \frac{1}{4!}E_t^{\mathcal{Q}}\left[R\left(t,\tau\right)^4\right] \tag{7}$$

The approximation is due to the exclusion of higher-order terms in the expansion of the exponential function. This approximation serves in a similar manner to the convexity adjustment term in the addition of higher-order moments, for the BKM method, this is in the form of volatility ($R\left(t,\tau\right)^2$), cubic ($R\left(t,\tau\right)^3$), and quadratic ($R\left(t,\tau\right)^4$) payoff contracts. These contracts are not standard; therefore, to calculate their values, the contracts are decomposed into standard, European options, bonds, and shares. Equation 1 of Carr and Madan (2001a) shows that any twice-differentiable payoff function with bounded expectation can be spanned by a continuum of out-of-the-money (OTM) European options, bonds, and shares. For payoff function $H\left(x\right) \in \mathscr{C}^2$ and some constant x_0, the decomposed payoff function is given by

$$H(x) = H(x_0) + H_x(x_0)(x - x_0) + \int_0^{x_0} H_{xx}(K) \max(K - x, 0)\, dK \\ + \int_{x_0}^{\infty} H_{xx}(K) \max(x - K, 0)\, dK \tag{8}$$

Following BKM in using Equation (8), setting $S_T = S$, the dependent variable, and setting x_0 to S_t, the following payoff

$$H(S) = \begin{cases} R(t,\tau)^2 & \text{volatility contract } (V) \\ R(t,\tau)^3 & \text{cubic contract } (W) \\ R(t,\tau)^4 & \text{quartic contract } (X) \end{cases} \tag{9}$$

results in $H(S_t) = 0$, $H_x(S_t) = 0$, and

$$H_{xx}(K) = \frac{n}{K^2}\left[(n-1)\left[\ln\left(\frac{K}{S_t}\right)\right]^{n-2} - \left[\ln\left(\frac{K}{S_t}\right)\right]^{n-1}\right]. \tag{10}$$

Finally, the expected value of each contract is given by

$$E_t^{\mathcal{Q}}\left[e^{-r\tau} R(t,\tau)^n\right] = \int_0^{\infty} \frac{n}{K^2}\left[(n-1)\left[\ln\left(\frac{K}{S_t}\right)\right]^{n-2} - \left[\ln\left(\frac{K}{S_t}\right)\right]^{n-1}\right] Q(K)\, dK \tag{11}$$

where n specifies the type of power contract and $Q(K)$ corresponds to the OTM option with strike K. If there exists both put and call at the at-the-money point, then the average of the two is taken. $n = \{2, 3, 4\}$ corresponds to the volatility (V), cubic (W), and quadratic (X) payoff contracts, respectively. With these payoff contracts, the skewness can be calculated using Equation (6). More details are shown in appendix C.

2.3 Testing for Errors and Convergence

There are many different numerical integration techniques; a commonly used technique is the trapezium rule (viz. trapezoidal integration). Another is Simpson's rule. The trapezium rule essentially adds the area of piecewise-linear lines and Simpson's rule adds the area of quadratic curves.

As data from the options market is not continuous, the integral calculation of Equation (11) must be solved numerically. Using the trapezium rule, the integral can be discretised to

$$E_t^{\mathbb{Q}}\left[e^{-r\tau}R\left(t,\tau\right)^n\right]=\sum_{i=1}^{\infty}\frac{n}{K_i^2}\left[\left(n-1\right)\left[\ln\left(\frac{K_i}{S_t}\right)\right]^{n-2}-\left[\ln\left(\frac{K_i}{S_t}\right)\right]^{n-1}\right]Q\left[K_i\right]\Delta K_i \quad (12)$$

where

$$\Delta K_i=\frac{1}{2}\begin{cases}K_2-K_1, & i=1\\ K_{i+1}-K_{i-1}, & 1<i<m\\ K_m-K_{m-1}, & i=m\end{cases} \quad (13)$$

and m is the number of strikes. This is slightly different from CBOE's discretisation methods as shown in appendix D.

The trapezium rule has been used by many, for example Chang, Christoffersen, and Jacobs (2013), Chatrath, Miao, Ramchander, and Wang (2016), Conrad, Dittmar, and Ghysels (2013), Dennis and Mayhew (2002), Jiang and Tian (2005), Neumann and Skiadopoulos (2013), Ruan and Zhang (2018), and Stilger, Kostakis, and Poon (2017). Stilger, Kostakis, and Poon (2017) uses the trapezium rule as a robustness check for their primary numerical integration method - Simpson's rule.

Jiang and Tian (2005) begin their study of truncation errors, discretisation errors, errors caused by using the spot prices rather than the forward prices and vice versa, and the limited availability of strike prices. Some of these errors are quantified and used to analyse the CBOE VIX in Jiang and Tian (2007). Similarly, Chang, Christoffersen, Jacobs, and Vainberg (2011) applies the same errors and extends it to option implied skewness.

The truncation and discretisation errors are the two sources of errors that were tested.

1. Truncation errors

$$\int_0^{\infty}\cdots dK\to\int_{K_{\min}}^{K_{\max}}\cdots dK \quad (14)$$

$$\text{as } K\in(0,\infty)\to K\in[K_{\min},K_{\max}] \quad (15)$$

The range of strikes are finite, therefore, the range of the integral is truncated to the strikes that are available. This is tested by defining $K_{\min}$ and $K_{\max}$ as

$$[K_{\min}, K_{\max}] := \left[F_t^T \times a, F_t^T / a\right] \tag{16}$$

where $a \in (0,1)$ is the boundary controlling factor. So as $a \to 0$, $K_{\min} \to 0$ and $K_{\max} \to \infty$ and as $a \to 1$, $K_{\min}, K_{\max} \to F_t^T$. For calculations, the maximum and minimum strike prices are rounded to the nearest dollar and a is varied between 0.05 and 0.95 in 0.01 intervals.

2. Discretisation errors

$$\int_{K_{\min}}^{K_{\max}} \cdots dK \to \sum_{K_{\min}}^{K_{\max}} \cdots \Delta K_i \tag{17}$$

To compute integrals numerically, the integrand and region must first be discretised. This can be done using the trapezium rule. This is not the only reason why discretising is required. The other reason is that the strikes provided in the market is not continuous, but rather, usually in fixed intervals of \$1, \$5, \$25, and \$50.[1] The step size, ΔK, has been chosen to vary from 1 to 50, in increments of 1.

For each combination of a and ΔK, of which there are 4,550, the mesh of points (pairs of skewness and kurtosis) within the valid Gram-Charlier region have been used to create virtual options (as described in Section 2.1). From this, the discretised BKM method is applied to each set of options to find the risk-neutral moments. The estimation error is defined as

$$\text{Estimation Error} \ := \ \text{Estimated Moment} - \text{True Moment} \tag{18}$$

The error for each point in the valid Gram-Charlier region for a specific combination of a and ΔK is averaged to find the average error for the whole valid region. The maximum of the absolute value of the error over the valid region was also recorded.

The specification of each parameter is shown in Table I and the skewness-kurtosis

[1] Generally, minimum strike price intervals are as follows: (1) \$0.50 where the strike price is less than \$15, (2) \$1 where the strike price is less than \$200, and (3) \$5 where the strike price is greater than \$200. (`http://www.cboe.com/products/vix-index-volatility/vix-options-and-futures/vix-options/vix-options-specs`)

points used in the valid Gram-Charlier region are shown in Figure 2.

[Insert Table I and Figure 2 about here.]

An example of the error calculation for skewness and kurtosis done for the points from Figure 2 are shown in Figure 3. From this, both the average of the errors and the maximum absolute error for each region are plotted at their respective boundary controlling factor and step size.

[Insert Figure 3 about here.]

2.4 Calculating the Volatility Indices

The CBOE VIX can be calculated analytically using the Gram-Charlier series with the following proposition.

Proposition 1. *Suppose that stock price is described by*

$$S_T = F_t^T e^{\left(-\frac{1}{2}\sigma^2+\mu_c\right)\tau+\sigma\sqrt{\tau}y},$$

where y is an extension of the standard normal distribution to include higher moments using the Gram-Charlier series and μ_c is the convexity adjustment term. Then the VIX is given by

$$VIX = 100\sqrt{\sigma^2 - 2\mu_c} \tag{19}$$

Proof. See appendix E. □

This proposition allows the variance swap to be calculated directly from the distribution of the returns, specifically, the moments, and the time to maturity. For a stock price model with moments no higher than kurtosis, the VIX is given by

$$\mathrm{VIX} = 100\sqrt{\sigma^2 + \frac{2}{\tau}\ln\left[1 + \frac{\lambda_1}{3!}\left(\sigma\sqrt{\tau}\right)^3 + \frac{\lambda_2}{4!}\left(\sigma\sqrt{\tau}\right)^4\right]} \tag{20}$$

A simpler form of Equation (20) can be obtained by approximating the log and square root, as shown here

$$\mathrm{VIX} = 100\sigma\left[1 + \frac{\lambda_1}{3!}\sigma\sqrt{\tau} + \frac{\lambda_2}{4!}\sigma^2\tau - \frac{\lambda_1^2}{72}\sigma^2\tau + o\left(\sigma^2\tau\right)\right] \tag{21}$$

By suppressing kurtosis, this formula shows that introducing negative skewness to returns will result in the VIX becoming smaller than the standard deviation. Originally, without kurtosis and skewness, the two were simply linked by a scaling factor.

The CBOE VIX is used as a benchmark against the VIX calculated using the BKM method. This VIX can be calculated with

$$\mathrm{VIX} \approx 100\sqrt{-\frac{2}{\tau}\mu + 2r}. \tag{22}$$

2.5 Testing the Convexity Adjustment Term

The approximation made by BKM (Equation (7)) can be analysed analytically by entering the Black-Scholes stock price model into the approximation[1]

$$\begin{aligned} E_t^{\mathbb{Q}}\left[R\left(t,\tau\right)\right] \approx &\left(r - \frac{1}{2}\sigma^2\right)\tau \\ &+ \frac{\left(32\left(r\tau\right)^3 - 24\left(r\tau\right)^2\sigma^2\tau + 8r\tau\left(\sigma^2\tau\right)^2 + 48r\tau\sigma^2\tau - \left(\sigma^2\tau\right)^3 - 16\left(\sigma^2\tau\right)^2\right)\sigma^2\tau}{384} \end{aligned} \tag{23}$$

and comparing the BKM return to Black-Scholes expected log return $\left(r - \frac{1}{2}\sigma^2\right)\tau$. Clearly, the returns are not the same. This additional term corresponds to the convexity adjustment term μ_c as calculated by the BKM method. This is found by comparing the expected log returns of the BKM method and Gram-Charlier series return

$$E_t^{\mathbb{Q}}\left[R\left(t,\tau\right)\right] = E_t^{\mathbb{Q}}\left[\ln\frac{S_T}{S_t}\right] = \left(r - \frac{1}{2}\sigma^2 + \mu_c\right)\tau \tag{24}$$

$$\implies \mu_{c,BKM} = \frac{1}{\tau}E_t^{\mathbb{Q}}\left[R\left(t,\tau\right)\right] - \left(r - \frac{1}{2}\sigma^2\right) \tag{25}$$

[1] The details are shown in appendix F.

Using Equation (24) in place of the BKM μ, the errors caused by BKM's approximation can be calculated.

3 Data

As this paper uses virtual options to test for estimation errors, the data required is minimal and can be set arbitrarily. However, to direct the results towards US markets, specifically the S&P 500, the risk-free rate, volatility, and time to maturity, have been set to 2.4%,[1] 0.20, and one month, respectively. These values partially reflect the current market conditions in April 2019. The current market volatility based on the CBOE VIX[2] is oddly low, so the standard deviation has been chosen to be 0.20. A one month time to maturity has been chosen as these tend to be the most liquid (shortest term contracts).

4 Numerical Results

4.1 BKM Estimation

The mean errors of the VIX, standard deviation, skewness, and kurtosis estimators are shown in Table II. The mean error is the mean of the estimation error (Equation (18)) calculated over each valid Gram-Charlier region. These errors show that, as expected, the smaller boundary controlling factor (a), the smaller the approximation error. Similarly, a smaller step size (ΔK) corresponds to a smaller discretisation error. As these two approaches zero, the discretisation and truncation of the BKM method tend towards an integral over an infinite range. For the average of skewness errors to be less than 10^{-3} the boundary controlling factor and step size, as shown in Table II, must be less than 0.75 and \$20, respectively. A more robust way to limit errors to be below 10^{-3} is done by using the maximum value of the absolute error rather than the mean. This is shown in

[1] Using the Treasure bill rates from `https://home.treasury.gov/`
[2] `https://www.cboe.com/VIX`

Table III. The constraint is stronger and is reflected in the allowable boundary controlling factor and step size. The boundary controlling factor and step size to ensure that the errors are bounded by 10^{-3} must also be bounded by 0.75 and \$2, respectively. The step size is dependent on the forward price, to generalize this result, the step size must be less than 0.1% of the forward price. Comparing the errors of the second to fourth moments shows that the BKM estimator, for these values, is more accurate for lower-order moment estimators.

[Insert Tables II and III about here.]

A visual form of Table II is presented in Figure 4. This figure shows that the errors caused by truncation of the integral causes less predictable behaviour, whereas changes in the step size do not seem to affect the error as unpredictably. Although the truncation error is less predictable, when the boundary controlling factor is below 0.75, the behaviour is stable.

[Insert Figure 4 about here.]

Figure 5 shows the projection of Figure 4 to show just the errors with respect to the step size. Due to the large errors caused by truncation, Figure 5 shows errors with boundary controlling factor values of 0.05, 0.40, and 0.75. The error boundary of 10^{-3} is shown in red. The same has been done for variance and VIX and is shown in Figure 7. Figures 6 and 8 show the same errors but with respect to the boundary controlling factors for step sizes of 1, 10, and 25.

[Insert Figures 5, 6, 7 and 8 about here.]

The skewness errors can easily be restricted below 10^{-3} by fixing $\Delta K = \$2$ and $a \leq 0.75$. Kurtosis, however, have much larger errors.

The standard deviation, skewness, and kurtosis using the BKM method and the VIX have been calculated for a skewness and kurtosis of -1 and 2.5, respectively. This has been done for a boundary controlling factor value of 0.05, 0.40, and 0.75 and step size of 1 to 50. This is shown in Figure 9. Figure 10 shows the same errors with respect to the boundary controlling factors.

[Insert Figures 9 and 10 about here.]

To examine the independence of the skewness estimator from kurtosis, the skewness is estimated when the skewness, the boundary controlling factor and step size are set to -1, 0.25 and 1, respectively, whilst adjusting the kurtosis within the valid Gram-Charlier region. The same has been done for kurtosis when it is set to 2.5. As neither the true skewness nor the true kurtosis has been chosen to be zero, the relative error for each estimator can, therefore, be calculated. The (absolute) relative error is defined as Relative Error $= \left|\frac{x-x_{true}}{x_{true}}\right|$, where x is the estimated value, and x_{true} is the true value. The maximum(minimum) relative error for skewness and kurtosis were found to be $4.476 \times 10^{-04}(4.476 \times 10^{-04})$ and $1.272 \times 10^{-03}(1.245 \times 10^{-03})$, respectively. These relative error curves are monotonically increasing and decreasing, as shown in Figure 11. These values cannot be used to compare the sensitivity of skewness and kurtosis directly. The difference of the maximum and minimum values for skewness and kurtosis are 5.866×10^{-08} and 2.663×10^{-05}, respectively. From this, the effects of kurtosis on the estimation of skewness and vice versa are shown to be insignificant. Therefore, for this case, the estimation of skewness is independent of kurtosis and vice versa.

[Insert Figure 11 about here.]

4.2 In-Depth Error Analysis

A closer inspection of errors shows that the current level of granularity of the step size $\Delta K = 1$ is not sufficient to show the overall form of the relationship between the errors and step size. By reducing the step size to $\Delta K = 0.001$, this reveals a form which is closer to the true form. This is shown in Figure 12.

[Insert Figure 12 about here.]

The error does not decrease exponentially with respect to the step size, but more so quadratically. The error envelope, positive (+) and negative (−), was found using a least-squares approximation of a relatively simple parsimonious quadratic curve

$$\beta_1 \Delta K + \beta_2 (\Delta K)^2. \tag{26}$$

The coefficients β_1^+, β_2^+, β_1^-, and β_2^- were found to be 2.984×10^{-04}, 1.969×10^{-05}, -1.639×10^{-04}, and -8.959×10^{-06}, respectively. Using these values, the asymmetric envelope can be decomposed into two components, a symmetric envelope, and a trend term. The symmetric envelope has parameters β_1^s and β_2^s equal to 2.311×10^{-04} and 1.433×10^{-05}, respectively. The trend component has parameters β_1^t and β_2^t equal to 6.726×10^{-05} and 5.366×10^{-06}, respectively. The error is oscillating with an increasing period, to capture this behaviour, Equation (26) seems to be suitable. The increasing period increases approximately quadratically with β_1^ω and β_2^ω equal to 6.980×10^{-04} and 4.967×10^{-04}, respectively. This model is also presented in Figure 12. The methodology used to obtain these parameters are presented in appendix G.

As the risk-neutral moments are composed of multiple contracts, which in itself produces errors, the errors of each contract with respect to the step size is shown in Figure 13.[1]

[Insert Figure 13 about here.]

The errors of the volatility contract can be modelled the same way as the skewness estimator. The same error model is not appropriate for the other contracts. The coefficients for the volatility contract (relative) errors are $\beta^s = \begin{bmatrix} 1.487 \times 10^{-04} & 8.976 \times 10^{-06} \end{bmatrix}^T$, $\beta^t = \begin{bmatrix} 2.539 \times 10^{-05} & 4.631 \times 10^{-06} \end{bmatrix}^T$, and $\beta^\omega = \begin{bmatrix} 2.773 \times 10^{-05} & 5.113 \times 10^{-04} \end{bmatrix}^T$. The relative error is defined as

$$\text{Relative Error} := \frac{\text{Estimated} - \text{True}}{\text{True}}. \tag{27}$$

[1] The details of true contract values are shown in appendix H.

4.3 Alternative Convexity Adjustment Term

The errors of using the Black-Scholes return in place of BKM's approximation have been tabulated in Tables IV and V and presented in Figures 14 and 15.

[Insert Tables IV and V and Figures 14 and 15 about here.]

Tables IV and V have been combined with Tables II and III in Tables VI and VII, respectively, to allow for ease of comparison.

[Insert Tables VI and VII about here.]

In general, there is a slight decrease in both mean and absolute maximum errors for skewness when the boundary controlling factor is less than 0.75. For larger boundary controlling factors, the alternate convexity adjustment term increases the errors.

5 Implications

The implications of this papers are that the discretised BKM methodology is unable to accurately mainly due to the interval of strikes being too large. The truncation errors do cause some issues; however, not as much as the step size (for the intervals used). Many papers have studied the smoothing and extrapolation techniques which can be used to improve the accuracy of the BKM estimators. This paper provides a way to compare the estimator with the true underlying moment. Using this methodology, smoothing techniques as well as extrapolating techniques can be remeasured using true moment values.

6 Conclusion

This paper shows a method to create virtual options using the Gram-Charlier series. These virtual options are created by specifying the mean, variance, skewness, and kurtosis. By doing so, risk-neutral moment estimators can be compared with the true values. The BKM method is used to calculate the CBOE SKEW, this paper finds that to estimate skewness within 10^{-3} of the true value, the range of strikes $(K_{\min}, K_{\max})$ must contain at least 3/4 to 4/3 of the forward price and have a step size (ΔK) of no more than 0.1% of the forward price. Rather than using the absolute error as the measure, if the average error is used, then the step size restriction can be relaxed to 1% of the forward price. Under the same boundary controlling factor and step size specification, the absolute errors of the kurtosis, standard deviation and VIX are bounded by 5×10^{-3}, 10^{-4}, and 5×10^{-3}, respectively.

The errors of skewness were found to oscillate with respect to the step size. Increasing the granularity of the step size decreases the error approximately quadratically, not exponentially.

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Appendix

A Gram-Charlier Region Derivation

The Gram-Charlier expansion up to the kurtosis term, which is given by Equation (1), can also be expressed in terms of Hermite polynomials and the standard normal distribution probability density function rather than the normal distribution and its derivatives. That is,

$$f(x) = n(x)\left[1 + \frac{\lambda_1}{3!}He_3(x) + \frac{\lambda_2}{4!}He_4(x)\right] \tag{A.1}$$

where He, the Hermite polynomial, is defined as

$$He_n(x) := (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} = \left(x - \frac{d}{dx}\right)^n \cdot 1 \tag{A.2}$$

which gives $He_2(x) = x^2 - 1$, $He_3(x) = x^3 - 3x$, and $He_4(x) = x^4 - 6x^2 + 3$.

A necessary but not sufficient condition for f to be a valid probability density is that f must be positive semi-definite. As $n(x)$ is already a valid density function, this condition is inherited by the Gram-Charlier series, that is,

$$1 + \frac{\lambda_1}{3!}He_3(x) + \frac{\lambda_2}{4!}He_4(x) \geq 0, \quad \forall x \tag{A.3}$$

Following Jondeau and Rockinger (2001), the valid Gram-Charlier region can be found by finding λ_1 and λ_2 which satisfies

$$1 + \frac{\lambda_1}{3!}He_3(x) + \frac{\lambda_2}{4!}He_4(x) = 0 \tag{A.4}$$

and

$$\frac{\lambda_1}{2!}He_2(x) + \frac{\lambda_2}{3!}He_3(x) = 0 \tag{A.5}$$

for all x. The λ_1 and λ_2 must simultaneously satisfy Equation (A.4) to ensure that $p_4(x) = 0$ and Equation (A.5) to ensure that adjacent values of λ_1 and λ_2 will also satisfy $p_4(x) = 0$ for (infinitesimally) small variations of x. Equation (A.5) can be found by taking the derivative of Equation (A.4) with respect to x. The explicit equations for λ_1 and λ_2, found from simultaneously solving Equation (A.4) and Equation (A.5), are

$$\lambda_1(x) = -24\frac{He_3(x)}{4He_3^2(x) - 3He_2(x)He_4(x)} \tag{A.6}$$

$$\lambda_2(x) = 72\frac{He_2(x)}{4He_3^2(x) - 3He_2(x)He_4(x)}. \tag{A.7}$$

These equations are used to plot the Gram-Charlier region in Figures 1, 2 and 3.

B Virtual Options Derivation

Using the methodology laid out by Zhang and Xiang (2008) the Gram-Charlier series can be used to create virtual options with known variance, skewness, and kurtosis.

With the underlying stock price, S_T, modelled by

$$S_T = F_t^T e^{\left(-\frac{1}{2}\sigma^2+\mu_c\right)\tau+\sigma\sqrt{\tau}y}, \tag{2}$$

in the risk-neutral world, where μ_c is the convexity adjustment term and y is a random number with mean zero, variance 1, skewness λ_1, and kurtosis λ_2.

Using the Martingale condition ($F_t^T = E_t^{\mathcal{Q}}[S_T]$), the convexity adjustment term can be found to be

$$\mu_c = -\frac{1}{\tau}\ln\left[1+\frac{\lambda_1}{3!}\left(\sigma\sqrt{\tau}\right)^3+\frac{\lambda_2}{4!}\left(\sigma\sqrt{\tau}\right)^4\right] \tag{B.1}$$

Zhang and Xiang (2008) then uses Harrison and Kreps (1979) and Harrison and Pliska (1981) to compute the European call option

$$\begin{aligned} c_t &= e^{-r\tau}E_t^{\mathcal{Q}}\left[\max(S_T-K,0)\right] \\ &= e^{-r\tau}\int_{-d_2}^{\infty}\left(F_t^T e^{\left(-\frac{1}{2}\sigma^2+\mu_c\right)\tau+\sigma\sqrt{\tau}y}-K\right)f(y)dy \end{aligned}$$

where

$$d_2 = \frac{\ln(F_t^T/K)+\left(-\frac{1}{2}\sigma^2+\mu_c\right)\tau}{\sigma\sqrt{\tau}}, d_1 = d_2+\sigma\sqrt{\tau}$$

Using basic integration techniques, Equation (4) can be obtained. Equation (4) is different to the call price presented in the appendix of Zhang and Xiang (2008) due to a minor error, the call price could be further simplified.

C BKM Derivation

The standardised skewness is given by Equation (6) when $n = 3$. Expanding this, the BKM formula for risk-neutral skewness can be found. Similarly, the standardised kurtosis can be found when $n = 4$. BKM defines μ, V, W, and X as $E_t^{\mathcal{Q}}[R(t,\tau)]$, $E_t^{\mathcal{Q}}\left[e^{-r\tau}R(t,\tau)^2\right]$, $E_t^{\mathcal{Q}}\left[e^{-r\tau}R(t,\tau)^3\right]$, and $E_t^{\mathcal{Q}}\left[e^{-r\tau}R(t,\tau)^4\right]$, respectively.

$$\text{Skewness} = \frac{E_t^{\mathcal{Q}}\left[\left(R(t,\tau)-\mu\right)^3\right]}{\left\{E_t^{\mathcal{Q}}\left[\left(R(t,\tau)-\mu\right)^2\right]\right\}^{\frac{3}{2}}} \tag{C.1}$$

$$= \frac{E_t^{\mathcal{Q}}\left[R(t,\tau)^3-3\mu R(t,\tau)^2+3\mu^2R(t,\tau)-\mu^3\right]}{\left\{E_t^{\mathcal{Q}}\left[R(t,\tau)^2-2\mu R(t,\tau)+\mu^2\right]\right\}^{\frac{3}{2}}} \tag{C.2}$$

$$= \frac{e^{r\tau}W-3\mu e^{r\tau}V+3\mu^2\mu-\mu^3}{\left[e^{r\tau}V-2\mu\mu+\mu^2\right]^{\frac{3}{2}}} = \frac{e^{r\tau}W-3\mu e^{r\tau}V+2\mu^3}{\left[e^{r\tau}V-\mu^2\right]^{\frac{3}{2}}} \tag{C.3}$$

$$\text{Kurtosis} = \frac{E_t^{\mathbb{Q}}\left[\left(R\left(t,\tau\right)-\mu\right)^4\right]}{\left\{E_t^{\mathbb{Q}}\left[\left(R\left(t,\tau\right)-\mu\right)^2\right]\right\}^2} \tag{C.4}$$

$$= \frac{E_t^{\mathbb{Q}}\left[R\left(t,\tau\right)^4 - 4\mu R\left(t,\tau\right)^3 + 6\mu^2 R\left(t,\tau\right)^2 - 4\mu^3 R\left(t,\tau\right) + \mu^4\right]}{\left\{E_t^{\mathbb{Q}}\left[R\left(t,\tau\right)^2 - 2\mu R\left(t,\tau\right) + \mu^2\right]\right\}^2} \tag{C.5}$$

$$= \frac{e^{r\tau}X - 4\mu e^{r\tau}W + 6\mu^2 e^{r\tau}V - 4\mu^3\mu + \mu^4}{\left[e^{r\tau}V - 2\mu\mu + \mu^2\right]^2} \tag{C.6}$$

$$= \frac{e^{r\tau}X - 4\mu e^{r\tau}W + 6\mu^2 e^{r\tau}V - 3\mu^4}{\left[e^{r\tau}V - \mu^2\right]^2} \tag{C.7}$$

The (annualized) variance σ^2 is given by

$$\sigma^2 = \frac{1}{\tau}E_t^{\mathbb{Q}}\left[\left(R\left(t,\tau\right)-\mu\right)^2\right] = \frac{e^{r\tau}V - \mu^2}{\tau} \tag{C.8}$$

D Numerical Integration

The traditional trapezium rule is given in the form of Equation (D.1). With a small rearrangement, Equation (D.2) can be obtained. This is the trapezium rule that is used for calculations.

$$\int_a^b f(x)dx = \sum_{i=1}^{n-1} \frac{f(x_i)+f(x_{i+1})}{2}\Delta x_i, \quad \Delta x_i = x_{i+1} - x_i \tag{D.1}$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x_i, \quad \Delta x_i = \frac{1}{2}\begin{cases} x_2 - x_1, & i = 1 \\ x_{i+1} - x_{i-1}, & 1 < i < n \\ x_n - x_{n-1}, & i = n \end{cases} \tag{D.2}$$

The CBOE uses Equation (D.3) which introduces a small error at the end points, $K_{\min}$ and $K_{\max}$. This method does, however, give equal weighting to each option - rather than half the weight given to end points if the traditional trapezium rule is used.

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i)\Delta x_i, \quad \Delta x_i = \begin{cases} x_2 - x_1, & i = 1 \\ \frac{x_{i+1}-x_{i-1}}{2}, & 1 < i < n \\ x_n - x_{n-1}, & i = n \end{cases} \tag{D.3}$$

$$= \int_a^b f(x)dx - \left(f(x_1)\frac{x_2 - x_1}{2} + f(x_n)\frac{x_n - x_{n-1}}{2}\right) \tag{D.4}$$

E VIX Derivation

E.1 Variance Swap Derivation

Demeterfi, Derman, Kamal, and Zou (1999) present a method to replicate variance swaps with European options. Following part of their procedure, for a differential stock price form of

$$\frac{dS_t}{S_t} = (r - q)\, dt + \sigma_t dW_t$$

where r, q and σ is risk-free rate, continuous dividend rate and the volatility parameters, respectively, the following can be obtained:

$$\mathrm{VIX}^2 = \frac{2}{\tau} E_t^{\mathcal{Q}} \left[\int_t^T \frac{dS_t}{S_t} - \ln \frac{S_T}{S_t} \right] \times 100^2 \tag{E.1}$$

With some algebraic manipulation, an intuitive formula can be found:

$$\mathrm{VIX} = 100 \sqrt{-\frac{2}{\tau} E_t^{\mathcal{Q}} \left[\ln \frac{S_T}{S_t e^{(r-q)\tau}} \right]} \tag{E.2}$$

E.2 CBOE VIX

Using the integral form of stock price derived using the Gram-Charlier region, with some algebra, the following can be obtained.

$$\begin{aligned}
\mathrm{VIX}^2 &= -\frac{2}{\tau} E_t^{\mathcal{Q}} \left[\ln \frac{S_T}{S_t e^{(r-q)\tau}} \right] \times 100^2 && \text{(E.3)}\\
&= -\frac{2}{\tau} E_t^{\mathcal{Q}} \left[\left(-\frac{1}{2}\sigma^2 + \mu_c \right) \tau + \sigma\sqrt{\tau} y \right] \times 100^2 && \text{(E.4)}\\
&= \left[\left(\sigma^2 - 2\mu_c \right) - \frac{2}{\tau} \sigma \sqrt{\tau} E_t^{\mathcal{Q}} \left[y \right] \right] \times 100^2 && \text{(E.5)}\\
&= \left[\sigma^2 - 2\mu_c \right] \times 100^2 && \text{(E.6)}\\
\Longrightarrow \ \mathrm{VIX} &= 100 \sqrt{\sigma^2 - 2\mu_c} && \text{(E.7)}
\end{aligned}$$

The volatility index can also calculated using the BKM method

$$\mathrm{VIX}^2 = -\frac{2}{\tau} E_t^{\mathcal{Q}} \left[\ln \frac{S_T}{S_t e^{r\tau}} \right] \times 100^2 = \left[-\frac{2}{\tau} E_t^{\mathcal{Q}} \left[R\left(t, \tau\right) \right] + 2r \right] \times 100^2 \tag{E.8}$$

$$\Longrightarrow \ \mathrm{VIX} \approx 100 \sqrt{-\frac{2}{\tau} \mu + 2r} \tag{E.9}$$

where

$$\mu(t, \tau) := e^{r\tau} - 1 - \frac{1}{2} E_t^{\mathcal{Q}} \left[R\left(t, \tau\right)^2 \right] - \frac{1}{3!} E_t^{\mathcal{Q}} \left[R\left(t, \tau\right)^3 \right] - \frac{1}{4!} E_t^{\mathcal{Q}} \left[R\left(t, \tau\right)^4 \right] \tag{E.10}$$

As μ, the approximation of $E_t^{\mathcal{Q}} \left[R\left(t, \tau\right) \right]$, is required, this method introduces errors, however, it does remain model-free.

F BKM and Black-Scholes Risk-Neutral Log Returns

Using the standard Black-Scholes stock price model,

$$S_T = S_t e^{\left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma W_\tau}, \tag{F.1}$$

where W_τ is a Wiener process, the expected value of log returns is

$$E_t^{\mathbb{Q}}\left[R\left(t,\tau\right)\right] = E_t^{\mathbb{Q}}\left[\left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma W_\tau\right] = \left(r - \frac{1}{2}\sigma^2\right)\tau. \tag{F.2}$$

From this, the BKM's approximation can be compared like so

$$E_t^{\mathbb{Q}}\left[R\left(t,\tau\right)\right] \approx e^{r\tau} - 1 - \frac{1}{2}E_t^{\mathbb{Q}}\left[R\left(t,\tau\right)^2\right] - \frac{1}{3!}E_t^{\mathbb{Q}}\left[R\left(t,\tau\right)^3\right] - \frac{1}{4!}E_t^{\mathbb{Q}}\left[R\left(t,\tau\right)^4\right].$$

As the BKM approximates the exponential term when used with $R(t,\tau)$, this is also done to $e^{r\tau}$. The result is

$$\begin{aligned} E_t^{\mathbb{Q}}\left[R\left(t,\tau\right)\right] \approx 1 + r\tau + \frac{1}{2}(r\tau)^2 + \frac{1}{3!}(r\tau)^3 + \frac{1}{4!}(r\tau)^4 - 1 \\ - \frac{1}{2}E_t^{\mathbb{Q}}\left[R\left(t,\tau\right)^2\right] - \frac{1}{3!}E_t^{\mathbb{Q}}\left[R\left(t,\tau\right)^3\right] - \frac{1}{4!}E_t^{\mathbb{Q}}\left[R\left(t,\tau\right)^4\right]. \end{aligned}$$

Expanding this further and simplifying, Equation (23) can be obtained.

G Error Envelope Derivation

Using a simple curve was sufficient to capture the main characteristics of the error.

$$Y = \text{Errors (Peaks (+ or } -))\tag{G.1}$$

$$x = \text{Corresponding } \Delta K \tag{G.2}$$

$$X = \begin{bmatrix} x & x^2 \end{bmatrix} \tag{G.3}$$

For $y = \beta_1 x + \beta_2 x^2$

$$Y = X\beta \tag{G.4}$$

$$\implies \beta = \left(X^T X\right)^{-1} X^T Y \tag{G.5}$$

For the positive envelope and negative envelope, the coefficients are assigned to β^+ and β^-, respectively. The symmetric envelope and trend can be found from $\beta^s = \frac{\beta^+ - \beta^-}{2}$ and $\beta^t = \frac{\beta^+ + \beta^-}{2}$, respectively. From this, the envelope component equations are given by

$$E^s = X\beta^s \tag{G.6}$$

$$E^t = X\beta^t \tag{G.7}$$

The positive and negative envelopes are therefore

$$E^+ = X\beta^s + X\beta^t, \qquad E^- = -X\beta^s + X\beta^t \tag{G.8}$$

To capture behaviour of the oscillations, the same quadratic model was used. Therefore, the coefficients were calculated the same way. The coefficient of the oscillations, denoted as β^ω, forms the following oscillating function

$$\Omega = \sin\left(\frac{2\pi}{X\beta^\omega}x\right) \tag{G.9}$$

Combining the envelope, trend and oscillating function forms the following model

$$\begin{aligned} \text{Error Model} &= E^s\Omega + E^t \\ &= X\beta^s \sin\left(\frac{2\pi}{X\beta^\omega}x\right) + X\beta^t \end{aligned} \tag{G.10}$$

As $X = \begin{bmatrix} x & x^2 \end{bmatrix}$, the model describes the error as changing quadratically in both the magnitude and period. Due to the specification of the model, when x (viz. ΔK) is zero, the error vanishes.

H Testing the Volatility, Cubic, and Quartic Contracts Directly

The BKM method utilizes three contracts formed by Carr and Madan's payoff decomposition function. To test the accuracy of the BKM estimators, the components within these estimators, the three contracts, can also be tested.

As the risk-neutral moments are known, the values of the contract can be calculated like so

$$e^{r\tau}V = \sigma^2\tau + \mu^2 \tag{H.1}$$

$$e^{r\tau}W = \lambda_1\left(\sigma^2\tau\right)^{\frac{3}{2}} + 3\mu\sigma^2\tau + \mu^3 \tag{H.2}$$

$$e^{r\tau}X = (\lambda_2 + 3)\left(\sigma^2\tau\right)^2 + 4\lambda_1\mu\left(\sigma^2\tau\right)^{\frac{3}{2}} + 6\mu^2\sigma^2\tau + \mu^4 \tag{H.3}$$

For this exact calculation, $\mu = \left(r - \frac{1}{2}\sigma^2 + \mu_c\right)\tau$, this uses the Black-Scholes and Gram-Charlier model (Equation (24)). This can be used to test the accuracy of the calculation of each contract.

Tables

Table I: Specification of Parameters.
The value(s) of each parameter used in calculations are specified in this table.

Parameter	Definition	Range
F_t^T	Forward Price at time t with maturity date T	\$2,000
r	Risk-Free Rate	2.4%
$\tau = T - t$	Time to Maturity	1 month
σ	Standard Deviation	0.20
λ_1	Skewness	$(-1.05, 1.05)$
λ_2	Excess Kurtosis	$[0, 4]$
$a = \dfrac{K_{\min}}{F_t^T} = \dfrac{F_t^T}{K_{\max}}$	Boundary Controlling Factor	$[0.05, 0.95]$
ΔK	Step Size of Strikes	$[1, 50]$

Table II: Lookup Table for Mean Errors.
This table shows the mean errors of the VIX, standard deviation, skewness, and kurtosis estimators which were calculated from the valid Gram-Charlier region. These values have been scaled. The options created have a forward price F_t^T, time to maturity τ, risk-free rate r, and standard deviation σ, have been arbitrarily set to \$2,000, one month, 2.4%, and 0.20, respectively. This has been done for various boundary controlling factor (a) values and step sizes (ΔK). The error for each a and ΔK is defined as Error := mean (Estimated Moment − True Moment).

ΔK	1	2	3	4	5	10	20	30	40	50
a	**VIX $\times 10$**									
0.05	-0.03	-0.06	-0.08	-0.10	-0.09	0.00	0.38	-0.49	-1.12	2.98
0.40	-0.03	-0.06	0.03	-0.10	-0.09	0.00	0.38	1.00	1.87	2.98
0.75	-0.03	-0.06	-0.03	-0.10	-0.09	0.00	0.38	-0.50	-1.12	2.98
0.80	-0.06	-0.09	-0.11	-0.13	-0.12	-0.03	0.35	-0.53	1.82	2.94
0.82	-0.17	-0.20	-0.12	-0.25	-0.24	-0.15	0.20	0.82	1.62	-0.26
0.84	-0.53	-0.55	-0.54	-0.60	-0.59	-0.49	-0.13	-1.07	1.22	-2.04
0.86	-1.44	-1.48	-1.52	-1.53	-1.50	-1.48	-1.11	-2.01	0.34	-3.11
0.88	-3.42	-3.45	-3.42	-3.50	-3.54	-3.45	-3.37	-2.49	-2.62	-3.93
0.90	-7.04	-7.06	-7.14	-7.21	-7.20	-7.11	-6.75	-7.71	-6.40	-5.34
0.92	-13.13	-13.24	-13.18	-13.29	-13.44	-13.35	-13.89	-14.01	-12.48	-16.77
0.94	-22.97	-23.14	-23.05	-23.49	-23.33	-23.99	-23.66	-23.10	-22.32	-34.92
0.95	-30.20	-30.42	-30.45	-30.48	-30.28	-31.18	-30.87	-36.88	-33.17	-28.67
a	**Risk-Neutral Standard Deviation $\times 1,000$**									
0.05	-0.03	-0.05	-0.08	-0.10	-0.09	0.00	0.38	-0.49	-1.12	2.98
0.40	-0.03	-0.05	0.03	-0.10	-0.09	0.00	0.38	1.00	1.87	2.98
0.75	-0.03	-0.06	-0.03	-0.10	-0.09	0.00	0.38	-0.50	-1.12	2.98
0.80	-0.06	-0.09	-0.11	-0.13	-0.12	-0.03	0.34	-0.52	1.82	2.94
0.82	-0.17	-0.20	-0.12	-0.25	-0.24	-0.15	0.20	0.82	1.63	-0.23
0.84	-0.53	-0.56	-0.54	-0.60	-0.59	-0.50	-0.13	-1.07	1.24	-2.04
0.86	-1.45	-1.49	-1.52	-1.53	-1.51	-1.48	-1.11	-2.01	0.33	-3.12
0.88	-3.43	-3.46	-3.42	-3.51	-3.54	-3.45	-3.34	-2.49	-2.51	-3.92
0.90	-7.05	-7.08	-7.14	-7.21	-7.20	-7.11	-6.75	-7.72	-6.30	-5.24
0.92	-13.13	-13.23	-13.18	-13.28	-13.42	-13.33	-13.80	-13.99	-12.39	-16.51
0.94	-22.96	-23.12	-23.03	-23.45	-23.30	-23.92	-23.58	-23.03	-22.25	-34.43
0.95	-30.19	-30.40	-30.43	-30.46	-30.27	-31.12	-30.80	-36.60	-33.10	-28.61
a	**Risk-Neutral Skewness $\times 1,000$**									
0.05	-0.02	-0.04	-0.07	-0.09	-0.07	0.09	0.75	-2.09	-1.99	5.42
0.40	-0.02	-0.04	0.02	-0.09	-0.07	0.09	0.75	1.84	3.40	5.42
0.75	-0.01	-0.03	-0.01	-0.08	-0.06	0.10	0.76	0.49	-1.97	5.43
0.80	0.76	0.73	0.71	0.69	0.71	0.88	1.58	-1.17	3.39	6.54
0.82	2.57	2.41	2.48	2.08	1.95	1.35	0.26	1.51	-1.56	-15.76
0.84	6.41	6.39	6.01	6.35	6.38	6.58	7.39	3.16	1.35	1.43
0.86	13.68	12.77	11.85	12.74	13.66	9.29	10.11	7.71	13.44	12.45
0.88	22.95	22.94	19.67	22.92	19.62	19.81	3.04	21.76	-33.26	11.79
0.90	33.81	33.80	28.57	28.54	28.55	28.70	29.27	29.62	-24.08	-21.35
0.92	39.19	35.34	39.18	35.32	27.55	27.68	-12.32	25.22	-9.41	-116.20
0.94	44.68	39.06	39.05	27.64	33.38	4.35	4.90	5.83	7.17	-312.59
0.95	48.46	41.53	41.53	41.52	48.46	13.17	13.59	-154.54	9.59	16.90
a	**Risk-Neutral Kurtosis $\times 100$**									
0.05	0.30	0.57	0.81	1.03	0.92	-0.05	-3.90	5.06	11.85	-29.91
0.40	0.30	0.57	-0.27	1.03	0.92	-0.05	-3.90	-10.23	-18.95	-29.91
0.75	0.25	0.52	0.30	0.99	0.87	-0.10	-3.95	5.18	11.79	-29.98
0.80	-2.72	-2.45	-2.20	-1.98	-2.10	-3.08	-6.96	1.86	-22.85	-33.19
0.82	-10.04	-9.86	-10.69	-9.59	-9.81	-11.28	-16.28	-22.69	-34.26	-18.72
0.84	-28.07	-27.82	-28.24	-27.38	-27.50	-28.46	-32.31	-26.23	-52.06	-15.07
0.86	-62.21	-62.36	-62.55	-61.95	-61.67	-64.59	-68.21	-60.58	-82.37	-51.65
0.88	-114.83	-114.61	-116.48	-114.24	-115.53	-116.33	-125.57	-124.74	-150.27	-122.66
0.90	-181.35	-181.16	-182.81	-182.33	-182.41	-183.05	-185.60	-179.00	-210.07	-217.06
0.92	-255.47	-256.14	-255.16	-255.86	-257.57	-258.03	-267.91	-254.03	-274.90	-278.01
0.94	-328.86	-329.59	-329.96	-331.04	-330.22	-334.64	-335.65	-337.35	-339.77	-342.57
0.95	-364.12	-364.86	-364.74	-364.63	-363.78	-368.12	-368.65	-376.61	-357.58	-372.80

Table III: Lookup Table for Maximum Absolute Errors.
This table shows the maximum values of the absolute errors of the VIX, standard deviation, skewness, and kurtosis estimators which were calculated from the valid Gram-Charlier region. These values have been scaled. The options created have a forward price F_t^T, time to maturity τ, risk-free rate r, and standard deviation σ, have been arbitrarily set to \$2,000, one month, 2.4%, and 0.20, respectively. This has been done for various boundary controlling factor (a) values and step sizes (ΔK). The error for each a and ΔK is defined as Error $:=$ max |Estimated Moment − True Moment|.

ΔK	**1**	**2**	**3**	**4**	**5**	**10**	**20**	**30**	**40**	**50**
a	**VIX ×10**									
0.05	0.03	0.06	0.08	0.10	0.09	0.01	0.39	0.49	1.13	3.01
0.40	0.03	0.06	0.03	0.10	0.09	0.01	0.39	1.01	1.89	3.01
0.75	0.03	0.06	0.04	0.10	0.09	0.01	0.39	0.50	1.13	3.01
0.80	0.09	0.12	0.14	0.16	0.15	0.06	0.38	0.56	1.87	2.99
0.82	0.29	0.31	0.23	0.37	0.36	0.28	0.37	0.99	1.85	0.51
0.84	0.92	0.94	0.93	0.99	0.98	0.89	0.52	1.52	1.81	2.48
0.86	2.51	2.56	2.61	2.61	2.57	2.59	2.23	3.15	1.68	4.30
0.88	5.82	5.85	5.85	5.89	5.97	5.88	5.98	4.93	5.74	6.61
0.90	11.45	11.48	11.60	11.67	11.66	11.57	11.21	12.20	11.39	10.30
0.92	19.89	20.03	19.95	20.08	20.29	20.19	20.94	20.84	19.43	24.53
0.94	31.46	31.65	31.55	32.02	31.85	32.57	32.18	31.55	30.66	44.52
0.95	38.99	39.22	39.25	39.28	39.07	39.98	39.61	45.94	41.88	37.03
a	**Risk-Neutral Standard Deviation ×1,000**									
0.05	0.03	0.05	0.08	0.10	0.09	0.00	0.38	0.49	1.12	2.98
0.40	0.03	0.05	0.03	0.10	0.09	0.00	0.38	1.00	1.87	2.98
0.75	0.03	0.06	0.03	0.10	0.09	0.00	0.38	0.50	1.12	2.98
0.80	0.09	0.12	0.14	0.16	0.15	0.06	0.38	0.55	1.87	2.97
0.82	0.29	0.32	0.24	0.37	0.36	0.28	0.37	0.99	1.85	0.45
0.84	0.94	0.96	0.95	1.01	1.00	0.91	0.54	1.52	1.81	2.49
0.86	2.56	2.60	2.64	2.65	2.62	2.61	2.25	3.17	1.67	4.34
0.88	5.90	5.93	5.91	5.98	6.03	5.94	5.93	5.00	5.47	6.59
0.90	11.60	11.62	11.71	11.78	11.77	11.68	11.32	12.31	11.17	10.09
0.92	20.07	20.18	20.13	20.23	20.38	20.29	20.82	20.93	19.30	23.86
0.94	31.66	31.82	31.72	32.12	31.99	32.53	32.15	31.51	30.62	43.31
0.95	39.19	39.39	39.42	39.45	39.28	40.01	39.64	45.26	41.90	37.05
a	**Risk-Neutral Skewness ×1,000**									
0.05	0.45	0.86	1.25	1.59	1.40	0.17	6.40	9.43	18.97	48.73
0.40	0.45	0.86	0.42	1.59	1.40	0.17	6.40	16.69	30.87	48.73
0.75	0.42	0.84	0.50	1.57	1.38	0.19	6.43	8.04	18.94	48.76
0.80	1.76	1.43	1.43	1.54	1.48	2.40	8.72	7.12	32.85	51.59
0.82	8.03	7.49	8.78	6.51	6.65	7.95	13.66	24.23	39.40	29.68
0.84	26.04	25.64	25.85	24.94	25.14	26.78	33.32	19.46	56.10	10.00
0.86	70.13	69.52	68.94	68.86	69.28	69.78	76.37	62.33	102.23	54.46
0.88	157.13	156.77	157.28	156.17	155.71	157.29	160.27	173.96	233.88	167.69
0.90	300.10	299.79	299.10	298.29	298.46	299.80	305.14	295.80	359.97	369.99
0.92	493.22	491.88	492.73	491.42	489.29	490.18	511.61	482.16	519.22	641.21
0.94	706.05	702.99	703.47	696.68	699.71	685.77	685.68	687.45	690.14	1041.57
0.95	805.46	800.84	800.68	800.51	804.98	781.73	781.60	954.29	760.23	781.10
a	**Risk-Neutral Kurtosis ×100**									
0.05	0.40	0.77	1.10	1.40	1.25	0.07	5.28	6.87	16.04	40.48
0.40	0.40	0.77	0.37	1.40	1.25	0.07	5.28	13.87	25.67	40.48
0.75	0.32	0.69	0.39	1.32	1.16	0.16	5.38	6.98	15.93	40.60
0.80	4.92	4.56	4.23	3.93	4.09	5.42	10.69	2.83	32.55	46.25
0.82	17.61	17.41	18.53	17.11	17.44	19.62	26.82	35.53	52.24	32.52
0.84	48.14	47.79	48.44	47.21	47.37	48.67	53.87	46.59	82.06	31.10
0.86	104.25	104.58	104.95	104.04	103.54	108.04	112.82	102.70	131.53	90.92
0.88	187.73	187.44	190.21	186.95	188.96	189.97	203.51	200.64	239.27	198.81
0.90	288.52	288.27	290.72	290.10	290.19	290.95	293.97	285.05	326.85	334.87
0.92	393.61	394.56	393.21	394.19	396.54	397.03	409.88	391.09	417.34	422.37
0.94	488.22	489.13	489.59	490.92	489.91	495.30	496.35	498.13	500.66	503.97
0.95	530.36	531.24	531.09	530.95	529.94	535.06	535.65	544.30	521.68	540.33

Table IV: Errors for Alternative Convexity Adjustment Term.
This table shows the mean errors of the VIX, standard deviation, skewness, and kurtosis estimators which were calculated from the valid Gram-Charlier region. The expected return of BKM has been replaced by $E_t^Q\left[R\left(t,\tau\right)\right] = \left(r - \frac{1}{2}\sigma^2 + \mu_c\right)\tau$. These values have been scaled. The options created have a forward price F_t^T, time to maturity τ, risk-free rate r, and standard deviation σ, have been arbitrarily set to \$2,000, one month, 2.4%, and 0.20, respectively. This has been done for various boundary controlling factor (a) values and step sizes (ΔK). The error for each a and ΔK is defined as Error $:=$ mean (Estimated Moment $-$ True Moment).

ΔK	1	2	3	4	5	10	20	30	40	50
a	**Risk-Neutral Standard Deviation $\times 1,000$**									
0.05	-0.03	-0.05	-0.08	-0.10	-0.09	0.00	0.38	-0.49	-1.12	2.98
0.40	-0.03	-0.05	0.03	-0.10	-0.09	0.00	0.38	1.00	1.87	2.98
0.75	-0.03	-0.06	-0.03	-0.10	-0.09	0.00	0.38	-0.50	-1.12	2.98
0.80	-0.06	-0.09	-0.11	-0.13	-0.12	-0.03	0.34	-0.52	1.82	2.94
0.82	-0.17	-0.20	-0.12	-0.25	-0.24	-0.15	0.20	0.82	1.63	-0.23
0.84	-0.53	-0.56	-0.54	-0.60	-0.59	-0.50	-0.13	-1.07	1.24	-2.04
0.86	-1.45	-1.49	-1.52	-1.53	-1.51	-1.48	-1.11	-2.01	0.33	-3.12
0.88	-3.43	-3.46	-3.42	-3.51	-3.54	-3.45	-3.34	-2.49	-2.51	-3.92
0.90	-7.05	-7.07	-7.14	-7.21	-7.20	-7.11	-6.75	-7.71	-6.30	-5.24
0.92	-13.12	-13.23	-13.18	-13.28	-13.42	-13.33	-13.79	-13.98	-12.39	-16.50
0.94	-22.95	-23.11	-23.02	-23.44	-23.29	-23.90	-23.57	-23.01	-22.24	-34.40
0.95	-30.17	-30.38	-30.41	-30.44	-30.25	-31.10	-30.78	-36.57	-33.08	-28.59
a	**Risk-Neutral Skewness $\times 1,000$**									
0.05	0.00	0.00	0.00	-0.00	0.01	0.09	0.42	-1.67	-1.01	2.86
0.40	0.00	0.00	0.00	-0.00	0.01	0.09	0.42	0.98	1.78	2.86
0.75	0.02	0.02	0.02	0.01	0.02	0.10	0.43	0.93	-1.00	2.87
0.80	0.81	0.81	0.81	0.81	0.82	0.91	1.28	-0.71	1.81	4.01
0.82	2.72	2.58	2.58	2.29	2.16	1.48	0.09	0.81	-2.96	-15.53
0.84	6.86	6.87	6.48	6.87	6.89	7.01	7.50	4.09	0.30	3.20
0.86	14.93	14.06	13.17	14.07	14.97	10.57	11.08	9.47	13.15	15.17
0.88	25.95	25.96	22.66	25.98	22.72	22.82	5.98	23.93	-30.97	15.24
0.90	40.03	40.04	34.88	34.91	34.92	34.98	35.23	36.45	-18.43	-16.65
0.92	50.99	47.24	51.03	47.27	39.65	39.69	0.20	37.85	1.80	-100.97
0.94	65.92	60.47	60.37	49.38	54.97	26.60	26.81	27.19	27.77	-279.03
0.95	76.99	70.30	70.33	70.35	77.08	42.73	42.82	-118.90	41.23	43.85
a	**Risk-Neutral Kurtosis $\times 100$**									
0.05	0.30	0.57	0.81	1.03	0.92	-0.05	-3.89	5.06	11.85	-29.91
0.40	0.30	0.57	-0.27	1.03	0.92	-0.05	-3.89	-10.23	-18.95	-29.91
0.75	0.25	0.52	0.30	0.99	0.87	-0.10	-3.95	5.18	11.79	-29.97
0.80	-2.72	-2.45	-2.20	-1.98	-2.10	-3.08	-6.96	1.86	-22.85	-33.18
0.82	-10.04	-9.87	-10.69	-9.60	-9.81	-11.28	-16.28	-22.69	-34.26	-18.72
0.84	-28.07	-27.82	-28.24	-27.38	-27.50	-28.46	-32.31	-26.24	-52.05	-15.08
0.86	-62.21	-62.37	-62.55	-61.96	-61.67	-64.59	-68.21	-60.58	-82.37	-51.66
0.88	-114.83	-114.61	-116.48	-114.24	-115.53	-116.33	-125.58	-124.75	-150.27	-122.67
0.90	-181.34	-181.15	-182.80	-182.32	-182.40	-183.04	-185.60	-178.99	-210.09	-217.07
0.92	-255.44	-256.11	-255.13	-255.82	-257.54	-258.01	-267.94	-254.01	-274.92	-278.24
0.94	-328.76	-329.50	-329.87	-330.98	-330.15	-334.64	-335.65	-337.35	-339.76	-343.96
0.95	-363.95	-364.71	-364.59	-364.48	-363.61	-368.08	-368.60	-377.34	-357.55	-372.74

Table V: Maximum Value of the Absolute Error for Alternative Convexity Adjustment Term.
This table shows the maximum values of the absolute errors of the VIX, standard deviation, skewness, and kurtosis estimators which were calculated from the valid Gram-Charlier region. The expected return of BKM has been replaced by $E_t^Q\left[R\left(t,\tau\right)\right] = \left(r - \frac{1}{2}\sigma^2 + \mu_c\right)\tau$. These values have been scaled. The options created have a forward price F_t^T, time to maturity τ, risk-free rate r, and standard deviation σ, have been arbitrarily set to \$2,000, one month, 2.4%, and 0.20, respectively. This has been done for various boundary controlling factor (a) values and step sizes (ΔK). The error for each a and ΔK is defined as Error $:=$ max |Estimated Moment − True Moment|.

ΔK	**1**	**2**	**3**	**4**	**5**	**10**	**20**	**30**	**40**	**50**
a	**Risk-Neutral Standard Deviation ×1,000**									
0.05	0.03	0.05	0.08	0.10	0.09	0.00	0.38	0.49	1.12	2.98
0.40	0.03	0.05	0.03	0.10	0.09	0.00	0.38	1.00	1.87	2.98
0.75	0.03	0.06	0.03	0.10	0.09	0.00	0.38	0.50	1.12	2.98
0.80	0.09	0.12	0.14	0.16	0.15	0.06	0.38	0.55	1.87	2.97
0.82	0.29	0.32	0.24	0.37	0.36	0.28	0.37	0.99	1.85	0.45
0.84	0.94	0.96	0.95	1.01	1.00	0.91	0.54	1.52	1.81	2.49
0.86	2.56	2.60	2.64	2.64	2.62	2.61	2.25	3.17	1.67	4.34
0.88	5.90	5.93	5.91	5.98	6.03	5.94	5.93	5.00	5.46	6.58
0.90	11.59	11.62	11.70	11.78	11.77	11.67	11.31	12.30	11.17	10.08
0.92	20.06	20.17	20.12	20.22	20.37	20.28	20.80	20.92	19.29	23.84
0.94	31.64	31.80	31.70	32.10	31.96	32.51	32.13	31.49	30.60	43.28
0.95	39.17	39.36	39.39	39.42	39.25	39.98	39.61	45.22	41.87	37.03
a	**Risk-Neutral Skewness ×1,000**									
0.05	0.43	0.83	1.18	1.50	1.34	0.16	6.07	9.01	18.00	46.15
0.40	0.43	0.83	0.40	1.50	1.34	0.16	6.07	15.81	29.24	46.15
0.75	0.44	0.83	0.52	1.50	1.34	0.18	6.10	8.47	17.96	46.18
0.80	1.82	1.53	1.55	1.68	1.61	2.44	8.43	6.66	31.25	49.06
0.82	8.23	7.71	8.92	6.77	6.84	8.06	13.46	23.48	40.68	29.30
0.84	26.46	26.08	26.27	25.43	25.62	27.18	33.39	20.33	54.92	12.10
0.86	71.30	70.71	70.15	70.09	70.50	70.93	77.20	63.96	101.80	57.07
0.88	159.92	159.58	160.02	159.01	158.56	160.06	162.76	175.88	231.40	170.68
0.90	305.91	305.62	304.93	304.18	304.33	305.59	310.62	302.15	354.32	365.29
0.92	504.25	502.95	503.81	502.54	500.44	501.25	499.51	493.84	508.41	625.21
0.94	726.13	723.16	723.56	717.04	719.99	704.95	705.76	707.12	709.05	1005.64
0.95	832.66	828.19	828.06	827.91	832.26	809.53	809.07	918.02	790.09	806.32
a	**Risk-Neutral Kurtosis ×100**									
0.05	0.40	0.77	1.10	1.40	1.25	0.07	5.28	6.87	16.04	40.47
0.40	0.40	0.77	0.37	1.40	1.25	0.07	5.28	13.87	25.66	40.47
0.75	0.32	0.69	0.39	1.32	1.16	0.16	5.37	6.98	15.93	40.59
0.80	4.92	4.56	4.23	3.94	4.09	5.42	10.69	2.82	32.54	46.24
0.82	17.61	17.41	18.53	17.12	17.44	19.62	26.82	35.52	52.28	32.50
0.84	48.14	47.80	48.44	47.21	47.37	48.68	53.87	46.59	82.05	31.19
0.86	104.26	104.59	104.96	104.05	103.54	108.05	112.83	102.71	131.53	90.93
0.88	187.72	187.43	190.21	186.94	188.96	189.97	203.52	200.64	239.15	198.82
0.90	288.48	288.23	290.69	290.06	290.16	290.91	293.93	285.02	326.87	334.93
0.92	393.51	394.47	393.11	394.09	396.47	396.97	409.96	391.03	417.40	422.37
0.94	488.02	488.95	489.41	490.79	489.75	495.28	496.33	498.11	500.64	505.50
0.95	530.08	530.98	530.84	530.70	529.65	534.97	535.55	545.30	521.61	540.23

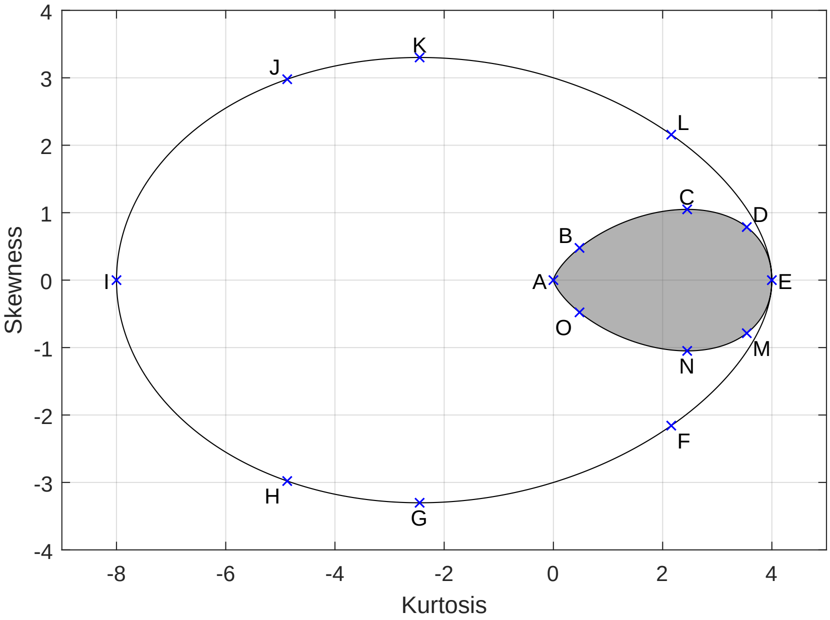
Table VI: Comparison of Errors for the Original and Alternative Convexity Adjustment Term.
This table shows a comparison of the mean errors of the standard deviation, skewness, and kurtosis estimators which were calculated from the valid Gram-Charlier region. The difference is that the expected return of BKM has been replaced by $E_t^{\mathbb{Q}}[R(t,\tau)] = \left(r - \frac{1}{2}\sigma^2 + \mu_c\right)\tau$. These values have been scaled. The options created have a forward price F_t^T, time to maturity τ, risk-free rate r, and standard deviation σ, have been arbitrarily set to \$2,000, one month, 2.4%, and 0.20, respectively. This has been done for various boundary controlling factor (a) values and step sizes (ΔK). The error for each a and ΔK is defined as Error $:=$ mean (Estimated Moment $-$ True Moment).

	$E_t^{\mathbb{Q}}[R(t,\tau)] = \text{BKM } \mu$					$E_t^{\mathbb{Q}}[R(t,\tau)] = \left(r - \frac{1}{2}\sigma^2 + \mu_c\right)\tau$				
ΔK	**1**	**2**	**3**	**4**	**5**	**1**	**2**	**3**	**4**	**5**
a	**Risk-Neutral Standard Deviation $\times 1,000$**									
0.05	-0.03	-0.05	-0.08	-0.10	-0.09	-0.03	-0.05	-0.08	-0.10	-0.09
0.40	-0.03	-0.05	0.03	-0.10	-0.09	-0.03	-0.05	0.03	-0.10	-0.09
0.75	-0.03	-0.06	-0.03	-0.10	-0.09	-0.03	-0.06	-0.03	-0.10	-0.09
0.80	-0.06	-0.09	-0.11	-0.13	-0.12	-0.06	-0.09	-0.11	-0.13	-0.12
0.82	-0.17	-0.20	-0.12	-0.25	-0.24	-0.17	-0.20	-0.12	-0.25	-0.24
0.84	-0.53	-0.56	-0.54	-0.60	-0.59	-0.53	-0.56	-0.54	-0.60	-0.59
0.86	-1.45	-1.49	-1.52	-1.53	-1.51	-1.45	-1.49	-1.52	-1.53	-1.51
0.88	-3.43	-3.46	-3.42	-3.51	-3.54	-3.43	-3.46	-3.42	-3.51	-3.54
0.90	-7.05	-7.08	-7.14	-7.21	-7.20	-7.05	-7.07	-7.14	-7.21	-7.20
0.92	-13.13	-13.23	-13.18	-13.28	-13.42	-13.12	-13.23	-13.18	-13.28	-13.42
0.94	-22.96	-23.12	-23.03	-23.45	-23.30	-22.95	-23.11	-23.02	-23.44	-23.29
0.95	-30.19	-30.40	-30.43	-30.46	-30.27	-30.17	-30.38	-30.41	-30.44	-30.25
a	**Risk-Neutral Skewness $\times 1,000$**									
0.05	-0.02	-0.04	-0.07	-0.09	-0.07	0.00	0.00	0.00	-0.00	0.01
0.40	-0.02	-0.04	0.02	-0.09	-0.07	0.00	0.00	0.00	-0.00	0.01
0.75	-0.01	-0.03	-0.01	-0.08	-0.06	0.02	0.02	0.02	0.01	0.02
0.80	0.76	0.73	0.71	0.69	0.71	0.81	0.81	0.81	0.81	0.82
0.82	2.57	2.41	2.48	2.08	1.95	2.72	2.58	2.58	2.29	2.16
0.84	6.41	6.39	6.01	6.35	6.38	6.86	6.87	6.48	6.87	6.89
0.86	13.68	12.77	11.85	12.74	13.66	14.93	14.06	13.17	14.07	14.97
0.88	22.95	22.94	19.67	22.92	19.62	25.95	25.96	22.66	25.98	22.72
0.90	33.81	33.80	28.57	28.54	28.55	40.03	40.04	34.88	34.91	34.92
0.92	39.19	35.34	39.18	35.32	27.55	50.99	47.24	51.03	47.27	39.65
0.94	44.68	39.06	39.05	27.64	33.38	65.92	60.47	60.37	49.38	54.97
0.95	48.46	41.53	41.53	41.52	48.46	76.99	70.30	70.33	70.35	77.08
a	**Risk-Neutral Kurtosis $\times 100$**									
0.05	0.30	0.57	0.81	1.03	0.92	0.30	0.57	0.81	1.03	0.92
0.40	0.30	0.57	-0.27	1.03	0.92	0.30	0.57	-0.27	1.03	0.92
0.75	0.25	0.52	0.30	0.99	0.87	0.25	0.52	0.30	0.99	0.87
0.80	-2.72	-2.45	-2.20	-1.98	-2.10	-2.72	-2.45	-2.20	-1.98	-2.10
0.82	-10.04	-9.86	-10.69	-9.59	-9.81	-10.04	-9.87	-10.69	-9.60	-9.81
0.84	-28.07	-27.82	-28.24	-27.38	-27.50	-28.07	-27.82	-28.24	-27.38	-27.50
0.86	-62.21	-62.36	-62.55	-61.95	-61.67	-62.21	-62.37	-62.55	-61.96	-61.67
0.88	-114.83	-114.61	-116.48	-114.24	-115.53	-114.83	-114.61	-116.48	-114.24	-115.53
0.90	-181.35	-181.16	-182.81	-182.33	-182.41	-181.34	-181.15	-182.80	-182.32	-182.40
0.92	-255.47	-256.14	-255.16	-255.86	-257.57	-255.44	-256.11	-255.13	-255.82	-257.54
0.94	-328.86	-329.59	-329.96	-331.04	-330.22	-328.76	-329.50	-329.87	-330.98	-330.15
0.95	-364.12	-364.86	-364.74	-364.63	-363.78	-363.95	-364.71	-364.59	-364.48	-363.61

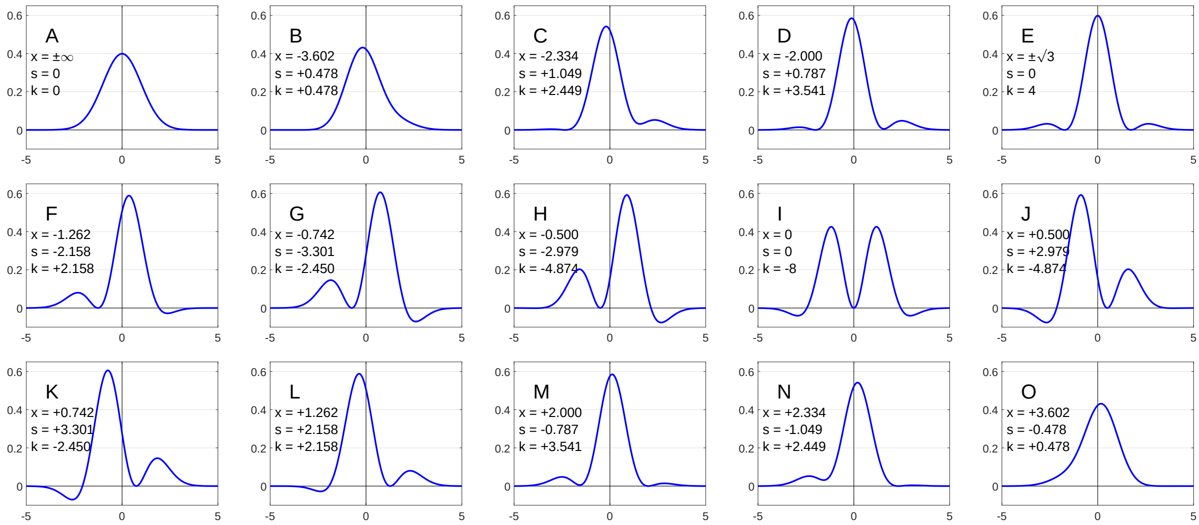
Table VII: Comparison of Maximum Value of the Absolute Error for the Original and Alternative Convexity Adjustment Term.
maximum values of the absolute errors of the standard deviation, skewness, and kurtosis estimators which were calculated from the valid Gram-Charlier region. The difference is that the expected return of BKM has been replaced by $E_t^Q\left[R\left(t,\tau\right)\right] = \left(r - \frac{1}{2}\sigma^2 + \mu_c\right)\tau$. These values have been scaled. The options created have a forward price F_t^T, time to maturity τ, risk-free rate r, and standard deviation σ, have been arbitrarily set to $2,000, one month, 2.4%, and 0.20, respectively. This has been done for various boundary controlling factor (a) values and step sizes (ΔK). The error for each a and ΔK is defined as Error := max |Estimated Moment − True Moment|.

	$E_t^Q\left[R\left(t,\tau\right)\right] = \text{BKM } \mu$					$E_t^Q\left[R\left(t,\tau\right)\right] = \left(r - \frac{1}{2}\sigma^2 + \mu_c\right)\tau$				
ΔK	**1**	**2**	**3**	**4**	**5**	**1**	**2**	**3**	**4**	**5**
a	**Risk-Neutral Standard Deviation $\times 1,000$**									
0.05	0.03	0.05	0.08	0.10	0.09	0.03	0.05	0.08	0.10	0.09
0.40	0.03	0.05	0.03	0.10	0.09	0.03	0.05	0.03	0.10	0.09
0.75	0.03	0.06	0.03	0.10	0.09	0.03	0.06	0.03	0.10	0.09
0.80	0.09	0.12	0.14	0.16	0.15	0.09	0.12	0.14	0.16	0.15
0.82	0.29	0.32	0.24	0.37	0.36	0.29	0.32	0.24	0.37	0.36
0.84	0.94	0.96	0.95	1.01	1.00	0.94	0.96	0.95	1.01	1.00
0.86	2.56	2.60	2.64	2.65	2.62	2.56	2.60	2.64	2.64	2.62
0.88	5.90	5.93	5.91	5.98	6.03	5.90	5.93	5.91	5.98	6.03
0.90	11.60	11.62	11.71	11.78	11.77	11.59	11.62	11.70	11.78	11.77
0.92	20.07	20.18	20.13	20.23	20.38	20.06	20.17	20.12	20.22	20.37
0.94	31.66	31.82	31.72	32.12	31.99	31.64	31.80	31.70	32.10	31.96
0.95	39.19	39.39	39.42	39.45	39.28	39.17	39.36	39.39	39.42	39.25
a	**Risk-Neutral Skewness $\times 1,000$**									
0.05	0.45	0.86	1.25	1.59	1.40	0.43	0.83	1.18	1.50	1.34
0.40	0.45	0.86	0.42	1.59	1.40	0.43	0.83	0.40	1.50	1.34
0.75	0.42	0.84	0.50	1.57	1.38	0.44	0.83	0.52	1.50	1.34
0.80	1.76	1.43	1.43	1.54	1.48	1.82	1.53	1.55	1.68	1.61
0.82	8.03	7.49	8.78	6.51	6.65	8.23	7.71	8.92	6.77	6.84
0.84	26.04	25.64	25.85	24.94	25.14	26.46	26.08	26.27	25.43	25.62
0.86	70.13	69.52	68.94	68.86	69.28	71.30	70.71	70.15	70.09	70.50
0.88	157.13	156.77	157.28	156.17	155.71	159.92	159.58	160.02	159.01	158.56
0.90	300.10	299.79	299.10	298.29	298.46	305.91	305.62	304.93	304.18	304.33
0.92	493.22	491.88	492.73	491.42	489.29	504.25	502.95	503.81	502.54	500.44
0.94	706.05	702.99	703.47	696.68	699.71	726.13	723.16	723.56	717.04	719.99
0.95	805.46	800.84	800.68	800.51	804.98	832.66	828.19	828.06	827.91	832.26
a	**Risk-Neutral Kurtosis $\times 100$**									
0.05	0.40	0.77	1.10	1.40	1.25	0.40	0.77	1.10	1.40	1.25
0.40	0.40	0.77	0.37	1.40	1.25	0.40	0.77	0.37	1.40	1.25
0.75	0.32	0.69	0.39	1.32	1.16	0.32	0.69	0.39	1.32	1.16
0.80	4.92	4.56	4.23	3.93	4.09	4.92	4.56	4.23	3.94	4.09
0.82	17.61	17.41	18.53	17.11	17.44	17.61	17.41	18.53	17.12	17.44
0.84	48.14	47.79	48.44	47.21	47.37	48.14	47.80	48.44	47.21	47.37
0.86	104.25	104.58	104.95	104.04	103.54	104.26	104.59	104.96	104.05	103.54
0.88	187.73	187.44	190.21	186.95	188.96	187.72	187.43	190.21	186.94	188.96
0.90	288.52	288.27	290.72	290.10	290.19	288.48	288.23	290.69	290.06	290.16
0.92	393.61	394.56	393.21	394.19	396.54	393.51	394.47	393.11	394.09	396.47
0.94	488.22	489.13	489.59	490.92	489.91	488.02	488.95	489.41	490.79	489.75
0.95	530.36	531.24	531.09	530.95	529.94	530.08	530.98	530.84	530.70	529.65

Figures



(a) Gram-Charlier Curve.



(b) Probability Densities.

Figure 1: Gram-Charlier Valid Region.
This figure shows the region (shaded area) in which pairs of skewness and kurtosis values will yield valid probability density function using the Gram-Charlier series. The curve represents the skewness and kurtosis required to make the density function vanish. Point A corresponds to the origin and points B to O, correspond to various points where the gradient is $-\infty$, -1, 0, 1, or ∞.

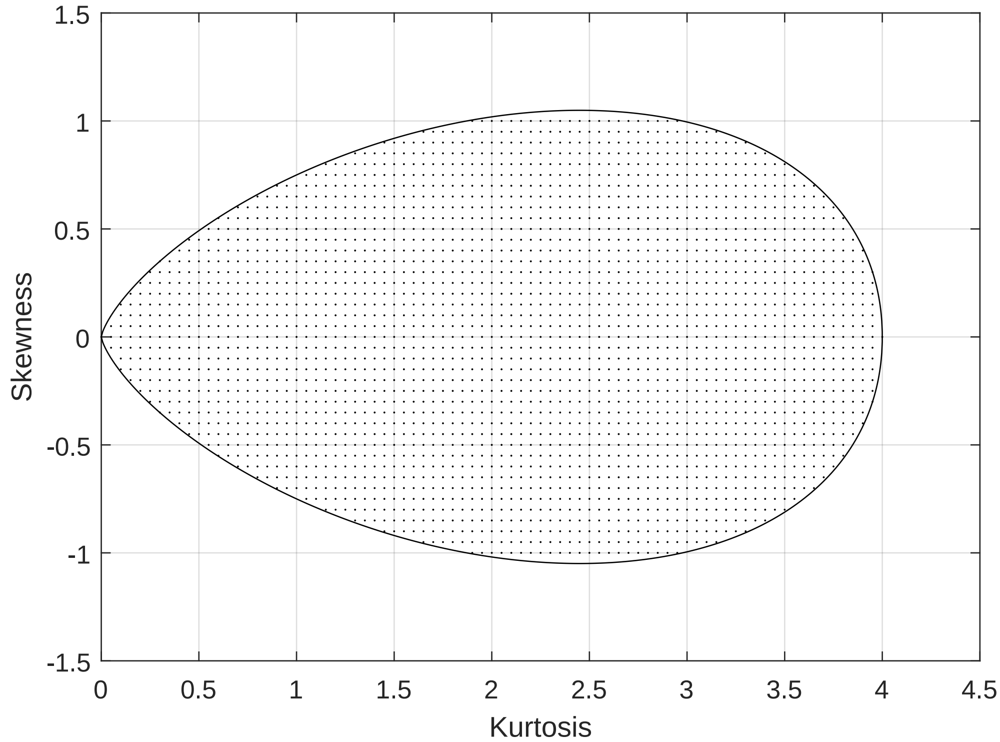
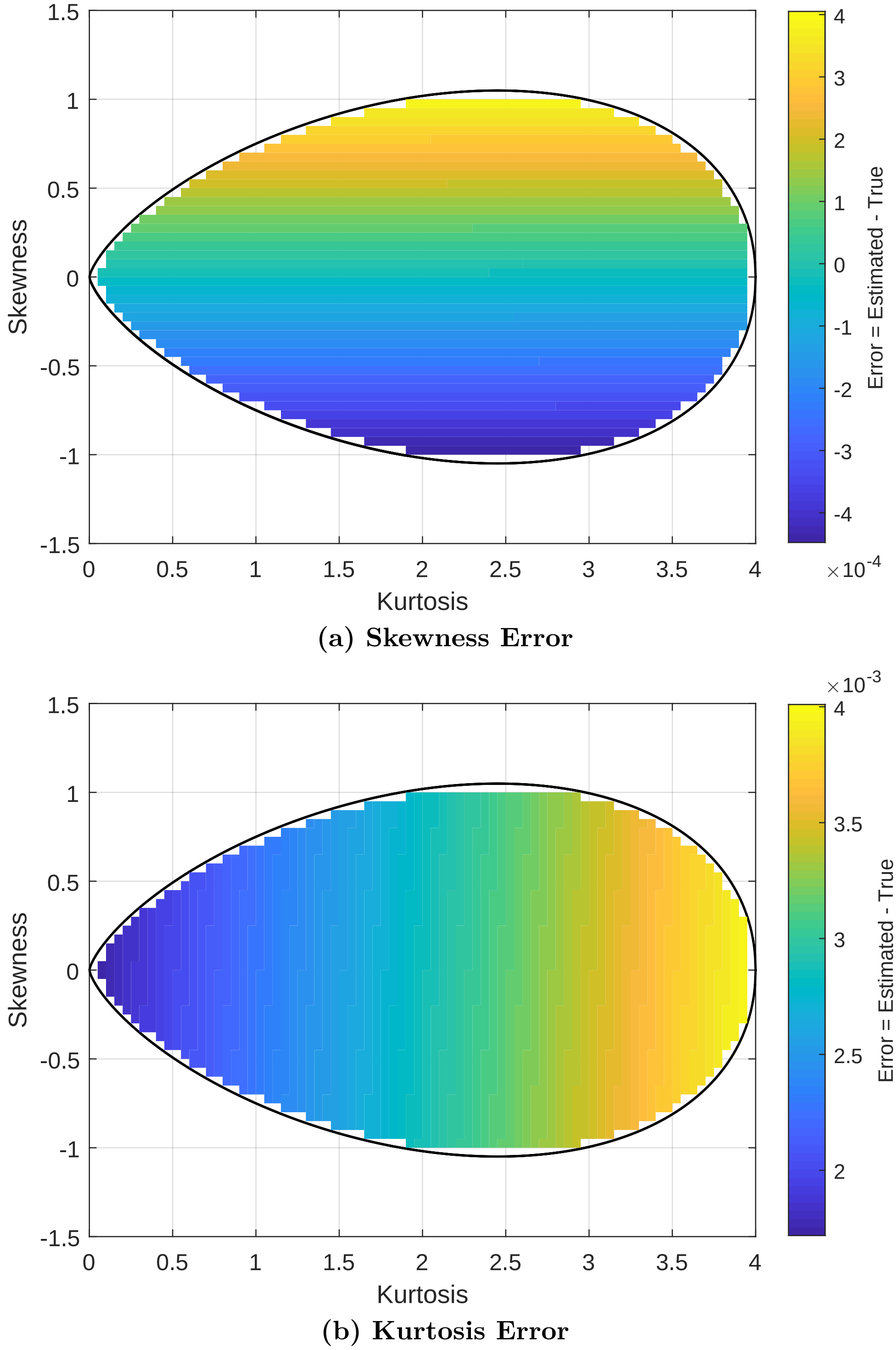


Figure 2: Points in the Gram-Charlier Region used for Calculations.
Each point used for calculations has been marked with a black dot. The number of points is 2,515. Skewness ranges between -1.05 and 1.05 and kurtosis between 0 and 4.



(a) Skewness Error

(b) Kurtosis Error

Figure 3: Example of the Errors for Skewness and Kurtosis.
These figures show the errors between the estimated skewness (a) and kurtosis (b) (using the BKM method) and their true values. Based on the valid Gram-Charlier region, of which 2,515 pairs of true skewness and kurtosis values were used to calculated the error surface. For this set of figures, $K \in [500, 8000]$ $(a = 0.25)$ and $\Delta K = 1$.

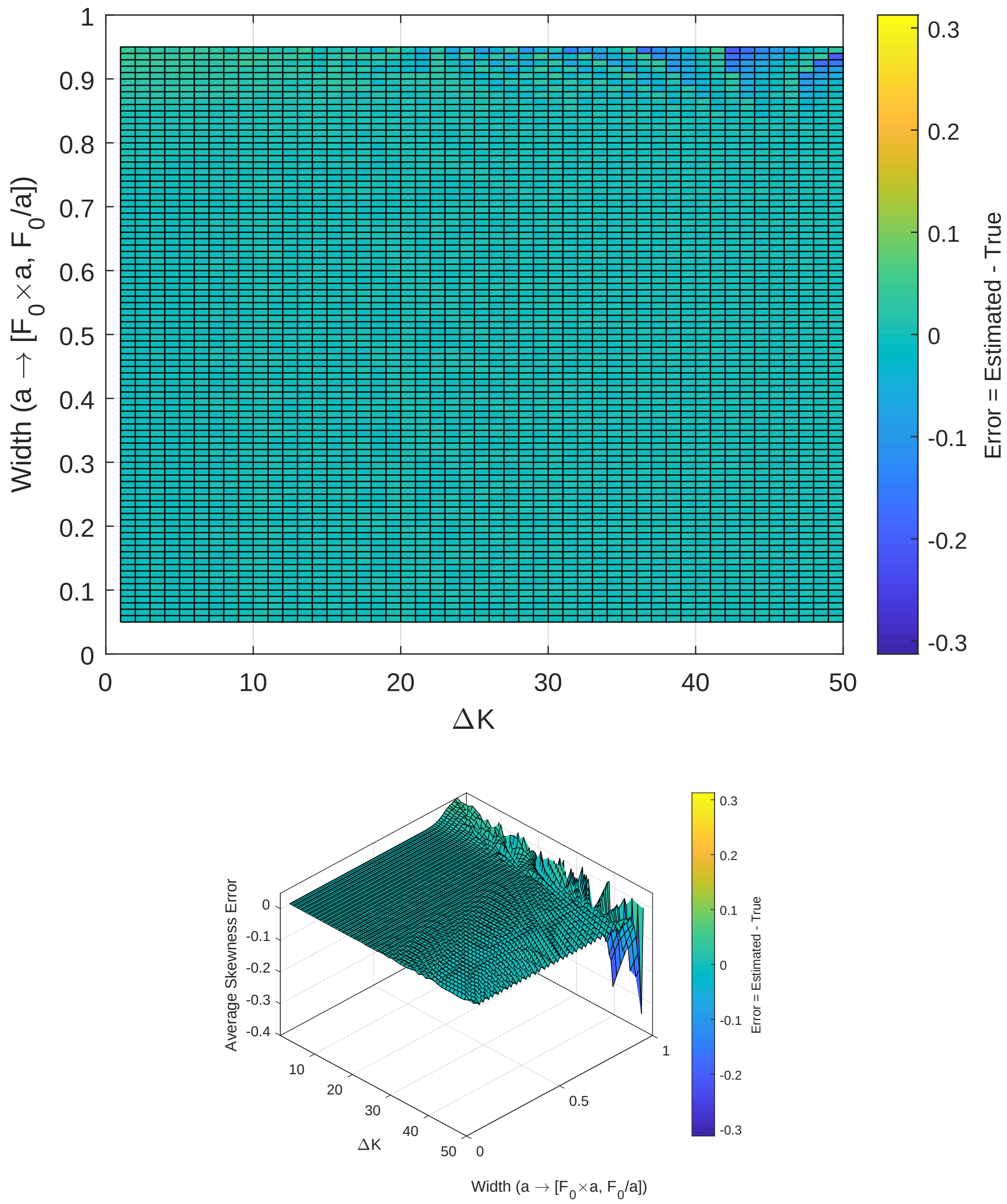
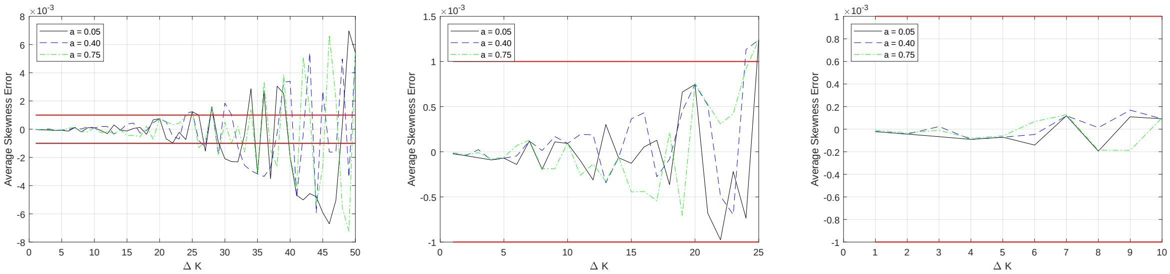
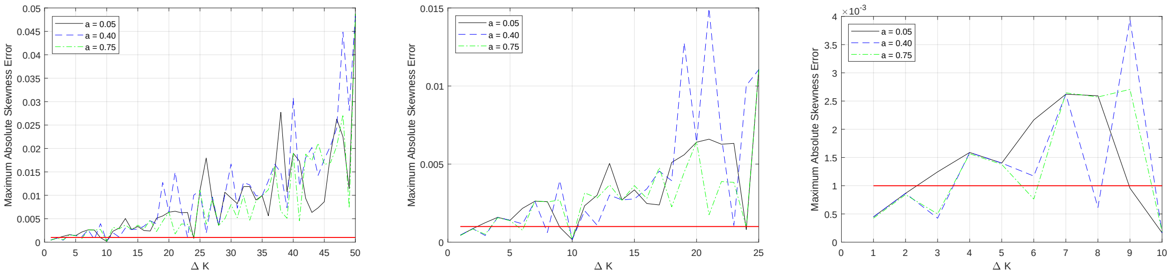


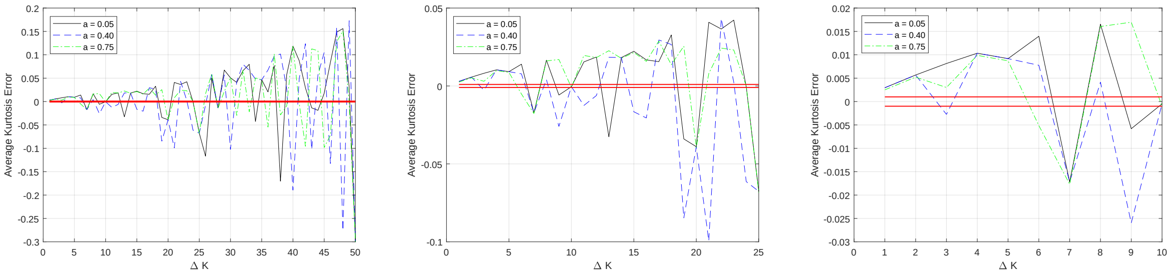
Figure 4: Sensitivity of the Average Error for Skewness.
Using the Gram-Charlier series, virtual options with known standard deviation σ, skewness λ_1, and kurtosis λ_2 were created for a forward price F_t^T, time to maturity τ, risk-free rate r, and standard deviation σ, have been arbitrarily set to \$2,000, one month, 2.4%, and 0.20, respectively. In total 2,515 different pairs of skewness and kurtosis (within the Gram-Charlier region) were used. The errors are shown for a varying degree of fineness of strike discretisation ($\Delta K \in [1, 50]$). The range of strikes used is also varied symmetrically by adjusting $a \in [0.05, 0.95]$ for $[K_{\min}, K_{\max}] = [F_t^T \times a, F_t^T / a]$ (rounded to the nearest dollar). These errors are the averaged errors of the entire valid Gram-Charlier region.



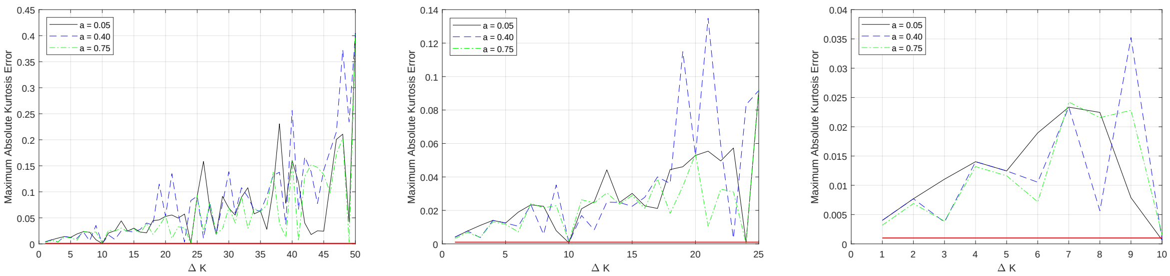
(a) Mean Skewness Error.



(b) Maximum Absolute Skewness Error.



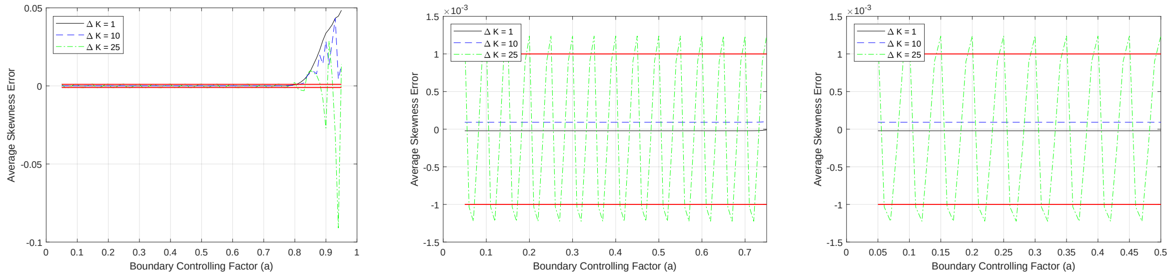
(c) Mean Kurtosis Error.



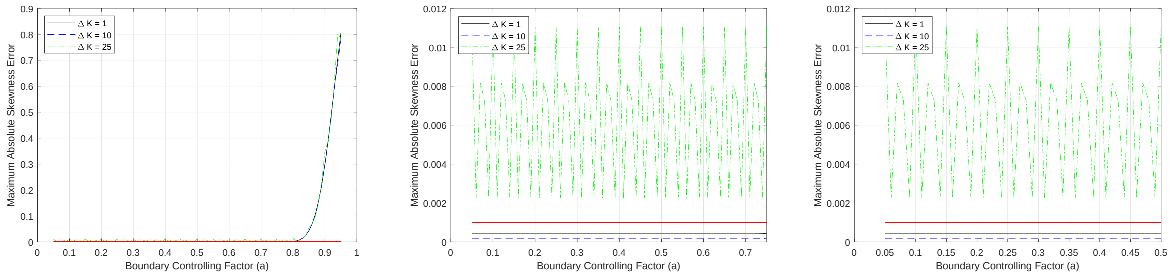
(d) Maximum Absolute Kurtosis Error.

Figure 5: Errors w.r.t. $\boldsymbol{\Delta K}$.

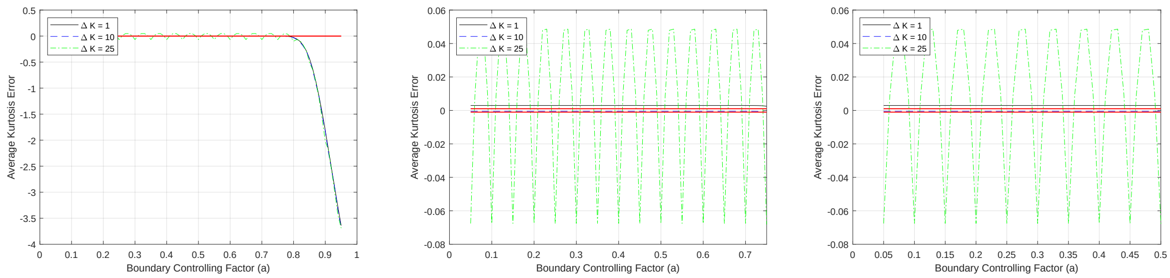
These figures have been created for boundary controlling factor values of 0.05, 0.40, and 0.75 and for step sizes of ΔK from 1 to 50, to 25, and to 10. The error boundary of 10^{-3} is shown in red.



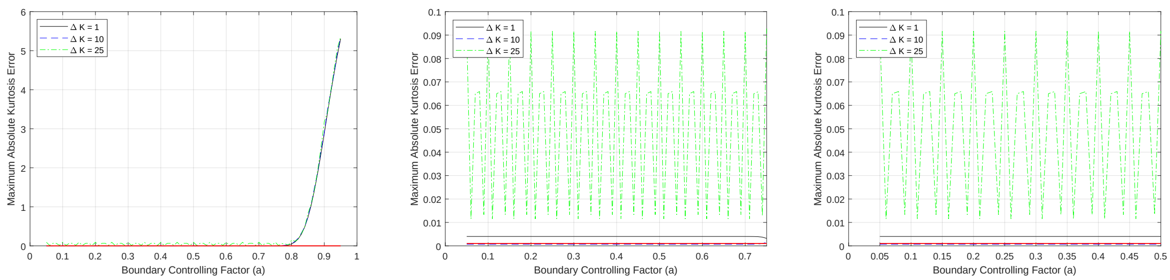
(a) **Mean Skewness Error.**



(b) **Maximum Absolute Skewness Error.**

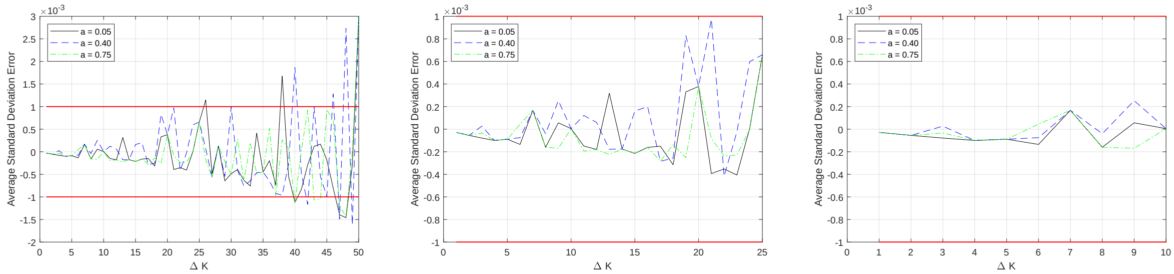


(c) **Mean Kurtosis Error.**

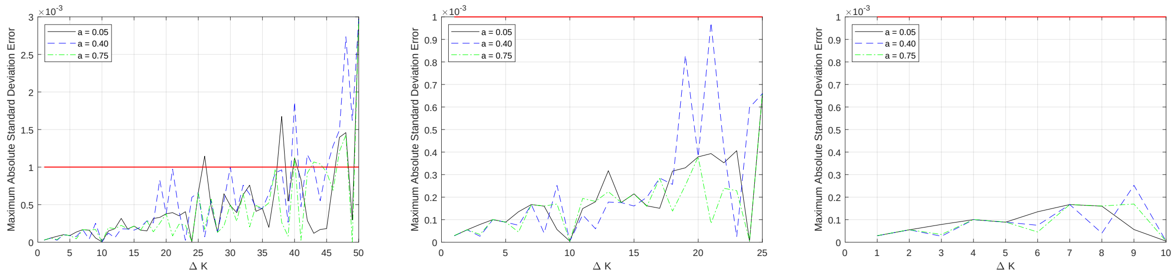


(d) **Maximum Absolute Kurtosis Error.**

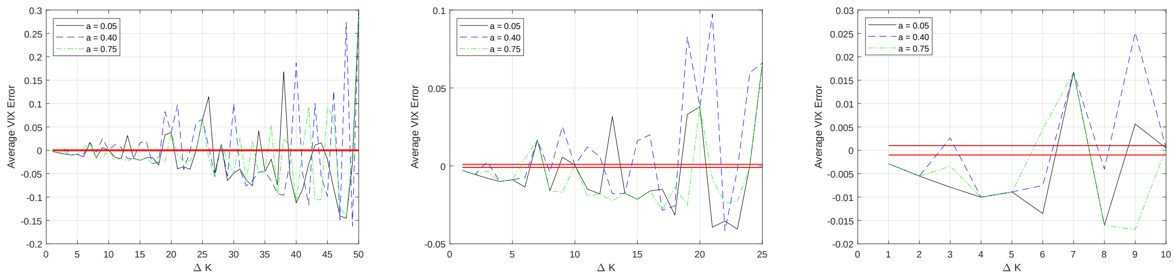
Figure 6: Errors w.r.t. $\boldsymbol{a}$**.**
These figures have been created for step sizes of 1, 10, and 25 and for boundary controlling factors of a from 0.05 to 0.95, to 0.75, and to 0.50. The error boundary of 10^{-3} is shown in red.



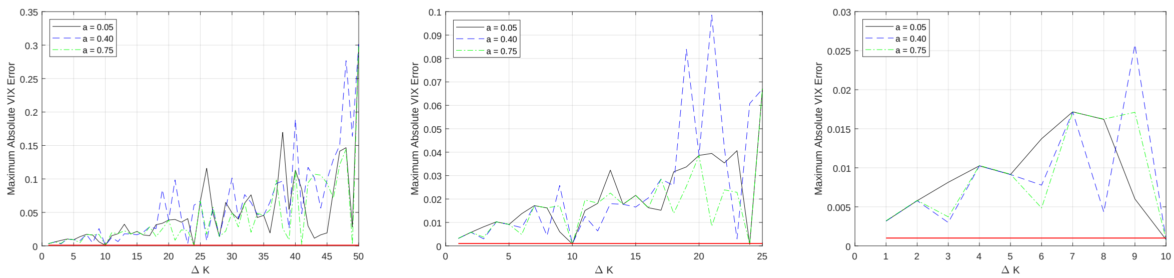
(a) **Mean Standard Deviation Error.**



(b) **Maximum Absolute Standard Deviation Error.**

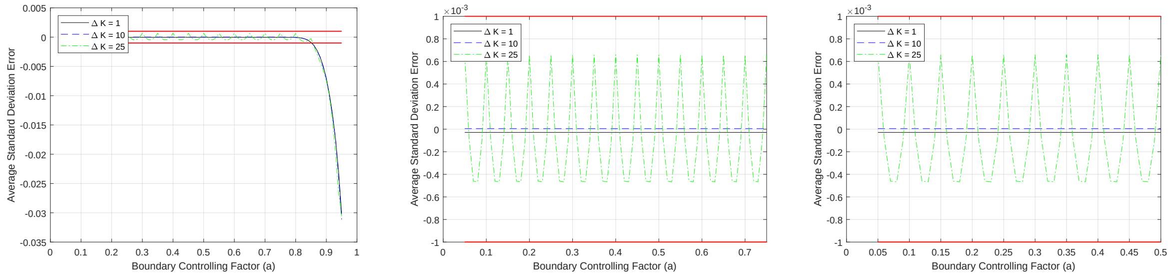


(c) **Mean VIX Error.**

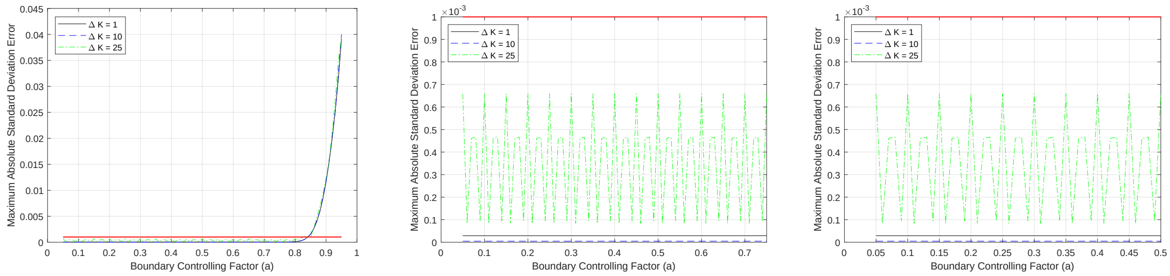


(d) **Maximum Absolute VIX Error.**

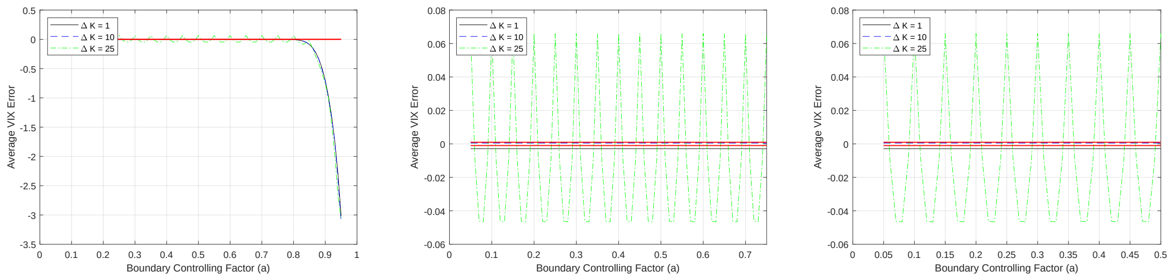
Figure 7: Standard Deviation Errors w.r.t. ΔK.
These figures have been created for boundary controlling factor values of 0.05, 0.40, and 0.75 and for step sizes of ΔK from 1 to 50, to 25, and to 10. The error boundary of 10^{-3} is shown in red.



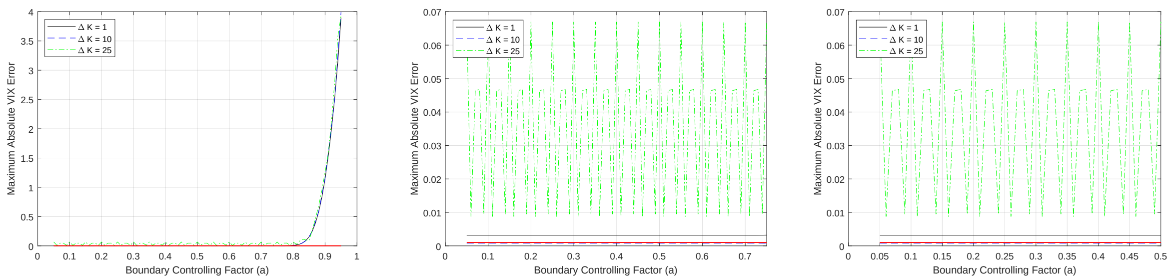
(a) Mean Standard Deviation Error.



(b) Maximum Absolute Standard Deviation Error.

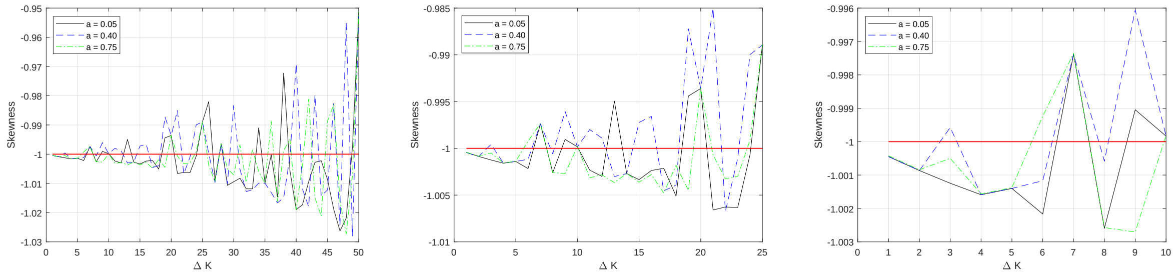


(c) Mean VIX Error.

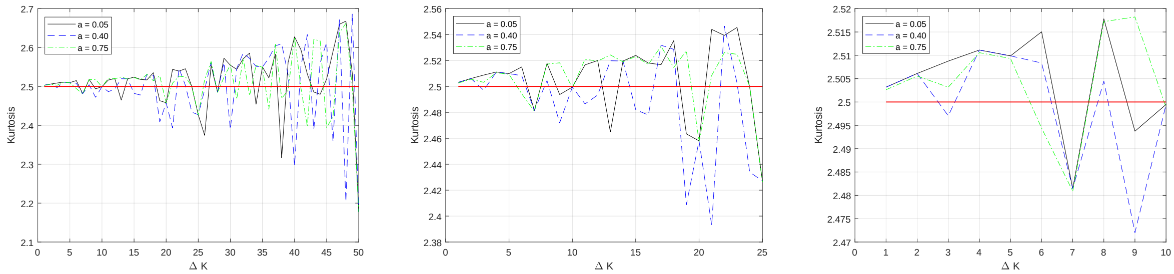


(d) Maximum Absolute VIX Error.

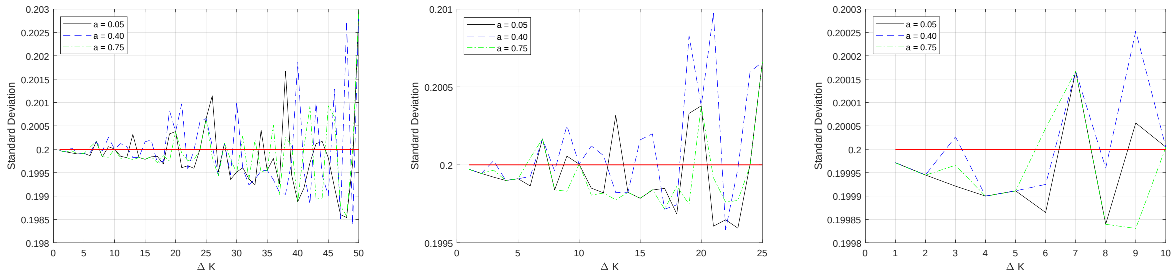
Figure 8: Standard Deviation Errors w.r.t. a.
These figures have been created for step sizes of 1, 10, and 25 and for boundary controlling factors of a from 0.05 to 0.95, to 0.75, and to 0.50. The error boundary of 10^{-3} is shown in red.



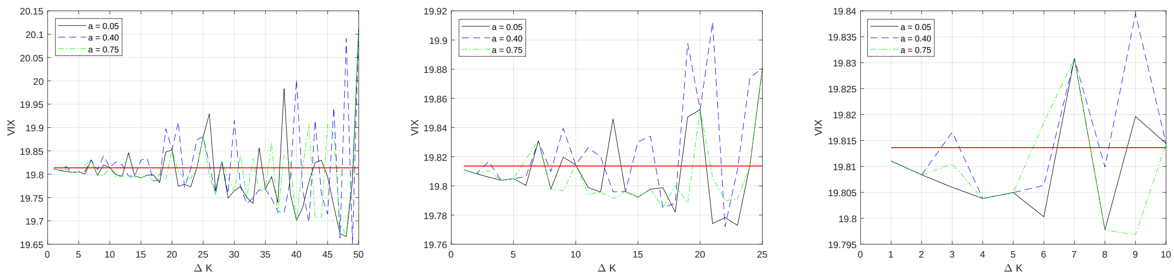
(a) Skewness.



(b) Kurtosis.

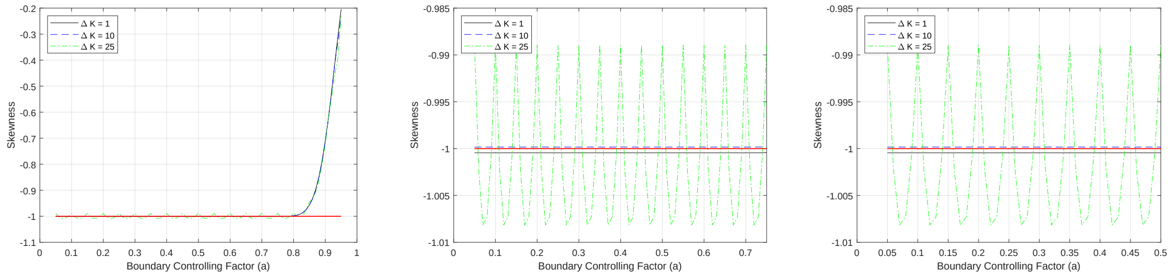


(c) Standard Deviation.

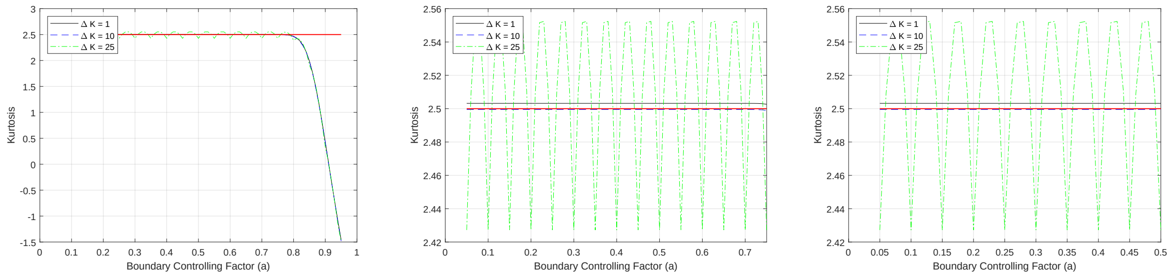


(d) VIX.

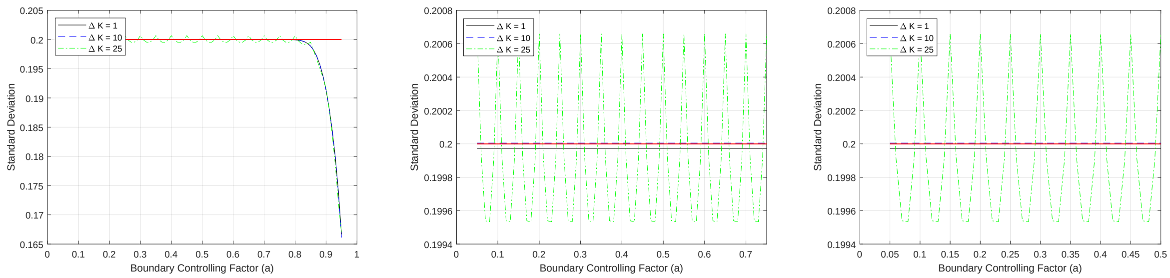
Figure 9: Values w.r.t. ΔK.
These figures have been created for boundary controlling factor values of 0.05, 0.40, and 0.75 and for step sizes of ΔK from 1 to 50, to 25, and to 10. The true value is shown in red. These figures have been plotted for a standard deviation (σ), skewness (λ_1) and kurtosis (λ_2) of 0.20, -1, and 2.5, respectively. The corresponding VIX value is 19.8136.



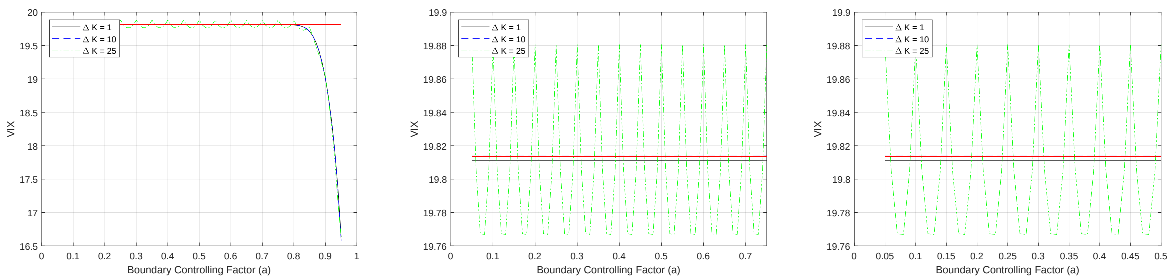
(a) Skewness.



(b) Kurtosis.

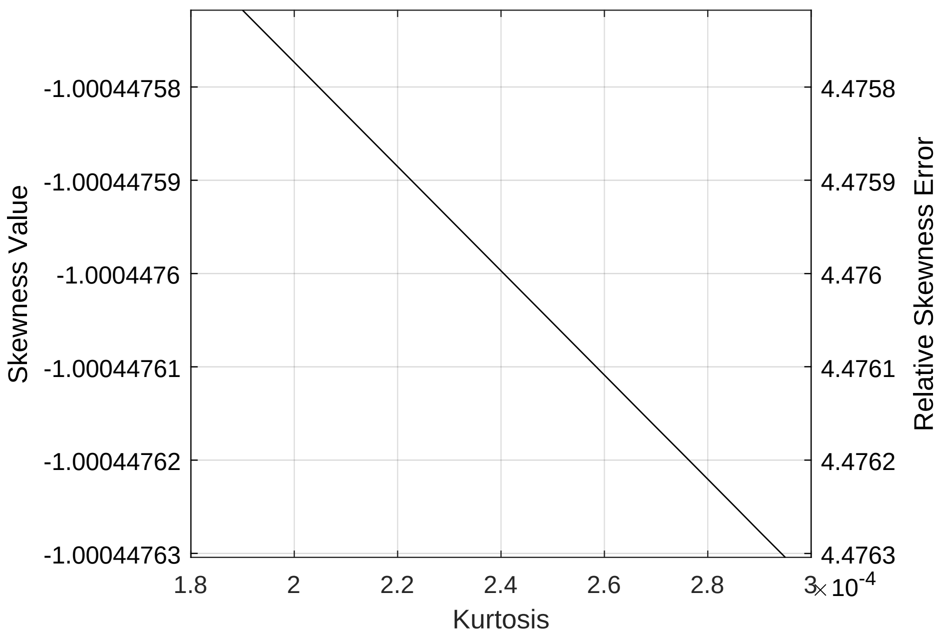


(c) Standard Deviation.

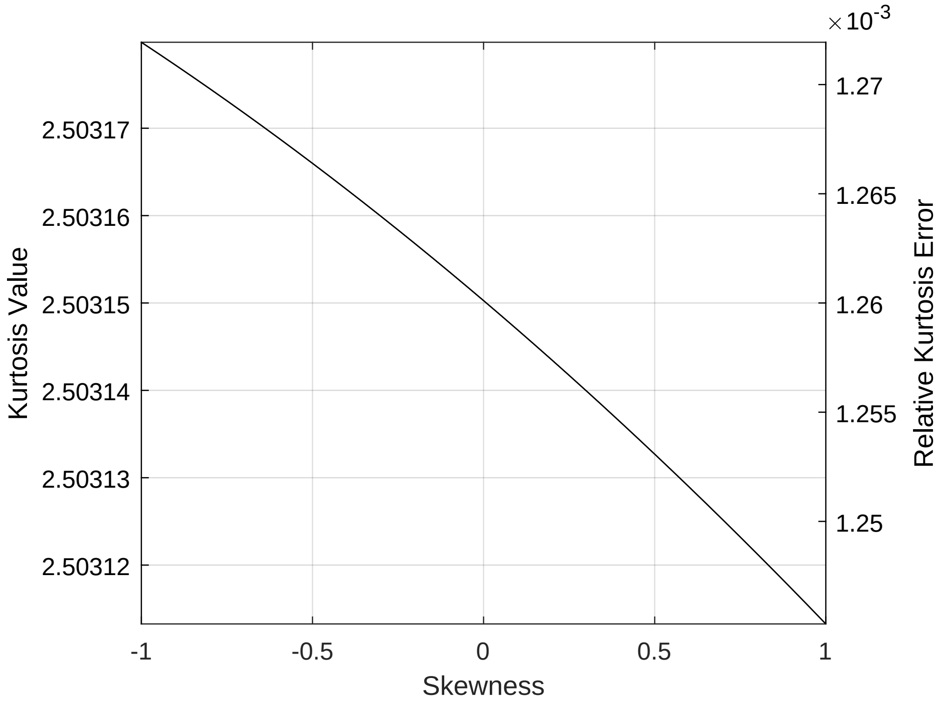


(d) VIX.

Figure 10: Values w.r.t. *a*.
These figures have been created for step sizes of 1, 10, and 25 and for boundary controlling factors of a from 0.05 to 0.95, to 0.75, and to 0.50. The true value is shown in red. These figures have been plotted for a standard deviation (σ), skewness (λ_1) and kurtosis (λ_2) of 0.20, -1, and 2.5, respectively. The corresponding VIX value is 19.8136.



(a) Skewness Values and Relative Errors.



(b) Kurtosis Values and Relative Errors.

Figure 11: Independence of Skewness and Kurtosis.
These figure shows to values and relative errors of skewness when kurtosis is changed (a), and kurtosis when skewness is changed (b). The boundary controlling factor and step size are set to 0.25 and 1, respectively. When testing skewness, the true skewness is set to -1. For kurtosis, the true value is set to 2.5.

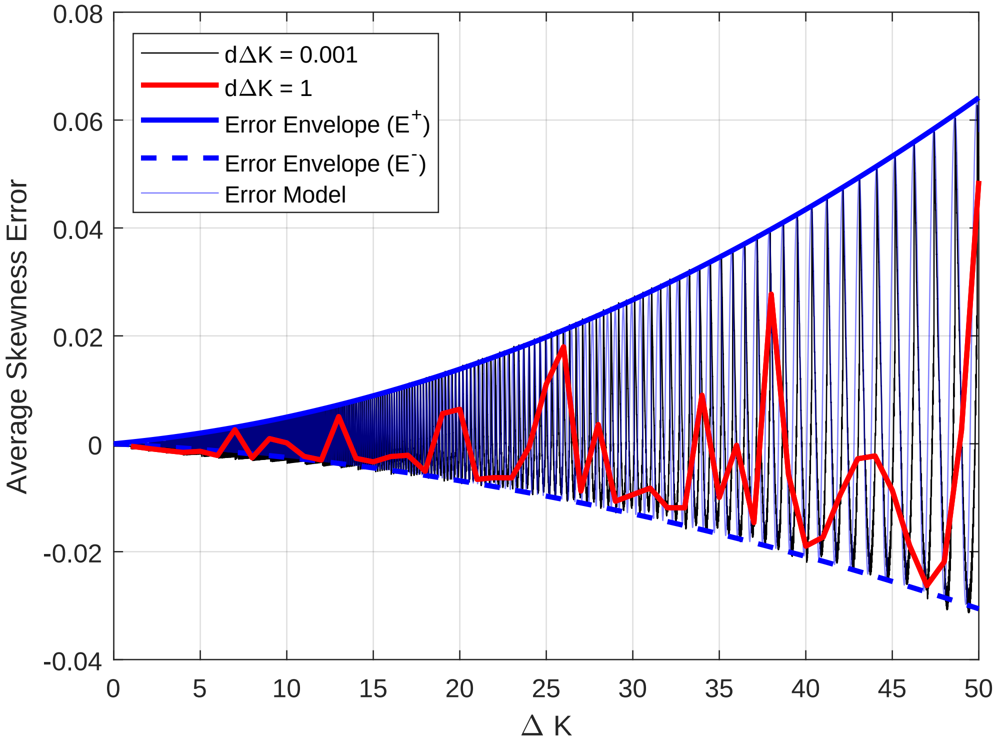
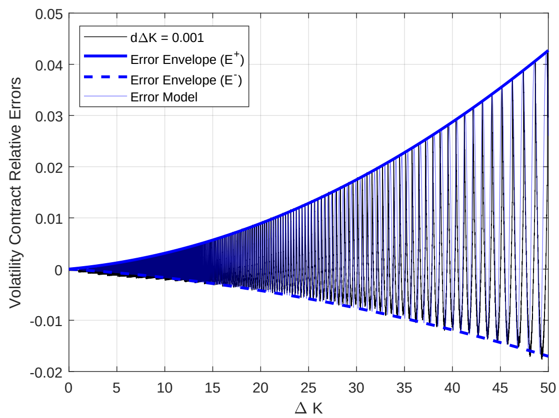


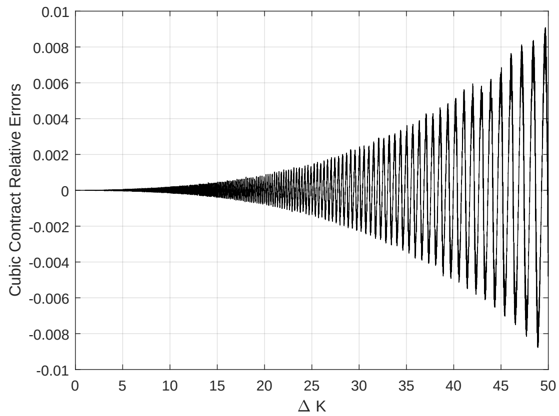
Figure 12: Granularity of Step Size ΔK.
The errors for skewness for a boundary controlling factor of $a = 0.05$ with varing step sizes between 1 and 50 shows that, in general, as the step size gets larger, so do the errors. When the fineness of the granularity is increased from $\Delta K = 1$ (black line) to $\Delta K = 0.001$ (red line), the shape of the relationship between the error and the step size is reavealed. The forward price F_t^T, time to maturity τ, risk-free rate r, standard deviation σ, skewness λ_1, and kurtosis λ_2, are equal to \$2,000, one month, 2.4%, 0.20, -1, and 2.5, respectively. The error model and envelope $(E^{\pm})$ is given by

$$\text{Error Model} = X\beta^s \sin\left(\frac{2\pi}{X\beta^\omega}\Delta K\right) + X\beta^t \text{ and } E^{\pm} = X\left(\beta^t \pm \beta^s\right)$$

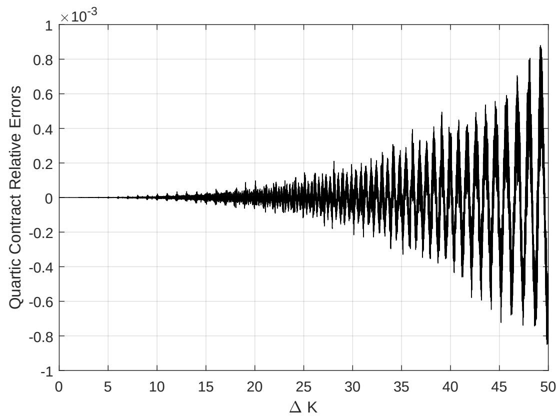
where $\beta^s = \left[2.311 \times 10^{-04} \quad 1.433 \times 10^{-05}\right]^T$, $\beta^t = \left[6.726 \times 10^{-05} \quad 5.366 \times 10^{-06}\right]^T$, $\beta^\omega = \left[6.980 \times 10^{-04} \quad 4.967 \times 10^{-04}\right]^T$, and $X = \left[\Delta K \quad (\Delta K)^2\right]$.



(a) Volatility Contract Relative Error.

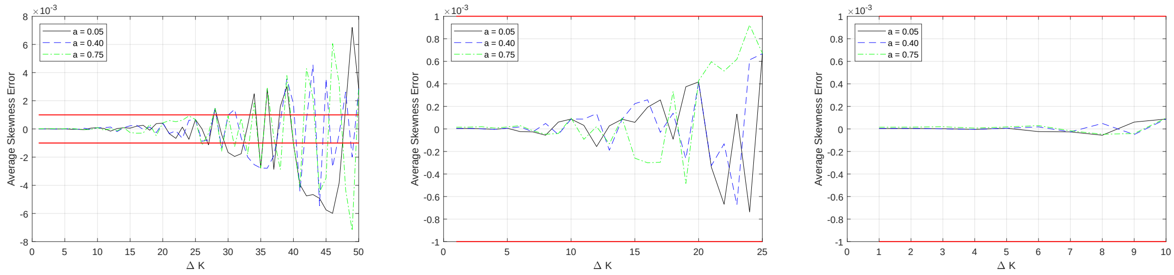


(b) Cubic Contract Relative Error.

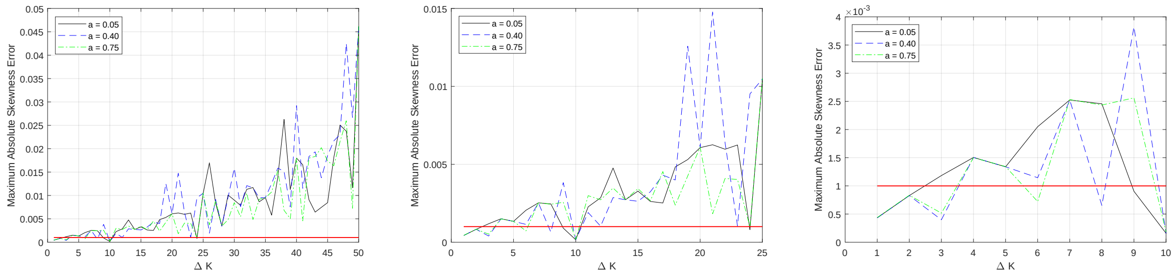


(c) Quartic Contract Relative Error.

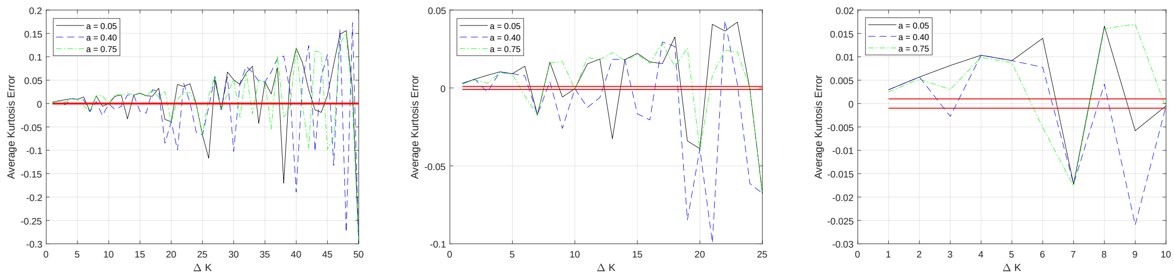
Figure 13: Contract Errors w.r.t. ΔK.
The relative errors for the volatility, cubic, and quartic contracts for a boundary controlling factor of $a = 0.05$ with varing step sizes between 1 and 50 shows that, in general, as the step size gets larger, so do the errors. The forward price F_t^T, time to maturity τ, risk-free rate r, standard deviation σ, skewness λ_1, and kurtosis λ_2, are equal to \$2,000, one month, 2.4%, 0.20, -1, and 2.5, respectively. The value of the volatility, cubic, and quartic contracts are 3.327×10^{-03}, -1.884×10^{-04}, and 6.071×10^{-05}, respectively. The relative error is defined as Relative Error $:= \frac{\text{Estimated}-\text{True}}{\text{True}}$.



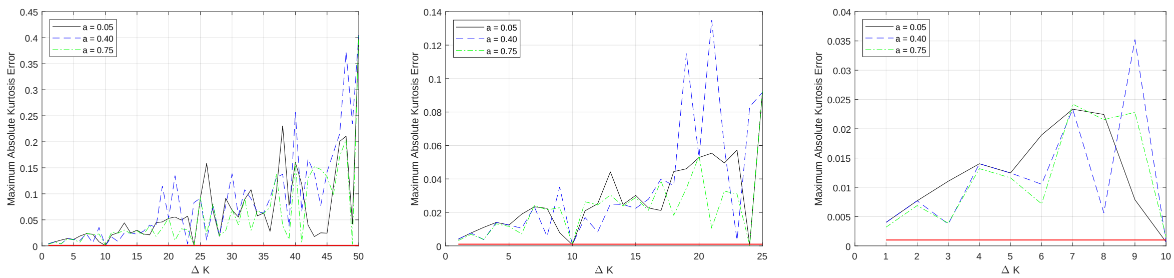
(a) Mean Skewness Error.



(b) Maximum Absolute Skewness Error.

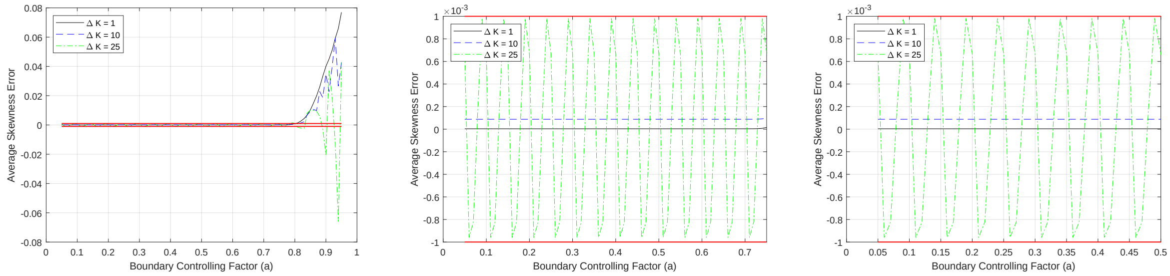


(c) Mean Kurtosis Error.

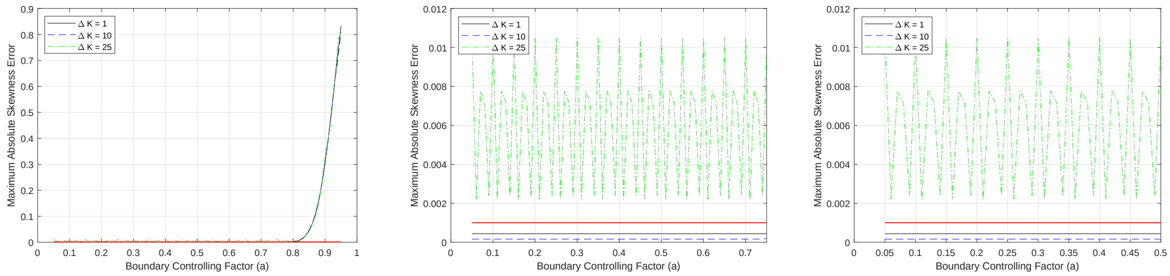


(d) Maximum Absolute Kurtosis Error.

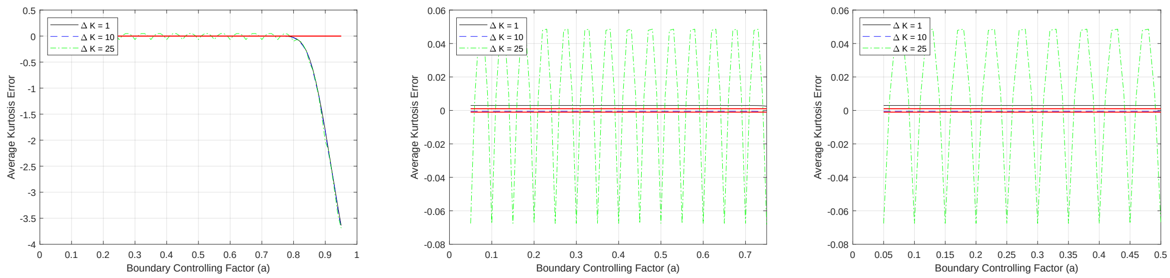
Figure 14: Errors w.r.t. ΔK with Alternative Convexity Adjustment Term. These figures have been created for boundary controlling factor values of 0.05, 0.40, and 0.75 and for step sizes of ΔK from 1 to 50, to 25, and to 10. The error boundary of 10^{-3} is shown in red.



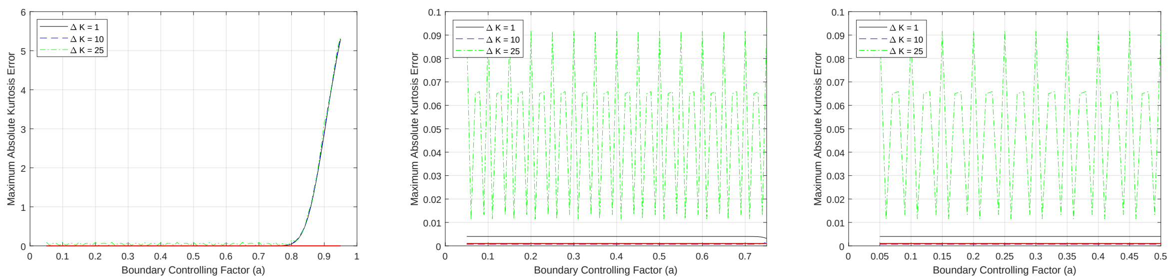
(a) Mean Skewness Error.



(b) Maximum Absolute Skewness Error.



(c) Mean Kurtosis Error.



(d) Maximum Absolute Kurtosis Error.

Figure 15: Errors w.r.t. a with Alternative Convexity Adjustment Term. These figures have been created for step sizes of 1, 10, and 25 and for boundary controlling factors of a from 0.05 to 0.95, to 0.75, and to 0.50. The error boundary of 10^{-3} is shown in red.