Can Small Transaction Costs Have Non–Vanishing Effect on Equilibrium Prices? The Affirmative Answer

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Abstract

This paper finds a linear-quadratic equilibrium in the stock market with a lot of small investors where a bid–ask spread faced by one investor increases with the number of traded shares. The latter effect is modeled by means of transaction costs which are convex in the number of traded shares. It is expected that in the limit of very small transaction costs optimal strategies and the stock price should converge to those in the market without these costs. I show, however, that the latter does not have to be the case and very small transitory price impacts can have a dramatic effect on the equilibrium stock price. This effect emerges because optimal allocations of investors depend on the ratio, rather than sizes, of transaction costs paid by different types of investors.

Keywords: Equilibrium; Asset Pricing; Transaction Costs
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1 Introduction

A competitive market assumes that supply of limit order has a perfect elasticity. That is, a market order of any size is met by a matching limit order for a given price. In practice, however, the latter is not the case even in the presence of large number of small investors. In particular, there is a significant probability that a market order of a given size will be split between a few limit orders with different prices. Consequently, a small investor exerts an impact on her trading price. This paper proposes a model which takes into account these deviations from the stock market competitiveness. It determines an equilibrium price in the stock market where investors impose transitory impacts on their trading prices. Furthermore, it shows that the presence of transitory price impacts may have such a strong effect on an equilibrium stock price that it remains significant even when this impact is very small. A non–vanishing effect of transitory price impacts on the equilibrium price will be called a zero–order effect.

I consider the economy with a lot of small investors. Market orders from investors are matched with limit orders from market makers. The quoted bid–ask spread has a limited elasticity of supply of limit orders resulting in a positive relation between the size of a market order and the actual bid–ask spread paid by an investor. Meantime, the average (or equilibrium) price between bid and ask prices is not affected by the number of shares traded by one investor due to the offsetting trades. The assumed transitory price impact faced by one investor will be called local. It will be shown that the presence of the local transitory price impact results in the macro transitory price impact where an equilibrium stock price undergoes temporal changes even in the absence of information shocks.

This study models the local transitory price impact by means of transaction costs. Moreover, the dependence of a bid–ask spread from the number of traded shares implies a convexity of transaction costs in the number of shares. The transaction costs are set to be quadratic for the purpose of tractability. Furthermore, the notions of a local transitory price impact and a transaction cost are synonymous in this paper and will be used interchangeably.

Traditionally, a bid–ask spread is modeled with proportional transaction costs which assume a perfect elasticity of supply of limit orders for a given bid–ask spread. An investor is then considered in a partial equilibrium framework[1] or in a general equilibrium framework usually in the presence of other investors.[2] It appears that both approaches typically result in the prediction that realistically small transaction costs should have a negligible effect on security prices. The exception from this conclusion is the paper by Lo, Mamaysky, and Wang (2004) which shows that a small fixed transaction costs κ can cause a change in the stock price of the order of $\sqrt{\kappa}$. Given that κ converges to zero much faster than $\sqrt{\kappa}$, the effect of transaction costs on prices *relatively* to the size of transaction costs could be much higher. The conclusion of Lo, Mamaysky, and Wang (2004) agrees with empirical studies according to which transaction costs can have a strong impact on

[1] See, for example, Constantinides (1986), He and Mamaysky (2005), Jang, Koo, Liu, and Loewenstein (2007).

[2] See, for example, Vayanos (1998), Lo, Mamaysky, and Wang (2004), Buss, Uppal, and Vilkov (2013), and Buss and Dumas (2017).

security prices.[3] This paper makes one step further and predicts that transaction costs can have zero–order effect on prices.

The paper considers the economy in continuous time. Therefore, convexity of costs in the number of shares becomes convexity of costs in the rate of share trading. As a consequence, an investor endogenously chooses to trade a finite volume and is not able to adjust her allocations quickly enough to respond to information shocks. In practice local transitory price impact implies that investors have to split their orders and spend more time on updating their portfolios. It follows that convex transaction costs make investors trade with delays. Hence, the predictions of the model will also hold in the presence of delays in capital allocations arising for reasons other than a local transitory price impact.

Investors of two types borrow and lend at a fixed interest rate and receive non–hedgeable endowments. Following Lo, Mamaysky, and Wang (2004), I make an endowment process short-term unpredictable in order to create strong trading incentives. Then linear–quadratic equilibria in the stock market in the economy with no transaction costs and in the economy where investors pay transaction costs are found. The solution of each equilibrium is reduced to finding the roots of nonlinear equation which can be done only numerically. The equilibrium with transaction costs can be found for an arbitrary size of these costs. Furthermore, I consider an approximate solution of this equilibrium in the limit of small transaction costs. It appears that the volatility and the risk premium of stock returns changes with decreasing coefficient in transaction costs proportionally to the square root of this coefficient. It follows that, like in the model of Lo, Mamaysky, and Wang (2004), the effect of transaction costs relatively to the size of transaction costs could be very strong when the costs are small.

Surprisingly, I find that the limits to which the moments of stock returns converge when transaction costs are becoming smaller are significantly different from the corresponding moments in the economy with no transaction costs. It follows that even small transaction costs result in a disproportionally strong impact on the stock price. Decreasing these costs will have a relatively small effect on prices in comparison to their total removal. The presence of convex transaction costs makes investors adjust their trading volumes to optimize the cost–investment opportunities trade–off. Consequently, investors trade volumes which are inversely proportional to their transaction costs and proportional to their coefficients of risk aversions. The requirement for the stock market to clear makes trading volumes be the same across the two types of investors. Therefore, the equilibrium allocations of investors depend on the ratio of coefficients of risk aversions and the ratio of transaction costs rather than their sizes. It follows that, as transaction costs decrease, the optimal allocations toward which investors trade do not converge to those in the market without transaction costs. Consequently, investors demand a higher unconditional risk premium than in the economy without transaction costs. Therefore, the stock price is considerably different from its counterpart in the economy with no transaction costs.

A zero–order effect on prices from transaction costs is possible only in the presence of heterogeneous agents. It follows that this effect cannot be predicted if the impact of transaction

[3] See, for example, Amihud and Mendelson (1986).

costs on prices is studied in the partial equilibrium framework. Furthermore, I find that for any distribution of endowment across two investors, if the ratio of coefficients of risk aversion of the two types of investors are equal to the ratio of transaction costs for these investors then a zero–order effect on prices from transaction costs is absent. Also, if this effect is absent then this relation between the coefficients of risk aversion and transaction costs must hold. These conclusions are due to a competition between investors. The impact of investors of one type on the conditional risk premium decreases as the transaction cost paid by these investors increases and/or their risk aversion falls. It follows that if the ratio of the coefficient in transaction costs to the coefficient of risk aversion are the same across the two types of investors then they have offsetting effects on prices resulting in the absence of zero–order effect.

For better understanding the effects from small transaction costs I consider an economy where transaction costs have an arbitrary convex shape defined by parameter ε. These costs become very close to linear when ε is very small. By expanding the utility functions of investors with respect to small transaction costs and also with respect to small deviations of stock allocations from the optimal ones I show heuristically that stock prices in the economy without transaction costs and in the economy with small transaction costs are significantly different when the cost convexity, ε, is not negligible. The difference could be very small only under the calibration described above when transaction costs are quadratic. The limit where parameter ε is very small or zero is left for a future research. Importantly, adding a negligible convex term to whether linear or fixed transaction costs will lead to a zero–order effect on stock prices through its influence on the boundaries of non–transaction zone.

The found results suggests that equilibrium is a discontinuous function of transaction costs where discontinuity is located at the point of zero costs. This conclusion casts a shadow on application of the perturbation analysis about the no-transaction-cost solution for approximate price calculation in equilibrium approach.[4] The idea of this approach is to find equilibrium utility functions and prices in the absence of transaction costs and then expand the solution of equilibrium with transaction costs with respect to a power of the coefficient in transaction costs assuming that this coefficient is small and using an equilibrium with no transaction costs as a zero–order approximation. Discontinuity of equilibria with respect to transaction costs makes this perturbation analysis inapplicable.

An equilibrium with fixed transaction costs is considered in Lo, Mamaysky, and Wang (2004) who find no zero order effects on prices. Furthermore, a number of studies consider an equilibrium with proportional transaction costs. See, for example, Vayanos (1998), Buss, Uppal, and Vilkov (2013), Buss and Dumas (2017). These studies also report an absence of zero order effects on prices from transaction costs. Notice, however, that the calibrations used by Lo, Mamaysky, and Wang (2004), Vayanos (1998), and Buss and Dumas (2017) formally satisfy the condition for the absence of a zero–order effect specified above. Furthermore, Buss, Uppal, and Vilkov (2013) and Buss and Dumas (2017) assume a very low trading frequency with low trading incentives which

[4]For the use of perturbation theory in the partial equilibrium analysis see, for example, Atkinson and Willmot (1995) and Atkinson and Al-Ali (1997).

significantly weakens a zero–order effect. The equilibrium with convex trading costs can be found in Isaenko (2013, 2015), Vayanos and Woolley (2013), and Sannikov and Skrzypacz (2016). These studies assume that trading costs are significant and do not report zero–order effects on prices.

The rest of this article is organized as follows. Sections 2 and 3 describes the model. Section 4 presents the solutions and discussions of the equilibria in economies with and without transaction costs. A numerical example is found in Section 5, while Section 6 delivers a discussion of the equilibrium with a more general shape of transaction costs. Section 7 concludes. The proofs of Theorems, Propositions and Corollary 3 are found in Appendix.

2 Economic Setting

I consider a Markovian economy with an infinite horizon. Uncertainty is resolved continuously, driven by a 3–dimensional Brownian motion (W_0, W_Y, W_D). The correlation coefficient between W_i and W_j, $i \neq j$, is equal to constant $\rho_{i,j}$, where $i, j \in \{0, Y, D\}$.

There is a stock that pays dividend flow D and has a post–dividend price S that follows the processes

$$dS_t + D_t dt = \mu_{St} dt + \sigma_{S_D t}\, dW_{Dt} + \sigma_{S_Y t}\, dW_{Yt}. \tag{1}$$

The dividend flow is given by an exogenous mean–reverting process

$$dD_t = \kappa_D(\bar{D} - D_t)dt + \sigma_D dW_{Dt}, \tag{2}$$

where κ_D, $\bar{D}$, σ_D are positive constants. Investors can borrow and lend at a constant interest rate r.

There are a lot of investors of two types who trade in the stock market. An investor places a market order which could be bigger than a size of the best limit order. Hence, an order size affects the trading price causing an investor to loose money. Therefore, an investor includes her impact on trading price into losses which she adds to transaction costs. Consequently, transaction costs become convex in the number of traded shares. In particular, I assume that the stock price faced by a buying investor increases with a number of traded shares, ΔN, linearly as $S_a = S_0 + \psi + \chi\Delta N$ where S_0 is an average between ask and bid prices and ψ, $\hat{\alpha}$ are positive constants. It is assumed that S_0 is independent from the number of traded shares due to a large number of investors in the market. It follows that the bid price to sell ΔN shares is $S_b = S_0 - \psi - \chi\Delta N$ and the transaction cost per trading ΔN shares is equal to $\psi|\Delta N| + \frac{1}{2}\chi(\Delta N)^2$.[5] This transaction cost is retained by a market maker and dissipates from the economy. The first term in these costs will be neglected for the purpose of tractability.

The presence of only two types of investors in the economy suggests that investors of the same type should place identical orders at the same time. This paradigm would result in lining up of identical market orders and investors facing different prices for identical orders placed at the same

[5]For the purpose of tractability transaction costs are assumed to be independent from the current stock price.

time. This situation rarely happens in practice due to heterogeneity between all small investors. I remedy this controversy by assuming a small heterogeneity between investors within each type which is significant enough to avoid piling up of identical orders at the same time but small enough to be neglected for the purpose of the equilibrium calculation.[6] Overall, it is assumed that the effects from overlapping of different orders are minimal.

Investors trade at the same small time–intervals τ which I approximate with continuous trading at rate u. Trading ΔN_n shares of the stock by investor of type $n\,(n = 1, 2)$ imposes transaction costs on her equal to $\alpha_n(u_n)^2\tau$ dollars, where $\alpha_n = \frac{1}{2}\,\chi_n\tau$ is constant and u_n is an average rate of trading by investor n within this time interval. Under approximation of continuous trading, the cost per time dt becomes $\alpha_n(u_n)^2\,dt$, where u_n is an instantaneous trading rate. Coefficient α_n survives the transition to continuous time to assure that impact on trading price stays under approximation of continuous trading.

Coefficients α_1 and α_2 can be different. This assumption is not crucial for validity of the model predictions and is set for generality. The difference between α's results in heterogeneous local transitory price impacts which may arise, for example, from asymmetry of information between investors perceived by market makers and/or from ability of some investors to place their orders on the venues with a better bid–ask spread. Furthermore, the difference between these coefficients is expected if they are used to model delays in capital allocations to be discussed below.

Trading costs do not allow the rate of trading to be infinite due to their convex shape. Consequently, share holding of the stock by investor n becomes absolutely continuous and can be written as

$$dN_{nt} = u_{nt}dt, \qquad n = 1, 2. \tag{3}$$

It follows that investors change their allocations at a finite rate. This implies that the rate of trading becomes a control variable of an investor, while an allocation becomes her state variable. The approach of modeling trading imperfections in continuous time by means of convex trading costs was pioneered by practitioners [see, for examples, Grinold and Kahn (2000)] and is also used by Almgren (2003), Isaenko (2010), Rogers and Singh (2010), Sannikov and Skrzypacz (2016), and others.

As investors cannot trade continuously in practice, one has to take a trading strategy u_{nt} derived in this model and multiply it by a trading time–interval τ to find a number of shares to be traded in a given state of economy. The convexity in transaction costs requires investors split their orders to avoid paying excessive transaction costs. Splitting orders decreases their trading volume and has to be balanced against missed investment opportunities. Furthermore, convex transaction costs make investors trade with delays. Trading with delayed does not have to result from impact on a trading price and can also follow from the costs of search for trading counterparties and security analysis, time to raise capital by intermediaries, opportunity costs from time delays caused by valuable alternative activities, difficulties in borrowing, cost of learning, and behavioral. It follows that the

[6] Alternatively, one can consider a lot of atomic investors or a continuum of investors defined by a distribution over one or a few parameters. These approaches will not affect the main conclusion of the paper but will render the economy unsolvable.

predictions of the model will also hold in the presence of delays in capital allocations arising from reasons other than small impact on trading price.

Similar to Lo, Mamaysky, and Wang (2004), I assume that investor n receives a non-tradable cumulative endowment $I_{nt} = \int_0^t Y_{ns} dW_{0s}$, where $Y_n = \beta_n Y$ and β_n is constant. Process Y is mean–reverting:

$$dY_t = \kappa_Y(\bar{Y} - Y_t)dt + \sigma_Y dW_{Yt}, \tag{4}$$

where κ_Y, $\bar{Y}$, and σ_Y are constant and κ_Y, σ_Y are positive. Stochastic endowment makes investors trade dynamically in the long run. Moreover, a diffusion term in the cumulative endowment is kept due to its significance for generating trading incentives and a drift term is dropped for the purpose of tractability. It follows that trading of investors is powered by the process Y which will be called as an endowment volatility.

3 Portfolio Choice by Investor

An investor of type $n \in \{1, 2\}$ faces the dynamic budget constraint and maximizes her expected CARA utility function that supports intermediate consumption with infinite horizon:

$$\max_{c_n \in R, u_n \in R} E_0 \int_0^\infty \left(-\frac{1}{\gamma_n} \exp(-\zeta_n t - \gamma_n c_{nt}) \right) dt, \tag{5}$$

$$\begin{aligned} dX_{nt} &= \left[rX_{nt} + N_{nt}RP_t - c_{nt} - \alpha_n(u_{nt})^2 \right] dt + Y_{nt} dW_{0t} \qquad (6) \\ &+ N_{nt}(\sigma_{S_D t} dW_{Dt} + \sigma_{S_Y t} dW_{Yt}), \end{aligned}$$

where $\gamma_n > 0$ is a coefficient of absolute risk aversion, ζ_n is a time discount rate, $RP_t \equiv \mu_{St} - rS_t$ is a conditional risk premium, while X_n stands for an investor's wealth.

The portfolio selection problem faced by an investor is solved in the dynamic programming approach. It is assumed that each investor of a given type has the same initial allocation to the stock. Hence, the rates of trading and stock's allocations are identical across all investors of a given type at any time. It follows that the state variables for an investor's problem should include processes X_n, Y, and N_n.

The dynamic programming problem is formulated in Appendix. Solving this problem will result in the indirect utility function of investor n, $V^n(t, X_n, N_n, Y)$. Appendix shows that $V^n(t, X_n, N_n, Y) = -\frac{1}{\gamma_n} \exp[-\zeta_n t - \gamma_n r X_n + g^n(N_n, Y)]$ and that the trading rate of investor n is given by

$$u_n(N_n, Y) = -\frac{g^n_{N_n}(N_n, Y)}{2\alpha_n r \gamma_n}, \tag{7}$$

where $g^n_{N_n}$ is a partial derivative of function g^n with respect to variable N_n. Function g^n is described in the section below. The presence of transaction costs forces an investor to be in a state in which her expected utility is not maximal for a given value of Y. Therefore, an investor would like to change her allocations even in the absence of information shocks in the economy.

4 Equilibria

This section describes the rational expectation equilibria in the economy without transaction costs and in the economy where investors pay transaction costs. It is assumed that equilibria are competitive.[7] A linear-quadratic equilibrium is found in each economy and then it is illustrated numerically using an example in Section 5.

4.1 Equilibrium with no Transaction Costs

As a benchmark case, I first consider an equilibrium in the economy with no trading costs so that investors can allocate their capital instantly. In this economy investor n controls her exposure to the risky security, N_n, rather than the trading rate, u_n.

Definition 1 *An equilibrium is a price system $(\mu_S, \sigma_{S_D}, \sigma_{S_Y})$ and a set of trading strategies (N_1^0, N_2^0) of investors of type 1 and 2, respectively, such that (i) individual agents choose their optimal portfolio strategies and (ii) the stock market clears, that is $\forall t \in [0, \infty)$*

$$N_{1t}^0 + N_{2t}^0 = 1. \tag{8}$$

Appendix proves the theorem below that describes the equilibrium.

Theorem 1 *Assume that coefficient H_2 defined in equation (A-26) is positive. Let also equation (A-27) have a solution given by C^{1*}. Moreover, assume that matrix $\boldsymbol{\omega}$ defined by equations (A-34) and (A-35) is not singular. Then a linear–quadratic equilibrium exists. In this equilibrium functions g^n $(n = 1, 2)$, trading strategies, consumption rates, the stock price, its conditional risk premium and volatilities are given by*

$$\begin{aligned}
g^n &= A^n + B^n Y + C^n Y^2, && (9)\\
N_n^0 &= G_1^n + G_2^n Y, && (10)\\
c_n &= -\frac{1}{\gamma_1}[\ln(r) + A^n + B^n Y + C^n Y^2] + rX_n, && (11)\\
S &= \frac{D}{r + \kappa_D} + H_0 + H_2 Y, && (12)\\
RP &= R_0 + R_2 Y, && (13)\\
\sigma_{S_D} &= \frac{\sigma_D}{r + \kappa_D}, && (14)\\
\sigma_{S_Y} &= \sigma_Y H_2, && (15)
\end{aligned}$$

where C^1 is equal to C^{1*} and the rest of the coefficients is given by equations (A-9), (A-10),(A-16),(A-22),(A-24), (A-29)–(A-32).[8,9]

[7] More complex equilibria are possible. See for example Sannikov and Skrzypacz (2016) who solves an equilibrium in the presence of large investors facing quadratic holding costs by assuming a uniform-price conditional double auction.

[8] These equations should be applied to calculate the coefficients in the following order: C^1, C^2, H_2, G_2^1, G_2^2, B^1, B^2, G_1^1, G_1^2, R_0, H_0, R_2, A^1, A^2.

[9] There exists two equivalent solutions to the equilibrium problem which differ by the signs of coefficients in front of process Y but have identical predictions. I choose the solution that has a positive correlation between the stock price and the endowment volatility.

Investors exhibit myopic behavior due to their ability to react instantly to trading opportunities. Moreover, the stock returns follow a 2-factor model where the factors are the dividend rate and the endowment volatility. Both investors disregard dividend risk in their stock allocations due to their constant absolute risk aversions. In general , the equilibrium coefficients cannot be found in a closed form and require solving a non–linear equation (A-27) numerically.

Theorem 1 leads to

Corollary 1 *The risk premium and the volatility of the stock market are*

$$\lim_{t\to\infty} E_0(RP_t) = R_0 + R_2\bar{Y}, \tag{16}$$

$$\sigma_S = \sqrt{\frac{\sigma_D^2}{(r+\kappa_D)^2} + (H_2)^2\sigma_Y^2 + 2\,\frac{\sigma_D}{r+\kappa_D}H_2\sigma_Y\rho_{DY}}, \tag{17}$$

where $E_0(\cdot)$ is an expectation conditioned on information at time zero.

4.2 Equilibrium with Transaction Costs

This section considers the case where both investors pay transaction costs.

Definition 2 *An equilibrium is a price system $(\mu_S, \sigma_{S_D}, \sigma_{S_Y})$ and a set of trading strategies $(u_1,\, u_2)$ of investors of type 1 and 2, respectively, such that (i) individual agents choose their optimal portfolio strategies and (ii) the stock market clears, that is $\forall t \in [0,\infty)$*

$$u_{1t} + u_{2t} = 0. \tag{18}$$

Appendix proves the following theorem that describes the equilibrium with transaction costs

Theorem 2 *Assume that coefficients H_2, A_2^1, B_1^1, and C_0^2 defined below are positive, the nonlinear algebraic equation (A-68) has a solution C_0^{1*}, and the coefficient (A-72) is not zero. Then a linear–quadratic equilibrium exists. In this equilibrium functions g^n $(n = 1, 2)$, trading strategies, consumption rates, the stock price, its conditional risk premium and volatilities are given by*

$$g^n = A_0^n + A_1^n N_n + \frac{1}{2}A_2^n (N_n)^2 + (B_0^n + B_1^n N_n)Y + C_0^n Y^2, \tag{19}$$

$$u_n = -\frac{A_2^n}{2\alpha_n r\gamma_n}(N_n - \hat{N}_n), \tag{20}$$

$$\hat{N}_n = -\frac{A_1^n + B_1^n Y}{A_2^n}, \tag{21}$$

$$\begin{aligned} c_n &= -\frac{1}{\gamma_n}[\ln(r) + A_0^n + A_1^n N_n + \frac{1}{2}A_2^n (N_n)^2 + (B_0^n + B_1^n N_n)Y + C_0^n Y^2] \\ &+ rX_n, \end{aligned} \tag{22}$$

$$S = \frac{D}{r+\kappa_D} + H_0 + H_1 N_1 + H_2 Y, \tag{23}$$

$$RP = R_0 + R_1 N_1 + R_2 Y, \tag{24}$$

$$\sigma_{S_D} = \frac{\sigma_D}{r+\kappa_D}, \tag{25}$$

$$\sigma_{S_Y} = \sigma_Y H_2, \tag{26}$$

where C_0^1 is equal to C_0^{1*} and the rest of coefficients is given by equations (A-45), (A-53), (A-61)–(A-26), (A-69)–(A-71).[10]

Similar to the case with no transaction costs, a closed form solution for the equilibrium coefficients does not exist. Finding these coefficients requires solving nonlinear equation (A-68) numerically. Equations (19)–(22) show that the welfare and the trading strategy of an investor depend on the current level of the endowment volatility as well as on her stock allocations. Moreover, the stock returns follow a 3-factor model where the factors are the dividend rate, the endowment volatility, and the stock holdings. The dependence of the stock price from the aggregate stock allocations introduces a transitory price impact for an equilibrium price. An information shock makes investors instantly adjust the stock price which then undergoes further change due to delayed trading. It follows that the presence of a local transitory price impact when trading of one investor moves her trading price but does not affect the average between bid and ask prices (or an equilibrium price) results in the presence of a macro transitory price impact.

Equation (20) suggests that an investor buys the stock when $N_n < \hat{N}_n$ and sells it when $N_n > \hat{N}_n$. I will call $\hat{N}_n$ as optimal allocation for investor n. Equation (20) also implies that the rate at which the stock allocation of investor n converges to its optimal position is given by coefficient $\xi = \frac{A_2^n}{2\alpha_n r \gamma_n}$. The market clearing condition and equations (A-56)–(A-58) in Appendix imply that this coefficient is the same for both investors.

The following proposition presents the stationary limit of the variance of misallocations, $N_{nt} - \hat{N}_{nt}$, conditioned on information at time zero:

Proposition 1 *The stationary limit of the variance of misallocation is equal to*

$$\lim_{t\to\infty} Var_0(N_{nt} - \hat{N}_{nt}) = \frac{(b_n \sigma_Y)^2}{2(\xi + \kappa_Y)}, \tag{27}$$

where $b_n = -\frac{A_1^n}{B_1^n}$.

The next corollary follows from theorem 2:

Corollary 2 *The risk premium and the standard deviation of the stock market returns are given by*

$$\lim_{t\to\infty} E_0(RP_t) = R_0 + R_2\bar{Y} - R_1 \frac{A_1^1 + B_1^1\bar{Y}}{A_2^1}, \tag{28}$$

$$\sigma_S = \sqrt{\frac{\sigma_D^2}{(r+\kappa_D)^2} + (H_2)^2\sigma_Y^2 + 2\,\frac{\sigma_D}{r+\kappa_D} H_2 \sigma_Y \rho_{DY}}. \tag{29}$$

A comment about multiplicity of equilibria is in order. Generally, an equilibrium with transaction costs allows at least sixteen equilibria subject to the choice of signs in front of the radicals in the solutions of quadratic equations for the equilibrium coefficients. However, most

[10] These equations should be applied to find the coefficients in the following order: C_0^1, C_0^2, B_1^1, B_1^2, H_2, A_2^1, A_2^2, R_1, H_1, R_2, A_1^1, A_1^2, B_0^1, B_0^2, R_0, A_0^1, A_0^2, H_0.

of these equilibria can be dismissed based on boundary conditions and the existence of solution under a chosen calibration. In particular, it is required to set coefficients A_2^n positive so that investors sell shares when their holdings are excessive. With A_2^n being positive, I find numerically that the economy allows two equilibria if transaction costs are small. These equilibria are different in the signs of coefficients multiplying process Y, but have the same unconditional moments of stock returns and the same intuition behind conditional returns. I choose the equilibrium that is consistent with the equilibrium in the absence of transaction costs by requiring coefficients H_2, B_1^1, and C_0^2 be positive.

4.2.1 Small Transaction Costs

This section assumes that the coefficients in transaction costs are small. Moreover, it is set from now on that $\alpha_1 = \alpha$ and $\alpha_2 = \delta\alpha$, where δ is a ratio of transaction costs. Unfortunately a closed form solution of the equilibrium coefficients is not available even for small transaction costs. The following corollary of Theorem 2 establishes the behavior of equilibrium coefficients at small α:

Corollary 3 *If α is small and decreases then equilibrium coefficients A_1^n, B_1^n, A_2^n, $n = 1, 2$, and H_1 converge to zero proportionally to $\sqrt{\alpha}$. Equilibrium coefficients C_0^n, B_0^n, A_0^n, $n = 1, 2$ and H_0, H_2, R_0, R_1, R_2 do not converge to zero and remain functions of δ even when α becomes zero:*

$$R_0 = R_0^{(0)} + \sqrt{\alpha}R_0^{(1)} + o(\sqrt{\alpha}), \quad H_0 = H_0^{(0)} + \sqrt{\alpha}H_0^{(1)} + o(\sqrt{\alpha}), \tag{30}$$

$$R_1 = R_1^{(0)} + \sqrt{\alpha}R_1^{(2)} + o(\sqrt{\alpha}), \quad H_1 = \sqrt{\alpha}\,H_1^{(1)} + \alpha\,H_1^{(2)} + o(\alpha), \tag{31}$$

$$R_2 = R_2^{(0)} + \sqrt{\alpha}R_2^{(1)} + o(\sqrt{\alpha}), \quad H_2 = H_2^{(0)} + \sqrt{\alpha}\,H_2^{(1)} + o(\sqrt{\alpha}), \tag{32}$$

$$A_0^1 = A_0^{1(0)} + \sqrt{\alpha}\,A_0^{1(1)} + o(\sqrt{\alpha}), \quad A_0^2 = A_0^{2(0)} + \sqrt{\alpha}\,A_0^{2(1)} + o(\sqrt{\alpha}), \tag{33}$$

$$A_1^1 = \sqrt{\alpha}\,A_1^{1(1)} + \alpha\,A_1^{1(2)} + o(\alpha), \quad A_1^2 = \sqrt{\alpha}\,A_1^{2(1)} + \alpha\,A_1^{2(2)} + o(\alpha), \tag{34}$$

$$A_2^1 = \sqrt{\alpha}\,A_2^{1(1)} + \alpha\,A_2^{1(2)} + o(\alpha), \quad A_2^2 = \sqrt{\alpha}\,A_2^{2(1)} + \alpha\,A_2^{2(2)} + o(\alpha), \tag{35}$$

$$B_0^1 = B_0^{1(0)} + \sqrt{\alpha}\,B_0^{1(1)} + o(\sqrt{\alpha}), \quad B_0^2 = B_0^{2(0)} + \sqrt{\alpha}\,B_0^{2(1)} + o(\sqrt{\alpha}), \tag{36}$$

$$B_1^1 = \sqrt{\alpha}\,B_1^{1(1)} + \alpha\,B_1^{1(2)} + o(\alpha), \quad B_1^2 = \sqrt{\alpha}\,B_1^{2(1)} + \alpha\,B_1^{2(2)} + o(\alpha), \tag{37}$$

$$C_0^1 = C_1^{0(0)} + \sqrt{\alpha}\,C_1^{0(1)} + o(\sqrt{\alpha}), \quad C_0^2 = C_0^{2(0)} + \sqrt{\alpha}\,C_0^{2(1)} + o(\sqrt{\alpha}), \tag{38}$$

where coefficients with upper indices $^{(0)}$, $^{(1)}$, and $^{(2)}$ depend on δ and do not dependent on α.

I conclude that, like in the model of Lo, Mamaysky, and Wang (2004), small transaction costs create effects on prices which are proportional to $\sqrt{\alpha}$. These effects are significant in comparison to the expected values of these costs. Furthermore, the statistical characteristics of stock returns depends on δ even when contribution of coefficient α into equilibrium coefficients become negligible. Hence, the effects from δ make the influence of transaction costs on the stock prices remain significant even when these costs become virtually absent. The rest of this section will be devoted to the analysis of this prediction.

First, I confirm that zero–order effect on the stock price does not arise from suboptimal allocations of investors. Indeed, if the coefficients in transaction costs decrease then investors trade faster and approach their optimal allocations closer. Investors spend more time close to their optimal allocations even though these allocations have infinite first order variation. Formally, according to proposition 1, the variance of the stock misallocation in the limit of small α becomes

$$\lim_{t\to\infty} Var_0(N_{nt} - \hat{N}_{nt}) = \sqrt{\alpha}\,\frac{r\gamma_1(b_1^{(0)}\sigma_Y)^2}{A_2^{1(1)}} + o(\sqrt{\alpha}), \tag{39}$$

where I took into account that $b^1 = -b^2$, $b_1^{(0)} = -\frac{A_1^{1(1)}}{B_1^{1(1)}}$, while coefficients $B_1^{1(1)}$, $A_2^{1(1)}$ and $A_1^{1(1)}$ are given by equations (A-103), (A-105), and (A-109), respectively. It follows that stock allocations approach the optimal position according to $\alpha^{1/4}$ rule. Furthermore, it follows from equation (20) that in the limit of small coefficient in transaction costs the current cost for investor n becomes

$$\left(\frac{A_2^{n(1)}}{2r\gamma_n}\right)^2 (N_n - \hat{N}_n)^2. \tag{40}$$

The last formula implies that the expected transaction costs paid by investors decrease according to $\alpha^{1/2}$ rule.

Even though the expected transaction costs decrease and investors spend more time close to their optimal allocations as coefficient α subsides, *the optimal stock allocations of investors in the market with transaction costs do not converge to the optimal stock allocations in the market without these costs*. Therefore, the average stock allocations of investors across the two economies remain substantially different even when transaction costs become negligible. Hence, the aggregate stock demand is different across the two economies resulting in different unconditional risk premia and stock prices.

The stock market clearing condition requires the ratio of trading volumes of the two investors to be equal to one. In turn, trading volumes are inversely proportional to the coefficients in transaction costs and proportional to the coefficients of risk aversions. The latter makes market clearing and, therefore, the equilibrium stock allocations, depend on the ratio of transaction costs and the ratio of coefficients of risk aversions. The last conclusion holds even when transaction costs are small. The effect on the optimal allocations from transaction costs could be seen, for example, from comparison of the optimal allocations of investor 1 in the market with no transaction costs, $N_1^0 = G_1^1 + G_2^1 Y$, and the optimal allocations of this investor in the market with transaction costs in zero order approximation, $\hat{N}_1^{(0)} = \frac{A_1^{1(1)} + B_1^{1(1)} Y}{A_2^{1(1)}}$. The two allocations can be the same only if

$$G_2^1 = \frac{B_1^{1(1)}}{A_2^{1(1)}}.$$

By using equations (A-103), (A-105) and assuming that $C^1 = C_0^{1(0)}$ one can confirm that the equality above depends on ratio δ and it holds only if $\delta = \frac{\gamma_2}{\gamma_1}$.

The heterogeneity in transaction costs, risk aversion and endowments creates incentives to trade different volumes across investors. Investor with smaller transaction costs can only trade the volume which meets a trading capacity of her counterpart. Investors will trade at the same rate only if an investor who has to trade slower is compensated by favorable adjustments in the conditional risk premium. Furthermore, the presence of transaction costs creates delays in capital allocations and, therefore, hinders investor's ability to control her risk exposure. This disability strongly interferes with risk aversions of investors. Consequently, investors with higher risk aversion demands significantly higher risk premium than in the absence of transaction costs. These investors exert stronger influence on the stock prices than the other investors do. Hence, a relatively higher coefficient of risk aversion and a relatively smaller coefficient in transaction costs defines a marginal investor in the stock market. A marginal investor will exists even if transaction costs are small since her existence depends on the ratio of these costs rather than on their values. This investor trades based on her needs to diversify endowment risk and take advantage of investment opportunities. These incentives require her to choose stock allocations adjusted for her relatively lower trading volume and the resulting excessive risk exposure.

The stock market clearing condition implies that the conditional risk premium is proportional to stock holdings of investor 1 with coefficient R_1 that does not converge to zero when transaction costs become smaller. This behavior of the conditional risk premium results from agents' inhomogeneity and their inability to adjust their allocations instantly. It follows from equation (40) that in the limit of small coefficient in transaction costs the current cost for the marginal investor could be very high due to a significant deviation from the optimal allocations. It prevents her from changing the stock allocations instantly. The situation aggravates further due to high portfolio volatility in these states. Therefore, the conditional risk premium demanded by the marginal investor is significant and does not vanishes when the coefficient in transaction costs becomes very small.[11]

Formally, the non-disappearance of coefficient R_1 at very small α follows from equations (A-48) and (A-49) from Appendix:

$$\begin{aligned} 0 &= -rA_2^1 + (r\gamma_1)^2\sigma_S^2 - 2r\gamma_1 R_1 + \sigma_Y^2 (B_1^1)^2 - \frac{(A_2^1)^2}{2\alpha_1 r\gamma_1} \\ &- 2\sigma_Y r\gamma_1(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})B_1^1, \qquad (41) \\ 0 &= -rA_2^2 + (r\gamma_2)^2\sigma_S^2 + 2r\gamma_2 R_1 + \sigma_Y^2 (B_1^2)^2 - \frac{(A_2^2)^2}{2\alpha_2 r\gamma_2} \\ &- 2\sigma_Y r\gamma_2(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})B_1^2. \qquad (42) \end{aligned}$$

Indeed, according to corollary 3 these equations can be written as

$$0 = (r\gamma_1)^2(\sigma_S^{(0)})^2 - 2r\gamma_1 R_1^{(0)} - \frac{(A_2^{1(1)})^2}{2\alpha_1 r\gamma_1} + O(\sqrt{\alpha}) \qquad (43)$$

$$0 = (r\gamma_2)^2(\sigma_S^{(0)})^2 + 2r\gamma_2 R_1^{(0)} - \frac{(A_2^{2(1)})^2}{2\alpha_2 r\gamma_2} + O(\sqrt{\alpha}), \qquad (44)$$

where $(\sigma_S^{(0)})^2 = \frac{\sigma_D^2}{(r+\kappa_D)^2} + (H_2^{(0)})^2\sigma_Y^2 + 2\frac{\sigma_D}{r+\kappa_D}H_2^{(0)}\sigma_Y\rho_{DY}$. Taking into account equation (A-57)

[11] The analysis of this paper applies to short–term returns. If longer–term, for example quarterly, returns are considered then the impact of current stock allocations on the conditional risk premium weakens.

that $\frac{A_2^{1(1)}}{\alpha_1\gamma_1} = \frac{A_2^{2(1)}}{\alpha_2\gamma_2}$ one arrives at relation

$$R_1^{(0)} = \frac{1}{2}\, r(\sigma_S^{(0)})^2\, \frac{\gamma_1\alpha_2 - \alpha_1\gamma_2}{\alpha_1 + \alpha_2} + O(\sqrt{\alpha}). \tag{45}$$

It follows from the last equation that a zero–order effect on coefficient R_1 is present only if $\frac{\alpha_1}{\alpha_2} \neq \frac{\gamma_1}{\gamma_2}$. The case when the two ratios are equal will be considered below.

Corollary 2 presents unconditional risk premium. This premium explicitly depends on the average optimal allocations. This dependence does not disappear in the limit of very small transaction costs due to zero–order effect of transaction costs on optimal allocations. The shift in aggregate demand for the stock across the two economies generates a significant difference in the corresponding unconditional risk premia.

A nontrivial relation between the conditional risk premium and stock allocations of investor 1 makes the stock price differ from its counterpart in the economy without transaction costs. A very small probability of significant deviations from the optimal allocations makes the current stock price barely depend from the allocations. However, the non-vanishing coefficients in the stock price are significantly different from their counterparts in the economy without transaction costs, since the stock price is an integral over realized returns and the latter are significantly different across the two economies. The impact of stock allocations on stock prices is formally seen from the proof of Theorem 2 in Appendix. Based on equations (A-53)–(A-55) one can write

$$H_0 = \frac{1}{r}\left[\frac{\kappa_D\bar{D}}{r+\kappa_D} - \frac{A_1^1 H_1}{2\alpha_1 r\gamma_1} + \kappa_Y\bar{Y}H_2 - R_0\right], \tag{46}$$

$$H_1 = -\frac{R_1}{r + \frac{A_2^1}{2\alpha_1 r\gamma_1}}, \tag{47}$$

$$H_2 = -\frac{\frac{H_1\,B_1^1}{2\alpha_1 r\gamma_1} + R_2}{r+\kappa_Y}. \tag{48}$$

In the limit of small transaction costs the last system leads to the equations

$$H_0^{(0)} = \frac{1}{r}\left[\frac{\kappa_D\bar{D}}{r+\kappa_D} + \frac{R_1^{(0)}}{A_2^{1(1)}}\left(A_1^{1(1)} + \frac{\kappa_Y\bar{Y}}{r+\kappa_Y}B_1^{1(1)}\right) - \frac{\kappa_Y\bar{Y}}{r+\kappa_Y}R_2^{(0)} - R_0^{(0)}\right], \tag{49}$$

$$H_1^{(1)} = -\frac{2r\gamma_1 R_1^{(0)}}{A_2^{1(1)}}, \tag{50}$$

$$H_2^{(0)} = \frac{R_1^{(0)}\,B_1^{1(1)} - R_2^{(0)}A_2^{1(1)}}{(r+\kappa_Y)A_2^{1(1)}}, \tag{51}$$

where equation (50) is used to replace coefficient $H_1^{(1)}$ in equations (49) and (51). The last equations reveal that equilibrium coefficients in the stock price are strongly affected by the presence of coefficient $R_1^{(0)}$. These effects do not have to be linear since equilibrium coefficients in utility functions of investors are also influenced by this coefficient.

Equation (45) helps to establish the necessary and sufficient conditions for a zero–order effect on prices from transaction costs to be absent.

Proposition 2 *A zero–order effect on prices from transaction costs is absent if and only if the ratio of coefficients in transaction costs is equal to the ratio of coefficients of risk aversion for the two types of investors:*

$$\frac{\alpha_1}{\alpha_2} = \frac{\gamma_1}{\gamma_2}. \tag{52}$$

See Appendix for the proof. The last proposition presents the condition at which an effect on prices from stock allocations of one type of investors is offset by an effect from stock allocations of the other type of investors at small transaction costs. Indeed, an investor with higher transaction costs has less flexibility in adjusting her stock allocations in response to information shocks and has to give up on more investment opportunities than the other investor leading to a decreasing influence on the conditional risk premium. Therefore, the conditional risk premium decreases if allocations to the stock market by this investor increase. Furthermore, an investor with higher risk aversion demands extra conditional risk premium implying a rise in the conditional risk premium if the stock allocations by this investor increase. Hence, a stronger impact on the conditional risk premium from the stock allocations due to high risk aversion is counterbalanced by a weak impact due to high transaction costs. If condition (52) holds then none of the two investors is marginal in the presence of transaction costs. It follows from equation (45) that zero–order effect on prices is absent if and only if coefficient $R_1^{(0)}$ is zero. I will use this criteria to study zero–order effect in a more general case where transaction costs are convex but not necessarily quadratic.

Finally, I conclude with the remarks on using perturbation method about the no-transaction-cost solution to find optimal portfolios and equilibrium prices. It is tempting to find an optimal portfolio for an investor by first solving her portfolio selection problem in the absence of transaction costs, then adding a series of terms proportional to powers of the coefficient in transaction costs, and then solving a sequence of portfolio selection problems for each term in the series. It follows from corollary 3 that small transaction costs have negligible effect on utility functions and consumption rates *conditioned on stock prices*. Therefore, the analysis of this paper validates the perturbation approach about the no-transaction-cost solution for a portfolio selection problem. Similar, one can try to use the perturbation method for finding equilibrium prices by assuming that one can expand the stock price and utility functions with respect to powers of the coefficient in transaction costs where zero power corresponds to the equilibrium without transaction costs. However, according to the analysis above, an equilibrium is a discontinuous function of the coefficient in transaction costs when this coefficient is zero. It follows that using perturbation method about the no-transaction-cost solution for finding prices should be carried with caution. It is justified only in a special case when condition (52) holds.

5 Example

This section illustrates the findings above by using the numerical example based on the calibration below.

5.1 Calibration

I choose the following calibration of the model: $\gamma_1 = \gamma_2 = 5$, $\beta_1 = 2.0$, $\beta_2 = 0.0$, $\delta = 1/3$, $r = 0.02$, $\zeta_1 = \zeta_2 = 0.05$, $\kappa_Y = 0.5$, $\bar{Y} = 0$, $\sigma_Y = 0.75$, $\kappa_D = 0.5$, $\bar{D} = 1$, $\sigma_D = 5.0$, $\rho_{DY} = \rho_{0D} = \rho_{0Y} = 0$.

The investors are assumed to be homogeneous in the risk aversion and patience for the purpose of tractability. Investor 1 pays three times the transaction cost of what investor 2 does. They are heterogeneous in transaction costs for the zero–order effect on prices to exist. For better tractability, I also assume that investor 2 does not receive endowment. Moreover, volatility of dividend flow is set to make the Sharpe ratio of the stock returns be close to its historical value in the US stock market.

I set parameters β_1 and σ_Y to create significant short–term trading incentives for investors. The values of correlation coefficients ρ_{DY}, ρ_{0D}, and ρ_{0Y} depend on the origin of endowment process. Given a broad interpretation of this process and a stylized nature of my model, the choice for these coefficients is motivated by tractability of results.

5.2 Equilibrium with no Transaction Costs

Using the calibration introduced above, I find that the risk premium, the volatility, and the Sharpe ratio, $\frac{\lim_{t\to\infty} E_0(RP_t)}{\sigma_S}$, of the stock market are equal to 4.62 \$, 10.84 \$, and 0.43, respectively.[12] Furthermore, the equilibrium coefficients suggest that investor 1 trades against the market and takes advantage of a long–term reversion of the stock price while investor 2 trades along the stock market and benefits from short–term investment opportunities.

5.3 Equilibrium with Transaction Costs

Figure 1 presents the risk premium and the volatility of stock returns for small values of α. As transaction costs decreases, so does their impact on the moments of stock returns. As seen from the graph, the decrease is proportional to $\sqrt{\alpha}$ which agrees with corollary 3. Furthermore, comparison of the left limits on this figure with the corresponding risk premium and the volatility of stock returns in the market with no transaction costs reveals that these stock characteristics are substantially different across the two economies.

Consider, for example, the case where $\alpha = 3 \times 10^{-6}$.[13] Equations (21) and (24) suggest that

[12] The equilibrium coefficients are found to be the following: $A^1 = -4.429$, $A^2 = 23.696$, $B^1 = B^2 = 0.334$, $C^1 = -0.038$, $C^2 = 0.962$, $G_1^1 = G_1^2 = 0.5$, $G_2^1 = -G_2^2 = -0.320$, $H_0 = -183.062$, $H_2 = 6.686$, $R_0 = 4.623$, $R_2 = -3.477$. The economy allows two equivalent equilibria which are different by the signs of coefficients multiplying variable Y in the formulas for utility functions and prices. I choose the equilibrium with positive coefficient H_2 so that the stock price has a positive correlation with Y. Coefficient C^1 is chosen negative since investor 1 is trading against the stock market and is better off when Y is large in magnitude, while coefficient C^2 is positive since investor 2 trades against the stock market and is worse off when Y is large in magnitude. The choice of signs in coefficients C^1, C^2, and H_2 is dictated by the boundary conditions.

[13] I calculate the following values of the equilibrium coefficients in this case: $A_0^1 = -83.3112$, $A_0^2 = 32.6304$, $A_1^1 = -0.0043$, $A_1^2 = -0.0010$, $A_2^1 = 0.0074$, $A_2^2 = 0.0025$, $B_0^1 = 7.175$, $B_0^2 = 2.908$, $B_1^1 = 0.0023$, $B_1^2 = -0.00076$, $C_0^1 = -1.562$, $C_0^2 = 1.677$, and $R_0 = 112.439$, $R_1 = -152.807$, $R_2 = -100.907$, $H_0 = -1156.729$, $H_1 = 0.0124$, $H_2 = 103.427$. Coefficients A_2^1 and A_2^2 are set positive so that utility functions of investors decrease when the stock

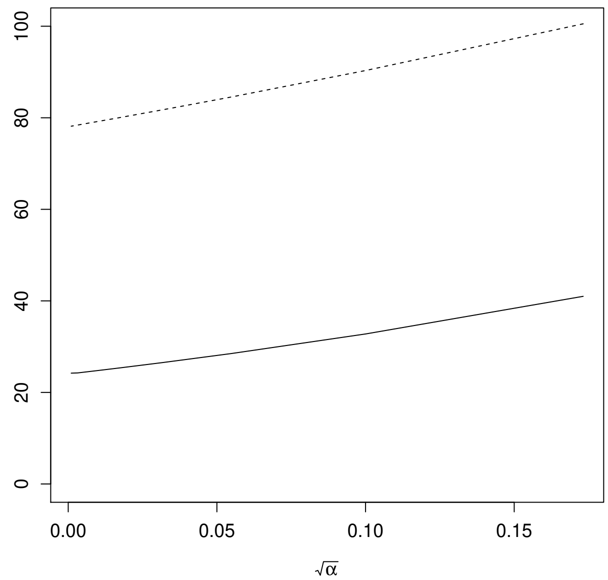

Figure 1: The figure shows the risk premium (solid line) and volatility (dashed line) of stock returns versus $\sqrt{\alpha}$. It is assumed that $\gamma_1 = \gamma_2 = 5$, $\beta_1 = 2.0$, $\beta_2 = 0.0$, $r = 0.02$, $\zeta_1 = \zeta_2 = 0.05$, $\kappa_Y = 0.5$, $\bar{Y} = 0$, $\delta = 1/3$, $\kappa_D = 0.5$, $\bar{D} = 1$, $\sigma_D = 5.0$, $\sigma_Y = 0.75$, $\rho_{DY} = \rho_{0D} = \rho_{0Y} = 0$.

the optimal allocation of investor 2 and the conditional risk premium are

$$\hat{N}_2 = 0.42 + 0.30\,Y, \tag{53}$$

$$RP = -40.37 + 152.81\,N_2 - 100.91\,Y. \tag{54}$$

It follows that a positive shock in Y will increase the stock price and will lift an optimal stock allocation of the marginal investor. Hence, adding transaction costs does not change the trading direction of investor 2 who continues to trade along the stock market. This investor faces smaller transaction costs but has to slow down her trading speed in order to meet trading capacities of investor 1. In turn, her role in price making increases and she demands a higher risk premium than in the market with no transaction costs. Moreover, her trading along the stock market substantially increases the volatility of the stock returns. Interestingly, the risk premium and the stock return volatility in the market with transaction costs is a few times higher than in the market with no such costs. This observation suggests that combination of high trading incentives and very small transaction costs can help to resolve the risk premium puzzle and the excess volatility puzzle. This conclusion is further developed in Isaenko (2013) who studies the effects from high trading incentives and delays in capital allocations on stock returns.

exposure $|N_n|$ is high and rises. Like in the market with no transaction costs, coefficient C_0^1 is negative and coefficient C_0^2 is positive since investor 1 follows bad news, while investor 2 follows good news. Coefficient H_2 is positive to be consistent with the market without transaction costs. The choice of these coefficient is consistent with the boundary conditions for utility functions and the stock price.

It follows from corollary 3 that the ratio of transaction costs δ has a very strong influence on zero–order effects on prices from transaction costs. Panel A of Figure 2 shows the risk premium and volatility of stock returns versus δ when α_1 is fixed at very small value. Both moments of stock returns increase with smaller δ. The impact on prices from investor 2 strengthens with decreasing δ since investor 1 has to pay relatively higher transaction costs. A stronger role in price making causes the volatility of stock returns to rise. Furthermore, her limited ability to trade with investor 1 makes her demand higher risk premium with smaller δ. It appears that the impact of investor 2 on the stock return volatility increases faster than her impact on the risk premium when transaction costs are close to each other, while the risk premium increases faster with decreasing δ when transaction costs are substantially different.

Equation (54) shows that if investor 2 is long and state variable Y decreases, then the conditional risk premium rises and the stock price falls. Therefore, investor 2 has an incentive to buy cheap stock shares in these states due to the substitution effect. On the other hand, she loses money on her current position in the stock market. The resulting losses make investor 2 sell her stock shares. Because investor 2 follows the last strategy, I conclude that taking care of current allocations is dominant for this investor, while investor 1 prefers to follow the substitution effect. Panel B of Figure 2 shows the expected stock holdings of investor 1 (dashed line), investor 2 (solid line), and the Sharpe ratio of stock returns (dotted line) versus δ. The expected allocations of investors are substantially different from those in the economy without transaction costs, where $\lim_{t\to\infty} E_0(N^0_{1t}) = \lim_{t\to\infty} E_0(N^0_{2t}) = 0.5$. In agreement with the discussion above, investor 1 follows closely the substitution effect and takes the longest average stock position in the economy with the highest Sharpe ratio which happens to be when the ratio of transaction costs is the smallest. Clearly, investors allocate to the stock based on the ratio of transaction costs rather than the costs themselves.

6 Equilibrium with Small Transaction Costs. General Case.

In this section I consider an equilibrium with small transaction costs when transaction cost paid by investor n is equal to $\alpha_n|u_n|^{1+\varepsilon}dt$, where $\varepsilon > 0$ and $\alpha_1 = \alpha$, $\alpha_2 = \delta\,\alpha$. These transaction costs become very close to linear in the limit of small ε. I study whether the zero order term proportional to allocations of investor 1 in the conditional risk premium will survive the limit of very small transaction costs. The survival will result in zero–order impact of transaction costs on stock prices.

One can show by following the steps of Appendix that the utility function of investor n can be

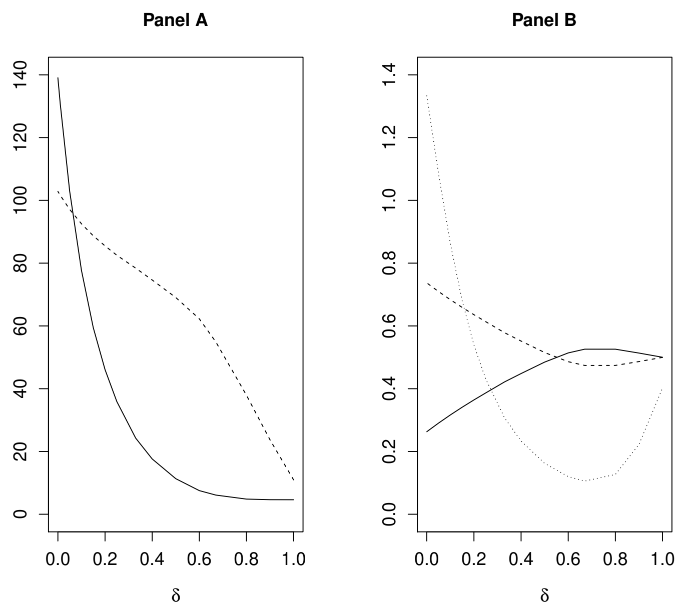

Figure 2: Panel A shows the risk premium (solid line) and volatility (dashed line) of stock returns versus $\delta = \alpha_2/\alpha_1$. Panel B shows the expected stock holdings of investor 1 (dashed line), investor 2 (solid line), and the Sharpe ratio of stock returns (dotted line) versus δ. It is assumed that $\alpha_1 = 3 \times 10^{-4}$, $\gamma_1 = \gamma_2 = 5$, $\beta_1 = 2.0$, $\beta_2 = 0.0$, $r = 0.02$, $\zeta_1 = \zeta_2 = 0.05$, $\kappa_Y = 0.5$, $\bar{Y} = 0$, $\kappa_D = 0.5$, $\bar{D} = 1$, $\sigma_D = 5.0$, $\sigma_Y = 0.75$, $\rho_{DY} = \rho_{0D} = \rho_{0Y} = 0$.

written as $V^n(X_n, N_n, Y) = -\frac{1}{\gamma_n}\exp[-\gamma_n r X_n + g^n(N_n, Y)]$, where g^n solves the following PDE:

$$
\begin{aligned}
0 \;&=\; \frac{(r\gamma_n)^2}{2}\Big[(Y_n)^2 + 2Y_nN_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) + (N_n)^2\sigma_S^2\Big] \\
&-\; r[\ln(r) + g^{n(0)}] - \zeta_n + r - r\gamma_n\, N_n\, RP + \frac{1}{2}\sigma_Y^2[g_{YY}^n + (g_Y^n)^2] \\
&-\; \varepsilon\left(\frac{|g_N^n|^{1+\varepsilon}}{\alpha_n r\gamma_n(1+\varepsilon)^{1+\varepsilon}}\right)^{\frac{1}{\varepsilon}} + \mu_Y g_Y^n - \sigma_Y r\gamma_n[Y_n\rho_{0Y} + N_n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]g_Y^n
\end{aligned}
\tag{55}
$$

with the control process given by

$$u_n = -sign(g_{N_n}^n)\left(\frac{|g_{N_n}^n|}{\gamma_n r\alpha_n(1+\varepsilon)}\right)^{\frac{1}{\varepsilon}}.$$

It follows from the stock market clearing condition that

$$\frac{|g_{N_1}^1|}{\alpha_1\gamma_1} = \frac{|g_{N_2}^2|}{\alpha_2\gamma_2}. \tag{56}$$

Let me assume that the transaction coefficient α is small enough so that one can use the following expansion:

$$g^n(N_n, Y) = g^{n(0)}(Y) + \alpha^{\frac{1}{1+\varepsilon}} g^{n(1)}(Y, N_n) + o(\alpha^{\frac{1}{1+\varepsilon}}). \tag{57}$$

With the help of the last expansion, one can rewrite equation (55) while keeping only zero–order terms in α:

$$
\begin{aligned}
0 &= \frac{(r\gamma_n)^2}{2}\Big[(Y_n)^2 + 2Y_nN_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma^{(0)}_{S_Y}) + (N_n)^2(\sigma^{(0)}_S)^2\Big] \\
&- r[\ln(r) + g^{n(0)}] - \zeta_n + r - r\gamma_n RP^{(0)}\,N_n + \frac{1}{2}\sigma^2_Y[g^{n(0)}_{YY} + (g^{n(0)}_Y)^2] \qquad (58)\\
&- \varepsilon\Big(\frac{|g^{n(1)}_{N_n}|^{1+\varepsilon}}{\delta_n r\gamma_n(1+\varepsilon)^{1+\varepsilon}}\Big)^{\frac{1}{\varepsilon}} + \mu_Y g^{n(0)}_Y - \sigma_Y r\gamma_n[Y_n\rho_{0Y} + N_n(\rho_{DY}\sigma_{S_D} + \sigma^{(0)}_{S_Y})]g^{n(0)}_Y,
\end{aligned}
$$

where $\delta_1 = 1$, $\delta_2 = \delta$ and $\sigma^{(0)}_{S_Y} = \sigma_Y\,H^{(0)}_2$. Taking into account that $N_1+N_2 = 1$ and using condition (56), the last two equations can be applied to find $RP^{(0)}$:

$$
\begin{aligned}
RP^{(0)} &= \frac{1}{r}\sum_{n=1,2}\frac{1}{\gamma_n}\Big(\frac{(r\gamma_n)^2}{2}\Big[(Y_n)^2 + 2Y_nN_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) + (N_n)^2\sigma^2_S\Big] \\
&- r[\ln(r) + g^{n(0)}] - \zeta_n + r + \frac{1}{2}\sigma^2_Y[g^{n(0)}_{YY} + (g^{n(0)}_Y)^2] + \mu_Y g^{n(0)}_Y \qquad (59)\\
&- \sigma_Y r\gamma_n[Y_n\rho_{0Y} + N_n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]g^{n(0)}_Y\Big) - \varepsilon(1+\delta)\Big(\frac{|g^{1(1)}_{N_1}|}{r\gamma_1(1+\varepsilon)}\Big)^{\frac{1+\varepsilon}{\varepsilon}}.
\end{aligned}
$$

For a better understanding when RP is not a function of N_1 in zero order approximation I take into account that function $g^n(N_n, Y)$ must have an extremum in N_n at which

$$
g^n_{N_n}(N_n, Y) = 0.
$$

Assuming that $\hat{N}_n(Y)$ is the solution of the last equation and taking into account that most of the time allocation N_n is close to the optimal allocation $\hat{N}_n$ when α is small, one can apply an additional approximation:

$$
\begin{aligned}
g^n(N_n, Y) &= g^{n(0)}(Y) + \alpha^{\frac{1}{1+\varepsilon}}[g^{n(1)}(Y, \hat{N}_n) + \frac{1}{2}g^{n(1)}_{N_nN_n}(Y, \hat{N}_n)(N_n - \hat{N}_n)^2 \\
&+ o[(N_n - \hat{N}_n)^2]] + o(\alpha^{\frac{1}{1+\varepsilon}}), \qquad (60)
\end{aligned}
$$

where $g^{n(1)}_{N_nN_n}(Y, \hat{N}_n) > 0$. I conclude that function $g^n(N_n, Y)$ is approximately quadratic in N_n. Hence, equation (59) can be written as

$$
\begin{aligned}
RP^{(0)} &= \frac{1}{r}\sum_{n=1,2}\frac{1}{\gamma_n}\Big(\frac{(r\gamma_n)^2}{2}\Big[(Y_n)^2 + 2Y_nN_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) + (N_n)^2\sigma^2_S\Big] \\
&- r[\ln(r) + g^{n(0)}] - \zeta_n + r + \frac{1}{2}\sigma^2_Y[g^{n(0)}_{YY} + (g^{n(0)}_Y)^2] + \mu_Y g^{n(0)}_Y \qquad (61)\\
&- \sigma_Y r\gamma_n[Y_n\rho_{0Y} + N_n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]g^{n(0)}_Y\Big) - \varepsilon(1+\delta)\Big(\frac{g^{1(1)}_{N_1N_1}|(N_1 - \hat{N}_1)|}{r\gamma_1(1+\varepsilon)}\Big)^{1+\frac{1}{\varepsilon}},
\end{aligned}
$$

where the powers of $N_n - \hat{N}_n$ in g_n higher than 2 are neglected. The last term in equation (61) can be canceled by the remaining terms only if $\varepsilon = 1$, since $g^{(0)}_n$ does not depend on N_n and N_1 is an independent variable. It follows that generally RP is a function of N_1 even in zero order in α.

The only exception when RP may not be a function of N_1 is when $\varepsilon = 1$. In the latter case the last term on the right side of the last equation becomes quadratic in N_1 and may cancel with the remaining terms which are quadratic in N_1.

Assuming that $S(D, Y, N_1) = \frac{D}{r+\kappa_D} + h(Y, N_1)$, one can use equation (A-40) for function $h(Y, N_1)$:

$$RP + rh = \frac{\kappa_D \bar{D}}{r + \kappa_D} + h_{N_1} u_1 + h_Y \mu_Y + \frac{1}{2}\sigma_Y^2 h_{YY}. \quad (62)$$

I use the following expansion for h:

$$h(N_1, Y) = h^{(0)}(Y) + \alpha^{\frac{1}{1+\varepsilon}} h^{(1)}(Y, N_1) + o(\alpha^{\frac{1}{1+\varepsilon}}), \quad (63)$$

to arrive to the PDE for h in zero order in α:

$$RP^{(0)} + rh^{(0)} = \frac{\kappa_D \bar{D}}{r + \kappa_D} - sign(g^{1(1)}_{N_1})\, h^{(1)}_{N_1} \left(\frac{|g^{1(1)}_{N_1}|}{\gamma_1 r(1+\varepsilon)} \right)^{\frac{1}{\varepsilon}} + h^{(0)}_Y \mu_Y + \frac{1}{2}\sigma_Y^2 h^{(0)}_{YY}. \quad (64)$$

Or,

$$RP^{(0)} + rh^{(0)} = \frac{\kappa_D \bar{D}}{r + \kappa_D} - sign(N_1 - \hat{N}_1)\, h^{(1)}_{N_1} \left(\frac{g^{1(1)}_{N_1 N_1} |(N_1 - \hat{N}_1)|}{\gamma_1 r(1+\varepsilon)} \right)^{\frac{1}{\varepsilon}} + h^{(0)}_Y \mu_Y + \frac{1}{2}\sigma_Y^2 h^{(0)}_{YY}.$$

The dependence of $RP^{(0)}$ from N_1 causes function $h^{(1)}_{N_1}$ to be non–trivial. In turn, the non–triviality of $h^{(1)}_{N_1}$ makes function $h^{(0)}$ and, therefore, volatility of stock returns, be different from their counterparts in the economy without transaction costs. Furthermore, it follows that finding an equilibrium in this economy requires a simultaneous solving of equations (58) for $n = 1, 2$ and (64) with a help of market clearing condition. The latter is generally not possible since they depend on six unknown functions: $g^{n(0)}$, $g^{n(1)}, n = 1, 2$, and $h^{(0)}$, $h^{(1)}$. It becomes possible if $RP^{(0)}$ is independent from N_1 which in turn makes the second term on the right side of equation (64) disappear. In the latter case functions $g^{n(0)}$ can be found by using the economy without transaction costs and a perturbation analysis about the no-transaction-cost solution can be applied.

It is interesting to see if the zero–order effect on the stock prices is present in the limit when convexity in transaction costs vanishes and the costs become linear. Importantly, adding a small convex term to a linear or fixed transaction costs is likely to result in the presence of a zero–order effect on prices. Indeed, market clearing outside of no–transaction zone will make investors trade toward the boundaries of this zone which location depend on the ratio of transaction costs and the ratio of coefficients of risk aversions. This should affect the utility–maximizing stock allocations and the stock prices.

7 Conclusion

The paper presents a model where small investors exerts transitory price impacts but trade competitively by including their price impact in transaction costs. This makes the costs become convex. The model predicts that it is possible that small convex transaction costs will have zero–order effects on stock returns. The presence of zero–order effects makes it impossible to solve an equilibrium by using the perturbation method about the no-transaction-cost solution.

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Appendix

In this Appendix I prove Theorems 1 and 2, Propositions, and corollary 3.

Proof of Theorem 1. It follows from portfolio selection problem (5), (6) that investor n has to solve the following HJB equation for her value function $V^n(X_n, Y)$

$$\begin{aligned} 0 &= \max_{c_n, N_n^0 \in R^2} \Bigg\{ -\frac{1}{\gamma_n} \exp(-\gamma_n c_n) - \zeta_n V^n + \frac{1}{2}\Bigg[(Y_n)^2 + 2Y_n N_n^0(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) \\ &+ (N_n^0)^2 \sigma_S^2\Bigg] V^n_{X_n X_n} + \frac{1}{2}\sigma_Y^2 V^n_{YY} + \sigma_Y[Y_n\rho_{0Y} + N_n^0(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]V^n_{X_n Y} + \mu_Y V^n_Y \\ &+ \Bigg[RP\, N_n^0 + rX_n - c_n\Bigg] V^n_{X_n}\Bigg\}, \end{aligned} \tag{A-1}$$

where $\sigma_S^2 = \sigma_{S_D}^2 + 2\rho_{DY}\sigma_{S_D}\sigma_{S_Y} + \sigma_{S_Y}^2$.

I conjecture that $V^n(X_n, Y) = -\frac{1}{\gamma_n}\exp[-\gamma_n r X_n + g^n(Y)]$. Then

$$c_n = -\frac{1}{\gamma_n}[\ln(r) + g^n] + rX_n$$

and g^n solves the following ordinary differential equation (ODE)

$$\begin{aligned} 0 &= -r[\ln(r) + g^n] - \zeta_n + r + \frac{(r\gamma_n)^2}{2}\Bigg[(Y_n)^2 + 2Y_n N_n^0(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) \\ &+ (N_n^0)^2\sigma_S^2\Bigg] - r\gamma_n RP\, N_n^0 + \mu_Y g^n_Y + \frac{1}{2}\sigma_Y^2[g^n_{YY} + (g^n_Y)^2] \\ &- \sigma_Y r\gamma_n[Y_n\rho_{0Y} + N_n^0(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]g^n_Y, \end{aligned} \tag{A-2}$$

where N_n^0 is given by

$$N_n^0 = \frac{RP + \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})g^n_Y - Y_n r\gamma_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y})}{r\gamma_n\sigma_S^2}. \tag{A-3}$$

Now let me consider a trial solution

$$g^n = A^n + B^n Y + C^n Y^2 \tag{A-4}$$

and assume that volatility of the stock returns is state independent, while

$$RP = R_0 + YR_2, \tag{A-5}$$

where A^n, B^n, C^n, R_0, R_2 are the coefficients to be determined. Similar, I assume that

$$N_n^0 = G_1^n + YG_2^n \tag{A-6}$$

with G_1^n, G_2^n being constants.

Next I determine the stock price. The drift of the stock returns is given by

$$rS + RP = D + S_D\mu_D + S_Y\mu_Y + \frac{1}{2}\sigma_D^2 S_{DD} + \frac{1}{2}\sigma_Y^2 S_{YY} + \rho_{DY}\sigma_D\sigma_Y S_{DY}. \tag{A-7}$$

I conjecture that

$$S = H_0 + \frac{D}{r+\kappa_D} + H_2 Y,$$

where H_0, H_2 are constants. Then equation (A-7) becomes

$$r(H_0 + \frac{D}{r+\kappa_D} + H_2Y) + R_0 + R_2Y \quad = \quad D + \frac{\mu_D}{r+\kappa_D} + H_2\mu_Y. \tag{A-8}$$

Matching the coefficients results in

$$H_0 \quad = \quad \frac{1}{r}\left(-R_0 + \frac{\kappa_D\bar{D}}{r+\kappa_D} - \kappa_Y\bar{Y}\frac{R_2}{r+\kappa_Y}\right), \tag{A-9}$$

$$H_2 \quad = \quad -\frac{R_2}{r+\kappa_Y}. \tag{A-10}$$

It follows that $\sigma_{S_D} = \frac{\sigma_D}{r+\kappa_D}$, $\sigma_{S_Y} = H_2\sigma_Y = -\frac{R_2}{r+\kappa_Y}\sigma_Y$.

I conclude that the number of unknown coefficients to be determined is 12. The stock market clearing condition implies $G_1^1 + YG_2^1 + G_1^2 + YG_2^2 = 1$. The latter identity suggests the relations

$$G_1^2 \quad = \quad 1 - G_1^1 \tag{A-11}$$

$$G_2^2 \quad = \quad -G_2^1. \tag{A-12}$$

Next, equations (A-3) lead to two more equations:

$$\begin{aligned} 0 \quad &= \quad r\gamma_n\left[(G_1^n + YG_2^n)\sigma_S^2 + Y\beta_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y})\right] \\ &- \quad (R_0 + YR_2) - \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})(B^n + 2C^nY). \end{aligned} \tag{A-13}$$

Matching the coefficients in the last equation provides the following relations

$$0 \quad = \quad r\gamma_n G_1^n\sigma_S^2 - R_0 - \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})B^n, \tag{A-14}$$

$$0 \quad = \quad r\gamma_n\left[G_2^n\sigma_S^2 + \beta_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y})\right] - R_2 - 2\sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})C^n. \tag{A-15}$$

Moreover, ODE (A-2) now can be written as

$$\begin{aligned}
0 &= -r[\ln(r) + A^n + B^nY + C^nY^2] - \zeta_n + r + \frac{(r\gamma_n)^2}{2}\Bigg[(Y_n)^2 \\
&+ 2Y_n(G_1^n + YG_2^n)(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) + (G_1^n + YG_2^n)^2\sigma_S^2\Bigg] \\
&- r\gamma_n(R_0 + YR_2)(G_1^n + YG_2^n) + \frac{1}{2}\sigma_Y^2[2C^n + (B^n + 2C^nY)^2] \\
&+ \Bigg(\mu_Y - \sigma_Y r\gamma_n[Y_n\rho_{0Y} + (G_1^n + YG_2^n)(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]\Bigg)(B^n + 2C^nY).
\end{aligned}$$

Matching the coefficients gives me 6 more equations:

$$\begin{aligned}
A^n &= 1 - \ln(r) - \frac{\zeta_n}{r} + \frac{r\gamma_n^2}{2}(G_1^n)^2\sigma_S^2 - \gamma_n R_0 G_1^n + \frac{1}{2r}\,\sigma_Y^2[2C^n + (B^n)^2] \\
&+ [\kappa_Y\bar{Y} - \sigma_Y r\gamma_n G_1^n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]\frac{B^n}{r}, \qquad \text{(A-16)} \\
0 &= -rB^n + (r\gamma_n)^2\Bigg[\beta_n G_1^n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) + G_1^nG_2^n\sigma_S^2\Bigg] + 2\sigma_Y^2B^nC^n \\
&- r\gamma_n(R_0G_2^n + G_1^nR_2) + 2\Bigg(\kappa_Y\bar{Y} - \sigma_Y r\gamma_n G_1^n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})\Bigg)C^n \\
&- \Bigg(\kappa_Y + \sigma_Y r\gamma_n[\beta_n\rho_{0Y} + G_2^n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]\Bigg)B^n, \qquad \text{(A-17)} \\
0 &= \frac{(r\gamma_n)^2}{2}\Bigg[(\beta_n)^2 + 2\beta_nG_2^n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) + (G_2^n)^2\sigma_S^2\Bigg] - r\gamma_nR_2G_2^n \\
&+ 2\sigma_Y^2(C^n)^2 - 2\Bigg(\kappa_Y + \sigma_Y r\gamma_n[\beta_n\rho_{0Y} + G_2^n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]\Bigg)C^n - rC^n. \qquad \text{(A-18)}
\end{aligned}$$

The rest of the proof is allocated to finding the unknown coefficients G_1^1, G_2^1, A^n, B^n, C^n, $n = 1, 2$, R_0, and H_2 by solving equations (A-14)–(A-18). Notice that with the help of equation (A-15) equation (A-18) can be rewritten as

$$0 = \frac{(r\gamma_n)^2}{2}\Bigg[(\beta_n)^2 - (G_2^n)^2\sigma_S^2\Bigg] + 2\sigma_Y^2(C^n)^2 - [r + 2(\kappa_Y + \sigma_Y r\gamma_n\beta_n\rho_{0Y})]C^n. \qquad \text{(A-19)}$$

I use equations (A-15) to find H_2 in terms of C^1, C^2:

$$H_2 = \frac{r\gamma_1\gamma_2(\beta_1 + \beta_2)\rho_{0D}\sigma_{S_D} - 2\sigma_Y\rho_{DY}\sigma_{S_D}(\gamma_2C^1 + \gamma_1C^2)}{2\sigma_Y^2(\gamma_2C^1 + \gamma_1C^2) - (r + \kappa_Y)(\gamma_1 + \gamma_2) - r\gamma_1\gamma_2(\beta_1 + \beta_2)\rho_{0Y}\sigma_Y}. \qquad \text{(A-20)}$$

Next, relation (A-19) leads to a quadratic equation for C^2 in terms of C^1:

$$\begin{aligned}
0 &= \frac{1}{2}[(\beta_1)^2 - (\beta_2)^2] + 2\sigma_Y^2\Bigg[\frac{(C^1)^2}{(r\gamma_1)^2} - \frac{(C^2)^2}{(r\gamma_2)^2}\Bigg] \qquad \text{(A-21)} \\
&+ [r + 2(\kappa_Y + \sigma_Y r\gamma_2\beta_2\rho_{0Y})]\frac{C^2}{(r\gamma_2)^2} - [r + 2(\kappa_Y + \sigma_Y r\gamma_1\beta_1\rho_{0Y})]\frac{C^1}{(r\gamma_1)^2}.
\end{aligned}$$

The last quadratic equation can be easily solved for C^2 as a function of C^1:

$$C^2(C^1) = \frac{1}{(2\sigma_Y)^2}\Bigg[r + 2(\kappa_Y + \sigma_Y r\gamma_2\beta_2\rho_{0Y}) + \Bigg\{[r + 2(\kappa_Y + \sigma_Y r\gamma_2\beta_2\rho_{0Y})]^2 \quad \text{(A-22)}$$

$$+ \quad 8(\sigma_Y r\gamma_2)^2\left[\frac{1}{2}[(\beta_1)^2 - (\beta_2)^2] + 2\sigma_Y^2\frac{(C^1)^2}{(r\gamma_1)^2} - [r + 2(\kappa_Y + \sigma_Y r\gamma_1\beta_1\rho_{0Y})]\frac{C^1}{(r\gamma_1)^2}\right]\Bigg\}^{1/2}\Bigg],$$

where I set a positive sign in front of the radical to make sure that investor 2 has a low utility function when Y is high. This choice of sign leads to investor 2 trading along changes in Y. Equation (A-15) implies:

$$G_2^n = \frac{R_2 + 2\sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})C^n - r\gamma_n\beta_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y})}{r\gamma_n\sigma_S^2}. \quad \text{(A-23)}$$

Then nonlinear equation (A-19) is solved for G_2^n:

$$G_2^n = (-1)^n\frac{1}{r\gamma_n\sigma_S}\sqrt{(r\gamma_n\beta_n)^2 + 4\sigma_Y^2(C^n)^2 - 2[r + 2(\kappa_Y + \sigma_Y r\gamma_n\beta_n\rho_{0Y})]C^n}, \quad \text{(A-24)}$$

where G_2^1 is set negative so that investor 1 trades against changes in Y. It follows from equations (A-23) and (A-24) that

$$(\sigma_{S_D}^2 + 2\rho_{DY}\sigma_{S_D}\sigma_Y H_2 + \sigma_Y^2 H_2^2)\left[(r\gamma_n\beta_n)^2 + 4\sigma_Y^2(C^n)^2 - 2[r + 2(\kappa_Y + \sigma_Y r\gamma_n\beta_n\rho_{0Y})]C^n\right]$$

$$= \left[H_2(2\sigma_Y^2 C^n - r\sigma_Y\gamma_n\beta_n\rho_{0Y} - r - \kappa_Y) + 2\sigma_Y\rho_{DY}\sigma_{S_D}C^n - r\gamma_n\beta_n\rho_{0D}\sigma_{S_D}\right]^2. \quad \text{(A-25)}$$

Equation (A-25) is not singular only if C^n is outside of the range $]C_-^n, C_+^n[$, where

$$C_\pm^n = \frac{r + 2(\kappa_Y + \sigma_Y r\gamma_n\beta_n\rho_{0Y}) \pm \sqrt{(r + 2(\kappa_Y + \sigma_Y r\gamma_n\beta_n\rho_{0Y}))^2 - (2r\gamma_n\beta_n\sigma_Y)^2}}{4\sigma_Y^2}.$$

Next, I use equation (A-25) for $n = 1$ to obtain H_2 in terms of C^1:

$$H_2 = \frac{-\epsilon_2 + \sqrt{\epsilon_2^2 - \epsilon_1\epsilon_3}}{\epsilon_1}, \quad \text{(A-26)}$$

where

$$\begin{aligned}
\epsilon_1 &= \sigma_Y^2[(r\gamma_1\beta_1)^2 + 2rC^1] - (r\sigma_Y\gamma_1\beta_1\rho_{0Y} + r + \kappa_Y)^2, \\
\epsilon_2 &= \sigma_Y\sigma_{S_D}(r\gamma_1\beta_1)^2(\rho_{DY} - \rho_{0D}\rho_{0Y}) - (r + \kappa_Y)r\gamma_1\beta_1\rho_{0D}\sigma_{S_D} \\
&+ 2\sigma_Y^2\sigma_{S_D}r\gamma_1\beta_1(\rho_{0D} - \rho_{0Y}\rho_{DY})C^1, \\
\epsilon_3 &= (2\sigma_Y\sigma_{S_D}C^1)^2(1 - \rho_{DY}^2) - 2\sigma_{S_D}^2[r + 2(\kappa_Y + \sigma_Y r\gamma_1\beta_1(\rho_{0Y} - \rho_{0D}\rho_{DY}))]C^1 \\
&+ (r\gamma_1\beta_1\sigma_{S_D})^2(1 - \rho_{0D}^2).
\end{aligned}$$

The sign in front of a radical symbol in equation (A-26) is chosen to make H_2 positive. Matching the last expression for H_2 with (A-20) and replacing C^2 in terms of C^1 provides me with a nonlinear equation for C^1:

$$\frac{r\gamma_1\gamma_2(\beta_1 + \beta_2)\rho_{0D}\sigma_{S_D} - 2\sigma_Y\rho_{DY}\sigma_{S_D}(\gamma_2 C^1 + \gamma_1 C^2(C^1))}{2\sigma_Y^2(\gamma_2 C^1 + \gamma_1 C^2(C^1)) - (r + \kappa_Y)(\gamma_1 + \gamma_2) - r\gamma_1\gamma_2(\beta_1 + \beta_2)\rho_{0Y}\sigma_Y} = \frac{-\epsilon_2 + \sqrt{\epsilon_2^2 - \epsilon_1\epsilon_3}}{\epsilon_1}. \quad \text{(A-27)}$$

Now let me consider a special case when $\rho_{DY} = (\beta_1+\beta_2)\rho_{0D} = 0$. It follows that numerator in equation (A-20) becomes zero which is possible only if

$$2\sigma_Y^2(\gamma_2 C^1 + \gamma_1 C^2) - (r+\kappa_Y)(\gamma_1+\gamma_2) - r\gamma_1\gamma_2(\beta_1+\beta_2)\rho_{0Y}\sigma_Y = 0 \tag{A-28}$$

By using equations (A-22) and (A-28) I find coefficients C^1 and C^2:

$$\begin{aligned}
C^1 &= \frac{[(r+\kappa_Y)(\gamma_1+\gamma_2) + r\gamma_1\gamma_2\rho_{0Y}\sigma_Y(\beta_1+\beta_2)][r\gamma_2 + \kappa_Y(\gamma_2-\gamma_1) + r\gamma_1\gamma_2\rho_{0Y}\sigma_Y(\beta_1-\beta_2)]}{2\sigma_Y^2 r\gamma_2(\gamma_1+\gamma_2)} \\
&- \frac{r\gamma_1^2\gamma_2(\beta_1^2-\beta_2^2)}{2(\gamma_1+\gamma_2)}, \\
C^2 &= \frac{[(r+\kappa_Y)(\gamma_1+\gamma_2) + r\gamma_1\gamma_2\rho_{0Y}\sigma_Y(\beta_1+\beta_2)][r\gamma_1 + \kappa_Y(\gamma_1-\gamma_2) + r\gamma_1\gamma_2\rho_{0Y}\sigma_Y(\beta_2-\beta_1)]}{2\sigma_Y^2 r\gamma_1(\gamma_1+\gamma_2)} \\
&- \frac{r\gamma_2^2\gamma_1(\beta_2^2-\beta_1^2)}{2(\gamma_1+\gamma_2)}.
\end{aligned}$$

H_2 is found from equation (A-26). Moreover, in the special case when $\rho_{DY} = \rho_{0D} = 0$, H_2 becomes

$$H_2 = \sqrt{-\frac{\epsilon_3}{\epsilon_1}},$$

where

$$\begin{aligned}
\epsilon_1 &= \sigma_Y^2[(r\gamma_1\beta_1)^2 + 2rC^1] - (r\sigma_Y\gamma_1\beta_1\rho_{0Y} + r + \kappa_Y)^2, \\
\epsilon_3 &= \sigma_{S_D}^2[(2\sigma_Y C^1)^2 - 2[r + 2(\kappa_Y + \sigma_Y r\gamma_1\beta_1\rho_{0Y})]C^1 + (r\gamma_1\beta_1)^2].
\end{aligned}$$

Now I return to a general case where the double equality $\rho_{DY} = 0$, $(\beta_1+\beta_2)\rho_{0D} = 0$ does not have to hold. The proof of the theorem is finished by finding coefficients G_1^1, R_0, B^1, B^2, A^1, and A^2. First, equations (A-14) are utilized to obtain G_1^1 and R_0 in terms of B^1 and B^2:

$$R_0 = \frac{r\gamma_1\gamma_2}{\gamma_1+\gamma_2}\sigma_S^2 - \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})\frac{\gamma_2 B^1 + \gamma_1 B^2}{\gamma_1+\gamma_2}, \tag{A-29}$$

$$G_1^1 = \frac{\gamma_2}{\gamma_1+\gamma_2} + \frac{\sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})(B^1 - B^2)}{r(\gamma_1+\gamma_2)\sigma_S^2}. \tag{A-30}$$

Hereafter, I use equations (A-14), (A-15) and rewrite the two linear equations (A-17) as

$$0 = -(r\gamma_n\sigma_S)^2 G_1^n G_2^n + (2\sigma_Y^2 C^n - r - \kappa_Y - \rho_{0Y}\sigma_Y r\gamma_n\beta_n)B^n + 2\kappa_Y\bar{Y}C^n$$

and solve them to find B^1 and B^2:

$$B^1 = \frac{d_2\omega_{12} - d_1\omega_{22}}{\omega_{11}\omega_{22} - \omega_{12}\omega_{21}}, \tag{A-31}$$

$$B^2 = \frac{d_1\omega_{21} - d_2\omega_{11}}{\omega_{11}\omega_{22} - \omega_{12}\omega_{21}}, \tag{A-32}$$

where

$$d_n = 2\kappa_Y\bar{Y}C^n - \frac{\gamma_j}{\gamma_1+\gamma_2}(r\gamma_n\sigma_S)^2 G_2^n, \tag{A-33}$$

$$\omega_{nn} = 2\sigma_Y^2 C^n - r - \kappa_Y - \rho_{0Y}\sigma_Y r\gamma_n\beta_n - \frac{r(\gamma_n)^2}{\gamma_1+\gamma_2}\sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})G_2^n, \tag{A-34}$$

$$\omega_{nj} = \frac{r(\gamma_n)^2}{\gamma_1+\gamma_2}\sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})G_2^n, \quad n \neq j. \tag{A-35}$$

Finally, equations (A-16) are used to derive A^1 and A^2. Q.E.D.

Proof of Theorem 2. The indirect utility function of an investor of type n, $V^n(X_n, N_n, Y)$, solves

$$\begin{aligned} 0 &= \max_{c_n,\, u^n \in R^2} \Bigg\{ -\zeta_n V^n + \frac{1}{2}\Big[(Y_n)^2 + 2Y_n N_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) + (N_n)^2\sigma_S^2 \Big] V^n_{X_n X_n} \\ &+ \frac{1}{2}\sigma_Y^2 V^n_{YY} + \sigma_Y[Y_n\rho_{0Y} + N_n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]V^n_{X_n Y} \\ &+ u_n V^n_{N_n} + \mu_Y V^n_Y + \Big(RP\, N_n - \alpha_n (u_n)^2 + rX_n - c_n \Big) V^n_{X_n} - \frac{1}{\gamma_n}\exp(-\gamma_n c_n) \Bigg\}. \end{aligned}$$

It follows that

$$c_n = -\frac{1}{\gamma_n}\ln(rV^n_{X_n}), \quad u_n = \frac{V^n_{N_n}}{2\alpha_n V^n_{X_n}}. \tag{A-36}$$

I conjecture that $V^n(X_n, N_n, Y) = -\frac{1}{\gamma_n}\exp[-\gamma_n r X_n + g^n(N_n, Y)]$, then

$$c_n = -\frac{1}{\gamma_n}[\ln(r) + g^n] + rX_n, \quad u_n = -\frac{g^n_N}{2\alpha_n r\gamma_n} \tag{A-37}$$

and g^n solves the following PDE

$$\begin{aligned} 0 &= \frac{(r\gamma_n)^2}{2}\Big[(Y_n)^2 + 2Y_n N_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) + (N_n)^2\sigma_S^2 \Big] \\ &- r[\ln(r) + g^n] - \zeta_n + r - r\gamma_n RP\, N_n + \frac{1}{2}\sigma_Y^2[g^n_{YY} + (g^n_Y)^2] \\ &- \frac{(g^n_N)^2}{4\alpha_n r\gamma_n} + \mu_Y g^n_Y - \sigma_Y r\gamma_n[Y_n\rho_{0Y} + N_n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]g^n_Y. \end{aligned} \tag{A-38}$$

Because the set of state variables for the stock price in this economy includes D, N_1, and Y, I find from the Itô formula that $\sigma_{S_D} = S_D\sigma_D$ and $\sigma_{S_Y} = S_Y\sigma_Y$.

The drift of the stock returns is given by

$$\mu_S = D + S_D\mu_D + S_{N_1}u_1 + S_Y\mu_Y + \frac{1}{2}\sigma_D^2 S_{DD} + \frac{1}{2}\sigma_Y^2 S_{YY} + \rho_{DY}\sigma_D\sigma_Y S_{DY}. \tag{A-39}$$

Assuming that $S(D, Y, N_1) = \frac{D}{r+\kappa_D} + h(Y, N_1)$, I find the PDE for h:

$$RP + rh = \frac{\kappa_D\bar{D}}{r+\kappa_D} + h_{N_1}u_1 + h_Y\mu_Y + \frac{1}{2}\sigma_Y^2 h_{YY}, \tag{A-40}$$

where RP is the risk premium of the stock market and $\sigma_{S_D} = \frac{\sigma_D}{r+\kappa_D}$, $\sigma_{S_Y} = h_Y\sigma_Y$.

I conjecture the following solution of the equilibrium

$$g^n = A^n_0 + A^n_1 N_n + \frac{1}{2}A^n_2(N_n)^2 + (B^n_0 + B^n_1 N_n)Y + C^n_0 Y^2, \tag{A-41}$$

$$h = H_0 + H_1 N_1 + H_2 Y, \tag{A-42}$$

$$RP = R_0 + R_1 N_1 + R_2 Y, \tag{A-43}$$

where all coefficients are independent from N's and Y. It follows that $\sigma_{S_Y} = \sigma_Y H_2$. I derive from equation (A-38):

$$
\begin{aligned}
&- \quad \zeta_n - r[\ln(r) - 1 + A_0^n + A_1^n N_n + \frac{1}{2} A_2^n (N_n)^2 + (B_0^n + B_1^n N_n) Y + C_0^n Y^2] \\
&+ \quad \frac{(r\gamma_n)^2}{2} \left[(Y_n)^2 + 2 Y_n N_n (\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) + (N_n)^2 \sigma_S^2 \right] \\
&- \quad r\gamma_n (R_0 + R_1 N_1 + R_2 Y) N_n + \frac{1}{2}\sigma_Y^2 [2C_0^n + (B_0^n + B_1^n N_n + 2C_0^n Y)^2] \\
&+ \quad (B_0^n + B_1^n N_n + 2C_0^n Y)\Big(\mu_Y - \sigma_Y r\gamma_n [Y_n \rho_{0Y} + N_n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})] \Big) \qquad \text{(A-44)} \\
&- \quad \frac{(A_1^n + A_2^n N_n + B_1^n Y)^2}{4\alpha_n r\gamma_n} = 0.
\end{aligned}
$$

The last equation implies the following set of equations:

$$
\begin{aligned}
0 &= -rA_0^n + r[1 - \ln(r)] - \zeta_n + \frac{1}{2}\sigma_Y^2[2C_0^n + (B_0^n)^2] - \frac{(A_1^n)^2}{4\alpha_n r\gamma_n} + \kappa_Y \bar{Y} B_0^n, && \text{(A-45)} \\
0 &= -rA_1^1 - r\gamma_1 R_0 + \sigma_Y^2 B_0^1 B_1^1 - \frac{1}{2r\gamma_1}\frac{A_1^1 A_2^1}{\alpha_1} + \kappa_Y \bar{Y} B_1^1 && \text{(A-46)} \\
&- \sigma_Y r\gamma_1(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y}) B_0^1, \\
0 &= -rA_1^2 - r\gamma_2(R_0 + R_1) + \sigma_Y^2 B_0^2 B_1^2 - \frac{1}{2r\gamma_2}\frac{A_1^2 A_2^2}{\alpha_2} + \kappa_Y \bar{Y} B_1^2 \\
&- \sigma_Y r\gamma_2(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y}) B_0^2, && \text{(A-47)} \\
0 &= -rA_2^1 + (r\gamma_1)^2\sigma_S^2 - 2r\gamma_1 R_1 + \sigma_Y^2 (B_1^1)^2 - \frac{(A_2^1)^2}{2\alpha_1 r\gamma_1} \\
&- 2\sigma_Y r\gamma_1(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y}) B_1^1, && \text{(A-48)} \\
0 &= -rA_2^2 + (r\gamma_2)^2\sigma_S^2 + 2r\gamma_2 R_1 + \sigma_Y^2 (B_1^2)^2 - \frac{(A_2^2)^2}{2\alpha_2 r\gamma_2} \\
&- 2\sigma_Y r\gamma_2(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y}) B_1^2, && \text{(A-49)} \\
0 &= (2\sigma_Y^2 C_0^n - r - \kappa_Y - \sigma_Y r\gamma_n\beta_n\rho_{0Y}) B_0^n - \frac{A_1^n B_1^n}{2\alpha_n r\gamma_n} + 2\kappa_Y C_0^n \bar{Y}, && \text{(A-50)} \\
0 &= -(r + 2\kappa_Y + 2\sigma_Y r\gamma_n\beta_n\rho_{0Y}) C_0^n + \frac{(r\gamma_n\beta_n)^2}{2} + 2\sigma_Y^2 (C_0^n)^2 - \frac{(B_1^n)^2}{4\alpha_n r\gamma_n}, && \text{(A-51)} \\
0 &= -(r + \kappa_Y + \sigma_Y r\gamma_n\beta_n\rho_{0Y}) B_1^n + (r\gamma_n)^2\beta_n(\rho_{0D}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}) - r\gamma_n R_2 \\
&+ 2\sigma_Y^2 B_1^n C_0^n - 2\sigma_Y r\gamma_n C_0^n(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y}) - \frac{A_2^n B_1^n}{2\alpha_n r\gamma_n}. && \text{(A-52)}
\end{aligned}
$$

Similar, I find from equation (A-40):

$$
R_0 + R_1 N_1 + R_2 Y + r(H_0 + H_1 N_1 + H_2 Y) = \frac{\kappa_D \bar{D}}{r + \kappa_D} - H_1 \frac{A_1^1 + A_2^1 N_1 + B_1^1 Y}{2\alpha_1 r\gamma_1} + H_2 \mu_Y.
$$

The last identity implies the following relations:

$$R_0 + rH_0 = \frac{\kappa_D \bar{D}}{r+\kappa_D} - H_1 \frac{A_1^1}{2\alpha_1 r\gamma_1} + \kappa_Y \bar{Y} H_2, \tag{A-53}$$

$$R_1 + rH_1 = -H_1 \frac{A_2^1}{2\alpha_1 r\gamma_1}, \tag{A-54}$$

$$R_2 + rH_2 = -H_1 \frac{B_1^1}{2\alpha_1 r\gamma_1} - \kappa_Y H_2. \tag{A-55}$$

Now I consider the stock market clearing condition:

$$\frac{A_1^1 + A_2^1 N_1 + B_1^1 Y}{\alpha_1\gamma_1} + \frac{A_1^2 + A_2^2 N_2 + B_1^2 Y}{\alpha_2\gamma_2} = 0,$$

implying the following relations

$$\frac{A_1^1}{\alpha_1\gamma_1} + \frac{A_1^2 + A_2^2}{\alpha_2\gamma_2} = 0, \tag{A-56}$$

$$\frac{A_2^1}{\alpha_1\gamma_1} - \frac{A_2^2}{\alpha_2\gamma_2} = 0, \tag{A-57}$$

$$\frac{B_1^1}{\alpha_1\gamma_1} + \frac{B_1^2}{\alpha_2\gamma_2} = 0. \tag{A-58}$$

Next, the system of equations above is solved for the unknown equilibrium coefficients. First, I use equations (A-51) to derive B_1^n as a function of C_0^n:

$$B_1^n = (-1)^{n+1} 2\sqrt{\alpha_n r\gamma_n \left[2\sigma_Y^2 (C_0^n)^2 - (r + 2\kappa_Y + 2\sigma_Y r\gamma_n\beta_n\rho_{0Y})C_0^n + \frac{(r\gamma_n\beta_n)^2}{2}\right]}, \tag{A-59}$$

where B_1^1 is set positive so that investor 1 sells stock shares when her endowment is excessive. Note that the last expression is well defined only if $C_0^n \le C_0^{n-}$, $C_0^n \ge C_0^{n+}$, where

$$C_0^{n\pm} = \frac{r + 2\kappa_Y + 2\sigma_Y r\gamma_n\beta_n\rho_{0Y} \pm \sqrt{(r + 2\kappa_Y + 2\sigma_Y r\gamma_n\beta_n\rho_{0Y})^2 - (2r\sigma_Y\gamma_n\beta_n)^2}}{4\sigma_Y^2}.$$

Equations (A-58) and (A-59) imply

$$\begin{aligned} & 2\sigma_Y^2 (C_0^2)^2 - (r + 2\kappa_Y + 2\sigma_Y r\gamma_2\beta_2\rho_{0Y})C_0^2 + \frac{(r\gamma_2\beta_2)^2}{2} \\ - & \frac{\gamma_2\alpha_2}{\gamma_1\alpha_1}\left[2\sigma_Y^2 (C_0^1)^2 - (r + 2\kappa_Y + 2\sigma_Y r\gamma_1\beta_1\rho_{0Y})C_0^1 + \frac{(r\gamma_1\beta_1)^2}{2}\right] = 0. \end{aligned} \tag{A-60}$$

Or,

$$\begin{aligned} C_0^2 = (4\sigma_Y^2)^{-1}\Bigg[&(r + 2\kappa_Y + 2\sigma_Y r\gamma_2\beta_2\rho_{0Y}) + \Bigg\{(r + 2\kappa_Y + 2\sigma_Y r\gamma_2\beta_2\rho_{0Y})^2 - (2r\sigma_Y\gamma_2\beta_2)^2 \\ + & 8\sigma_Y^2 \frac{\gamma_2\alpha_2}{\gamma_1\alpha_1}\left(2\sigma_Y^2 (C_0^1)^2 - (r + 2\kappa_Y + 2\sigma_Y r\gamma_1\beta_1\rho_{0Y})C_0^1 + \frac{(r\gamma_1\beta_1)^2}{2}\right)\Bigg\}^{1/2}\Bigg], \end{aligned} \tag{A-61}$$

where a positive sign in front of the radical is set to make sure that coefficient C_0^2 is consistent with coefficient C^2 in the economy without transaction costs. Now I apply equations (A-48) and (A-49) to determine A_2^n and R_1 as functions of C_0^1 and H_2:

$$\begin{aligned} 0 &= \frac{(\alpha_1+\alpha_2)}{2\alpha_1 r\gamma_1}(A_2^1)^2 + r(\alpha_1+\alpha_2)A_2^1 - r^2\gamma_1\alpha_1(\gamma_1+\gamma_2)\sigma_S^2 \\ &+ 2r(\gamma_1\alpha_1-\gamma_2\alpha_2)\,\sigma_Y(\rho_{DY}\sigma_{S_D}+\sigma_{S_Y})B_1^1 - \sigma_Y^2\frac{\gamma_1\alpha_1^2+\gamma_2\alpha_2^2}{\gamma_1\alpha_1}(B_1^1)^2, \end{aligned}$$

implying

$$\begin{aligned} A_2^1 &= \alpha_1 r\gamma_1\Bigg\{-r+\Bigg[r^2+\frac{2}{\alpha_1 r\gamma_1(\alpha_1+\alpha_2)}\Bigg(2r\sigma_Y(\gamma_2\alpha_2-\gamma_1\alpha_1) \qquad \text{(A-62)} \\ &\times (\rho_{DY}\sigma_{S_D}+\sigma_{S_Y})B_1^1 + r^2\gamma_1\alpha_1(\gamma_1+\gamma_2)\sigma_S^2 + \sigma_Y^2\frac{\gamma_1\alpha_1^2+\gamma_2\alpha_2^2}{\gamma_1\alpha_1}(B_1^1)^2\Bigg)\Bigg]^{1/2}\Bigg\} \\ R_1 &= -\frac{(A_2^1)^2}{4\alpha_1(r\gamma_1)^2} - \frac{A_2^1}{2\gamma_1} + \sigma_Y^2\frac{(B_1^1)^2}{2r\gamma_1} - \sigma_Y(\rho_{DY}\sigma_{S_D}+\sigma_{S_Y})B_1^1 + \frac{1}{2}r\gamma_1\sigma_S^2, \qquad \text{(A-63)} \end{aligned}$$

where the sign in front of the radical is set positive so that A_2^1 is positive as well.

Next, equations (A-54) and (A-55) are used to find H_1 and R_2, respectively:

$$\begin{aligned} H_1 &= -\frac{2\alpha_1 r\gamma_1 R_1}{2\alpha_1 r^2\gamma_1 + A_2^1}, \qquad \text{(A-64)} \\ R_2 &= -\frac{H_1 B_1^1}{2\alpha_1 r\gamma_1} - (r+\kappa_Y)H_2. \qquad \text{(A-65)} \end{aligned}$$

Afterward, equations (A-52) for $n = 1, 2$ are applied to obtain C_0^1 and H_2. First, coefficient H_2 is found in terms of C_0^1 and then C_0^1 is found numerically. In more details, equations (A-52) imply:

$$\begin{aligned} &\left[2\sigma_Y^2(C_0^1\alpha_1+C_0^2\alpha_2) - (\kappa_Y+\frac{r}{2})(\alpha_1+\alpha_2) - \sigma_Y\rho_{0Y}(r\gamma_1\beta_1\alpha_1+r\gamma_2\beta_2\alpha_2)\right]\frac{B_1^1}{r\gamma_1} \\ - &\ 2\alpha_1\sigma_Y(\rho_{DY}\sigma_{S_D}+\sigma_{S_Y})(C_0^1-C_0^2) + \alpha_1 r(\gamma_1\beta_1-\gamma_2\beta_2)(\rho_{0D}\sigma_{S_D}+\rho_{0Y}\sigma_{S_Y}) \\ = &\ \frac{B_1^1}{2r\gamma_1}(\alpha_1+\alpha_2)\Bigg[r^2 - \frac{2}{\alpha_1 r\gamma_1(\alpha_1+\alpha_2)}\Bigg(2r(\gamma_1\alpha_1-\gamma_2\alpha_2)\,\sigma_Y(\rho_{DY}\sigma_{S_D}+\sigma_{S_Y})B_1^1 \\ - &\ r^2\gamma_1\alpha_1(\gamma_1+\gamma_2)\sigma_S^2 - \sigma_Y^2\frac{\gamma_1\alpha_1^2+\gamma_2\alpha_2^2}{\gamma_1\alpha_1}(B_1^1)^2\Bigg)\Bigg]^{1/2}. \end{aligned}$$

Or,

$$\begin{aligned} 0 &= \frac{(2r\gamma_1)^2}{[B_1^1(\alpha_1+\alpha_2)]^2}\Bigg(\Bigg[2\sigma_Y^2(C_0^1\alpha_1+C_0^2\alpha_2) - (\kappa_Y+\frac{r}{2})(\alpha_1+\alpha_2) \qquad \text{(A-66)} \\ &- \sigma_Y\rho_{0Y}(r\gamma_1\beta_1\alpha_1+r\gamma_2\beta_2\alpha_2)\Bigg]\frac{B_1^1}{r\gamma_1} - 2\alpha_1\sigma_Y\rho_{DY}\sigma_{S_D}(C_0^1-C_0^2) + \\ &+ \alpha_1 r(\gamma_1\beta_1-\gamma_2\beta_2)\rho_{0D}\sigma_{S_D} + \alpha_1[r(\gamma_1\beta_1-\gamma_2\beta_2)\rho_{0Y} - 2\sigma_Y(C_0^1-C_0^2)]H_2\sigma_Y\Bigg)^2 \\ &+ \frac{2}{\alpha_1 r\gamma_1(\alpha_1+\alpha_2)}\Bigg(2r(\gamma_1\alpha_1-\gamma_2\alpha_2)\,\sigma_Y(\rho_{DY}\sigma_{S_D}+H_2\sigma_Y)B_1^1 \\ &- r^2\gamma_1\alpha_1(\gamma_1+\gamma_2)(\sigma_{S_D}^2+2\rho_{DY}\sigma_{S_D}\sigma_Y H_2+\sigma_Y^2H_2^2) - \sigma_Y^2\frac{\gamma_1\alpha_1^2+\gamma_2\alpha_2^2}{\gamma_1\alpha_1}(B_1^1)^2\Bigg) - r^2. \end{aligned}$$

The solution of the last equation with respect to H_2 is obvious:

$$H_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \tag{A-67}$$

where the sign in front of the radical is chosen to make H_2 it consistent with its counterpart in the economy without transaction costs and the expressions for coefficients a, b, and c follow from equation (A-66) and are not presented for the purpose of compactness.

Next, C_0^1 is calculated from the following nonlinear equation which is derived from equation (A-52):

$$\begin{aligned}(\alpha_1 + \alpha_2)R_2 &= 2\sigma_Y^2\left[\frac{\alpha_2 B_1^1}{r\gamma_1}(C_0^1 - C_0^2) - H_2(C_0^1\alpha_2 + C_0^2\alpha_1)\right] \\ &+ \sigma_Y\rho_{0Y}(\gamma_2\beta_2 - \gamma_1\beta_1)\frac{\alpha_2 B_1^1}{\gamma_1} + r(\alpha_2\gamma_1\beta_1 + \alpha_1\gamma_2\beta_2)(\rho_{OD}\sigma_{S_D} + \rho_{0Y}\sigma_{S_Y}),\end{aligned} \tag{A-68}$$

where B_1^1 and C_0^2 are given by equations (A-59) and (A-61), respectively, while R_2 is found from equation (A-65).

I have determined coefficients B_1^n, C_0^n, A_2^n, $n = 1, 2$, H_1, H_2, R_1, and R_2. The remaining equilibrium coefficients are found in the rest of the proof. I use equation (A-56) to determine A_1^2 versus A_1^1:

$$A_1^2(A_1^1) = -\frac{\gamma_2\alpha_2}{\gamma_1\alpha_1}(A_2^1 + A_1^1).$$

Equations (A-50) suggests

$$B_0^n = \frac{\frac{A_1^n B_1^n}{2\alpha_n r\gamma_n} - 2\kappa_Y\bar{Y}C_0^n}{2\sigma_Y^2 C_0^n - r - \kappa_Y - \sigma_Y\rho_{0Y}r\gamma_n\beta_n}. \tag{A-69}$$

Next, linear equations (A-46) and (A-47) lead to A_1^1 and R_0:

$$R_0 = -\frac{A_1^1}{\gamma_1} + \frac{\sigma_Y^2}{r\gamma_1}B_0^1 B_1^1 - \frac{A_1^1 A_2^1}{2\alpha_1(r\gamma_1)^2} + \frac{\kappa_Y\bar{Y}}{r\gamma_1}B_1^1 - \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})B_0^1, \tag{A-70}$$

where

$$\begin{aligned}A_1^1 &= \left[\frac{A_2^1}{\gamma_1\alpha_1}\alpha_2 + \frac{(A_2^1)^2}{2(r\gamma_1\alpha_1)^2}\alpha_2 - R_1 - \kappa_Y\bar{Y}\frac{B_1^1}{r\gamma_1}\left(1 + \frac{\alpha_2}{\alpha_1}\right)\right. \\ &+ \left. z_1[\sigma_Y^2\frac{B_1^1}{r\gamma_1} - \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})] + z_2[\sigma_Y^2\frac{B_1^1}{r\gamma_1} + \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})]\right]\pi^{-1},\end{aligned} \tag{A-71}$$

and

$$\begin{aligned}\pi &= x_1[\sigma_Y^2\frac{B_1^1}{r\gamma_1} - \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})] + x_2[\sigma_Y^2\frac{B_1^1\alpha_2}{r\gamma_1\alpha_1} + \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y})] \\ &- \frac{(\alpha_1 + \alpha_2)}{\gamma_1\alpha_1} - \frac{A_2^1}{2(r\gamma_1\alpha_1)^2}(\alpha_1 + \alpha_2), \quad x_n = \frac{(-1)^{n+1}B_1^n}{2\alpha_1 r\gamma_1(2\sigma_Y^2 C_0^n - r - \kappa_Y - \sigma_Y\rho_{0Y}r\gamma_n\beta_n)}, \\ z_1 &= \frac{2\kappa_Y\bar{Y}C_0^1}{2\sigma_Y^2 C_0^1 - r - \kappa_Y - \sigma_Y\rho_{0Y}r\gamma_1\beta_1}, \quad z_2 = \frac{\frac{A_2^1 B_1^2}{2\alpha_1 r\gamma_1} + 2\kappa_Y\bar{Y}C_0^2}{2\sigma_Y^2 C_0^2 - r - \kappa_Y - \sigma_Y\rho_{0Y}r\gamma_2\beta_2}.\end{aligned} \tag{A-72}$$

Finally, coefficients A_0^n and H_0 are determined from equations (A-45) and (A-53), respectively. Q.E.D.

Proof of Proposition 1.

Consider

$$\begin{aligned} E_0(N_{nt}) &= \int_0^t E_0(u_{ns})ds = -\vartheta t - \int_0^t E_0(\varphi Y_s + \xi N_{ns})ds \\ &= -\vartheta_n t - \int_0^t \{\varphi_n[Y_0 e^{-\kappa_Y s} + \bar{Y}(1-e^{-\kappa_Y s})] + \xi E_0(N_{ns})\}ds, \end{aligned}$$

where $\vartheta_n = \frac{A_1^n}{2\alpha_n r\gamma_n}$, $\varphi_n = \frac{B_1^n}{2\alpha_n r\gamma_n}$. The last integral equation can be written as

$$X_n'(t) = -\vartheta - \varphi[Y_0 e^{-\kappa_Y t} + \bar{Y}(1-e^{-\kappa_Y t})] - \xi X_n(t),$$

where $X_n(t) = E_0(N_{nt})$. The solution of this equation is given by

$$X_n(t) = N_{n0}e^{-\xi t} + \varphi_n \frac{Y_0 - \bar{Y}}{\xi - \kappa_Y}(e^{-\xi t} - e^{-\kappa_Y t}) + \frac{\vartheta_n + \varphi_n \bar{Y}}{\xi}(e^{-\xi t} - 1).$$

Hence,

$$\lim_{t\to\infty} X_n(t) = -\frac{\vartheta_n + \varphi_n \bar{Y}}{\xi} = E_0(\hat{N}_n).$$

It follows that

$$\lim_{t\to\infty} Var_0(N_t - \hat{N}_t) = \lim_{t\to\infty} E_0(N_t - \hat{N}_t)^2.$$

Consider

$$E_0(N_t - \hat{N}_t)^2 = E_0(N_t^2) - 2E_0(N_t\hat{N}_t) + E_0(\hat{N}_t^2), \tag{A-73}$$

where index n is dropped for simplicity of notations. Finding the terms on the right side of the last equation is easy if I calculate function $g(t,t') \equiv E_0(N_t N_{t'})$:

$$\begin{aligned} g(t,t') &= N_0^2 + N_0\int_0^t E_0(u(s))ds + N_0\int_0^{t'} E_0(u(s))ds + \int_0^t\int_0^{t'} E_0(u(s)u(s'))dsds' \\ &= N_0^2 - \xi N_0\int_0^t E_0(N_s - \hat{N}_s)ds - \xi N_0 \int_0^{t'} E_0(N_s - \hat{N}_s)ds \\ &+ \xi^2 \int_0^t\int_0^{t'} E_0[(N_s - \hat{N}_s)(N_{s'} - \hat{N}_{s'})]dsds' \\ &= N_0^2 - \xi N_0\left[\frac{N_0}{\xi}(1-e^{-\xi t}) + \varphi\frac{Y_0 - \bar{Y}}{\xi - \kappa_Y}\left(\frac{1-e^{-\xi t}}{\xi} - \frac{1-e^{-\kappa_Y t}}{\kappa_Y}\right)\right] \\ &- \xi N_0\left[\frac{N_0}{\xi}(1-e^{-\xi t'}) + \varphi\frac{Y_0 - \bar{Y}}{\xi - \kappa_Y}\left(\frac{1-e^{-\xi t'}}{\xi} - \frac{1-e^{-\kappa_Y t'}}{\kappa_Y}\right)\right] \\ &+ \xi^2\int_0^t\int_0^{t'}[g(s,s') - f(s,s') - f(s',s) + h(s,s')]dsds', \end{aligned} \tag{A-74}$$

where $f(s,s') \equiv E_0(N_s \hat{N}_{s'})$, $h(s,s') \equiv E_0(\hat{N}_s \hat{N}_{s'})$. It is expected that the initial conditions should not affect function g in a stationary limit. Therefore, I simplify the last equation by assuming that $N_0 = 0$ and $Y_0 = \bar{Y}$:

$$g(t,t') = \xi^2 \int_0^t \int_0^{t'} [g(s,s') - f(s,s') - f(s',s) + h(s,s')] ds ds', \quad \text{(A-75)}$$

The last integral equation can also be reduced to a partial differential equation (PDE):

$$g_{tt'}(t,t') = \xi^2 [g(t,t') - f(t,t') - f(t',t) + h(t,t')]. \quad \text{(A-76)}$$

Solving the last PDE requires finding functions f and h. I start from function h:

$$\begin{aligned} h(t,t') &= E_0(\hat{N}_t \hat{N}_{t'}) = E_0[(a + bY_t)(a + bY_{t'})] \\ &= a^2 + ab[E_0(Y_t) + E_0(Y_{t'})] + b^2 E_0(Y_t Y_{t'})], \end{aligned} \quad \text{(A-77)}$$

where $a = -\frac{A_1^n}{A_2^n}$, $b = -\frac{B_1^n}{A_2^n}$. It follows that

$$\begin{aligned} &h(t,t') = a^2 + ab[Y_0(e^{-\kappa_Y t} + e^{-\kappa_Y t'}) + \bar{Y}(2 - e^{-\kappa_Y t} - e^{-\kappa_Y t'})] \quad \text{(A-78)} \\ &+ \; b^2 \left[\frac{\sigma_Y^2}{2\kappa_Y} (e^{-\kappa_Y |t-t'|} - e^{-\kappa_Y (t+t')}) + [Y_0 e^{-\kappa_Y t} + \bar{Y}(1 - e^{-\kappa_Y t})][Y_0 e^{-\kappa_Y t'} + \bar{Y}(1 - e^{-\kappa_Y t'})] \right]. \end{aligned}$$

Or,

$$h(t,t') = (a + b\bar{Y})^2 + b^2 \left[\frac{\sigma_Y^2}{2\kappa_Y} (e^{-\kappa_Y |t-t'|} - e^{-\kappa_Y (t+t')}) \right]. \quad \text{(A-79)}$$

Now I consider function $f(t,t')$:

$$\begin{aligned} f(t,t') &= N_0 E_0(\hat{N}_{t'}) + \int_0^t E_0(u(s)\hat{N}_{t'}) ds = N_0[a + b(Y_0 e^{-\kappa_Y t'} + \bar{Y}(1 - e^{-\kappa_Y t'}))] \\ &- \; \xi \int_0^t E_0[(N_s - \hat{N}_s)\hat{N}_{t'}] ds = N_0[a + b(Y_0 e^{-\kappa_Y t'} + \bar{Y}(1 - e^{-\kappa_Y t'}))] \\ &- \; \xi \int_0^t [f(s,t') - h(s,t')] ds. \end{aligned} \quad \text{(A-80)}$$

The last integral equation can be written as

$$f_t(t,t') = -\xi[f(t,t') - h(t,t')]. \quad \text{(A-81)}$$

The solution of the last ODE is straightforward:

$$\begin{aligned} f(t,t') &= N_0 E_0(\hat{N}_{t'}) e^{-\xi t} + \xi \int_0^t h(s,t') e^{-\xi(t-s)} ds = N_0[a + b(Y_0 e^{-\kappa_Y t} + \bar{Y}(1 - e^{-\kappa_Y t}))] e^{-\xi t} \\ &+ \; (a + b\bar{Y})^2 (1 - e^{-\xi t}) + \xi \frac{(b\sigma_Y)^2}{2\kappa_Y} \left[\frac{1}{\kappa_Y - \xi} (e^{-\kappa_Y (t+t')} - e^{-\kappa_Y t' - \xi t}) + \phi(t,t') \right], \end{aligned}$$

or,

$$f(t,t') = (a + b\bar{Y})^2 (1 - e^{-\xi t}) + \xi \frac{(b\sigma_Y)^2}{2\kappa_Y} \left[\frac{1}{\kappa_Y - \xi} (e^{-\kappa_Y (t+t')} - e^{-\kappa_Y t' - \xi t}) + \phi(t,t') \right] \quad \text{(A-82)}$$

where

$$\phi(t,t') = \int_0^t e^{|s-t'|}e^{-\xi(t-s)}ds = \frac{1}{\xi+\kappa_Y}\left(e^{-\xi(t-t')}I(t\geq t') + e^{-\kappa_Y(t'-t)}I(t<t') - e^{-\kappa_Y t'-\xi t)}\right)$$
$$+ \frac{1}{\xi-\kappa_Y}\left(e^{-\kappa_Y(t-t')} - e^{-\xi(t-t'))}\right)I(t\geq t') \tag{A-83}$$

It follows that

$$f(t,t') + f(t',t) = (a+b\bar{Y})^2(2-e^{-\xi t}-e^{-\xi t'}) + \xi\frac{(b\sigma_Y)^2}{\kappa_Y(\xi^2-\kappa_Y^2)}\Big[-(\xi+\kappa_Y)e^{-\kappa_Y(t+t')}$$
$$+ \kappa_Y(e^{-\kappa_Y t'-\xi t} + e^{-\kappa_Y t-\xi t'}) + \left(\xi e^{-\kappa_Y(t-t')} - \kappa_Y e^{-\xi(t-t')}\right)I(t\geq t')$$
$$+ \left(\xi e^{-\kappa_Y(t'-t)} - \kappa_Y e^{-\xi(t'-t)}\right)I(t<t')\Big] \tag{A-84}$$

Now let me conjecture the following solution of equation (A-75):

$$g(t,t') = ce^{-\xi(t+t')} + a_0 + a_1e^{-\kappa_Y(t+t')} + a_2(e^{-\xi t}+e^{-\xi t'}) + a_3(e^{-\kappa_Y t'-\xi t} + e^{-\kappa_Y t-\xi t'})$$
$$+ a_4[e^{-\kappa_Y(t-t')}I(t\geq t') + e^{-\kappa_Y(t'-t)}I(t<t')] + a_5[e^{-\xi(t-t')}I(t\geq t') + e^{-\xi(t'-t)}I(t<t')],$$

where c, a_0, a_1, a_2, a_3, a_4, a_5 are the constants to be determined. Matching the coefficients suggest:

$$a_0 = (a+b\bar{Y})^2, \tag{A-85}$$
$$a_1 = -\frac{(b\sigma_Y\xi)^2}{2\kappa_Y(\xi-\kappa_Y)^2}, \tag{A-86}$$
$$a_2 = -(a+b\bar{Y})^2, \tag{A-87}$$
$$a_3 = \frac{(b\sigma_Y\xi)^2}{(\xi-\kappa_Y)^2(\xi+\kappa_Y)}, \tag{A-88}$$
$$a_4 = \frac{(b\sigma_Y\xi)^2}{2\kappa_Y(\xi^2-\kappa_Y^2)}, \tag{A-89}$$
$$a_5 = -\frac{\xi(b\sigma_Y)^2}{2(\xi^2-\kappa_Y^2)}. \tag{A-90}$$

Coefficient c is found from the boundary conditions:

$$g(0,t') = g(t,0) = 0.$$

Or,

$$ce^{-\xi t} + a_0 + a_1e^{-\kappa_Y t} + a_2(e^{-\xi t}+1) + a_3(e^{-\xi t}+e^{-\kappa_Y t}) + a_4e^{-\kappa_Y t} + a_5e^{-\xi t} = 0. \tag{A-91}$$

The last equation can hold only if the following equations hold:

$$a_0 + a_2 = 0, \tag{A-92}$$
$$a_1 + a_3 + a_4 = 0, \tag{A-93}$$
$$c + a_2 + a_3 + a_5 = 0. \tag{A-94}$$

It is straightforward to verify that equations (A-92) and (A-93) hold. Equation (A-93) allows to find coefficient c:

$$c \;=\; (a+b\bar{Y})^2 - \frac{\xi(b\sigma_Y)^2}{2(\xi-\kappa_Y)^2}. \tag{A-95}$$

The comparison of equations (A-73) and (A-76) suggests that

$$E_0(N_t-\hat{N}_t)^2 = g(t,t) - 2f(t,t) + h(t,t) = \frac{1}{\xi} g_{t,t'}(t,t) \tag{A-96}$$

$$= \frac{1}{\xi^2}\Bigg(\xi^2 c e^{-2\xi t} + \kappa_Y^2 a_1 e^{-2\kappa_Y t} + 2a_3\kappa_Y \xi e^{-(\kappa_Y+\xi)t} - a_4\kappa_Y^2 - a_5\xi^2\Bigg). \tag{A-97}$$

It follows that in the stationary limit

$$Var(N_t-\hat{N}_t) = E_0(N_t-\hat{N}_t)^2 = -\frac{1}{\xi^2}(a_4\kappa_Y^2 + a_5\xi^2) = \frac{(b\sigma_Y)^2}{2(\xi+\kappa_Y)}. \tag{A-98}$$

Q.E.D.

Proof of corollary 3

Let me conjecture that equations (30) hold. Also, let

$$\Omega_1 = \{A_0^1,\, A_0^2,\, B_0^1,\, B_0^2,\, C_0^1,\, C_0^2,\, R_0,\, H_0,\, R_1,\, R_2,\, H_2\}$$

be a set of coefficients which zero–order terms are generally not equal to zero and

$$\Omega_2 = \{A_1^1,\, A_2^1,\, B_1^1,\, H_1\}$$

be a set of coefficients which zero–order terms are equal to zero. Equations (A-45)-(A-55) can be written symbolically as

$$F_i(A_0^1, ..., H_2;\; A_1^1,\, A_2^1,\, B_1^1,\, H_1) + \frac{f_i(A_1^1,\, A_2^1,\, B_1^1,\, H_1)}{\alpha} = 0, \tag{A-99}$$

where i is an index for an equation, f_i is proportional to a product of two coefficients from set Ω_2 and coefficients A_1^2, A_2^2, B_1^2 are excluded from equations based on conditions (A-56)–(A-58). The last equation can be rewritten as

$$F_i\Bigg(A_0^{1(0)} + \sqrt{\alpha}\,A_0^{1(1)} + o(\sqrt{\alpha}), ..., H_2^{(0)} + \sqrt{\alpha}\,H_2^{(1)} + o(\sqrt{\alpha});\; \sqrt{\alpha}\,A_1^{1(1)} + \alpha\,A_1^{1(2)} + o(\alpha), ...,$$
$$\sqrt{\alpha}\,H_1^{(1)} + \alpha\,H_1^{(2)} + o(\alpha)\Bigg) + \frac{f_i(\sqrt{\alpha}\,A_1^{1(1)} + \alpha\,A_1^{1(2)} + o(\alpha), ..., \sqrt{\alpha}\,H_1^{(1)} + \alpha\,H_1^{(2)} + o(\alpha))}{\alpha}$$
$$= 0.$$

Or,

$$F_i\Bigg(A_0^{1(0)} + \sqrt{\alpha}\,A_0^{1(1)} + o(\sqrt{\alpha}), ..., H_2^{(0)} + \sqrt{\alpha}\,H_2^{(1)} + o(\sqrt{\alpha});\; \sqrt{\alpha}\,A_1^{1(1)} + \alpha\,A_1^{1(2)} + o(\alpha), ...,$$
$$\sqrt{\alpha}\,H_1^{(1)} + \alpha\,H_1^{(2)} + o(\alpha)\Bigg) + f_i(\,A_1^{1(1)} + \sqrt{\alpha}\,A_1^{1(2)} + o(\sqrt{\alpha}), ..., H_1^{(1)} + \sqrt{\alpha}\,H_1^{(2)} + o(\sqrt{\alpha}))$$
$$= 0. \tag{A-100}$$

It follows that in zero–order approximation in $\sqrt{\alpha}$ one has to solve the system of equations with respect to zero–order coefficients from set Ω_1 and the first–order coefficients from set Ω_2:

$$F_i(A_0^{1(0)}, ..., H_2^{(0)}; 0, ..., 0) + f_i(A_1^{1(1)}, ..., H_1^{(1)}) = 0. \quad \text{(A-101)}$$

The system of equations (A-101) cannot be solved in a closed form. I find its solution numerically by taking a limit $\alpha \to 0$ in a general solution for an arbitrary value of α. After expanding equations (A-100) with respect to $\sqrt{\alpha}$ and taking into account the system of equations (A-101) I arrive to an additional set of equations in the first order in $\sqrt{\alpha}$:

$$\sum_{X \in \Omega_1 \bigcup \Omega_2} F_{iX}(A_0^{1(0)}, ..., H_2^{(0)}; 0, ..., 0) X^{(1)} + \sum_{X \in \Omega_2} f_{iX}(A_1^{1(1)}, ..., H_1^{(1)}) X^{(2)} = 0, \quad \text{(A-102)}$$

where summation is taken with respect to each coefficient from a given set and subindex X stands for a derivative with respect to X. The last system of equations is non–homogeneous linear with respect to coefficients $A_0^{1(1)}, ..., H_2^{(1)}$; $A_1^{1(2)}, ..., H_1^{(2)}$. Its solution is straightforward assuming that the system is not degenerate. Derivation of the equilibrium coefficients of higher order in $\sqrt{\alpha}$ is similar.

Finally, it is straightforward to show that equilibrium coefficients in zero–order approximation in $\sqrt{\alpha}$ can be written as

$$B_1^{1(1)} = 2\sqrt{r\gamma_1 \left[2\sigma_Y^2 (C_0^{1(0)})^2 - (r + 2\kappa_Y + 2\sigma_Y r \gamma_1 \beta_1 \rho_{0Y}) C_0^{1(0)} + \frac{(r\gamma_1\beta_1)^2}{2} \right]}, \quad \text{(A-103)}$$

$$R_1^{(0)} = \frac{r(\sigma_S^{(0)})^2}{2(1+\delta)}(\delta\gamma_1 - \gamma_2), \quad \text{(A-104)}$$

$$A_2^{1(1)} = \sqrt{\frac{2r^3\gamma_1^2(\sigma_S^{(0)})^2(\gamma_1+\gamma_2)}{1+\delta}}, \quad \text{(A-105)}$$

$$A_2^{2(1)} = \delta \frac{\gamma_2}{\gamma_1} A_2^{1(1)}, \quad \text{(A-106)}$$

$$H_1^{(1)} = -\sqrt{\alpha r (\sigma_S^{(0)})^2}\, \frac{\delta\gamma_1 - \gamma_2}{\delta\sqrt{2(\gamma_1+\gamma_2)(1+\delta)}}, \quad \text{(A-107)}$$

$$R_2^{(0)} = -\frac{H_1^{(1)} B_1^{1(1)}}{2r\gamma_1} - (r+\kappa_Y) H_2^{(0)}, \quad \text{(A-108)}$$

$$A_1^{1(1)} = \left[\frac{(A_2^{1(1)})^2}{2(r\gamma_1)^2}\,\delta - R_1^{(0)} + (z_2^{(0)} - z_1^{(0)})\sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y}^{(0)})] \right] (\pi^{(0)})^{-1}, \quad \text{(A-109)}$$

$$\pi^{(0)} = (x_2^{(0)} - x_1^{(0)})\sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_{S_Y}^{(0)}) - \frac{(1+\delta)}{\gamma_1} - \frac{A_2^{1(1)}}{2(r\gamma_1)^2}(1+\delta), \quad \text{(A-110)}$$

$$x_n = \sqrt{\alpha} x_n^{(0)} + o(\sqrt{\alpha}), \quad x_n^{(0)} = \frac{(-1)^{n+1} B_1^{n(1)}}{2r\gamma_1(2\sigma_Y^2 C_0^{n(0)} - r - \kappa_Y - \sigma_Y \rho_{0Y} r \gamma_n \beta_n)},$$

$$z_n = z_n^{(0)} + O(\sqrt{\alpha}), \quad z_1^{(0)} = \frac{2\kappa_Y \bar{Y} C_0^{1(0)}}{2\sigma_Y^2 C_0^{1(0)} - r - \kappa_Y - \sigma_Y \rho_{0Y} r \gamma_1 \beta_1}, \quad \text{(A-111)}$$

$$z_2^{(0)} = \frac{\frac{A_2^{1(1)} B_1^{2(1)}}{2r\gamma_1} + 2\kappa_Y \bar{Y} C_0^{2(0)}}{2\sigma_Y^2 C_0^{2(0)} - r - \kappa_Y - \sigma_Y \rho_{0Y} r \gamma_2 \beta_2}, \quad \text{(A-112)}$$

$$B_0^{n(0)} = \frac{\frac{A_1^{n(1)} B_1^{n(1)}}{2\delta_n r\gamma_n} - 2\kappa_Y \bar{Y} C_0^{n(0)}}{2\sigma_Y^2 C_0^n - r - \kappa_Y - \sigma_Y \rho_{0Y} r\gamma_n \beta_n}, \tag{A-113}$$

$$R_0^{(0)} = -\frac{A_1^{1(1)} A_2^{1(1)}}{2(r\gamma_1)^2} - \sigma_Y(\rho_{DY}\sigma_{S_D} + \sigma_Y H_2^{(0)}) B_0^{1(0)}, \tag{A-114}$$

$$A_0^{n(0)} = 1 - \ln(r) + [-\zeta_n + \frac{1}{2}\sigma_Y^2(2C_0^{n(0)} + (B_0^{n(0)})^2) - \frac{(A_1^{n(1)})^2}{4\delta_n r\gamma_n} + \kappa_Y \bar{Y} B_0^{n(0)}]/r, \tag{A-115}$$

$$H_0^{(0)} = \frac{1}{r}\left[\frac{\kappa_D \bar{D}}{r + \kappa_D} - \frac{A_1^{1(1)} H_1^{(1)}}{2r\gamma_1} + \kappa_Y \bar{Y} H_2^{(0)} - R_0^{(0)}\right], \tag{A-116}$$

where $\delta_1 = 1$, $\delta_2 = \delta$ and I used the approximation $\sigma_S^2 = (\sigma_S^{(0)})^2 + 2\sqrt{\alpha}\sigma_Y^2 H_2^{(0)} H_2^{(1)}$, in which $(\sigma_S^{(0)})^2 = \sigma_D^2 + \sigma_Y^2 (H_2^{(0)})^2 + 2\sigma_D \sigma_Y H_2^{(0)} \rho_{DY}$. Q.E.D.

Proof of Proposition 2

If relation $\delta\gamma_1 = \gamma_2$ holds then it follows from equations (A-104) and (A-107) that

$$R_1^{(0)} = H_1^{(1)} = 0. \tag{A-117}$$

The last equality implies that

$$R_2^{(0)} = -(r + \kappa_Y) H_2^{(0)}. \tag{A-118}$$

Now I consider the main equations used in the derivation of equilibrium coefficients, rewrite them using equations (A-103)–(A-108), and keep only the terms of zero order in α. Comparison of equations (A-22) and (A-61) shows that they are identical in zero order in α. Similar, equations (A-52) for $n = 1, 2$ are identical to equations (A-25) for $n = 1, 2$, respectively. Equations (A-22), (A-52) for $n = 1, 2$ have three unknown coefficients C^1, C^2, H_2, while equations (A-61), (A-25) for $n = 1, 2$ have three unknown coefficients C_0^1, C_0^2, H_2 in the market with transaction costs. It follows that the corresponding coefficients are the same across the two markets. It follows from equations (A-103)–(A-108) that coefficients $R_2^{(0)}$, $R_0^{(0)}$, $H_0^{(0)}$, $B_0^{1(0)}$, $B_0^{2(0)}$, $A_0^{1(0)}$, $A_0^{2(0)}$ are equal to the coefficients R_2, R_0, H_0, B^1, B^2, A^1, A^2, respectively, while the remaining coefficients are equal to zero in zero order approximation in α. Further comparison of equations (A-113) for $n = 1, 2$ and equations (A-31), (A-32) shows identity between B^1, B^2 and $B_0^{1(0)}$, $B_0^{2(0)}$, respectively, after equation (A-109) is taken into account. Similar, equations (A-115) and (A-16), with (A-116) and (A-9) are identical suggesting equality between $A_0^{n(0)}$ and A^n as well as between $H_2^{(0)}$ and H_2.

Now assume that zero–order coefficients in the economy with transaction costs are equal to the corresponding coefficients in the economy without transaction costs. It follows that coefficients C^1 and C^2 satisfy both equations (A-22) and (A-61). Comparison of these two equations implies the identity:

$$\frac{(\gamma_2)^2}{(\gamma_1)^2}\left(\frac{(r\beta_1\gamma_1)^2}{2}+2\sigma_Y^2(C^1)^2-[r+2(\kappa_Y+\sigma_Y r\gamma_1\beta_1\rho_{0Y})]C^1\right)$$
$$=\frac{\gamma_2\alpha_2}{\gamma_1\alpha_1}\left(\frac{(r\gamma_1\beta_1)^2}{2}+2\sigma_Y^2(C^1)^2-(r+2\kappa_Y+2\sigma_Y r\gamma_1\beta_1\rho_{0Y})C^1\right), \quad \text{(A-119)}$$

which is possible only if $\gamma_2/\gamma_1=\alpha_2/\alpha_1$. Q.E.D.