Changing Probability Measures in GARCH Option Pricing Models

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Abstract

In this paper, we consider several option pricing models with stochastic volatility in the context of the generalized autoregressive conditional heteroskedastic (GARCH) processes. We propose a globally risk-neutral valuation relationship (GRNVR) to derive the model dynamics under risk-neutral measure and obtain the corresponding closed-form pricing formula for the Chicago Board Options Exchange Volatility Index (CBOE VIX). The parameters of the proposed models are then calibrated using the S&P 500 returns data and the CBOE VIX. Based on the empirical pricing performances, we observe that the proposed GRNVR generally performs better than the locally risk-neutral valuation relationship (LRNVR). We also provide theoretical justification of the proposed GRNVR.

Keywords: GARCH process, Stochastic volatility, CBOE VIX

JEL Classifications: G13, C52.
1. Introduction

The study of finance largely concerns about the trade-off between risk and expected return. An important source of risk is the uncertainty of the volatility of equity indices, where volatility is understood as the standard deviation of the return of a financial instrument with a specific time horizon. Since the Chicago Board Options Exchange (CBOE) introduced standardized derivative contracts on S&P 500 implied volatility index (VIX), e.g., VIX options and VIX futures, volatility derivatives have been traded actively in the CBOE market. As the interest in understanding and forecasting the VIX index grows, it is important to develop reliable mathematical models for the term structure of VIX. The literature of using stochastic volatility model to evaluate VIX and its derivatives has grown rapidly in the last decade. The first stochastic volatility model to study the VIX index and VIX futures were proposed in Zhang and Zhu (2006). Zhu and Zhang (2007) extended the model in Zhang and Zhu (2006) by changing the long-term mean variance to be time-dependent. Lin (2007) used the affine jump-diffusion model to describe the VIX with jumps in both stock and volatility processes. The term structure of VIX was analyzed in Luo and Zhang (2012). Unfortunately, the parameter estimation of stochastic volatility models is often a challenging task as the models generally do not have the closed form likelihood function given the S&P 500 stock and VIX data.

On the other hand, GARCH models introduced by Bollerslev in Bollerslev (1986) are widely used to model volatility term structure in practice mainly because the like-
likelihood function of the parameters in the GARCH models can often be expressed in closed form in terms of the observed data. So it is possible to derive the maximum likelihood estimation (MLE) of the model parameters. Inspired by the success in describing volatility with GARCH models, Duan (1995) pioneered in applying GARCH models for S&P 500 option pricing by proposing a locally risk-neutral valuation relationship (LRNVR). Under the LRNVR, the one-period ahead conditional variance remains the same during the change of probability measure. This approach developed a link between the conditional variance in the physical measure and the risk neutral measure. Therefore, GARCH models is a natural candidate for calibration using the historical VIX data. Kannaiinen, Lin, and Yang (2014) evaluate S&P 500 options using three variant GARCH models using VIX data. They found empirical evidence that a joint maximum likelihood estimation using S&P 500 returns and VIX improved the performance of pricing S&P 500 options compared to traditional MLE using the returns data only. However, there exists some drawback in the calibration of the GARCH models under the LRNVR risk-neutral measure. The empirical evidence was obtained in Barone-Adesi, Engle, and Mancini (2008); Christoffersen, Heston, and Jacobs (2013) to show that the same model parameters in the conditional volatility of historical and risk-neutral pricing dynamics results in poor calibration results. Hao and Zhang (2013) also demonstrated that the GARCH-type models under the LRNVR could not capture the variance risk premium. They examined a family of GARCH models and derived the model-implied VIX index under the LRNVR. When the models were estimated
with returns data only, the model implied VIX was significantly lower than the market CBOE VIX. Even when the models were jointly estimated with both returns and VIX, the equity risk parameter distorted to be a large positive price, and the model implied VIX still underestimated the CBOE VIX, and did not match the statistical aspects of the CBOE VIX. They further provided the theoretical arguments by considering the diffusion limit of GARCH models. Inspired by the discussion in Hao and Zhang (2013), we propose a new globally risk-neutral valuation relationship (GRNVR) for the risk-neutral dynamics of GARCH models and display the advantage of the GRNVR compared to the LRNVR using theoretical and empirical evidences.

The article is structured as follows. Section 2 proposes a new way for changing physical probability measure for the GARCH(1,1) to the risk-neutral probability measure. In Section 3, we derive theoretical VIX formula for the GARCH(1,1) model under the risk-neutral measure, and extend the derivation idea to a broad class of GARCH models which include GARCH, TGARCH, AGARCH and EGARCH models. In Section 4, we calibrate these GARCH models using various combinations of time series of the S&P 500 index and the CBOE VIX. Section 4 compares the the CBOE VIX with the GARCH implied VIX obtained from the calibrated GARCH models. In Section 5, we analyze the diffusion limit of the GARCH process under the risk-neutral measure to demonstrate that the risk-neutral dynamics captures the variance risk premium. We then conclude the findings in Section 6.
2. GARCH model specification

In this paper, we consider the asset price as a discrete-time stochastic process and denote the asset price at time $t$ as $X_t$. It was proposed in Duan (1995) that the return of the asset follows a conditional lognormal distribution under the physical measure $P$ as

$$\ln \frac{X_t}{X_{t-1}} = r - \frac{1}{2} h_t + \lambda_1 \sqrt{h_t} + \epsilon_t,$$

where $r$ is the one-period risk-free interest rate, $\lambda_1$ is the asset risk premium, and $\epsilon_t$ follows a GARCH($p, q$) process introduced in Bollerslev (1986) with mean zero and conditional variance $h_t$

$$\epsilon_t|\phi_{t-1} \sim N(0, h_t) \quad \text{under measure P,}$$

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j}, \quad (1)$$

where $\phi_t$ is the information set of up to and including time $t$; $\alpha_0 \geq 0$, $\alpha_i \geq 0$ for $i = 1, 2, \ldots, q$ and $\beta_j \geq 0$ for $j = 1, 2, \ldots, p$. We focus on the GARCH(1,1) case, so the equation (1) simplifies to

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 h_{t-1}. \quad (2)$$

We propose an alternative risk-neutral valuation relationship to the locally risk-neutral valuation relationship (LRNVR) introduced by Duan (1995). We shall refer to
the proposed risk neutral-valuation relationship as the globally risk-neutral valuation relationship (GRNVR); and the dynamics of asset return in the risk-neutral pricing measure $Q$ under the GRNVR has the following form as

$$
\ln \frac{X_t}{X_{t-1}} = r - \frac{1}{2}h_t + \xi_t, \quad \xi_t \sim N(0, h_t) \quad \text{under measure } Q,
$$

$$
h_t = \alpha_0 + \alpha_1 \left( \xi_{t-1} - \lambda_1 \sqrt{h_{t-1}} \right)^2 + \beta_1^* h_{t-1}
$$

$$
= \alpha_0 + \alpha_1 \left( \xi_{t-1} - \lambda_1 \sqrt{h_{t-1}} \right)^2 + (\beta_1 - \sqrt{2}\alpha_1 \lambda_2)h_{t-1}. \quad (3)
$$

Note that the proposed GARCH(1,1) process (3) is different to the one derived by Duan (1995) under the LRNVR. Under the GRNVR the persistence parameter $\beta_1$ is designed to be different in the $P$ and $Q$ measures, whereas under the LRNVR the persistence parameter $\beta_1$ is the same in the $P$ and $Q$ measures. Specifically, for the dynamics of risk-neutral measure $Q$ under the GRNVR, the persistence parameter of conditional variance is $\beta_1^* = \beta_1 - \sqrt{2}\alpha_1 \lambda_2$, where $\lambda_2$ represents the variance risk premium of the asset. The motivation for the inclusion of the variance risk premium is discussed in Hao and Zhang (2013) where it was shown that there is no risk adjustment for the variance risk of the process in Duan (1995) from physical measure to the risk-neutral measure under the LRNVR. It was also discussed in Barone-Adesi et al. (2008); Christoffersen et al. (2013) that the restriction of conditional volatility of historical and risk-neutral pricing distributions with the same model parameters leads to poor calibration results.
in the empirical studies (cf. Chernov and Ghysels (2000); Christoffersen, Heston, and Jacobs (2006); Hao and Zhang (2013)). Therefore, it was suggested that the parameters of volatility dynamics of historical and risk-neutral pricing returns might be different in Barone-Adesi et al. (2008). We adopt the idea by modifying the persistence parameter in $Q$ to incorporate the variance risk premium in the model. The theoretical justification of the modification is further discussed in section 6.

3. VIX formulas of the GARCH models

The Chicago Board Options Exchange (CBOE) introduced a volatility index, named VIX, in 1993. The VIX index is calculated from the implied volatilities of the eight near-the-money, nearby, and second nearby S&P 100 index options based on the methodology by Whaley Whaley (1993). The VIX was a proxy of the implied volatility of 30 calendar days at-the-money (ATM) options. In 2003, the CBOE used another theory proposed in Carr and Madan (1998); Demeterfi, Derman, Kamal, and Zou (1999) to design a new methodology to compute the CBOE volatility index VIX. The new VIX is based on the prices of a portfolio of 30 calendar days out-of-the-money (OTM) S&P 500 index call and put options. The square of new VIX represents the S&P 500 30-day variance swap rate. The old VIX has been renamed to be VXO.

The VIX index reflects investors’ expectation of the volatility of the S&P 500 in
the next 30 calendar days or 21 trading days, which is calculated using its definition as

\[
\left( \frac{\text{VIX}_t}{100} \right)^2 = \mathbb{E}_t^Q \left( \frac{1}{\tau_0} \int_t^{t+\tau_0} \tilde{h}_s ds \right),
\]

where \( \tilde{h}_s \) denotes the annualized instantaneous variance of the return of S&P 500 and \( \tau_0 \) is 30 calendar days or 21 trading days. In this paper, VIX is computed as the mean value of the expected variance in the \( n \) sub-periods of the next 21 trading days, that is

\[
\left( \frac{\text{VIX}_t}{100} \right)^2 = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}_t^Q \left( h_{t+k} \right).
\]

In particular, we use the daily closing value data, so it implies \( \tau_0 = n \), and

\[
V_t = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}_t^Q \left( h_{t+k} \right),
\]

where the term \( V_t = \frac{1}{252} \left( \frac{\text{VIX}_t}{100} \right)^2 \) is defined as a function of VIX\(_t\) to measure the expected daily variance of S&P 500. The conditional mean of the future variance can be calculated in a broad class of GARCH models as discussed in Hao and Zhang (2013); Wang, Shen, Jiang, and Huang (2017).

We derive the implied VIX from the model (3) under \( Q \) by first rewriting the error
terms of the process using the standard normal distribution as

\[ \ln \frac{X_t}{X_{t-1}} = r - \frac{1}{2} h_t + \sqrt{h_t} \varepsilon_t, \]

\[ h_t = \alpha_0 + \alpha_1 h_{t-1} (\varepsilon_{t-1} - \lambda_1)^2 + (\beta_1 - \sqrt{2} \alpha_1 \lambda_2) h_{t-1}, \tag{4} \]

where \( \varepsilon_t \) is the standard normal random variable, conditional on the information set up to and including time \( t - 1 \) under \( Q \).

One can rewrite the GARCH(1, 1) process (4) as a special case of the square-root stochastic autoregressive volatility (SR-SARV(1)) models introduced in Meddahi and Renault (2004) with the following form

\[ h_{t+1} = \omega + \gamma h_t + \nu_t, \text{ with } \mathbb{E}[\nu_t | \phi_{t-1}] = 0, \tag{5} \]

\[ \omega = \alpha_0, \quad \gamma = \alpha_1 (1 + \lambda_1^2) + \beta_1 - \sqrt{2} \alpha_1 \lambda_2, \]

\[ \nu_t = \alpha_1 h_t (\varepsilon_t^2 - 1 - 2 \lambda_1 \varepsilon_t). \]

It was shown in Hao and Zhang (2013) that if the S&P 500 return follows a SR-SARV(p) process under the risk-neutral measure, then the implied daily variance \( V_t \) at time \( t \) is affine in the conditional variance \( h_{t+1} \). Following similar ideas, we can obtain the long term variance as \( \bar{h} = \lim_{m \to \infty} \mathbb{E}_t^Q [h_{t+m}] = \frac{\omega}{1 - \gamma} \) by noticing

\[ \bar{h} = \lim_{m \to \infty} \mathbb{E}_t^Q [h_{t+m}] = \lim_{m \to \infty} \mathbb{E}_t^Q [\omega + \gamma h_{t+m-1} + \nu_{t+m-1}] = \omega + \gamma \bar{h}. \]
Then the conditional expectation of the variance after two periods can be obtained via the long run variance

\[ E_t^Q [h_{t+2}] - \bar{h} = E_t^Q [\omega + \gamma h_{t+1} + \nu_{t+1}] - \frac{\omega}{1 - \gamma} = \omega + \gamma h_{t+1} - \frac{\omega}{1 - \gamma} = \gamma (h_{t+1} - \bar{h}). \]

So the conditional expectation of the variance after \( n \) periods is given by

\[ E_t^Q [h_{t+n}] = \bar{h} + \gamma^{n-1} (h_{t+1} - \bar{h}). \]

Therefore, we can represent the expected daily variance as an affine function of \( h_{t+1} \)

\[ V_t = \frac{1}{n} \sum_{k=1}^{n} E_t^Q (h_{t+k}) \]

\[ = \bar{h} + \frac{1}{n} \sum_{k=1}^{n} \gamma^{k-1} (h_{t+1} - \bar{h}) \]

\[ = \bar{h} + \frac{1 - \gamma^n}{n(1 - \gamma)} (h_{t+1} - \bar{h}) \]

\[ = \left( 1 - \frac{1 - \gamma^n}{n(1 - \gamma)} \right) \frac{\omega}{1 - \gamma} + \frac{1 - \gamma^n}{n(1 - \gamma)} h_{t+1} \]

\[ = A + Bh_{t+1}, \quad (6) \]

where \( A = \frac{(1-B)\omega}{1-\gamma} \) and \( B = \frac{1-\gamma^n}{n(1-\gamma)}. \)

Apart from the GARCH(1,1) model discussed above, we also consider the threshold GARCH(1,1) (TGARCH) model introduced in Glosten, Jagannathan, and Runkle (1993), the non-linear asymmetric GARCH(1,1) (AGARCH) model proposed in Engle
and Ng (1993) and the exponential GARCH(1,1) (EGARCH) model by Nelson (Nelson, 1991). The forms of the models in the physical measure $P$ and in the risk-neutral measure $Q$ under the GRNVR are as follows:

**TGARCH(1,1)**

**Physical measure:**

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \theta \epsilon_{t-1}^2 \mathbf{1}(\epsilon_{t-1} < 0) + \beta_1 h_{t-1},$$  

(7)

**GRNVR:**

$$h_t = \alpha_0 + \left( \xi_{t-1} - \lambda_1 \sqrt{h_{t-1}} \right)^2 \left( \alpha_1 + \theta \mathbf{1}(\xi_{t-1} - \lambda_1 \sqrt{h_{t-1}} < 0) \right) + (\beta_1 - \sqrt{2} \alpha_1 \lambda_2) h_{t-1}. $$

**AGARCH(1,1)**

**Physical measure:**

$$h_t = \alpha_0 + \alpha_1 \left( \epsilon_{t-1} - \theta \sqrt{h_{t-1}} \right)^2 + \beta_1 h_{t-1},$$  

(8)

**GRNVR:**

$$h_t = \alpha_0 + \alpha_1 \left( \xi_{t-1} - \lambda_1 \sqrt{h_{t-1}} - \theta \sqrt{h_{t-1}} \right)^2 + (\beta_1 - \sqrt{2} \alpha_1 \lambda_2) h_{t-1}. $$

**EGARCH(1,1)**

**Physical measure:**

$$\ln h_t = \alpha_0 + \beta_1 \ln h_{t-1} + \alpha_1 \frac{\epsilon_{t-1}}{\sqrt{h_{t-1}}} + \kappa \left( \left| \frac{\epsilon_{t-1}}{\sqrt{h_{t-1}}} \right| - \sqrt{\frac{2}{\pi}} \right),$$  

(9)

**GRNVR:**

$$\ln h_t = \alpha_0 + (\beta_1 - \sqrt{2} \alpha_1 \lambda_2) \ln h_{t-1} + \alpha_1 \left( \frac{\epsilon_{t-1}}{\sqrt{h_{t-1}}} - \lambda_1 \right) + \kappa \left( \left| \frac{\epsilon_{t-1}}{\sqrt{h_{t-1}}} - \lambda_1 \right| - \sqrt{\frac{2}{\pi}} \right).$$

As shown in Hao and Zhang (2013), these widely used models are special cases of SR-SARV($p$) models; and following similar derivation process as the GARCH(1,1) model,
we can obtain the implied VIX formula for different GARCH models analogous to the ones in Hao and Zhang (2013) as:

TGARCH(1,1)

\[ V_t = C + Dh_{t+1}, \quad (10) \]

where

\[
C = \frac{\alpha_0(1 - D)}{1 - \eta}, \\
D = \frac{1 - \eta^n}{n(1 - \eta)}, \\
\eta = \alpha_1(1 + \lambda_1^2) + (\beta_1 - \sqrt{2}\alpha_1\lambda_2) + \theta S, \\
S = \frac{\lambda_1}{\sqrt{2\pi}}e^{-\frac{\lambda_1^2}{2}} + (1 + \lambda_1^2)N(\lambda_1).
\]

Note that \( N(\cdot) \) denotes the cumulative function of the Normal distribution.

AGARCH(1,1)

\[ V_t = E + Fh_{t+1}, \quad (11) \]
where

\[
E = \frac{a_0(1 - F)}{1 - \eta}, \\
F = \frac{1 - \eta^n}{n(1 - \eta)}, \\
\eta = \alpha_1(1 + (\lambda_1 + \theta)^2) + (\beta_1 - \sqrt{2}\alpha_1\lambda_2).
\]

\[
E_{GARCH(1,1)}
\]

\[
V_t = \frac{1}{n} \left( h_{t+1} + \sum_{k=1}^{n-1} \left( \prod_{i=0}^{k-1} I_i \right) h_{t+1}^{(\beta_1 - \sqrt{2}\alpha_1\lambda_2)^k} \right), \quad (12)
\]

where

\[
I_i = e^{(\beta_1 - \sqrt{2}\alpha_1\lambda_2)^i (a_0 - \kappa \sqrt{2})}
\]

\[
\left( e^{-(\beta_1 - \sqrt{2}\alpha_1\lambda_2)^i (a_1 - \kappa)\lambda_1 + 0.5(\beta_1 - \sqrt{2}\alpha_1\lambda_2)^2 (a_1 - \kappa)^2} N(\lambda_1 - (\beta_1 - \sqrt{2}\alpha_1\lambda_2)^i (a_1 - \kappa)) \\
+ e^{-(\beta_1 - \sqrt{2}\alpha_1\lambda_2)^i (a_1 + \kappa)\lambda_1 + 0.5(\beta_1 - \sqrt{2}\alpha_1\lambda_2)^2 (a_1 + \kappa)^2} N((\beta_1 - \sqrt{2}\alpha_1\lambda_2)^i (a_1 + \kappa) - \lambda_1) \right)
\]

4. Data and estimation

It was shown in Hao and Zhang (2013) that under the LRNVR, the GARCH implied VIX does not fit the market data of CBOE VIX very well. The model was analyzed to display that the reason may be that variance risk premium and the volatility risk price were not present in the diffusion limit of the GARCH models under the LRNVR. In
the modified GARCH processes, we include the variance risk premium into the models under the GRNVR. And it is of interest to see whether the implied VIX in the modified GARCH models fit the CBOE VIX market values better. In this section, we will investigate this question by estimating the parameters in the modified GARCH models and calculating the corresponding GARCH implied VIX times series for comparison to the CBOE VIX.

The two time series data we use for the GARCH models calibration are the closing values of S&P 500 and the CBOE VIX ranging from 2nd January, 1990 to 30th June, 2017. For the daily risk-free interest rate, we use the 3-month treasury bill secondary market rate from the U.S. Federal Reserve website.

There are different methods to calibrate the models using market data. We will use the common maximum likelihood approach to estimate the parameters of the models. We can use only the S&P 500 returns data to obtain a maximum likelihood estimation of the GARCH processes under the physical measure $P$ and fix the variance risk premium parameter $\lambda_2 = 0$, since $\lambda_2$ is not included in the GARCH models under $P$ measure. For the S&P 500 returns data only, the log-likelihood function $\ln L_R$ for the GARCH models is given by

$$\ln L_R = -\frac{T \ln(2\pi)}{2} - \frac{1}{2} \sum_{t=1}^{T} \left( \ln(h_t) + \left( \ln \frac{X_t}{X_{t-1}} - r - \lambda_1 \sqrt{h_t} + \frac{h_t}{2} \right)^2 / h_t \right),$$

where the conditional variance $h_t$ is updated by corresponding processes using different forms of GARCH models. For the maximum likelihood estimation, the conditional
variance for the first period is set as the variance of S&P 500 returns over the whole sample period. The stationary conditions for the GARCH processes under physical and the risk-neutral measures are different, with the latter having more strict constraints on the parameters. Thus, we find the estimation of the parameters in the GARCH models by maximizing the corresponding log-likelihood function subject to the stationary conditions under the risk-neutral measures.

We may also calibrate the GARCH models by matching the model implied VIX to the market value of CBOE VIX, since the CBOE VIX series may contain additional information about the underlying S&P 500 return process. To utilize both time series, we follow the assumption in Hao and Zhang (2013) that the pricing differences between the CBOE VIX and the implied VIX on a daily basis come from a Normal distribution

\[ \text{VIX}^{\text{Mkt}} = \text{VIX}^{\text{Imp}} + \mu, \quad \mu \sim N(0, s^2), \]

where \( s^2 \) is estimated using the sample variance of pricing difference of \( \hat{s}^2 = \text{var}(\text{VIX}^{\text{Mkt}} - \text{VIX}^{\text{Imp}}) \). Under the above assumption, the log-likelihood function corresponding to the CBOE VIX data is

\[
\ln L_V = -\frac{T \ln(2\pi \hat{s}^2)}{2} - \sum_{t=1}^{T} \frac{(\text{VIX}^{\text{Mkt}} - \text{VIX}^{\text{Imp}})^2}{2\hat{s}^2}.
\]

(14)

In addition to use the S&P 500 returns data and CBOE VIX data for calibration of the GARCH models separately, we can also combine both time series to find a joint
maximum likelihood estimation of the models by maximizing the joint log-likelihood function

\[ \ln L_T = \ln L_R + \ln L_V. \]  

5. Numerical results

In this section, we compare the estimated parameters from different data used for the calibration. In particular, Table 1 displays the maximum likelihood estimates and the standard errors of GARCH(1,1) model. The values of the three log-likelihood functions (13, 14, 15) are also displayed in Table 1. Although the contributions from S&P 500 returns and CBOE VIX as well as the joint likelihood values are reported, we maximize \( \ln L_R \) when only S&P 500 returns are used, \( \ln L_V \) when only CBOE VIX data are used and \( \ln L_T \) when both time series are used.

From the output in Table 1, we can see that the equity risk premium \( \lambda_1 \) increases significantly from 0.0886 (return data used) to 0.2134 (both data used) and 0.2253 (VIX data used) in the GARCH(1,1) models when the CBOE VIX data is used for calibration. The variance risk premium \( \lambda_2 \) is negative and significantly different from zero as -0.3670 (both data used) and -0.3514 (VIX data used). The persistence of conditional variance, \( \beta_1 \) increases slightly from 0.8543 (return data used) to 0.9251 (both data used) and 0.9286 (VIX data used). There is a sizable decrease of the parameter value \( \alpha_1 \) from 0.0886 (return data used) to 0.0474 (both data used) and
Comparing the maximum likelihood result of the model under the GRNVR and the results under the LRNVR, we can see that the maximum likelihood value increases significantly from 54697 to 55921 (both data used) and from 33424 to 33662 (VIX data used).

Similar numerical results are also observed in the other types of GARCH models as displayed in Tables 2-4. Specifically, Table 2 shows that the equity risk premium $\lambda_1$ increases significantly from 0.0131 (return data used) to 0.1160 (both data used) and 0.0889 (VIX data used) in the TGARCH(1,1) model when the CBOE VIX data is used for calibration. The variance risk premium $\lambda_2$ is negative and significantly different from zero as -0.4412 (both data used) and -0.3978 (VIX data used). The persistence of conditional variance, $\beta_1$ increases significantly from 0.8338 (return data used) to 0.9561 (both data used) and 0.9553 (VIX data used). There is a decrease of the parameter value $\alpha_1$ from 0.0256 (return data used) to 0.0091 (both data used) and 0.0060 (VIX data used). Comparing the maximum likelihood result of the TGARCH(1,1) model under the GRNVR and the results under the LRNVR, we can see that the maximum likelihood value increases significantly from 55455 to 56282 (both data used) and from 33468 to 33795 (VIX data used).

Table 3 shows calibration results of the AGARCH(1,1) model using both returns and VIX data. If using VIX data only, it is not easy to distinguish the parameters $\theta$ and $\lambda_1$. Therefore, the numerical results using VIX data are not displayed in the table. From Table 3 we observe that the equity risk premium $\lambda_1$ increases significantly from
0.0255 (return data used) to 0.1158 (both data used) in the AGARCH(1,1) model when the CBOE VIX data and returns used for calibration. The variance risk premium $\lambda^2$ is negative and significantly different from zero as -0.3125 (both data used). The persistence of conditional variance, $\beta_1$ increases from 0.8810 (return data used) to 0.9316 (both data used). There is a big decrease of the parameter value $\alpha_1$ from 0.0841 (return data used) to 0.0380 (both data used). Comparing the maximum likelihood result of the AGARCH(1,1) model under the GRNVR and the results under the LRNVR, we can see that the maximum likelihood value increases significantly from 55483 to 56333 (both data used).

Table 4 shows that in the EGARCH(1,1) model the variance risk premium $\lambda^2$ is negative as -0.0567 (both data used) and -0.0483 (VIX data used), both significantly different than zero. The persistence of conditional variance, $\beta_1$ increases slightly from 0.9792 (return data used) to 0.9906 (both data used) and 0.9891 (VIX data used). Comparing the maximum likelihood result of the EGARCH(1,1) model under the GRNVR and the results under the LRNVR, we can see that the maximum likelihood value increases significantly from 56399 to 57105 (both data used) and from 33774 to 34303 (VIX data used).

From the comparisons in the GARCH, TGARCH, AGARCH and EGARCH models, we see that the maximum likelihood results under the GRNVR are generally better than those under the LRNVR. Table 5 measures how the implied VIX fits the CBOE VIX by computing a list of statistics and the results demonstrate that the implied VIX
under GRNVR fits the CBOE quite well.

After obtaining the estimates of the parameters in the models, we can then calculate the conditional variance $h_t$ and compute the corresponding GARCH implied VIX. Figure 1 shows the time series of the CBOE VIX and the implied VIX of the four GARCH models estimated using returns only. Figure 2 shows the time series of the CBOE VIX and the implied VIX of the GARCH(1,1) model estimated using VIX data only. Figure 3 shows the time series of the CBOE VIX and the implied VIX of the GARCH(1,1) model estimated using both returns and VIX. Similar comparison plots are obtained for other GARCH models. Specifically, the time series of the CBOE VIX and the implied VIX of the TGARCH(1,1) model estimated with VIX data only are displayed in Figure 4. The time series of the CBOE VIX and the implied VIX of the TGARCH(1,1) and AGARCH (1,1) model estimated with both returns and VIX data are shown in Figure 5 and Figure 6, respectively. For the EGARCH(1,1) model, Figure 7 shows the comparison between the CBOE VIX and model implied VIX with VIX data only, and Figure 8 displays the comparison between the CBOE VIX and model implied VIX with both returns and VIX data. From the list of graphs, we observe that the model implied VIX fits the CBOE VIX better under the GRNVR compared to the LRNVR in general.
6. Theoretical justification

Duan studied the bivariate diffusion limit of the GARCH(1,1) model as the length of the time period tends towards zero in Duan (1996, 1997). Applying Duan’s arguments, one can show that the limiting bivariate diffusion process of the approximating GARCH(1,1) process under the physical measure $P$ is given by

\begin{align}
\ln X_t &= \left( r - \frac{1}{2} h_t + \lambda_1 \sqrt{h_t} \right) dt + \sqrt{h_t} dW_{1t}, \quad (16) \\
 dh_t &= (\alpha_0 + (\alpha_1 + \beta_1 - 1)h_t)dt + \sqrt{2}\alpha_1 h_t dW_{2t}, \quad (17) \\
 &= (\alpha_0 + (\alpha_1 + \beta_1^* - 1)h_t)dt + \sqrt{2}\alpha_1 \lambda_2 h_t dt + \sqrt{2}\alpha_1 h_t dW_{2t}, \quad (18)
\end{align}

where the persistence parameter of conditional variance is defined as $\beta_1^* = \beta_1 - \sqrt{2}\alpha_1 \lambda_2$ under the GRNVR. The terms $dW_{1t}$ and $dW_{2t}$ are independent standard Brownian motions under the physical measure $P$. The limiting bivariate diffusion under the risk-neutral measure $Q$ is a re-parameterization of Hull and White’s bivariate diffusion model Hull and White (1987) as follows:

\begin{align}
\ln X_t &= \left( r - \frac{1}{2} h_t \right) dt + \sqrt{h_t} dZ_{1t} \quad (19) \\
 dh_t &= (\alpha_0 + (\alpha_1 + \beta_1^* - 1)h_t)dt + \sqrt{2}\alpha_1 h_t dZ_{2t}, \quad (20)
\end{align}

where $dZ_{1t} = dW_{1t} + \lambda_1 dt$ and $dZ_{2t} = dW_{2t} + \lambda_2 dt$ are independent standard Brownian motions under the GRNVR $Q$. Both equity risk premium $\lambda_1$ and variance risk premium
\(\lambda_2\) are present in the model under the risk-neutral measure \(Q\). The discrete-time GARCH(1,1) process (3) corresponding to the limiting diffusion process under the risk-neutral measure \(Q\).

7. Conclusion

In this paper, we follow the GARCH option pricing framework of Duan (1995) and propose a new way of changing from physical probability measure to risk-neutral probability measure. The new risk-neutral valuation is referred to as the GRNVR. The advantage of the GRNVR compared to the LRNVR commonly used in the literature Duan (1995); Hao and Zhang (2013); Wang et al. (2017) is that the variance risk premium is included in the risk-neutral dynamics under the GRNVR. The absence of variance risk premium in the risk-neutral dynamics under the LRNVR is noted in Hao and Zhang (2013), where it is shown that both empirical studies and theoretical results indicated that the GARCH models under the LRNVR did not capture the variance premium.

We then find the theoretical VIX squared value as the conditional risk-neutral expectation of the arithmetic mean variance over the next 21 trading days under the GRNVR. Specifically, the GARCH implied VIX formulas are derived using the features of square-root stochastic autoregressive volatility (SR-SARV) models. We apply several calibration methods to estimate the model parameters using various sets of time series data, and compare the theoretical formula performances with the market
data. Various combinations of time series of the daily closing price of S&P 500 index and the CBOE VIX are used to find the maximum likelihood estimation of the GARCH models. The corresponding implied VIX time series are then calculated from the calibrated model. Similar to the empirical evidences in Hao and Zhang (2013); Wang et al. (2017), when only S&P 500 returns are used for estimation, the GARCH implied VIX is consistently and significantly lower than the CBOE VIX. When the CBOE VIX is used for estimation, the implied VIX fits the statistical properties of the CBOE VIX and matches the CBOE VIX data better. The numerical results provide evidences that the GARCH option pricing under the GRNVR is more suitable to price volatility. In the case of GARCH(1,1), we compare the diffusion limit of the GARCH process under the physical measure and the GRNVR risk-neutral measure to show that variance premium is captured in the risk-neutral dynamics.
References


Duan, J., 1996, A unified theory of option pricing under stochastic volatility from GARCH to diffusion, Technical report, Hong Kong University of Science and Technology.


Table 1: Maximum likelihood estimates of GARCH(1,1) model using return data only, VIX data only or both return and VIX data. The bold values indicate the log-likelihood value which is being maximized. The standard errors are provided in the parentheses.

<table>
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<th>VIX only</th>
<th>Return &amp; VIX</th>
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<tbody>
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<td>GRNVR</td>
<td>LRNVR</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>2.474e-6 (0.089e-6)</td>
<td>1.678e-6 (0.044e-6)</td>
<td>1.641e-6 (0.031e-6)</td>
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<tr>
<td>$\alpha_1$</td>
<td>0.1256 (0.0045)</td>
<td>0.0382 (0.0006)</td>
<td>0.0456 (0.0015)</td>
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<tr>
<td>$\beta_1$</td>
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<td>0.9351 (0.0010)</td>
<td>0.9286 (0.0036)</td>
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<td>$\lambda_1$</td>
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<td>0.8144 (0.0171)</td>
<td>0.2253 (0.0122)</td>
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<tr>
<td>$\lambda_2$</td>
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<td>0</td>
<td>-0.3514 (0.0705)</td>
</tr>
<tr>
<td>$\ln L_R$</td>
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<td>20355</td>
<td>20593</td>
</tr>
<tr>
<td>$\ln L_V$</td>
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<td>33424</td>
<td>33662</td>
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<td>$\ln L_T$</td>
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<td>54255</td>
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Table 2: Maximum likelihood estimates of TGARCH(1,1) model using return data only, VIX data only or both return and VIX data. The bold values indicate the log-likelihood value which is being maximized. The standard errors are provided in the parentheses.

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<td>GRNVR</td>
<td>LRNVR</td>
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<td>$\alpha_0$</td>
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<td>1.476e-6 (0.075e-6)</td>
<td>1.520e-6 (0.018e-6)</td>
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<td>$\alpha_1$</td>
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<td>0.0020 (0.0015)</td>
<td>0.0060 (0.0010)</td>
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<tr>
<td>$\beta_1$</td>
<td>0.8338 (0.0031)</td>
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<td>0.9553 (0.0006)</td>
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<td>$\theta$</td>
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<td>0.0531 (0.0008)</td>
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<tr>
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<td>0.3094 (0.0155)</td>
<td>0.0889 (0.0087)</td>
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<tr>
<td>$\lambda_2$</td>
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<td>0</td>
<td>-0.3978 (0.0980)</td>
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<td>22242</td>
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<td>$\ln L_V$</td>
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<td><strong>33795</strong></td>
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<td>56037</td>
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Table 3: Maximum likelihood estimates of AGARCH(1,1) model using return data only or both return and VIX data. The bold values indicate the log-likelihood value which is being maximized. The standard errors are provided in the parentheses.

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<th>Return &amp; VIX</th>
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<td>0.8810 (0.0108)</td>
<td>0.9302 (0.0010)</td>
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<td>$\theta$</td>
<td>0.7861(0.0534)</td>
<td>0.7795 (0.0204)</td>
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<tr>
<td>$\lambda_1$</td>
<td>0.0255 (0.0119)</td>
<td>0.0120 (0.0095)</td>
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<tr>
<td>$\lambda_2$</td>
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<td>0</td>
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<td>$\ln L_R$</td>
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<td>$\ln L_T$</td>
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<td><strong>55483</strong></td>
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Table 4: Maximum likelihood estimates of EGARCH(1,1) model using return data only, VIX data only or both return and VIX data. The bold values indicate the log-likelihood value which is being maximized. The standard errors are provided in the parentheses.

<table>
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<th>Return only</th>
<th>VIX only</th>
<th>Return &amp; VIX</th>
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<tbody>
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<td></td>
<td>LRNVR</td>
<td>GRNVR</td>
<td>LRNVR</td>
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<td>(\alpha_0)</td>
<td>-0.1919 (0.0133)</td>
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<td>-0.0622 (0.0012)</td>
<td>-0.0638 (0.0012)</td>
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<td>(\beta_1)</td>
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<td>(\kappa)</td>
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<td>0.0879 (0.0012)</td>
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<td>(\lambda_1)</td>
<td>0.0189 (0.0106)</td>
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Table 5: The table displays the related statistics between the model implied VIX and the CBOE VIX for the GARCH models during the period from January 2, 1990 to June 30, 2017. The error is computed as the difference between CBOE VIX and the implied VIX. The mean error (ME) represents the mean daily difference between the implied VIX and the CBOE VIX. The standard error (Std.Err.) represents the standard deviation of the error. The mean absolute error (MAE) calculates the mean daily absolute difference between the implied VIX and the CBOE VIX. The mean squared error (MSE) computes the mean daily squared difference between the implied VIX and the CBOE VIX. The root mean squared error (RMSE) computes the square root of the mean squared error. The $P$-value is for the null hypothesis that the implied VIX and the CBOE VIX have the same mean values. Violation of one-sigma band stands for the probability that the implied VIX lies out of the one-standard-deviation band of the CBOE VIX. The correlation coefficient (Corr. Coef.) computes the linear correlation between the implied VIX and the CBOE VIX. Autocorrelation coefficients with lag of 1, 10, and 30 days and higher moments of implied VIX are also reported.

<table>
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<th>Model &amp; Data</th>
<th>ME</th>
<th>Std.Err.</th>
<th>MAE</th>
<th>MSE</th>
<th>RMSE</th>
<th>$P$-value</th>
<th>Violation of one-sigma band</th>
<th>Corr. Coef.</th>
<th>AR1</th>
<th>AR10</th>
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<tr>
<td>Returns</td>
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<td>4.03</td>
<td>2.69</td>
<td>16.86</td>
<td>4.11</td>
<td>0.0000</td>
<td>7.56%</td>
<td>0.90</td>
<td>0.9892</td>
<td>0.8879</td>
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<td>20.37</td>
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<td>2.99</td>
<td>2.29</td>
<td>8.96</td>
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<td>0.5223</td>
<td>1.68%</td>
<td>0.93</td>
<td>0.9956</td>
<td>0.9452</td>
<td>0.7945</td>
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<td>3.14</td>
<td>2.25</td>
<td>9.03</td>
<td>3.01</td>
<td>0.1012</td>
<td>2.27%</td>
<td>0.92</td>
<td>0.9950</td>
<td>0.9434</td>
<td>0.7853</td>
<td>56.65</td>
<td>2.69</td>
<td>15.70</td>
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<tr>
<td>Returns</td>
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<td>4.02</td>
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<td>4.70</td>
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<td>0.86</td>
<td>0.9596</td>
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<td>3.01</td>
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<td>1.80%</td>
<td>0.93</td>
<td>0.9954</td>
<td>0.9431</td>
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<td>0.9815</td>
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<tr>
<td>Returns</td>
<td>3.39</td>
<td>3.38</td>
<td>3.62</td>
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<td>4.79</td>
<td>0.0000</td>
<td>10.02%</td>
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<td>0.79%</td>
<td>0.94</td>
<td>0.9946</td>
<td>0.9395</td>
<td>0.7955</td>
<td>54.37</td>
<td>2.20</td>
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<td>2.72</td>
<td>2.10</td>
<td>7.41</td>
<td>2.72</td>
<td>0.7396</td>
<td>0.82%</td>
<td>0.94</td>
<td>0.9944</td>
<td>0.9373</td>
<td>0.7903</td>
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<td>0.8988</td>
<td>0.7580</td>
<td>61.68</td>
<td>2.10</td>
<td>10.70</td>
</tr>
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</table>
Figure 1: Comparison between CBOE VIX and implied VIX using return data only for four GARCH models.
Figure 2: Comparison between CBOE VIX and implied VIX of the GARCH(1,1) model using VIX data only with the upper panel showing the result under the GRNVR and the lower panel showing the result under the LRNVR.
Figure 3: Comparison between CBOE VIX and implied VIX of the GARCH(1,1) model using both return and VIX data with the upper panel showing the result under the GRNVR and the lower panel showing the result under the LRNVR.
Figure 4: Comparison between CBOE VIX and implied VIX of the TGARCH(1,1) model using VIX data only with the upper panel showing the result under the GRNVR and the lower panel showing the result under the LRNVR.
Figure 5: Comparison between CBOE VIX and implied VIX of the TGARCH(1,1) model using both return and VIX data with the upper panel showing the result under the GRNVR and the lower panel showing the result under the LRNVR.
Figure 6: Comparison between CBOE VIX and implied VIX of the AGARCH(1,1) model using both return and VIX data with the upper panel showing the result under the GRNVR and the lower panel showing the result under the LRNVR.
Figure 7: Comparison between CBOE VIX and implied VIX of the EGARCH(1,1) model using VIX data only with the upper panel showing the result under the GRNVR and the lower panel showing the result under the LRNVR.
Figure 8: Comparison between CBOE VIX and implied VIX of the EGARCH(1,1) model using both return and VIX data with the upper panel showing the result under the GRNVR and the lower panel showing the result under the LRNVR.