Changing Probability Measures in GARCH Option Pricing Models

Wenjun Zhang
Department of Mathematical Sciences
School of Engineering, Computer and Mathematical Sciences
Auckland University of Technology
Auckland 1142, New Zealand
Email: wenjun.zhang@aut.ac.nz

Jin E. Zhang
Department of Accountancy and Finance
Otago Business School, University of Otago
Dunedin 9054, New Zealand
Email: jin.zhang@otago.ac.nz

This Version: 10 August 2017

Abstract

In this paper, we consider several option pricing models with stochastic volatility in the context of the generalized autoregressive conditional heteroskedastic (GARCH) processes. We propose a globally risk-neutral valuation relationship (GRNVR) to derive the model dynamics under risk-neutral measure and obtain the corresponding closed-form pricing formula for the Chicago Board Options Exchange Volatility Index (CBOE VIX). The parameters of the proposed models are then calibrated using the S&P 500 returns data and the CBOE VIX. Based on the empirical pricing performances, we observe that the proposed GRNVR generally performs better than the locally risk-neutral valuation relationship (LRNVR). We also provide theoretical justification of the proposed GRNVR.

Keywords: GARCH process, Stochastic volatility, CBOE VIX

JEL Classifications: G13, C52.

1. Introduction

The study of finance largely concerns about the trade-off between risk and expected return. An important source of risk is the uncertainty of the volatility of equity indices, where volatility is understood as the standard deviation of the return of a financial instrument with a specific time horizon. Since the Chicago Board Options Exchange (CBOE) introduced standardized derivative contracts on S&P 500 implied volatility index (VIX), e.g., VIX options and VIX futures, volatility derivatives have been traded actively in the CBOE market. As the interest in understanding and forecasting the VIX index grows, it is important to develop reliable mathematical models for the term structure of VIX. The literature of using stochastic volatility model to evaluate VIX and its derivatives has grown rapidly in the last decade. The first stochastic volatility model to study the VIX index and VIX futures were proposed in Zhang and Zhu (2006). Zhu and Zhang (2007) extended the model in Zhang and Zhu (2006) by changing the long-term mean variance to be time-dependent. Lin (2007) used the affine jump-diffusion model to describe the VIX with jumps in both stock and volatility processes. The term structure of VIX was analyzed in Luo and Zhang (2012). Unfortunately, the parameter estimation of stochastic volatility models is often a challenging task as the models generally do not have the closed form likelihood function given the S&P 500 stock and VIX data.

On the other hand, GARCH models introduced by Bollerslev in Bollerslev (1986) are widely used to model volatility term structure in practice mainly because the likelihood function of the parameters in the GARCH models can often be expressed in closed form in terms of the observed data. So it is possible to derive the maximum likelihood estimation (MLE) of the model parameters. Inspired by the success in describing volatility with GARCH models, Duan (1995) pioneered in applying GARCH models for S&P 500 option pricing by proposing a locally risk-neutral valuation relationship (LRNVR). Under the LRNVR, the one-period ahead conditional variance remains the same during the change of probability measure. This approach developed a link between the conditional variance in the physical measure and the risk neutral measure. Therefore, GARCH models is a natural candidate for calibration using the historical VIX data. Kanniainen, Lin, and Yang (2014) evaluate S&P 500 options using three variant GARCH models using VIX data. They found empirical evidence that a joint maximum likelihood estimation using S&P 500 returns and VIX improved the performance of pricing S&P 500 options compared to traditional MLE using the returns data only. However, there exists some drawback in the calibration of the GARCH models under the LRNVR risk-neutral measure. The empirical evidence was obtained in Barone-Adesi, Engle, and Mancini (2008); Christoffersen, Heston, and Jacobs (2013) to show that the same model parameters in the conditional volatility of historical and risk-neutral pricing dynamics results in poor calibration results. Hao and Zhang (2013) also demonstrated that the GARCH-type models under the LRNVR could not capture the variance risk premium. They examined a family of GARCH models and derived the model-implied VIX index under the LRNVR. When the models were estimated with returns data only, the model implied VIX was significantly lower than the market CBOE VIX. Even when the models were jointly estimated with both returns and VIX, the equity risk parameter distorted to be a large positive price, and the model implied VIX still underestimated the CBOE VIX, and did not match the statistical aspects of the CBOE VIX. They further provided the theoretical arguments by considering the diffusion limit of GARCH models. Inspired by the discussion in Hao and Zhang (2013), we propose a new globally risk-neutral valuation relationship (GRNVR) for the risk-neutral dynamics of GARCH models and display the advantage of the GRNVR compared to the LRNVR using theoretical and empirical evidences.

The article is structured as follows. Section 2 proposes a new way for changing physical probability measure for the GARCH(1,1) to the risk-neutral probability measure. In Section 3, we derive theoretical VIX formula for the GARCH(1,1) model under the risk-neutral measure, and extend the derivation idea to a broad class of GARCH models which include GARCH, TGARCH, AGARCH and EGARCH models. In Section 4, we calibrate these GARCH models using various combinations of time series of the S&P 500 index and the CBOE VIX. Section 4 compares the the CBOE VIX with the GARCH implied VIX obtained from the calibrated GARCH models. In Section 5, we analyze the diffusion limit of the GARCH process under the risk-neutral measure to demonstrate that the risk-neutral dynamics captures the variance risk premium. We then conclude the findings in Section 6.

2. GARCH model specification

In this paper, we consider the asset price as a discrete-time stochastic process and denote the asset price at time t as X_t. It was proposed in Duan (1995) that the return of the asset follows a conditional lognormal distribution under the physical measure P as

$$\ln \frac{X_t}{X_{t-1}} = r - \frac{1}{2} h_t + \lambda_1 \sqrt{h_t} + \epsilon_t,$$

where r is the one-period risk-free interest rate, λ_1 is the asset risk premium, and ϵ_t follows a GARCH(p, q) process introduced in Bollerslev (1986) with mean zero and conditional variance h_t

$$\epsilon_t | \phi_{t-1} \sim N(0, h_t) \quad \text{under measure } P,$$

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j}, \tag{1}$$

where ϕ_t is the information set of up to and including time t; $\alpha_0 \geq 0$, $\alpha_i \geq 0$ for $i = 1, 2, \ldots, q$ and $\beta_j \geq 0$ for $j = 1, 2, \ldots, p$. We focus on the GARCH(1,1) case, so the equation (1) simplifies to

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 h_{t-1}. \tag{2}$$

We propose an alternative risk-neutral valuation relationship to the locally risk-neutral valuation relationship (LRNVR) introduced by Duan (1995). We shall refer to the proposed risk neutral-valuation relationship as the globally risk-neutral valuation relationship (GRNVR); and the dynamics of asset return in the risk-neutral pricing measure Q under the GRNVR has the following form as

$$\ln \frac{X_t}{X_{t-1}} = r - \frac{1}{2} h_t + \xi_t, \quad \xi_t | \phi_{t-1} \sim N(0, h_t) \quad \text{under measure } Q,$$

$$\begin{aligned} h_t &= \alpha_0 + \alpha_1 \left(\xi_{t-1} - \lambda_1 \sqrt{h_{t-1}} \right)^2 + \beta_1^* h_{t-1} \\ &= \alpha_0 + \alpha_1 \left(\xi_{t-1} - \lambda_1 \sqrt{h_{t-1}} \right)^2 + (\beta_1 - \sqrt{2} \alpha_1 \lambda_2) h_{t-1}. \end{aligned} \tag{3}$$

Note that the proposed GARCH(1,1) process (3) is different to the one derived by Duan (1995) under the LRNVR. Under the GRNVR the persistence parameter β_1 is designed to be different in the P and Q measures, whereas under the LRNVR the persistence parameter β_1 is the same in the P and Q measures. Specifically, for the dynamics of risk-neutral measure Q under the GRNVR, the persistence parameter of conditional variance is $\beta_1^* = \beta_1 - \sqrt{2}\alpha_1\lambda_2$, where λ_2 represents the variance risk premium of the asset. The motivation for the inclusion of the variance risk premium is discussed in Hao and Zhang (2013) where it was shown that there is no risk adjustment for the variance risk of the process in Duan (1995) from physical measure to the risk-neutral measure under the LRNVR. It was also discussed in Barone-Adesi et al. (2008); Christoffersen et al. (2013) that the restriction of conditional volatility of historical and risk-neutral pricing distributions with the same model parameters leads to poor calibration results in the empirical studies (cf. Chernov and Ghysels (2000); Christoffersen, Heston, and Jacobs (2006); Hao and Zhang (2013)). Therefore, it was suggested that the parameters of volatility dynamics of historical and risk-neutral pricing returns might be different in Barone-Adesi et al. (2008). We adopt the idea by modifying the persistence parameter in Q to incorporate the variance risk premium in the model. The theoretical justification of the modification is further discussed in section 6.

3. VIX formulas of the GARCH models

The Chicago Board Options Exchange (CBOE) introduced a volatility index, named VIX, in 1993. The VIX index is calculated from the implied volatilities of the eight near-the-money, nearby, and second nearby S&P 100 index options based on the methodology by Whaley Whaley (1993). The VIX was a proxy of the implied volatility of 30 calendar days at-the-money (ATM) options. In 2003, the CBOE used another theory proposed in Carr and Madan (1998); Demeterfi, Derman, Kamal, and Zou (1999) to design a new methodology to compute the CBOE volatility index VIX. The new VIX is based on the prices of a portfolio of 30 calendar days out-of-the-money (OTM) S&P 500 index call and put options. The square of new VIX represents the S&P 500 30-day variance swap rate. The old VIX has been renamed to be VXO.

The VIX index reflects investors' expectation of the volatility of the S&P 500 in the next 30 calender days or 21 trading days, which is calculated using its definition as

$$\left(\frac{\mathrm{VIX}_t}{100}\right)^2 = E_t^Q\left(\frac{1}{\tau_0}\int_t^{t+\tau_0} \tilde{h}_s ds\right),$$

where $\tilde{h}_s$ denotes the annualized instantaneous variance of the return of S&P 500 and τ_0 is 30 calendar days or 21 trading days. In this paper, VIX is computed as the mean value of the expected variance in the n sub-periods of the next 21 trading days, that is

$$\left(\frac{\mathrm{VIX}_t}{100}\right)^2 = \frac{1}{n}\sum_{k=1}^{n} E_t^Q\left(h_{t+\frac{\tau_0 k}{n}}\right).$$

In particular, we use the daily closing value data, so it implies $\tau_0 = n$, and

$$V_t = \frac{1}{n}\sum_{k=1}^{n} E_t^Q\left(h_{t+k}\right),$$

where the term $V_t = \frac{1}{252}\left(\frac{\mathrm{VIX}_t}{100}\right)^2$ is defined as a function of VIX_t to measure the expected daily variance of S&P 500. The conditional mean of the future variance can be calculated in a broad class of GARCH models as discussed in Hao and Zhang (2013); Wang, Shen, Jiang, and Huang (2017).

We derive the implied VIX from the model (3) under Q by first rewriting the error terms of the process using the standard normal distribution as

$$\begin{aligned}\ln \frac{X_t}{X_{t-1}} &= r - \frac{1}{2}h_t + \sqrt{h_t}\varepsilon_t, \\ h_t &= \alpha_0 + \alpha_1 h_{t-1}\left(\varepsilon_{t-1} - \lambda_1\right)^2 + (\beta_1 - \sqrt{2}\alpha_1\lambda_2)h_{t-1}, \end{aligned} \tag{4}$$

where ε_t is the standard normal random variable, conditional on the information set up to and including time $t-1$ under Q.

One can rewrite the GARCH$(1,1)$ process (4) as a special case of the square-root stochastic autoregressive volatility (SR-SARV(1)) models introduced in Meddahi and Renault (2004) with the following form

$$\begin{aligned} & h_{t+1} = \omega + \gamma h_t + \nu_t, \text{ with } \mathbb{E}\left[\nu_t | \phi_{t-1}\right] = 0, \\ & \omega = \alpha_0, \quad \gamma = \alpha_1(1+\lambda_1^2) + \beta_1 - \sqrt{2}\alpha_1\lambda_2, \\ & \nu_t = \alpha_1 h_t(\varepsilon_t^2 - 1 - 2\lambda_1\varepsilon_t). \end{aligned} \tag{5}$$

It was shown in Hao and Zhang (2013) that if the S&P 500 return follows a SR-SARV(p) process under the risk-neutral measure, then the implied daily variance V_t at time t is affine in the conditional variance h_{t+1}. Following similar ideas, we can obtain the long term variance as $\bar{h} = \lim_{m\to\infty} \mathbb{E}_t^Q[h_{t+m}] = \frac{\omega}{1-\gamma}$ by noticing

$$\bar{h} = \lim_{m\to\infty} \mathbb{E}_t^Q[h_{t+m}] = \lim_{m\to\infty} \mathbb{E}_t^Q[\omega + \gamma h_{t+m-1} + \nu_{t+m-1}] = \omega + \gamma\bar{h}.$$

Then the conditional expectation of the variance after two periods can be obtained via the long run variance

$$\mathbb{E}_t^Q\left[h_{t+2}\right]-\bar{h}=\mathbb{E}_t^Q\left[\omega+\gamma h_{t+1}+\nu_{t+1}\right]-\frac{\omega}{1-\gamma}=\omega+\gamma h_{t+1}-\frac{\omega}{1-\gamma}=\gamma(h_{t+1}-\bar{h}).$$

So the conditional expectation of the variance after n periods is given by

$$\mathbb{E}_t^Q\left[h_{t+n}\right]=\bar{h}+\gamma^{n-1}(h_{t+1}-\bar{h}).$$

Therefore, we can represent the expected daily variance as an affine function of h_{t+1}

$$\begin{aligned}V_t &= \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}_t^Q\left(h_{t+k}\right)\\ &= \bar{h}+\frac{1}{n}\sum_{k=1}^{n}\gamma^{k-1}(h_{t+1}-\bar{h})\\ &= \bar{h}+\frac{1-\gamma^n}{n(1-\gamma)}(h_{t+1}-\bar{h})\\ &= \left(1-\frac{1-\gamma^n}{n(1-\gamma)}\right)\frac{\omega}{1-\gamma}+\frac{1-\gamma^n}{n(1-\gamma)}h_{t+1}\\ &= A+Bh_{t+1}, \qquad (6)\end{aligned}$$

where $A=\frac{(1-B)\omega}{1-\gamma}$ and $B=\frac{1-\gamma^n}{n(1-\gamma)}$.

Apart from the GARCH(1,1) model discussed above, we also consider the threshold GARCH(1,1) (TGARCH) moder introduced in Glosten, Jagannathan, and Runkle (1993), the non-linear asymmetric GARCH(1,1) (AGARCH) model proposed in Engle and Ng (1993) and the exponential GARCH(1,1) (EGARCH) model by Nelson (Nelson, 1991). The forms of the models in the physical measure P and in the risk-neutral measure Q under the GRNVR are as follows:

TGARCH(1,1)

$$\text{Physical measure: } h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \theta \epsilon_{t-1}^2 \mathbf{1}(\epsilon_{t-1} < 0) + \beta_1 h_{t-1}, \tag{7}$$

$$\text{GRNVR: } h_t = \alpha_0 + \left(\xi_{t-1} - \lambda_1 \sqrt{h_{t-1}}\right)^2 \left(\alpha_1 + \theta \mathbf{1}(\xi_{t-1} - \lambda_1 \sqrt{h_{t-1}} < 0)\right) + (\beta_1 - \sqrt{2}\alpha_1 \lambda_2) h_{t-1}.$$

AGARCH(1,1)

$$\text{Physical measure: } h_t = \alpha_0 + \alpha_1 \left(\epsilon_{t-1} - \theta \sqrt{h_{t-1}}\right)^2 + \beta_1 h_{t-1}, \tag{8}$$

$$\text{GRNVR: } h_t = \alpha_0 + \alpha_1 \left(\xi_{t-1} - \lambda_1 \sqrt{h_{t-1}} - \theta \sqrt{h_{t-1}}\right)^2 + (\beta_1 - \sqrt{2}\alpha_1 \lambda_2) h_{t-1}.$$

EGARCH(1,1)

Physical measure:

$$\ln h_t = \alpha_0 + \beta_1 \ln h_{t-1} + \alpha_1 \frac{\epsilon_{t-1}}{\sqrt{h_{t-1}}} + \kappa \left(\left| \frac{\epsilon_{t-1}}{\sqrt{h_{t-1}}} \right| - \sqrt{\frac{2}{\pi}} \right), \tag{9}$$

GRNVR:

$$\ln h_t = \alpha_0 + (\beta_1 - \sqrt{2}\alpha_1 \lambda_2) \ln h_{t-1} + \alpha_1 \left(\frac{\epsilon_{t-1}}{\sqrt{h_{t-1}}} - \lambda_1 \right) + \kappa \left(\left| \frac{\epsilon_{t-1}}{\sqrt{h_{t-1}}} - \lambda_1 \right| - \sqrt{\frac{2}{\pi}} \right).$$

As shown in Hao and Zhang (2013), these widely used models are special cases of SR-SARV(p) models; and following similar derivation process as the GARCH(1,1) model, we can obtain the implied VIX formula for different GARCH models analogous to the ones in Hao and Zhang (2013) as:

TGARCH(1,1)

$$V_t = C + Dh_{t+1}, \tag{10}$$

where

$$\begin{aligned}
C &= \frac{\alpha_0(1-D)}{1-\eta}, \\
D &= \frac{1-\eta^n}{n(1-\eta)}, \\
\eta &= \alpha_1(1+\lambda_1^2) + (\beta_1 - \sqrt{2}\alpha_1\lambda_2) + \theta S, \\
S &= \frac{\lambda_1}{\sqrt{2\pi}} e^{-\frac{\lambda_1^2}{2}} + (1+\lambda_1^2)N(\lambda_1).
\end{aligned}$$

Note that $N(\cdot)$ denotes the cumulative function of the Normal distribution.

AGARCH(1,1)

$$V_t = E + Fh_{t+1}, \tag{11}$$

where

$$
\begin{aligned}
E &= \frac{\alpha_0(1-F)}{1-\eta}, \\
F &= \frac{1-\eta^n}{n(1-\eta)}, \\
\eta &= \alpha_1(1+(\lambda_1+\theta)^2)+(\beta_1-\sqrt{2}\alpha_1\lambda_2).
\end{aligned}
$$

EGARCH(1,1)

$$
V_t = \frac{1}{n}\left(h_{t+1} + \sum_{k=1}^{n-1}\left(\prod_{i=0}^{k-1} l_i\right) h_{t+1}^{(\beta_1-\sqrt{2}\alpha_1\lambda_2)^k}\right), \tag{12}
$$

where

$$
\begin{aligned}
& l_i = e^{(\beta_1-\sqrt{2}\alpha_1\lambda_2)^i\left(\alpha_0-\kappa\sqrt{\frac{2}{\pi}}\right)} \\
& \Big(e^{-(\beta_1-\sqrt{2}\alpha_1\lambda_2)^i(\alpha_1-\kappa)\lambda_1+0.5(\beta_1-\sqrt{2}\alpha_1\lambda_2)^{2i}(\alpha_1-\kappa)^2} N(\lambda_1-(\beta_1-\sqrt{2}\alpha_1\lambda_2)^i(\alpha_1-\kappa)) \\
& + e^{-(\beta_1-\sqrt{2}\alpha_1\lambda_2)^i(\alpha_1+\kappa)\lambda_1+0.5(\beta_1-\sqrt{2}\alpha_1\lambda_2)^{2i}(\alpha_1+\kappa)^2} N((\beta_1-\sqrt{2}\alpha_1\lambda_2)^i(\alpha_1+\kappa)-\lambda_1)\Big)
\end{aligned}
$$

4. Data and estimation

It was shown in Hao and Zhang (2013) that under the LRNVR, the GARCH implied VIX does not fit the market data of CBOE VIX very well. The model was analyzed to display that the reason may be that variance risk premium and the volatility risk price were not present in the diffusion limit of the GARCH models under the LRNVR. In the modified GARCH processes, we include the variance risk premium into the models under the GRNVR. And it is of interest to see whether the implied VIX in the modified GARCH models fit the CBOE VIX market values better. In this section, we will investigate this question by estimating the parameters in the modified GARCH models and calculating the corresponding GARCH implied VIX times series for comparison to the CBOE VIX.

The two time series data we use for the GARCH models calibration are the closing values of S&P 500 and the CBOE VIX ranging from 2nd January, 1990 to 30th June, 2017. For the daily risk-free interest rate, we use the 3-month treasury bill secondary market rate from the U.S. Federal Reserve website.

There are different methods to calibrate the models using market data. We will use the common maximum likelihood approach to estimate the parameters of the models. We can use only the S&P 500 returns data to obtain a maximum likelihood estimation of the GARCH processes under the physical measure P and fix the variance risk premium parameter $\lambda_2 = 0$, since λ_2 is not included in the GARCH models under P measure. For the S&P 500 returns data only, the log-likelihood function $\ln L_R$ for the GARCH models is given by

$$\ln L_R = -\frac{T\ln(2\pi)}{2} - \frac{1}{2}\sum_{t=1}^{T}\left(\ln(h_t) + \left(\ln\frac{X_t}{X_{t-1}} - r - \lambda_1\sqrt{h_t} + \frac{h_t}{2}\right)^2 / h_t\right), \qquad (13)$$

where the conditional variance h_t is updated by corresponding processes using different forms of GARCH models. For the maximum likelihood estimation, the conditional variance for the first period is set as the variance of S&P 500 returns over the whole sample period. The stationary conditions for the GARCH processes under physical and the risk-neutral measures are different, with the latter having more strict constraints on the parameters. Thus, we find the estimation of the parameters in the GARCH models by maximizing the corresponding log-likelihood function subject to the stationary conditions under the risk-neutral measures.

We may also calibrate the GARCH models by matching the model implied VIX to the market value of CBOE VIX, since the CBOE VIX series may contain additional information about the underlying S&P 500 return process. To utilize both time series, we follow the assumption in Hao and Zhang (2013) that the pricing differences between the CBOE VIX and the implied VIX on a daily basis come from a Normal distribution

$$\mathrm{VIX}^{\mathrm{Mkt}} = \mathrm{VIX}^{\mathrm{Imp}} + \mu, \quad \mu \sim N(0, s^2),$$

where s^2 is estimated using the sample variance of pricing difference of $\hat{s}^2 = \mathrm{var}(\mathrm{VIX}^{\mathrm{Mkt}} - \mathrm{VIX}^{\mathrm{Imp}})$. Under the above assumption, the log-likelihood function corresponding to the CBOE VIX data is

$$\ln L_V = -\frac{T \ln(2\pi \hat{s}^2)}{2} - \sum_{t=1}^{T} \frac{\left(\mathrm{VIX}^{\mathrm{Mkt}} - \mathrm{VIX}^{\mathrm{Imp}}\right)^2}{2\hat{s}^2}. \tag{14}$$

In addition to use the S&P 500 returns data and CBOE VIX data for calibration of the GARCH models separately, we can also combine both time series to find a joint maximum likelihood estimation of the models by maximizing the joint log-likelihood function

$$\ln L_T = \ln L_R + \ln L_V. \tag{15}$$

5. Numerical results

In this section, we compare the estimated parameters from different data used for the calibration. In particular, Table 1 displays the maximum likelihood estimates and the standard errors of GARCH(1,1) model. The values of the three log-likelihood functions (13, 14, 15) are also displayed in Table 1. Although the contributions from S&P 500 returns and CBOE VIX as well as the joint likelihood values are reported, we maximize $\ln L_R$ when only S&P 500 returns are used, $\ln L_V$ when only CBOE VIX data are used and $\ln L_T$ when both time series are used.

From the output in Table 1, we can see that the equity risk premium λ_1 increases significantly from 0.0886 (return data used) to 0.2134 (both data used) and 0.2253 (VIX data used) in the GARCH(1,1) models when the CBOE VIX data is used for calibration. The variance risk premium λ_2 is negative and significantly different from zero as -0.3670 (both data used) and -0.3514 (VIX data used). The persistence of conditional variance, β_1 increases slightly from 0.8543 (return data used) to 0.9251 (both data used) and 0.9286 (VIX data used). There is a sizable decrease of the parameter value α_1 from 0.0886 (return data used) to 0.0474 (both data used) and

0.0456 (VIX data used). Comparing the maximum likelihood result of the model under the GRNVR and the results under the LRNVR, we can see that the maximum likelihood value increases significantly from 54697 to 55921 (both data used) and from 33424 to 33662 (VIX data used).

Similar numerical results are also observed in the other types of GARCH models as displayed in Tables 2-4. Specifically, Table 2 shows that the equity risk premium λ_1 increases significantly from 0.0131 (return data used) to 0.1160 (both data used) and 0.0889 (VIX data used) in the TGARCH(1,1) model when the CBOE VIX data is used for calibration. The variance risk premium λ_2 is negative and significantly different from zero as -0.4412 (both data used) and -0.3978 (VIX data used). The persistence of conditional variance, β_1 increases significantly from 0.8338 (return data used) to 0.9561 (both data used) and 0.9553 (VIX data used). There is a decrease of the parameter value α_1 from 0.0256 (return data used) to 0.0091 (both data used) and 0.0060 (VIX data used). Comparing the maximum likelihood result of the TGARCH(1,1) model under the GRNVR and the results under the LRNVR, we can see that the maximum likelihood value increases significantly from 55455 to 56282 (both data used) and from 33468 to 33795 (VIX data used).

Table 3 shows calibration results of the AGARCH(1,1) model using both returns and VIX data. If using VIX data only, it is not easy to distinguish the parameters θ and λ_1. Therefore, the numerical results using VIX data are not displayed in the table. From Table 3 we observe that the equity risk premium λ_1 increases significantly from

0.0255 (return data used) to 0.1158 (both data used) in the AGARCH(1,1) model when the CBOE VIX data and returns used for calibration. The variance risk premium λ_2 is negative and significantly different from zero as -0.3125 (both data used). The persistence of conditional variance, β_1 increases from 0.8810 (return data used) to 0.9316 (both data used). There is a big decrease of the parameter value α_1 from 0.0841 (return data used) to 0.0380 (both data used). Comparing the maximum likelihood result of the AGARCH(1,1) model under the GRNVR and the results under the LRNVR, we can see that the maximum likelihood value increases significantly from 55483 to 56333 (both data used).

Table 4 shows that in the EGARCH(1,1) model the variance risk premium λ_2 is negative as -0.0567 (both data used) and -0.0483 (VIX data used), both significantly different than zero. The persistence of conditional variance, β_1 increases slightly from 0.9792 (return data used) to 0.9906 (both data used) and 0.9891 (VIX data used). Comparing the maximum likelihood result of the EGARCH(1,1) model under the GRNVR and the results under the LRNVR, we can see that the maximum likelihood value increases significantly from 56399 to 57105 (both data used) and from 33774 to 34303 (VIX data used).

From the comparisons in the GARCH, TGARCH, AGARCH and EGARCH models, we see that the maximum likelihood results under the GRNVR are generally better than those under the LRNVR. Table 5 measures how the implied VIX fits the CBOE VIX by computing a list of statistics and the results demonstrate that the implied VIX under GRNVR fits the CBOE quite well.

After obtaining the estimates of the parameters in the models, we can then calculate the conditional variance h_t and compute the corresponding GARCH implied VIX. Figure 1 shows the time series of the CBOE VIX and the implied VIX of the four GARCH models estimated using returns only. Figure 2 shows the time series of the CBOE VIX and the implied VIX of the GARCH(1,1) model estimated using VIX data only. Figure 3 shows the time series of the CBOE VIX and the implied VIX of the GARCH(1,1) model estimated using both returns and VIX. Similar comparison plots are obtained for other GARCH models. Specifically, the time series of the CBOE VIX and the implied VIX of the TGARCH(1,1) model estimated with VIX data only are displayed in Figure 4. The time series of the CBOE VIX and the implied VIX of the TGARCH(1,1) and AGARCH (1,1) model estimated with both returns and VIX data are shown in Figure 5 and Figure 6, respectively. For the EGARCH(1,1) model, Figure 7 shows the comparison between the CBOE VIX and model implied VIX with VIX data only, and Figure 8 displays the comparison between the CBOE VIX and model implied VIX with both returns and VIX data. From the list of graphs, we observe that the model implied VIX fits the CBOE VIX better under the GRNVR compared to the LRNVR in general.

6. Theoretical justification

Duan studied the bivariate diffusion limit of the GARCH(1,1) model as the length of the time period tends towards zero in Duan (1996, 1997). Applying Duan's arguments, one can show that the limiting bivariate diffusion process of the approximating GARCH$(1,1)$ process under the physical measure P is given by

$$
\begin{aligned}
d\ln X_t &= \left(r - \frac{1}{2}h_t + \lambda_1\sqrt{h_t}\right)dt + \sqrt{h_t}dW_{1t}, && (16)\\
dh_t &= (\alpha_0 + (\alpha_1 + \beta_1 - 1)h_t)dt + \sqrt{2}\alpha_1 h_t dW_{2t}, && (17)\\
&= (\alpha_0 + (\alpha_1 + \beta_1^* - 1)h_t)dt + \sqrt{2}\alpha_1\lambda_2 h_t dt + \sqrt{2}\alpha_1 h_t dW_{2t}, && (18)
\end{aligned}
$$

where the persistence parameter of conditional variance is defined as $\beta_1^* = \beta_1 - \sqrt{2}\alpha_1\lambda_2$ under the GRNVR. The terms dW_{1t} and dW_{2t} are independent standard Brownian motions under the physical measure P. The limiting bivariate diffusion under the risk-neutral measure Q is a re-parameterization of Hull and White's bivariate diffusion model Hull and White (1987) as follows:

$$
\begin{aligned}
d\ln X_t &= \left(r - \frac{1}{2}h_t\right)dt + \sqrt{h_t}dZ_{1t} && (19)\\
dh_t &= (\alpha_0 + (\alpha_1 + \beta_1^* - 1)h_t)dt + \sqrt{2}\alpha_1 h_t dZ_{2t}, && (20)
\end{aligned}
$$

where $dZ_{1t} = dW_{1t} + \lambda_1 dt$ and $dZ_{2t} = dW_{2t} + \lambda_2 dt$ are independent standard Brownian motions under the GRNVR Q. Both equity risk premium λ_1 and variance risk premium

λ_2 are present in the model under the risk-neutral measure Q. The discrete-time GARCH$(1,1)$ process (3) corresponding to the limiting diffusion process under the risk-neutral measure Q.

7. Conclusion

In this paper, we follow the GARCH option pricing framework of Duan (1995) and propose a new way of changing from physical probability measure to risk-neutral probability measure. The new risk-neutral valuation is referred to as the GRNVR. The advantage of the GRNVR compared to the LRNVR commonly used in the literature Duan (1995); Hao and Zhang (2013); Wang et al. (2017) is that the variance risk premium is included in the risk-neutral dynamics under the GRNVR. The absence of variance risk premium in the risk-neutral dynamics under the LRNVR is noted in Hao and Zhang (2013), where it is shown that both empirical studies and theoretical results indicated that the GARCH models under the LRNVR did not capture the variance premium.

We then find the theoretical VIX squared value as the conditional risk-neutral expectation of the arithmetic mean variance over the next 21 trading days under the GRNVR. Specifically, the GARCH implied VIX formulas are derived using the features of square-root stochastic autoregressive volatility (SR-SARV) models. We apply several calibration methods to estimate the model parameters using various sets of time series data, and compare the theoretical formula performances with the market data. Various combinations of time series of the daily closing price of S&P 500 index and the CBOE VIX are used to find the maximum likelihood estimation of the GARCH models. The corresponding implied VIX time series are then calculated from the calibrated model. Similar to the empirical evidences in Hao and Zhang (2013); Wang et al. (2017), when only S&P 500 returns are used for estimation, the GARCH implied VIX is consistently and significantly lower than the CBOE VIX. When the CBOE VIX is used for estimation, the implied VIX fits the statistical properties of the CBOE VIX and matches the CBOE VIX data better. The numerical results provide evidences that the GARCH option pricing under the GRNVR is more suitable to price volatility. In the case of GARCH(1,1), we compare the diffusion limit of the GARCH process under the physical measure and the GRNVR risk-neutral measure to show that variance premium is captured in the risk-neutral dynamics.

References

Barone-Adesi, G., R. Engle, and L. Mancini, 2008, A GARCH option pricing model with filtered historical simulation, *The Review of Financial Studies* 21, 1223–1258.

Bollerslev, T., 1986, Generalized autoregressive conditional heteroskedasticity, *Journal of Econometrics* 31, 307–327.

Carr, P., and D. Madan, 1998, Towards a theory of volatility trading, in Jarrow R., ed., *Risk Book on Volatility*, 417–427 (Risk, New York).

Chernov, M., and E. Ghysels, 2000, A study towards a unified approach to the joint estimation of objective and risk neutral measures for the purpose of options valuation, *Journal of Financial Economics* 56, 407–458.

Christoffersen, P., S. Heston, and K. Jacobs, 2006, Option valuation with conditional skewness, *Journal of Econometrics* 131, 253–284.

Christoffersen, P., S. Heston, and K. Jacobs, 2013, Capturing option anomalies with a variance-dependent pricing kernel, *The Review of Financial Studies* 26, 1962–2006.

Demeterfi, K., E. Derman, M. Kamal, and J. Zou, 1999, A guide to volatility swaps, *Journal of Derivatives* 7, 9–32.

Duan, J., 1995, The GARCH option pricing model, *Mathematical Finance* 5, 13–32.

Duan, J., 1996, A unified theory of option pricing under stochastic volatility from GARCH to diffusion, Technical report, Hong Kong University of Science and Technology.

Duan, J., 1997, Augmented GARCH(p, q) process and its diffusion limit, *Journal of Econometrics* 79, 97–127.

Engle, R., and V. Ng, 1993, Measuring and testing the impact of news on volatility, *Journal of Finance* 48, 1749–1778.

Glosten, L., R. Jagannathan, and D. Runkle, 1993, On the relation between the expected value and the volatility of the nominal excess return on stocks, *Journal of Finance* 48, 1779–1801.

Hao, J., and J. E. Zhang, 2013, GARCH option pricing models, the CBOE VIX and variance risk premium, *Journal of Financial Econometrics* 11, 556–580.

Hull, J., and A. White, 1987, The pricing of options on assets with stochastic volatilities, *Journal of Finance* 42, 281–299.

Kanniainen, J., B. Lin, and H. Yang, 2014, Estimating and using GARCH models with VIX data for option valuation, *Journal of Banking & Finance* 43, 200–211.

Lin, Y., 2007, Pricing VIX futures: Evidence from integrated physical and risk-neutral probability measures, *Journal of Futures Markets* 27, 1175–1217.

Luo, X., and J. E. Zhang, 2012, The term structure of VIX, *Journal of Futures Markets* 32, 1092–1123.

Meddahi, N., and E. Renault, 2004, Temporal aggregation of volatility models, *Journal of Econometrics* 116, 181–224.

Nelson, D., 1991, Conditional heteroskedasticity in asset returns: A new approach, *Econometrica* 59, 347–370.

Wang, T., Y. Shen, Y. Jiang, and Z. Huang, 2017, Pricing the CBOE VIX futures with the Heston-Nandi GARCH model, *Journal of Futures Market* 37, 641–659.

Whaley, R., 1993, Derivatives on market volatility hedging tools long overdue, *The Journal of Derivative* 1, 71–84.

Zhang, Jin E., and Y. Zhu, 2006, VIX futures, *Journal of Futures Markets* 26, 521–531.

Zhu, Y., and Jin E. Zhang, 2007, Variance term structure and VIX futures pricing, *International Journal of Theoretical and Applied Finance* 10, 111–127.

Table 1: Maximum likelihood estimates of GARCH(1,1) model using return data only, VIX data only or both return and VIX data. The bold values indicate the log-likelihood value which is being maximized. The standard errors are provided in the parentheses.

	Return only	VIX only		Return & VIX	
		LRNVR	GRNVR	LRNVR	GRNVR
α_0	2.474e-6 (0.089e-6)	1.678e-6 (0.044e-6)	1.641e-6 (0.031e-6)	1.699e-6 (0.034e-6)	1.675e-6 (0.032e-6)
α_1	0.1256 (0.0045)	0.0382 (0.0006)	0.0456 (0.0015)	0.0498 (0.0007)	0.0474 (0.0007)
β_1	0.8543 (0.0028)	0.9351 (0.0010)	0.9286 (0.0036)	0.9450 (0.0007)	0.9251 (0.0012)
λ_1	0.0886 (0.0116)	0.8144 (0.0171)	0.2253 (0.0122)	0.2976 (0.0085)	0.2134 (0.0085)
λ_2	0	0	-0.3514 (0.0705)	0	-0.3670 (0.0195)
$\ln L_R$	**22720**	20355	20593	21796	22597
$\ln L_V$	29221	**33424**	**33662**	32901	33324
$\ln L_T$	51941	53779	54255	**54697**	**55921**

Table 2: Maximum likelihood estimates of TGARCH(1,1) model using return data only, VIX data only or both return and VIX data. The bold values indicate the log-likelihood value which is being maximized. The standard errors are provided in the parentheses.

	Return only	VIX only		Return & VIX	
		LRNVR	GRNVR	LRNVR	GRNVR
α_0	3.454e-6 (0.090e-6)	1.476e-6 (0.075e-6)	1.520e-6 (0.018e-6)	1.477e-6 (0.021e-6)	1.480e-6 (0.015e-6)
α_1	0.0256 (0.0063)	0.0020 (0.0015)	0.0060 (0.0010)	0.0021 (0.0012)	0.0091 (0.0014)
β_1	0.8338 (0.0031)	0.9597 (0.0007)	0.9553 (0.0006)	0.9582 (0.0006)	0.9561 (0.0008)
θ	0.1336 (0.0154)	0.0595 (0.0026)	0.0531 (0.0008)	0.0636 (0.0020)	0.0553 (0.0022)
λ_1	0.0131 (0.0118)	0.3094 (0.0155)	0.0889 (0.0087)	0.1729 (0.0135)	0.1160 (0.0075)
λ_2	0	0	-0.3978 (0.0980)	0	-0.4112 (0.1376)
$\ln L_R$	**22817**	21941	22242	22324	22578
$\ln L_V$	30310	**33468**	**33795**	33131	33704
$\ln L_T$	53127	55409	56037	**55455**	**56282**

Table 3: Maximum likelihood estimates of AGARCH(1,1) model using return data only or both return and VIX data. The bold values indicate the log-likelihood value which is being maximized. The standard errors are provided in the parentheses.

	Return only	Return & VIX	
		LRNVR	GRNVR
α_0	1.8592e-6 (0.078e-6)	1.708e-6 (0.087e-6)	1.568e-6 (0.083e-6)
α_1	0.0841 (0.0032)	0.0415 (0.0011)	0.0380 (0.0005)
β_1	0.8810 (0.0108)	0.9302 (0.0010)	0.9316 (0.0012)
θ	0.7861(0.0534)	0.7795 (0.0204)	0.8012 (0.0174)
λ_1	0.0255 (0.0119)	0.0120 (0.0095)	0.1158 (0.0538)
λ_2	0	0	-0.3125 (0.0297)
$\ln L_R$	**22875**	22298	22690
$\ln L_V$	29940	33185	33643
$\ln L_T$	52815	**55483**	**56333**

Table 4: Maximum likelihood estimates of EGARCH(1,1) model using return data only, VIX data only or both return and VIX data. The bold values indicate the log-likelihood value which is being maximized. The standard errors are provided in the parentheses.

	Return only	VIX only		Return & VIX	
		LRNVR	GRNVR	LRNVR	GRNVR
α_0	-0.1919 (0.0133)	-0.0742 (0.0010)	-0.0795 (0.0011)	-0.0845 (0.0012)	-0.0840 (0.0012)
α_1	-0.1159 (0.0068)	-0.0622 (0.0012)	-0.0638 (0.0012)	-0.0598 (0.0016)	-0.0575 (0.0017)
β_1	0.9792 (0.0014)	0.9897 (0.0002)	0.9895 (0.0002)	0.9891 (0.0010)	0.9906 (0.0002)
κ	0.1239 (0.0081)	0.0869 (0.0013)	0.0879 (0.0012)	0.0795(0.0021)	0.0817 (0.0013)
λ_1	0.0189 (0.0106)	0.0378 (0.0201)	0.0294 (0.0114)	0.0048 (0.0068)	0.0108 (0.0041)
λ_2	0	0	-0.0483 (0.0083)	0	-0.0567 (0.0044)
$\ln L_R$	**22862**	22572	22642	22645	22813
$\ln L_V$	29304	**33774**	**34303**	33754	34292
$\ln L_T$	52166	56346	56945	**56399**	**57105**

Table 5: The table displays the related statistics between the model implied VIX and the CBOE VIX for the GARCH models during the period from January 2, 1990 to June 30, 2017. The error is computed as the difference between CBOE VIX and the implied VIX. The mean error (ME) represents the mean daily difference between the implied VIX and the CBOE VIX. The standard error (Std.Err.) represents the standard deviation of the error. The mean absolute error (MAE) calculates the mean daily absolute difference between the implied VIX and the CBOE VIX. The mean squared error (MSE) computes the mean daily squared difference between the implied VIX and the CBOE VIX. The root mean squared error (RMSE) computes the square root of the mean squared error. The P-value is for the null hypothesis that the implied VIX and the CBOE VIX have the same mean values. Violation of one-sigma band stands for the probability that the implied VIX lies out of the one-standard-deviation band of the CBOE VIX. The correlation coefficient (Corr. Coef.) computes the linear correlation between the implied VIX and the CBOE VIX. Autocorrelation coefficients with lag of 1, 10, and 30 days and higher moments of impliex VIX are also reported.

Model&Data	ME	Std.Err.	MAE	MSE	RMSE	P-value	Violation of one-sigma band	Corr.Coef.	AR1	AR10	AR30	Variance	Skewness	Kurtosis
GARCH														
Returns	2.76	4.03	2.69	16.86	4.11	0.0000	7.56%	0.90	0.9892	0.8879	0.6856	77.39	3.36	20.37
VIX	0.08	2.99	2.29	8.96	2.99	0.5223	1.68%	0.93	0.9956	0.9452	0.7945	56.04	2.61	15.01
Both	0.16	3.14	2.25	9.03	3.01	0.1012	2.27%	0.92	0.9950	0.9434	0.7853	56.65	2.69	15.70
TGARCH														
Returns	2.44	4.02	3.40	22.13	4.70	0.0000	9.47%	0.86	0.9596	0.7433	0.5327	53.21	3.33	21.08
VIX	0.08	2.96	2.27	8.80	2.96	0.5574	1.59%	0.93	0.9955	0.9473	0.8072	55.88	2.81	16.83
Both	0.22	3.00	2.26	9.03	3.01	0.1012	1.80%	0.93	0.9954	0.9431	0.7940	57.53	2.83	17.21
AGARCH														
Returns	3.08	3.32	3.47	20.49	4.53	0.0000	8.14%	0.91	0.9815	0.8495	0.6220	57.60	3.44	22.50
Both	0.19	2.98	2.27	8.93	2.99	0.1440	1.13%	0.93	0.9948	0.9383	0.7774	57.60	2.92	17.52
EGARCH														
Returns	3.39	3.38	3.62	22.96	4.79	0.0000	10.02%	0.92	0.9817	0.8498	0.6408	35.52	2.08	10.54
VIX	0.01	2.71	2.10	7.38	2.72	0.8865	0.79%	0.94	0.9946	0.9395	0.7955	54.37	2.20	11.12
Both	0.01	2.72	2.10	7.41	2.72	0.7396	0.82%	0.94	0.9944	0.9373	0.7903	54.62	2.15	10.81
CBOE VIX									0.9812	0.8988	0.7580	61.68	2.10	10.70

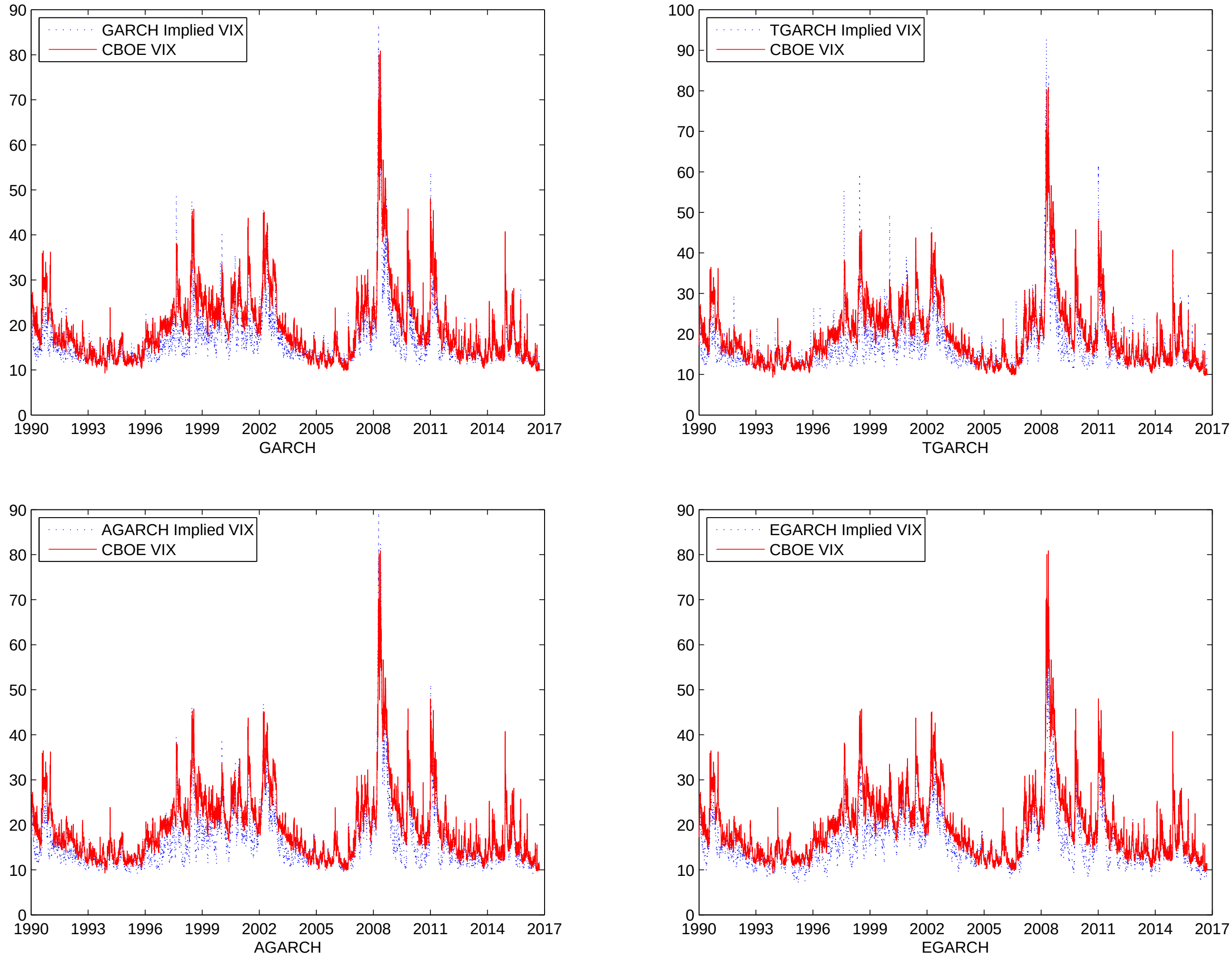


Figure 1: Comparison between CBOE VIX and implied VIX using return data only for four GARCH models.

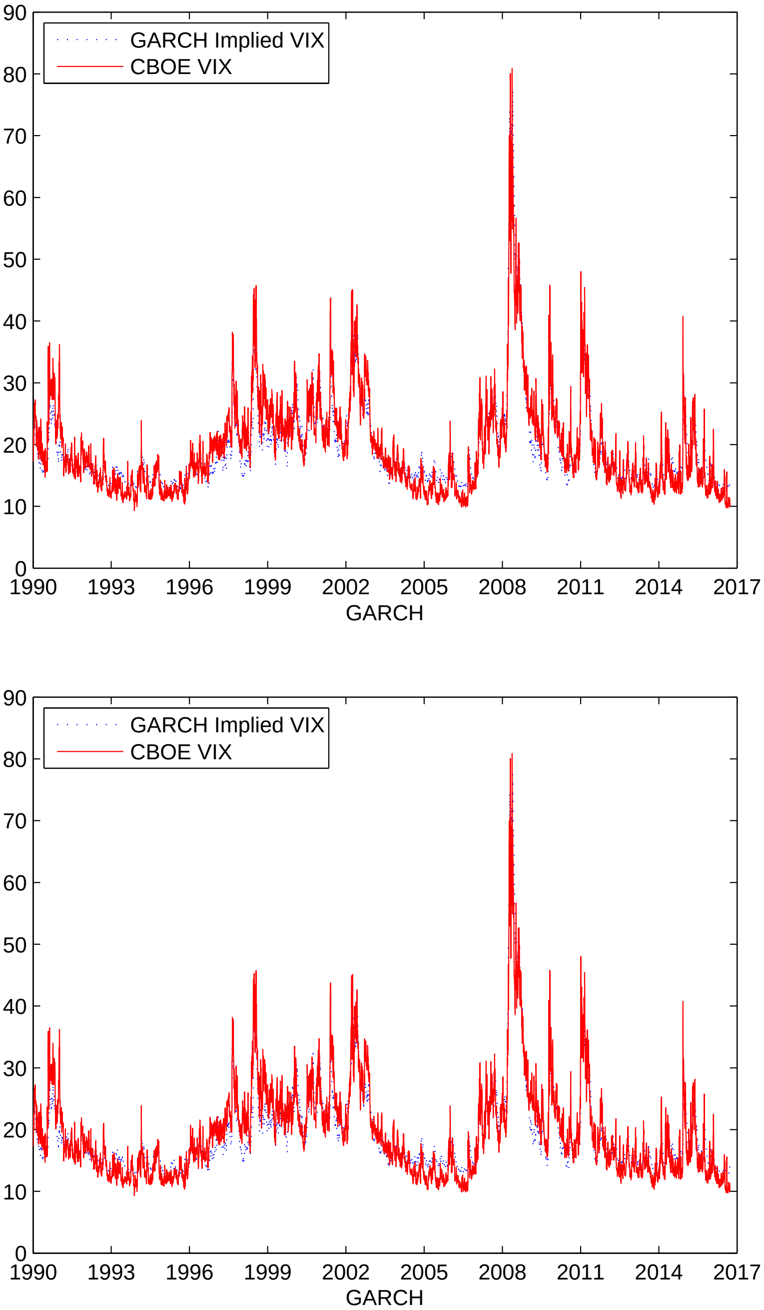
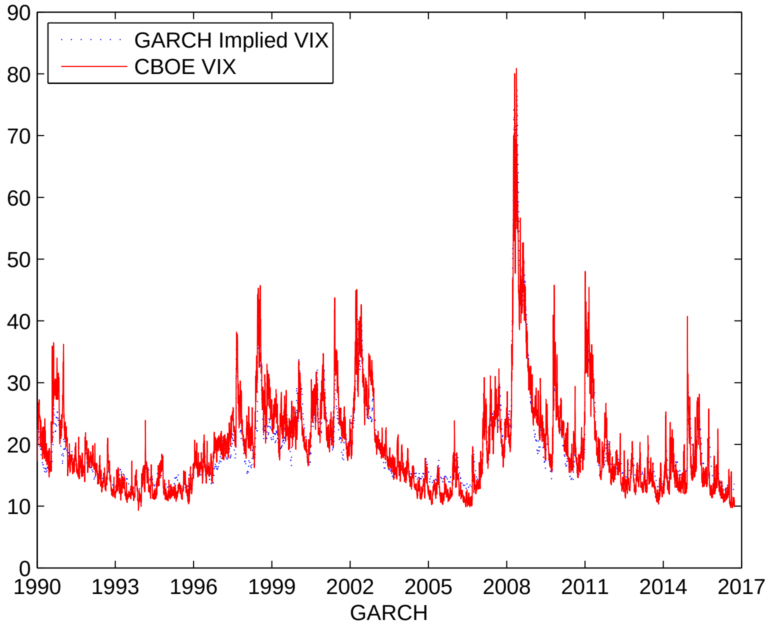


Figure 2: Comparison between CBOE VIX and implied VIX of the GARCH(1,1) model using VIX data only with the upper panel showing the result under the GRNVR and the lower panel showing the result under the LRNVR.



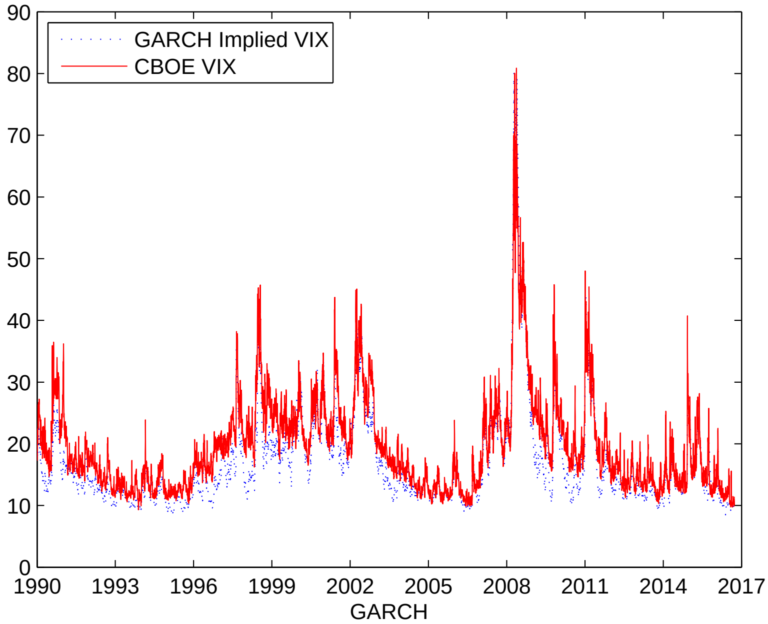


Figure 3: Comparison between CBOE VIX and implied VIX of the GARCH(1,1) model using both return and VIX data with the upper panel showing the result under the GRNVR and the lower panel showing the result under the LRNVR.

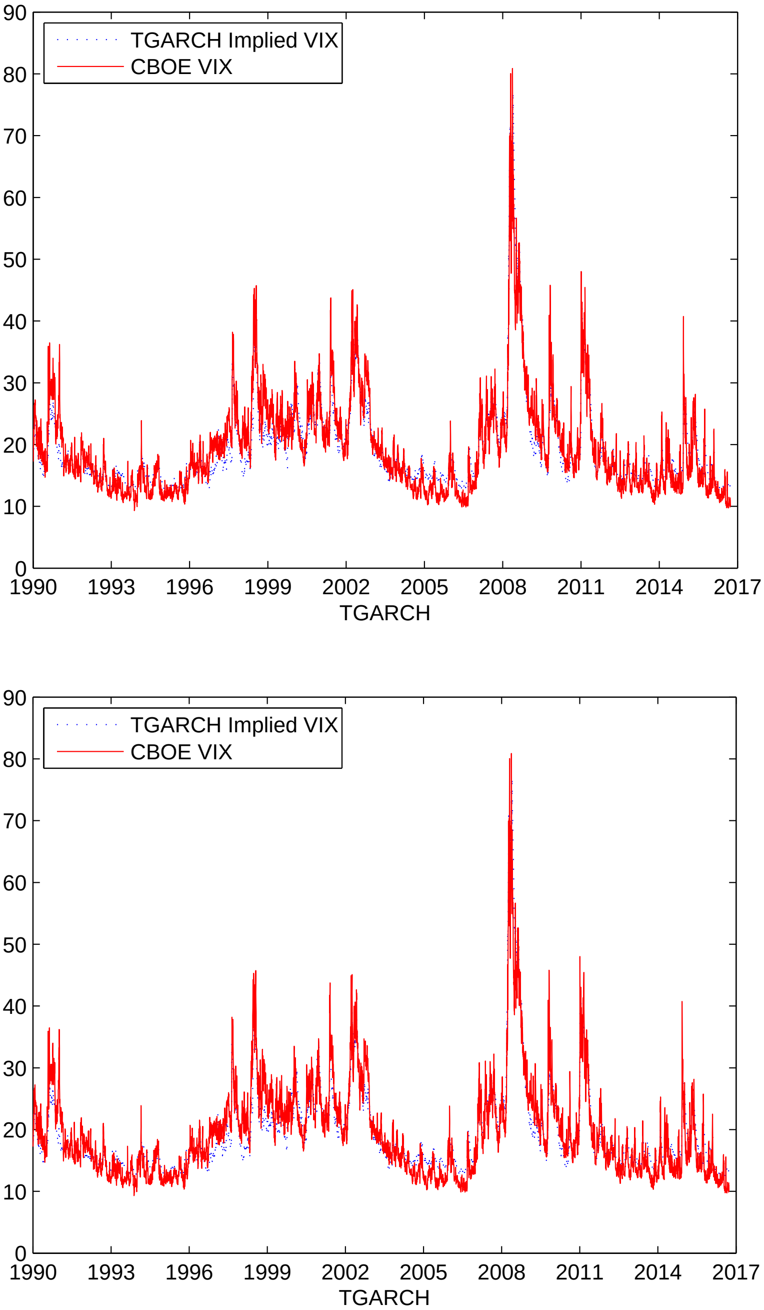


Figure 4: Comparison between CBOE VIX and implied VIX of the TGARCH(1,1) model using VIX data only with the upper panel showing the result under the GRNVR and the lower panel showing the result under the LRNVR.

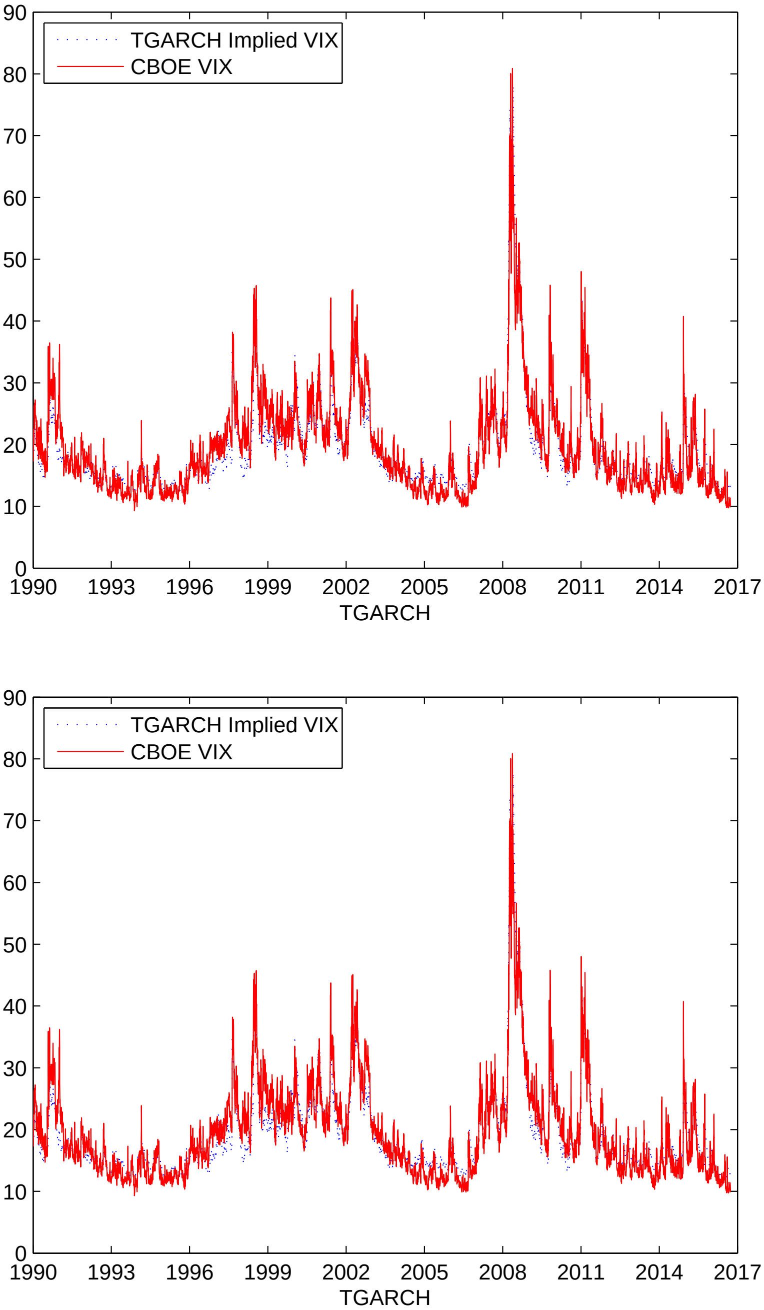


Figure 5: Comparison between CBOE VIX and implied VIX of the TGARCH(1,1) model using both return and VIX data with the upper panel showing the result under the GRNVR and the lower panel showing the result under the LRNVR.

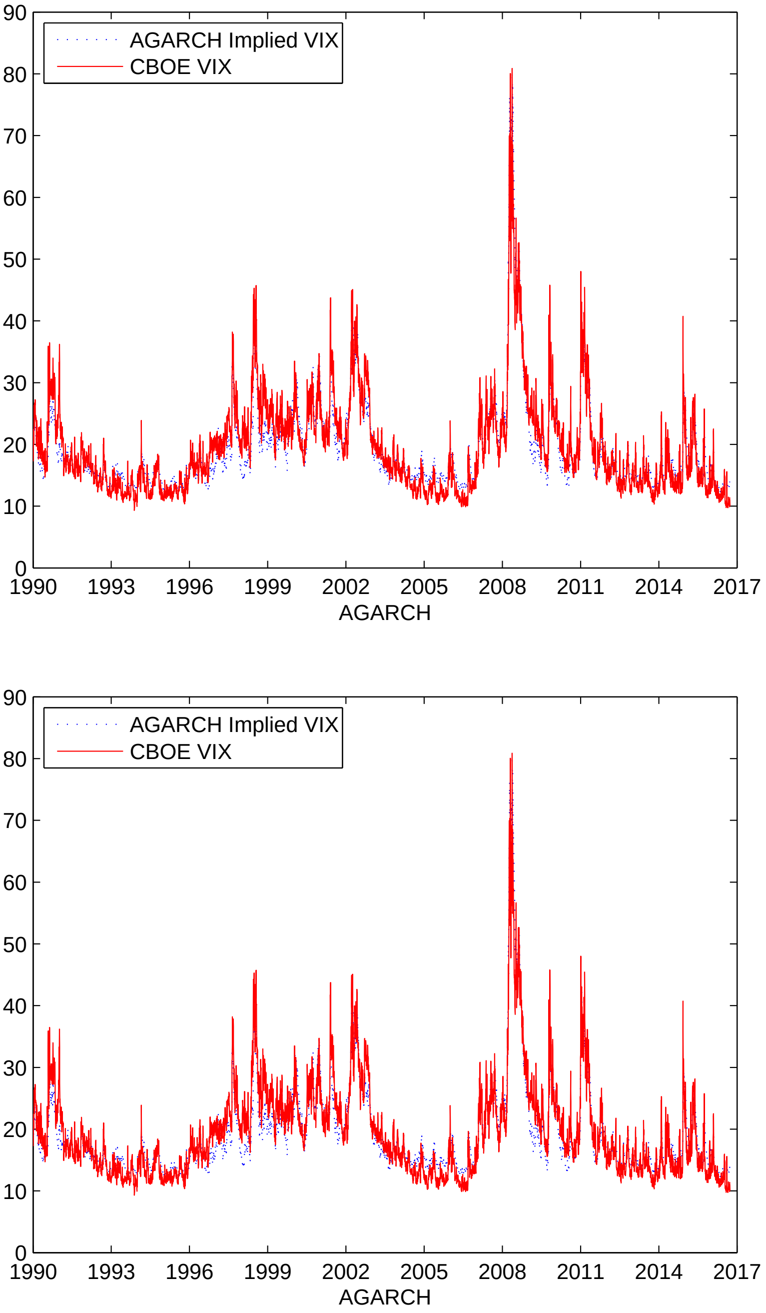
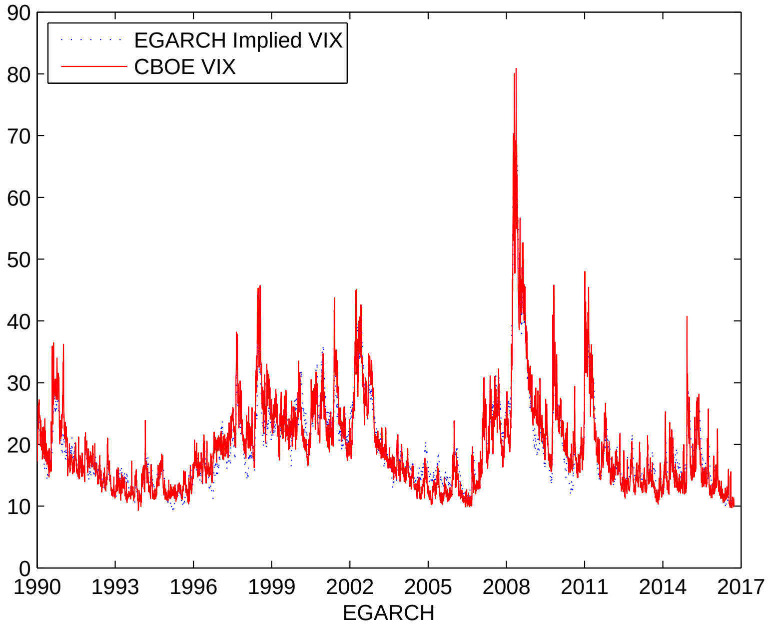


Figure 6: Comparison between CBOE VIX and implied VIX of the AGARCH(1,1) model using both return and VIX data with the upper panel showing the result under the GRNVR and the lower panel showing the result under the LRNVR.



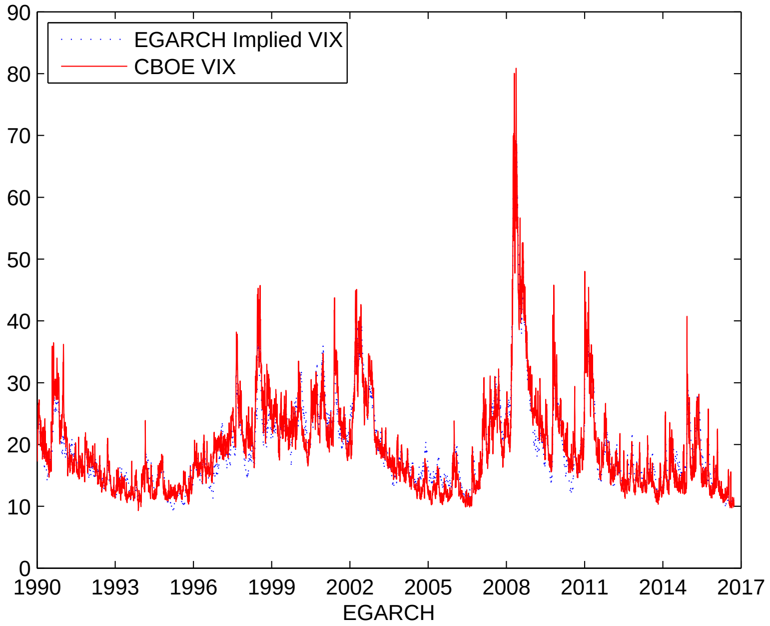


Figure 7: Comparison between CBOE VIX and implied VIX of the EGARCH(1,1) model using VIX data only with the upper panel showing the result under the GRNVR and the lower panel showing the result under the LRNVR.

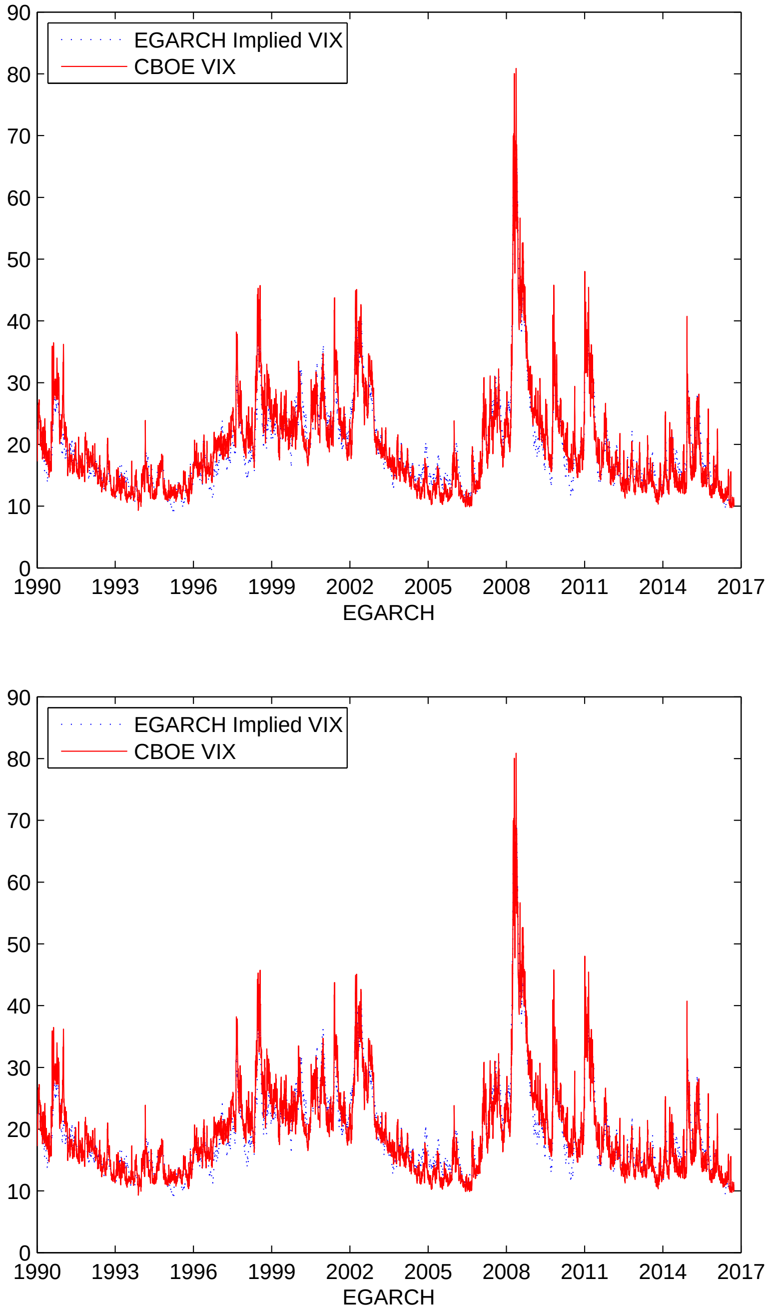


Figure 8: Comparison between CBOE VIX and implied VIX of the EGARCH(1,1) model using both return and VIX data with the upper panel showing the result under the GRNVR and the lower panel showing the result under the LRNVR.