ASSET PRICING UNDER KEEPING UP WITH THE JONESES AND TIME-VARYING SENTIMENT

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ABSTRACT. This paper studies the joint effect of "Keeping up with the Joneses" (KUJ) preferences, time-varying sentiment, and average pessimism in a two-agent equilibrium asset pricing model. We find that although the irrational agent does not survive in the long run, due to KUJ, sentiment continues to have a significant effect on market equilibrium. In particular, the model generates a procyclical price-dividend ratio, excess countercyclical stock volatility, and a large countercyclical equity premium, which are consistent with empirical observations. Moreover, the term-structure of real interest rates is upward (resp. downward) sloping when the short rate is relatively low (resp. high).

1. INTRODUCTION

Keeping up with the Joneses (KUJ) postulates that economic agents, when maximizing their expected utility, are not only concerned about their own level of (absolute) consumption, but also their consumption relative to other agents in the economy.[1] In this paper, we embed KUJ preferences in a dynamic differences-of-opinion (DO) equilibrium model with two constant-relative-risk-aversion (CRRA) agents, where one agent has rational expectations, while the other agent's subjective beliefs is driven by time-varying sentiment, i.e., waves of optimism and pessimism about the fundamental state of the economy. We solve the model in closed-form, and find that many stylized facts of the aggregate equity market including (i) a large countercyclical equity premium, (ii) an excess countercyclical stock volatility, and (iii) a procyclical and concave price-dividend (P/D) ratio can be explained under plausible parameterizations of the model.

We consider a one-tree Lucas (1978) economy populated by two economic agents with heterogeneous beliefs about the expected growth rate of aggregate endowment. Agent A believes that the expected growth rate follows a Ornstein–Uhlenbeck process, whereas Agent B believes the expected growth rate is constant. We assume the true data-generating process coincides with Agent B's perception. Therefore, Agent A is driven by sentiment, i.e., he mistakens noise for information, and Agent B has rational expectation. We assume Agent A is on average pessimistic relative to Agent B. Each agents maximizes, under their subjective probability measure, expected CRRA utility of his relative consumption to the other agent in the economy. The agents trade in a risk-free bond and a risky stock that represents a claim to the stream of dividends produced by the tree. The stock and bond prices are endogenously determined by the market clearing conditions. We solve the model in closed-form.

One issue that confronts DO equilibrium models is that effect of heterogeneity in beliefs does not persist in the long-run, because irrational agents are eventually driven out the market as their consumption share converge almost surely to zero. As a result, equilibrium converge to one under a rational expectation, representative-agent economy. We show that, under KUJ

[1] Micro data provides supportive evidence of the KUJ peer effect. Revina (2019) uses a sample of U.S. credit-card account holders and finds that a significant fraction of the reference group enters the utility function.

preferences, despite the fact that the irrational agent whose belief is driven by sentiment (*sentiment agent*) vanishes in the long run, the impact of sentiment on asset prices persists and leads to significant cyclical behavior of equilibrium quantities.

Intuition is as follows. Impact of sentiment on prices comes from two separate channels. First is the standard channel via the consumption share of sentiment agent, which gradually shuts down due to the market selection of rational agent in the long run. Second is the KUJ channel via relative consumption concern, which remains active even when the sentiment agent's consumption share is very small. To illustrate, let $\nu_t = c_t^A/(c_t^A + c_t^B)$ be Agent A's consumption share, where c_t^A and c_t^B are Agents' A and B's consumption, respectively. Suppose ν_t is close to zero, Agent B's relative consumption, $c_t^B/c_t^A = (1-\nu_t)/\nu_t$, remains sensitive to changes in ν_t. As a result, Agent B still cares about Agent A's subjective belief, which affects his marginal utility, thus the equilibrium state price density.

The second key ingredient is *time-varying* sentiment. Instead of constant pessimism/optimism, we assume sentiment is procyclical, i.e., it tends to be optimistic during boom and pessimistic during economic downturn. We find that time-varying sentiment plays a crucial role in generating cyclical variation in the P/D ratio. Intuitively, the propensity to consume, i.e., the inverse of wealth-to-consumption (W/C) ratio for agents with KUJ preferences depends on their expectation of future growth in the relative consumption rather than absolute consumption, which is independent of the current level of consumption share. Consequently, agents' W/C ratio is a function of sentiment only. Therefore, if sentiment is constant, W/C ratios are also constant, and since W/C ratios share very similar cyclical behavior, P/D also behaves similarly to the W/C ratios. Hence, the cyclical behavior of equilibrium quantities varies very little with respect to change in the level of consumption share.

The third ingredient is *average pessimism*, i.e, we assume sentiment has a pessimistic long-run mean. Giordani and Söderlind (2006) find using the Survey of Professional Forecasters (SPF) that there is on average an underestimate of expected GDP growth of 0.7% per annum, which approximately matches the average belief about the expected growth rate of aggregate endowment between the rational and sentiment agents in our model. In a rational-expectations, representative-agent model, this helps to increase the equity premium by 0.7%. However, since stock returns exhibits excess and countercyclical volatility in our model, a 0.7% average pessimism leads to much more significant increase in equity premium. In fact, the unconditional mean of equity premium is 3% p.a. in our model, which accounts for half of the empirically observed level in the U.S.

KUJ has been incorporated into habit models in consumption-based asset pricing literature to explain market anomalies, in particular the equity premium and risk-free rate puzzles.[2] In these models, the habit is either internal or external,[3] and agents may have different preferences but are homogeneous in their beliefs. Campbell and Cochrane (1999) use an external habit model with counter-cyclical variation in risk aversion to explain asset pricing puzzles in a model with a representative agent, in particular the equity premium and risk-free rate puzzles. Chan and Kogan (2002) argue that the variation in risk aversion can be due to the endogenous wealth fluctuations in an economy with multiple agent with heterogenous risk aversions. However, Xiouros and Zapatero (2010) show that the level of heterogeneity in risk aversion assumed in Chan and Kogan (2002) is unrealistically high, thus heterogeneity in risk aversions alone has only a negligible effect on market equilibrium. What we do differently in this paper is that we endogenize each agent's external consumption habit by using the optimal consumption plan of the other agent in the economy, and also we combine KUJ with time-varying sentiment to match key stylized facts in the aggregate equity market.

Other related works include Huang, Qiu and Tang (2013) and van Bilsen, Bovenberg and Laeven (2020). Huang et al. (2013) analyzes an equilibrium asset pricing model in which agents with heterogeneous beliefs care about relative performance. They show relative performance concern leads to similar trading, which can increase or decrease stock volatility depending on the level of consumption share. However, the magnitude of changes in volatility is not very significant. In contrast, in our model, KUJ and time-varying sentiment leads to a much stronger excess and countercyclical stock volatility. van Bilsen et al. (2020) develop a closed-form approach to solve consumption portfolio choice problem for an individual with Epstein-Zin preference, who derives utility from the ratio between his consumption and an endogenous habit. They take stock and bond price as exogenous processes rather than determined in equilibrium.

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[2] See, for example, Sundaresan (1989), Constantinides (1990), Abel (1990), Campbell and Cochrane (1999), and Chan and Kogan (2002)

[3] In the *internal habit* models such as Constantinides (1990) and Sundaresan (1989), habit is usually a weighted average of the agent's own past consumption, so that the agent's current consumption decision will affect his future habit. In the more popular *external habit* models such as Abel (1990) and Campbell and Cochrane (1999), habit can take any form of a stochastic process and will not be affected by agent's own consumption decisions.

The paper is structured as follows. In Section 2, we present the model and solve for the optimal portfolio/consumption decisions under KUJ preferences and heterogeneous beliefs. Section 3 analyzes the joint impact of KUJ and time-varying sentiment on the equilibrium quantities. In Section 4, we provide an extension to the baseline model in Section 2 and compare the results. Section 5 concludes. All proofs can be found in Appendix A.

2. THE MODEL

We consider an infinite-horizon continuous-time pure exchange economy with a dynamically complete financial market and a single consumption good. The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ on which an one-dimensional Brownian motion ω_t is defined. There is an exogenous aggregate endowment process Y_t, which follows a geometric Brownian motion,

$$\frac{dY_t}{Y_t} = \mu_Y dt + \sigma_Y d\omega_t, \;\; Y_0 > 0, \tag{1}$$

where the expected growth rate, μ_Y, and volatility, σ_Y are constants.

2.1. **Agents' Beliefs.**

There are two (groups of) agents in the economy, indexed by $i = A, B$.[4] We make the following assumption about agents' subjective beliefs.

(H1a) Agent i has a subjective belief, $\mu^i_{Y,t}$, about the expected growth rate of aggregate endowment, and believes that the process follows

$$\frac{dY_t}{Y_t} = \mu^i_{Y,t} dt + \sigma_Y d\omega^i_t, \tag{2}$$

where ω^i_t is a Brownian motion under Agent i's subjective probability measure $\mathbb{P}^i$, and $d\omega^i_t = d\omega_t - (\mu^i_{Y,t} - \mu_Y)/\sigma_Y dt$.

(H1b) Agent A is *irrational* and erroneously perceives that

$$d\mu^A_{Y,t} = -\zeta(\mu^A_{Y,t} - \bar{\mu}^A_Y)dt + \delta d\omega^A_{Y,t}, \qquad \mu^A_{Y,0} = \bar{\mu}^A_Y. \tag{3}$$

Agent B is *rational* and correctly perceives that $\mu^B_{Y,t} = \mu_Y$, therefore $d\omega^B_t = d\omega_t$.

[4] As in Scheinkman and Xiong (2003), Dumas, Kurshev and Uppal (2009), and Jouini and Napp (2011), we consider a two-agent economy in this paper. This is the simplest and most natural extension of the traditional representative economy to characterize the impact of heterogeneity on market equilibrium. For economy with many heterogeneous agents, we refer to Jouini and Napp (2006), Yan (2008), Cvitanić, Jouini., Malamud. and Napp (2012), and Muraviev (2013).

Assumption (H1a) implies that although agents observe the aggregate same endowment process, they agree to disagree about its expected growth rate, which is unobservable. While Agent B correctly perceives the expected growth rate as a constant, Agent A mistakenly perceives it to follow an Ornstein-Uhlenbeck (OU) process. As a result, Agent A updates his belief based on the realization of $d\omega_t^A = (dY_t/Y_t - \mu_{Y,t}^A)/\sigma_Y$, even though $d\omega_t^A$ is pure noise since the true expected growth rate, μ_Y, is a constant. We measure the level of *sentiment*, i.e., *optimism/pessimism* in Agent A's belief using

$$\theta_t \equiv \frac{\mu_{Y,t}^A - \mu_Y}{\sigma_Y},$$

which follows an OU process,

$$d\theta_t = -\zeta_\theta(\theta_t - \bar{\theta})dt + \delta_\theta d\omega_t, \tag{4}$$

under the objective probability measure, $\mathbb{P}$, where

$$\delta_\theta = \frac{\delta}{\sigma_Y}, \;\; \zeta_\theta = \zeta + \delta_\theta, \;\; \text{and } \bar{\theta} = \left(\frac{\zeta}{\zeta_\theta}\right)\frac{\bar{\mu}_Y^A - \mu_Y}{\sigma_Y}.$$

Thus, the probability belief of Agent A can be characterized by the martingale $M_t = d\mathbb{P}^A/d\mathbb{P}$, which satisfies

$$\frac{dM_t}{M_t} = \theta_t d\omega_t, \qquad M_0 = 1. \tag{5}$$

2.2. **Tradable Securities.** Agents can continuously trade a locally riskless bond and a long-lived risky security (stock). The stock is a claim to the aggregate endowment process, which pays an instantaneous dividend of $Y_t dt$, and is in positive unit supply. The locally riskless bond pays an instantaneous interest rate r_t, and is in zero net supply. The bond and stock prices have the following dynamics,

$$\frac{dB_t}{B_t} = r_t dt, \qquad B_0 = 1, \tag{6}$$

$$\begin{aligned} \frac{dS_t + Y_t dt}{S_t} &= \mu_{S,t} dt + \sigma_{S,t} d\omega_t, \\ &= \mu_{S,t}^i dt + \sigma_{S,t} d\omega_t^i, \qquad i = A, B. \end{aligned} \tag{7}$$

The triplet $\{r_t, \mu_{S,t}, \sigma_{S,t}\}$ are to be endogenously determined in equilibrium. Moreover, the subjective beliefs satisfy the consistency condition,

$$\frac{\mu^i_{S,t} - \mu_{S,t}}{\sigma_{S,t}} = \frac{\mu^i_{Y,t} - \mu_Y}{\sigma_Y} = \begin{cases} \theta_t, & i = A; \\ 0, & i = B. \end{cases} \tag{8}$$

Since the market is dynamically complete, in the sense that any contingent claims can be replicated, there exists a unique state price density process ξ_t under objective probability measure $\mathbb{P}$ that satisfies

$$\frac{d\xi_t}{\xi_t} = -r_t dt - \kappa_t d\omega_t, \qquad \xi_0 = 1, \tag{9}$$

where

$$\kappa_t \equiv \frac{\mu_{S,t} - r_t}{\sigma_{S,t}} \tag{10}$$

is the market price of risk or the Sharpe ratio.

We assume Agent i, $i = A, B$, is initially endowed with x^i_S share of the stock, where $x^A_S + x^B_S = 1$. Then, Agent i chooses a nonnegative consumption process c^i_t and invests a proportion, π^i_t, of wealth in the stock, thus his financial wealth process W^i_t is given by

$$dW^i_t = \left(r_t W^i_t - c^i_t\right) dt + W^i_t \pi^i_t \left[(\mu_{S,t} - r_t)dt + \sigma_{S,t} d\omega_t\right], \qquad W^i_0 = x^i_S S_0. \tag{11}$$

Due to market completeness, using the state price density process the dynamic budget constraint in (11) can be written as

$$\mathbb{E}\left[\int_0^\infty \xi_t c^i_t dt \middle| \mathcal{F}_0\right] \le W^i_0. \tag{12}$$

Next, we specify agents' preferences in order to derive the SPD process, ξ_t.

2.3. **Keeping Up with the Joneses (KUJ) Preferences.** We make the following assumption about agents' preferences.

(H2) Agents' instantaneous utilities are given by

$$U^A_t(c^A_t, c^B_t) \equiv \frac{e^{-\rho t}}{1-\gamma}\left(\frac{c^A_t}{c^B_t}\right)^{1-\gamma},$$

$$U^B_t(c^A_t, c^B_t) \equiv \frac{e^{-\rho t}}{1-\gamma}\left(\frac{c^B_t}{c^A_t}\right)^{1-\gamma}, \qquad \gamma > 1. \tag{13}$$

Thus, the optimization problem for the agents is given by

$$\max_{c_t^i} \mathbb{E}\left[\int_0^\infty M_t^i e^{-\rho t} U_t^i(c_t^A, c_t^B) dt \middle| \mathcal{F}_0\right], \qquad i = A, B, \tag{14}$$

subjected to the static budget constraint in (12), where $M_t^A = M_t$ and $M_t^B = 1$. Assumption (H2) implies that Agent A is concerned about his relative consumption to Agent B, and vice-versa, rather than the absolute level of his consumption. This is a reasonable assumption since households often compare with their neighbours in regards to social status. From the marginal utility of consumption we have

$$\frac{\partial U_t^A}{\partial c_t^A} = e^{-\rho t}(c_t^A)^{-\gamma}(c_t^B)^{\gamma-1}, \qquad \frac{\partial U_t^B}{\partial c_t^B} = e^{-\rho t}(c_t^A)^{\gamma-1}(c_t^B)^{-\gamma}.$$

Since $\gamma > 1$, a rise in Agent A's consumption increases the marginal utility of Agent B's consumption, and vice-versa, which is consistent with the idea of "keeping up with the Joneses" (KUJ). Note that when $\gamma = 1$, corresponding to logarithmic utility, each agent's marginal utility is independent of his habit.[5] We solve for agents' optimal consumption plans in the next proposition.

Proposition 1. *The optimal consumption for Agents A and B are given by*

$$\hat{c}_t^A = e^{-\rho t} M_t^{1-p} / (\bar{y}^A \xi_t); \tag{15}$$

and

$$\hat{c}_t^B = e^{-\rho t} M_t^p / (\bar{y}^B \xi_t), \tag{16}$$

respectively, where $p = (\gamma - 1)/(2\gamma - 1)$, $\bar{y}^A = (y^A)^{1-p}(y^B)^p$, $\bar{y}^B = (y^B)^{1-p}(y^A)^p$, y^A and y^B are the Lagrange multipliers corresponding to Agents A and B's budget constraints, respectively.

Proposition 1 shows that Agent A and B's optimal consumption resembles that of a logarithmic agent with characteristics M_t^{1-p} and M_t^p, respectively. However, note that the characteristics are not probability beliefs since they are super-martingales. More specifically, we can

[5]This is consistent with the finding in Kadaras (2010) that logarithmic utility is numeraire invariant. That is, when an agent has a logarithmic utility, the optimal portfolio and consumption are independent of the numeraire used to denominate the wealth/consumption of the agent.

separate each characteristic into a probability belief and a discount factor as follows,

$$M_t^{1-p} = \eta_t \hat{M}_t^A, \qquad M_t^p = \eta_t \hat{M}_t^B,$$

where

$$\frac{d\hat{M}_t^A}{\hat{M}_t^A} = (1-p)\theta_t d\omega_t, \qquad \frac{d\hat{M}_t^B}{\hat{M}_t^B} = p\theta_t d\omega_t, \qquad \frac{d\eta_t}{\eta_t} = -\frac{1}{2}p(1-p)\theta_t^2 dt. \tag{17}$$

Thus, Agent A and B's optimal consumption in (15) and (16) can re-written as

$$\hat{c}_t^A = (e^{-\rho t}\eta_t)\hat{M}_t^A/(\bar{y}^A\xi_t), \qquad \hat{c}_t^B = (e^{-\rho t}\eta_t)\hat{M}_t^B/(\bar{y}^B\xi_t).$$

Therefore, interestingly, agents with relative consumption concerns and disagreement behave as logarithmic agents with a stochastic time-discount factor, given by $e^{-\rho t}\eta_t$, which is positively related to disagreement. Intuition is as follows. Suppose the current level of disagreement is large, agents perceive an improvement in their investment opportunity set, which increases their propensity to consume since the intertemporal elasticity of substitution, IES=$1/\gamma < 1$.

Moreover, since Agent i behaves as a logarithmic agent, his *wealth-to-consumption* (W/C) ratio is independent of the state price density process, given by

$$\Phi_t^i \equiv \frac{W_t^i}{\hat{c}_t^i} = \mathbb{E}\left[\int_t^\infty e^{-\rho(u-t)}\frac{\eta_u \hat{M}_u^i}{\eta_t \hat{M}_t^i} du \middle| \mathcal{F}_t\right], \qquad i = A, B. \tag{18}$$

Furthermore, we can obtain agents' share of aggregate consumption. Let $\nu_t = \hat{c}_t^A/Y_t$ be Agent A's share of aggregate consumption, and we can show that

$$\frac{d\nu_t}{\nu_t} = \mu_{\nu,t}dt + \sigma_{\nu,t}d\omega_t, \tag{19}$$

where

$$\sigma_{\nu,t} = (1-\nu_t)(1-2p)\theta_t, \qquad \mu_{\nu,t} = -(1-\nu_t)(1-2p)^2\theta_t^2\left(\frac{p}{1-2p} + \nu_t\right).$$

In the next section, we derive the equilibrium quantities as functions of θ_t and ν_t, which we use as state variables.

2.4. **Equilibrium.** We define and solve for the equilibrium quantities, which are all functions of the two state variables, θ_t and ν_t, their dynamics are given by (4) and (19), respectively.

Recall that the equilibrium is given by the triplet $\{r_t, \mu_{S,t}, \sigma_{S,t}\}$ characterizing stock and bond prices, S_t and B_t, optimal consumption and investment $\{\hat{c}_t^i, \hat{\pi}_t^i\}$ solving optimization (14) for Agent i with the market clearing conditions

$$\hat{c}_t^A + \hat{c}_t^B = Y_t, \;\; \hat{\pi}_t^A W_t^A + \hat{\pi}_t^B W_t^B = S_t, \;\; (1 - \hat{\pi}_t^A) W_t^A + (1 - \hat{\pi}_t^B) W_t^B = 0. \tag{20}$$

Since the equilibrium condition requires that $\hat{c}_t^A + \hat{c}_t^B = Y_t$, from (15) and (16) we obtain state price density process under the objective probability $\mathbb{P}$,

$$\xi_t = e^{-\rho t} \bar{M}_t / Y_t, \qquad \bar{M}_t = \eta_t \left(\nu_0 \hat{M}_t^A + (1 - \nu_0) \hat{M}_t^B \right) = \nu_0 M_t^{1-p} + (1 - \nu_0) M_t^p, \tag{21}$$

where the initial consumption share of Agent A is given by

$$\nu_0 = \frac{1/\bar{y}^A}{1/\bar{y}^A + 1/\bar{y}^B}.$$

Thus, we have

$$\frac{\xi_u}{\xi_t} = e^{-\rho(u-t)} \left(\frac{Y_t}{Y_u} \right) \left[(1 - \nu_t) \left(\frac{M_u}{M_t} \right)^p + \nu_t \left(\frac{M_u}{M_t} \right)^{1-p} \right], \tag{22}$$

Moreover, from (21) we can obtain explicit expressions for market prices of risk and the risk-free rate by matching the coefficients in (9), which are given in the following corollary.

Corollary 2. *In equilibrium, the market price of risk is given by*

$$\kappa_t = \sigma_Y - \left[p(1 - \nu_t) + (1 - p)\nu_t \right] \theta_t, \tag{23}$$

and the risk-free rate is given by

$$r_t = \rho + \mu_Y - \sigma_Y^2 + \sigma_Y \left[p(1 - \nu_t) + (1 - p)\nu_t \right] \theta_t + \frac{1}{2} p(1 - p) \theta_t^2. \tag{24}$$

Next, the price-dividend (P/D) ratio can be obtained by

$$\begin{aligned} \Psi_t \equiv \frac{S_t}{Y_t} &= \int_t^\infty \mathbb{E} \left[\left(\frac{\xi_u}{\xi_t} \right) \frac{Y_u}{Y_t} \middle| \mathcal{F}_t \right] du \\ &= \int_t^\infty e^{-\rho(u-t)} \left\{ (1 - \nu_t) \mathbb{E} \left[\left(\frac{M_u}{M_t} \right)^p \middle| \mathcal{F}_t \right] + \nu_t \mathbb{E} \left[\left(\frac{M_u}{M_t} \right)^{1-p} \middle| \mathcal{F}_t \right] \right\} du. \end{aligned} \tag{25}$$

We conjecture the P/D ratio to be a function of the two state variables, i.e., $\Psi_t = \Psi(\theta_t, \nu_t)$.[6] Thus, stock volatility can be obtained by

$$\sigma_{S,t} = \sigma_S(\theta_t, \nu_t) = \sigma_Y + \frac{1}{\Psi_t}\left(\frac{\partial \Psi_t}{\partial \theta_t}\sigma_\theta + \frac{\partial \Psi_t}{\partial \nu_t}\nu_t\sigma_{\nu,t}\right) \tag{26}$$

Furthermore, the price of a zero-coupon bond with maturity T can be obtained by

$$\begin{aligned} B_{t,T} &= B(\theta_t, \nu_t, \tau) = \mathbb{E}\left[\frac{\xi_T}{\xi_t}\middle|\mathcal{F}_t\right] \\ &= e^{-\rho\tau}\left\{(1-\nu_t)\mathbb{E}\left[\left(\frac{M_u}{M_t}\right)^p \frac{Y_t}{Y_T}\middle|\mathcal{F}_t\right] + \nu_t\mathbb{E}\left[\left(\frac{M_u}{M_t}\right)^{1-p}\frac{Y_t}{Y_T}\middle|\mathcal{F}_t\right]\right\}, \qquad \tau = T - t. \end{aligned} \tag{27}$$

Corollary 2 shows that even when the irrational agent, Agent A, is driven out of the market in the long-run as $\nu_t \to 0$ almost surely, the impact of sentiment, θ_t, does not diminish. In fact, if we set $\nu_t = 0$ in (23) and (24), the market price of risk and risk-free rate become

$$\kappa(\theta_t) \equiv \kappa(\theta_t, 0) = \sigma_Y - p\theta_t, \qquad r(\theta_t) \equiv r(\theta_t, 0) = \rho + \mu_Y - \sigma_Y^2 + \sigma_Y p\theta_t + \frac{1}{2}p(1-p)\theta_t^2,$$

the P/D ratio becomes

$$\Psi(\theta_t) \equiv \Psi(\theta_t, 0) = \int_t^\infty e^{-\rho(u-t)}\mathbb{E}\left[\left(\frac{M_u}{M_t}\right)^p \middle|\mathcal{F}_t\right] du,$$

and the stock volatility is given by

$$\sigma_S(\theta_t) \equiv \sigma_S(\theta_t, 0) = \sigma_Y + \frac{1}{\Psi_t}\frac{\partial \Psi_t}{\partial \theta_t}\delta_\theta.$$

Therefore, distinct from standard asset pricing results under heterogeneous beliefs, the equilibrium quantities still exhibits cyclical behavior due to the state variable θ_t in the long run when the irrational agent is driven out of the market. The reason is because the price impact of sentiment (θ_t) is coming through two channels. The first is the standard channel via the consumption share of the irrational agent, i.e., Agent A. Intuitively, when Agent A has a larger consumption share, his belief matters more in determining equilibrium prices. The second channel is via relative consumption concern. Although Agent B is completely rational, he incorporates Agent A's belief into his optimal consumption plan as shown in Proposition 1. Therefore, in the long

[6]We verify this conjecture later when we explicitly solve for the P/D ratio in closed-form.

run as $\nu_t \to 0$, second channel remains active while the first channel is shut down. In fact, as we shall see in Section 3, where we explicitly solve for the P/D ratio and stock volatility, and examine the behavior of the equilibrium quantities more closely, most of the cyclical variation comes from the state variable θ_t rather than ν_t. In other words, market behavior does not change significantly as Agent B gradually dominates the market over time. This is a key feature when KUJ preferences are embedded in a differences-of-opinion type general equilibrium model.

3. ANALYSIS OF EQUILIBRIUM

In this section, we explicitly solve for the P/D ratio, stock volatility, equity risk premium, and the term structure of interest rates. We then examine their behavior as functions of the two state variables, θ_t and ν_t.

Proposition 3. *Define the function*

$$H(\theta_t, u-t; \alpha, \beta) \equiv \mathbb{E}\left[\left(\frac{M_u}{M_t}\right)^{\alpha}\left(\frac{Y_u}{Y_t}\right)^{\beta}\bigg|\mathcal{F}_t\right]. \tag{28}$$

It can be shown that

$$H(\theta_t, u-t; \alpha, \beta) = e^{\lambda_0(u-t)+\lambda_1(u-t)\theta_t+\frac{1}{2}\lambda_2(u-t)\theta_t^2}, \tag{29}$$

where

$$\lambda_2(u-t) = \frac{c\left(1-e^{-2q(u-t)}\right)}{(q-b)e^{-2q(u-t)}+b+q}, \tag{30}$$

$$\lambda_1(u-t) = \frac{(\cosh(q(u-t))-1)(b\,k_1+c\,k_2)+k_1 q\sinh(q(u-t))}{q(b\sinh(q(u-t))+q\cosh(q(u-t)))}, \tag{31}$$

$$\begin{aligned}\lambda_0(u-t) = {} & \left(\beta\mu_Y+\frac{1}{2}\beta(\beta-1)\sigma_Y^2\right)(u-t) \\ & +\frac{1}{2}\left((u-t)(b+q)-\ln\left((b+q)e^{2q(u-t)}-b+q\right)+\ln(2q)\right) \\ & +\int_t^u k_2\lambda_1(z)+\frac{a}{2}\lambda_1^2(z)dz;\end{aligned} \tag{32}$$

$$k_1=\alpha\beta\sigma_D,\ \ k_2=\beta\delta_\theta\sigma_D+\zeta_\theta\bar{\theta},\ \ a=\delta_\theta^2,\ \ b=\zeta_\theta-\alpha\delta_\theta,\ \ c=-\alpha(1-\alpha),\ \ q=\sqrt{b^2-ac}.$$

Hence, the P/D ratio can be characterized by

$$\Psi(\theta_t,\nu_t) = \nu_t\Phi^A(\theta_t)+(1-\nu_t)\Phi^B(\theta_t), \tag{33}$$

where Agents A and B's W/C ratios are given by

$$\Phi^A(\theta_t) = \int_t^{\infty} e^{-\rho(u-t)} H(\theta_t, u-t; 1-p, 0) du, \qquad \Phi^B(\theta_t) = \int_t^{\infty} e^{-\rho(u-t)} H(\theta_t, u-t; p, 0) du.$$

The stock volatility is given by

$$\sigma_S(\theta_t, \nu_t) = \sigma_Y + \frac{1}{\Psi(\theta_t, \nu_t)} \left\{ \nu_t \sigma_\nu(\nu_t) \left[\Phi^A(\theta_t) - \Phi^B(\theta_t) \right] + \delta_\theta \left[\nu_t (\Phi^A)'(\theta_t) + (1-\nu_t)(\Phi^B)'(\theta_t) \right] \right\}. \tag{34}$$

Moreover, the equilibrium price of a zero-coupon bond with maturity $T > t$ and face value of 1 is given by

$$B(\theta_t, \nu_t, \tau) = e^{-\rho(T-t)} \left[\nu_t H(\theta_t, \tau; 1-p, -1) + (1-\nu_t) H(\theta_t, \tau; p, -1) \right], \ \ \tau = T - t. \tag{35}$$

3.1. **Stock Prices.** In Figure 1, we plot the set of equilibrium quantities, i.e., P/D ratio $\Psi(\theta_t, \nu_t)$ in Panel (a), stock volatility $\sigma_S(\theta_t, \nu_t)$ in Panel (b), and equity risk premium $\kappa(\theta_t, \nu_t)\sigma_S(\theta_t, \nu_t)$ in Panel (c) as functions of the state variables, θ_t and ν_t. Since the sentiment level follows an OU process, in the long run, θ_t has a *stationary distribution*, which is normal with mean $\bar{\theta}$ and variance $\delta_\theta^2/(2\zeta_\theta)$. Therefore, we use

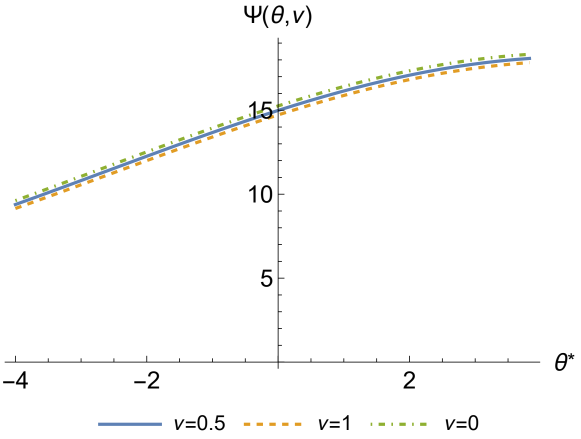
$$\theta_t^* \equiv \frac{\theta_t - \bar{\theta}}{\delta_\theta / \sqrt{2\zeta_\theta}} \tag{36}$$

to measure the number of standard deviations from the long-run mean, $\theta_t^* > 0$ characterizes good states of the economy whereas $\theta_t^* < 0$ characterizes bad states of the economy. We have the following observations from Figure 1.

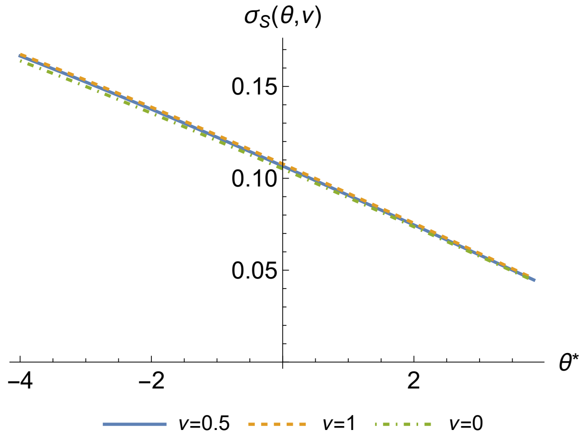
First, Panel (a) shows that the P/D ratio, $\Psi(\theta_t, \nu_t)$, is increasing and concave in θ_t^*. On the other hand, $\Psi(\theta_t, \nu_t)$ shows very little variation with respect to ν_t, which suggests that most of the cyclical variation comes from changes in sentiment, rather than changes in the consumption share. Therefore, market behavior is not significantly affected even as irrational agents' consumption share decreases towards zero in the long run.

To understand the behavior of the P/D ratio, it is helpful to first consider the limiting case where sentiment level is constant. Assume $\zeta_\theta \to \infty$, and $\theta_t = \bar{\theta}$, from Proposition 3, the C/W ratios or *propensities to consume* are give by

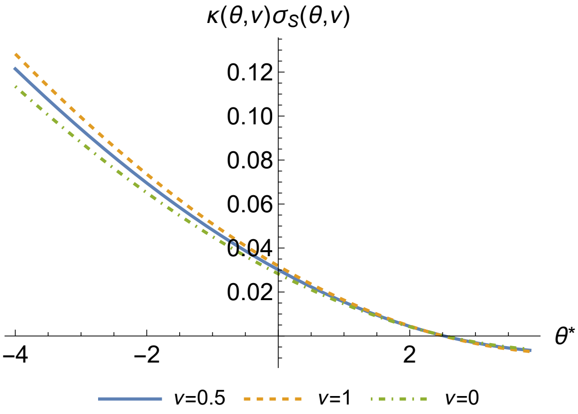
$$\frac{1}{\Phi^i(\theta)} = \rho + \frac{1}{2} p(1-p)\bar{\theta}^2, \qquad i = A, B. \tag{37}$$



(a) Price-Dividend Ratio



(b) Stock Volatility



(c) Equity Risk Premium

FIGURE 1. Equilibrium quantities as a function of the state variables θ^* and ν under KUJ preference and stochastic sentiment with $\gamma = 10$, $\rho = 0.02$, $\bar{\theta} = -0.5$, $\zeta_\theta = 0.4$, $\delta_\theta = 0.2$, $\mu_Y = 0.018$, $\sigma_Y = 0.032$.

Therefore, KUJ agents have a higher (constant) propensity of consumption than standard logarithmic agents without KUJ preferences whose propensity to consume is equal to ρ. Next, we explain why the propensity is *higher* than the time discount rate, ρ, under KUJ and *constant*, even though consumption shares are stochastic. Using the first order condition of Agent B's optimization problem, we can write the state price density process as

$$\xi_t = (y^B)^{-1} e^{-\rho t} \left(\frac{c_t^B}{c_t^A} \right)^{1-\gamma} (c_t^B)^{-1}.$$ [7]

Thus, Agent B's W/C ratio can be expressed as

$$\Phi^B(\theta) = \int_t^\infty e^{-\rho(u-t)} \mathbb{E}\left[\left(\frac{c_u^B / c_t^B}{c_u^A / c_t^A} \right)^{1-\gamma} \middle| \mathcal{F}_t \right] du.$$

[7]We can make the same argument using Agent A's optimization problem.

Intuitively, when determining the optimal propensity of consumption, Agent B anticipates the future (expected) growth in consumption *relative* to that of Agent A, which only depends on $\bar{\theta}^2$ (disagreement), but not the consumption share ν_t. This is because relative consumption growth is independent of the state price density, ξ_t, and only depends on the ratio of agents' probability beliefs, i.e.,

$$\frac{c_u^B/c_t^B}{c_u^A/c_t^A} = \left(\frac{M_u}{M_t}\right)^{1-2p}.$$

Moreover, disagreement increases propensity to consume. This is because, since $\gamma > 1$, when Agent B anticipates higher future growth in relative consumption due to disagreement, i.e., better investment opportunities, he increases current consumption due to the wealth effect. Note that when there is complete agreement, i.e., $\bar{\theta} = 0$, relative consumption growth becomes one, thus propensity to consume equals ρ for both agents. Furthermore, we observe from (37) that Agent B's W/C ratio is constant, which is due to the fact that sentiment level, $\theta_t = \bar{\theta}$, is constant.

In contrast, in a standard DO general equilibrium model with CRRA agents maximizing expected life-time utility of absolute consumption, the W/C ratios depend on both disagreement and consumption share, because absolute consumption growth depends on the state price density, ξ_t. For example, the rational agents would expected higher (resp. lower) consumption growth when the irrational agent's consumption share is larger (resp. small), since it provides more (resp. less) profitable speculative trading opportunities due to under/overpricing as measured by ξ_t.

Next, we examine the general case when level of sentiment, θ_t, follows an OU process. Same as in the special case of constant sentiment, the W/C ratio, as shown in Proposition 3 equation (33), depends only on θ_t, and not on ν_t. Moreover, to show why W/C ratios are increasing and concave functions of θ_t, we note that, as explained previously, the KUJ agents due to their relative consumption concerns have a greater propensity to consume when disagreement is large, which lowers the W/C ratio. Intuitively, the W/C ratio behaves similar to a (concave) quadratic function of θ_t, i.e., it decreases and agents consume a larger fraction of wealth when θ_t^2 becomes larger. So, why does Panel (a) show that the W/C and P/D ratios are always increasing? This is because sentiment is assumed to be on average pessimistic, i.e., $\bar{\theta} < 0$. Therefore, disagreement is not zero when $\theta_t = \bar{\theta}$ (or equivalently $\theta_t^* = 0$), which means that the stationary/maximum points of the W/C ratios occur at a larger (positive) value of θ^*. In summary, KUJ, time-varying sentiment, and average pessimism are the key ingredients for the increasing and concave W/C and P/D ratios.

Mathematically, by taking the first derivative of Agent B's W/C ratio, we obtain

$$(\Phi^B)'(\theta_t) = \int_t^\infty e^{-\rho(u-t)}[\lambda_1(u-t) + \lambda_2(u-t)\theta_t]H(\theta_t, u-t; p, 0)du.$$

We also note that, $\lambda_2(u-t) < 0$ in (30), and $\lambda_1(u-t)$ in (31) has the opposite sign as $\bar{\theta}$ since $\beta = 0$ (thus, $k_1 = 0$), and $\lambda_1(u-t)$ increases as $\bar{\theta}$ become more negative. Furthermore, the concavity condition,

$$(\Phi^B)''(\theta_t) = \int_t^\infty e^{-\rho(u-t)} \left\{[\lambda_1(u-t) + \lambda_2(u-t)\theta_t]^2 + \lambda_2(u-t)\right\} H(\theta_t, u-t; p, 0)du < 0,$$

is likely to be met if θ_t is not too large.

Moreover, we observe in Panel (a) that the P/D ratio is almost constant with respect to Agent A's consumption share, ν_t. The reason is because, due to KUJ, although agents' beliefs differ, their W/C ratios can share very similar cyclical behavior. More specifically, note that the W/C ratios for Agents A and B are given by

$$\Phi^A(\theta_t) = \int_t^\infty e^{-\rho(u-t)} \left(\frac{M_u}{M_t}\right)^{1-p} du \qquad \text{and} \qquad \Phi^B(\theta_t) = \int_t^\infty e^{-\rho(u-t)} \left(\frac{M_u}{M_t}\right)^{p} du,$$

respectively. In Figure 1, we set relative risk aversion $\gamma = 10$, thus, according to Proposition (1), p is close to a half, which make the W/C ratios almost identical. Intuitively, due to KUJ, agents' propensity to consume only depend on their probability beliefs, and not on the state price density. Furthermore, under KUJ, each agent uses a weighted average of his own and other agents' beliefs to make portfolio/consumption decisions. Thus, when the weighting, p, is close to a half, agents make very similar portfolio/consumption decisions despite their disagreement about the expected aggregate endowment growth rate.

Second, we discuss the cyclical behavior of stock volatility in Figure 1 Panel (b). We observe that stock returns exhibit both *excess* and *countercyclical* volatility. This is a direct consequence of P/D ratio's cyclical behavior in Panel (a). From (34), excess volatility can come from two channels. First channel is when the optimistic agent has a higher W/C ratio than the pessimistic agent. Second channel is when W/C ratios are more sensitive to changes in sentiment. The first channel is shut down since W/C ratios share very similar cyclical behavior. Therefore, effect mainly comes from the second channel. Intuitively, stock volatility is excessive because W/C and P/D ratios are increasing in sentiment level, i.e., stock has a higher valuation during economic boom when sentiment is optimistic than during economic downturn when sentiment is pessimistic. Also, stock volatility is countercyclical because W/C and P/D ratios are concave in sentiment, i.e., they are more (resp. less) sensitive to change in sentiment when sentiment is pessimistic (resp. optimistic).

Third, Panel (c) shows that the model is able to produce a sizeable equity premium, with $\kappa(\bar{\theta})\sigma_S(\bar{\theta}) \approx 2.8\%$ p.a. at $\nu_t = 0$, even though aggregate consumption volatility is low ($\sigma_Y = 0.032$). In comparison, in the absence of Agent A, if Agent B is the only (representative) CRRA agent in the economy, equity premium is given by $\gamma\sigma_Y^2$. On the other hand, if Agents A and B are both present, have KUJ preferences, and sentiment is constant ($\theta_t = \bar{\theta}$), equity premium in the limit Agent A vanishes, i.e., $\nu_t \to 0$, is equal to $(-p\bar{\theta} + \sigma_Y)\sigma_Y$. Based on the parameter values provided for Figure 1, equity premiums are 1.0% and 0.9% p.a. for the first and second cases, respectively. Therefore, we need all three ingredients, namely, KUJ preference, average pessimism, and time-varying sentiment, for the model to generate a large positive equity premium. Intuitively, the large equity premium is partially due to average pessimism, which help to boost up the market price of risk, and partially due to excess stock volatility.

Furthermore, we can compute the unconditional equity premium at the stead-state when $\nu_t \to 0$, i.e., $\mathbb{E}[\kappa(\theta)\sigma_S(\theta)]$ given that $\theta \sim \mathcal{N}(\bar{\theta}, \delta_\theta/\sqrt{2\zeta_\theta})]$ in the long-run. Figure 2 shows the probability distribution of the equity premium in the long-run. The unconditional mean of the distribution is approximately 3.0%, which is 0.2% higher than $\kappa(\bar{\theta})\sigma_S(\bar{\theta})$. This is because the market price of risk and stock volatility are both countercyclical, which means that equity premium is an asymmetric function of sentiment, therefore distribution in Figure 2 has a positive skewness, which increases the unconditional mean.

In summary, Figure 1 shows that the model is able to simultaneously match several key stylized facts about aggregate equity markets, including a procyclical and concave P/D ratio, excess and countercyclical stock volatility, and a large countercyclical equity premium. Admittedly, the matching of stylized facts is qualitative rather than quantitative, which is expected due to the parsimonious nature of the model. On the other hand, the advantage of having a parsimonious model is that, because it has less moving pieces, it allows us to understand the equilibrium outcomes and thus derive important economic insights more easily.

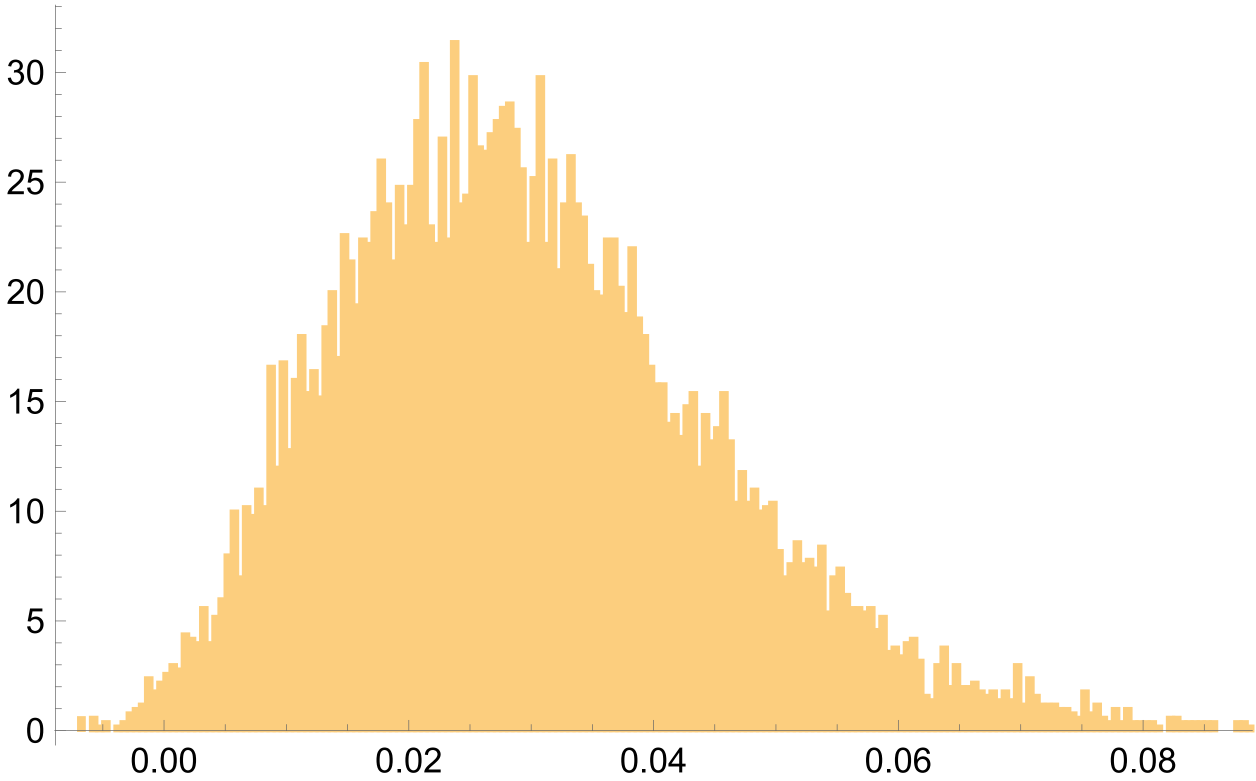


FIGURE 2. Probability distribution function of the equity premium, $\kappa(\theta)\sigma_S(\theta)$, we assume $\nu_t \to 0$, and sentiment has reached its stationary distribution, i.e., $\theta^* \sim \mathcal{N}(0,1)$. The mean of the distribution is approximately 3%. Parameter values are $\gamma = 10$, $\rho = 0.02$, $\bar{\theta} = -0.5$, $\zeta_\theta = 0.4$, $\delta_\theta = 0.2$, $\mu_Y = 0.018$, $\sigma_Y = 0.032$.

3.2. **Bond Prices.** Next, we investigate the effect of KUJ and time-varying sentiment on the term-structure of interest rates. Since the cyclical behavior of equilibrium quantities mainly comes from sentiment, θ_t, rather than consumption share, ν_t. For simplicity, we assume $\nu_t \to 0$, as Agent A vanishes in the long-run. Note that, this does not mean market equilibrium converge to that of a representative-CRRA-agent economy. In contrary, as we will see, sentiment has a significant effect on the yield curve.

We compute the zero-coupon bond prices according to (35) in Proposition 3 by setting $\nu_t = 0$, and plot the yield-to-maturity (YTM),

$$y(\theta_t, \tau) \equiv -\ln\left[B(\theta_t, 0, \tau)\right],$$

as a function of sentiment, θ_t, and years to maturity, $\tau = T - t$. Similar to Figure 1, we normalize sentiment by using θ_t^* as defined in 36, which in the long-run follows a standard normal distribution. In Figure 3, we plot $y(\theta^*, \tau)$ against τ for different levels of θ^*, we also plot the unconditional mean, $\bar{y}(\tau)$, by taking expectation over $\theta^* \sim \mathcal{N}(0,1)$.

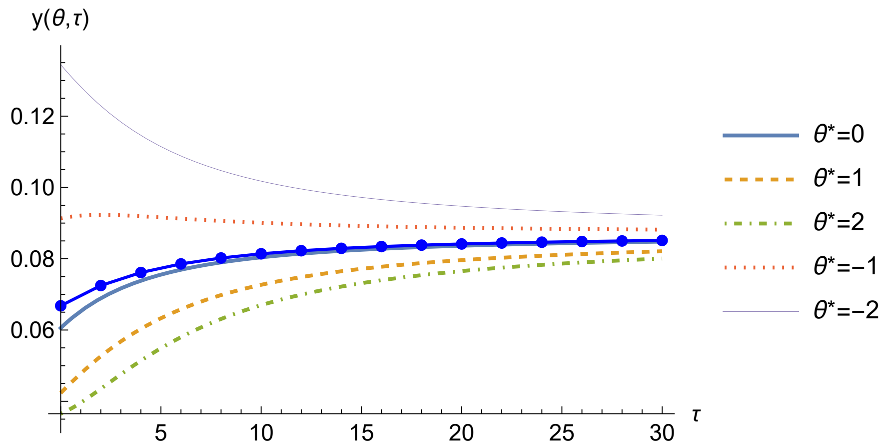


FIGURE 3. YTM p.a. of zero-coupon bonds, $y(\theta^*, \tau)$ as a function of the (normalized) state variable θ^* and time to maturity, we assume $\nu_t \to 0$. The blue-circled line represents the unconditional mean of YTM, $\bar{y}(\tau)$, over all possible θ^*, assuming it has reached its stationary distribution, i.e., $\theta^* \sim \mathcal{N}(0, 1)$. Parameter values are $\gamma = 10$, $\rho = 0.02$, $\bar{\theta} = -0.5$, $\zeta_\theta = 0.4$, $\delta_\theta = 0.2$, $\mu_Y = 0.018$, $\sigma_Y = 0.032$.

We observe that the yield curve tends to be upward sloping in good and neutral states of the market, and flat or downward sloping in bad states of the market. Moreover, the unconditional mean appears to be slightly upward sloping, with a term spread of $\mathbb{E}[y(\theta^*, 30) - r_t] = 1.8\%$ p.a.

So, how do these results stack up against empirical evidence? Ang, Bekaert and Wei (2008) examines the term structure of real interest rates in the U.S. and identifies four regimes based on level and volatility. They find that when the short rate (instantaneous risk-free rate) is relatively high, the yield curve is downward sloping. In contrast, when short rate is relatively low, the yield curve is upward sloping at the short end, and flat or downward sloping at the long end. Moreover, as their *Claim 1* states, "Unconditionally, the term structure of real rates assumes a fairly flat shape around 1.3%, with a slight hump, peaking at a 1-year maturity". They also claim that "there is no significant real term spread", which is mainly due to the hump-shape nature of the (unconditional) yield curve. Therefore, on the positive note, the model results in Figure 3 are consistent with the negative relationship between the level of short rate and slope of the yield curve. On the negative note, the model is unable to match the hump-shape of the unconditional yield curve, it also cannot match the low level of empirically observed real rates.

3.3. **Optimal Portfolios.** Next, we want to examine the joint impact of KUJ and sentiment on agents' optimal portfolios. Since Agent i's wealth, $W_t^i = \Phi_t^i \hat{c}_t^i$, we can obtain that

$$\frac{dW_t^i}{W_t^i} = \frac{d\Phi_t^i}{\Phi_t^i} + \frac{d\hat{c}_t^i}{\hat{c}_t^i} + \frac{d\Phi_t^i}{\Phi_t^i}\frac{d\hat{c}_t^i}{\hat{c}_t^i}, \qquad i = A, B,$$

where $\hat{c}_t^A = \nu_t Y_t$, $\hat{c}_t^B = (1-\nu_t)Y_t$, and the consumption share dynamics are given in (19). Therefore, we can derive the following expressions for agents' financial wealth processes,

$$\frac{dW_t^A}{W_t^A} = [\cdot]dt + \left[\sigma_Y + (\Phi^A)'(\theta_t)\frac{\delta_\theta}{\Phi_t^A} + (1-\nu_t)(1-2p)\theta_t\right] d\omega_t,$$

and

$$\frac{dW_t^B}{W_t^B} = [\cdot]dt + \left[\sigma_Y + (\Phi^B)'(\theta_t)\frac{\delta_\theta}{\Phi_t^B} - \nu_t(1-2p)\theta_t\right] d\omega_t.$$

Hence, agents' optimal portfolios are given by

$$\hat{\pi}^A(\theta_t,\nu_t) = \sigma_S(\theta_t,\nu_t)^{-1}\left[\sigma_Y + (\Phi^A)'(\theta_t)\frac{\delta_\theta}{\Phi^A(\theta_t)} + (1-\nu_t)(1-2p)\theta_t\right] \tag{38}$$

and

$$\hat{\pi}^B(\theta_t,\nu_t) = \sigma_S(\theta_t,\nu_t)^{-1}\left[\sigma_Y + (\Phi^B)'(\theta_t)\frac{\delta_\theta}{\Phi^B(\theta_t)} - \nu_t(1-2p)\theta_t\right]. \tag{39}$$

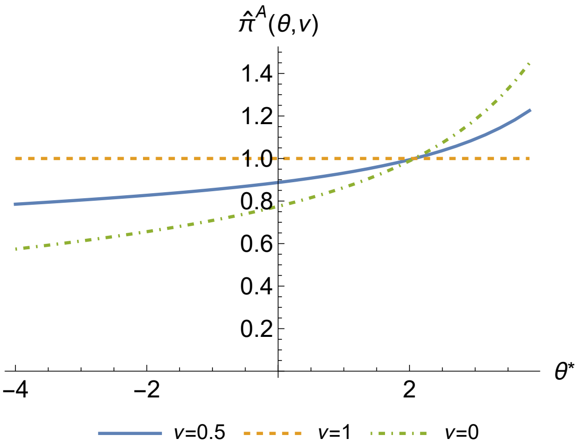
In Figure we plot agents' optimal portfolios, $\{\hat{\pi}^i(\theta_t^*,\nu_t)\}_{i\in\{A,B\}}$, for $\theta_t^* \in [-4,4]$ and $\nu_t \in \{0, 0.5, 1\}$. For comparison, we also compute and plot the myopic (optimal mean-variance) portfolios, which are given by

$$\tilde{\pi}^A(\theta_t,\nu_t) \equiv \frac{(1-p)\mu_{S,t}^A + p\mu_{S,t}^B - r_t}{\sigma_{S,t}^2} = \frac{\kappa(\theta_t,\nu_t) + (1-p)\theta_t}{\sigma_S(\theta_t,\nu_t)},$$

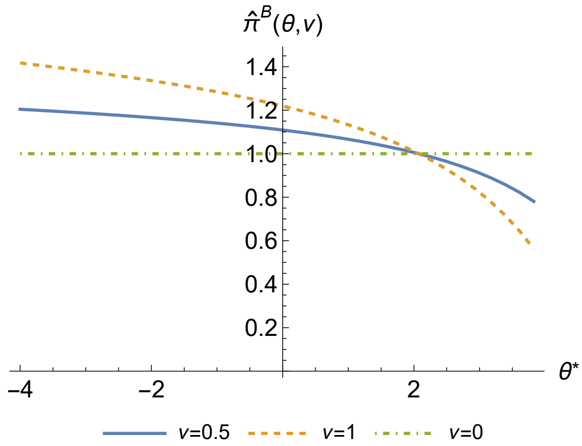
and

$$\tilde{\pi}^B(\theta_t,\nu_t) \equiv \frac{p\mu_{S,t}^A + (1-p)\mu_{S,t}^B - r_t}{\sigma_{S,t}^2} = \frac{\kappa(\theta_t,\nu_t) - p\theta_t}{\sigma_S(\theta_t,\nu_t)}.$$

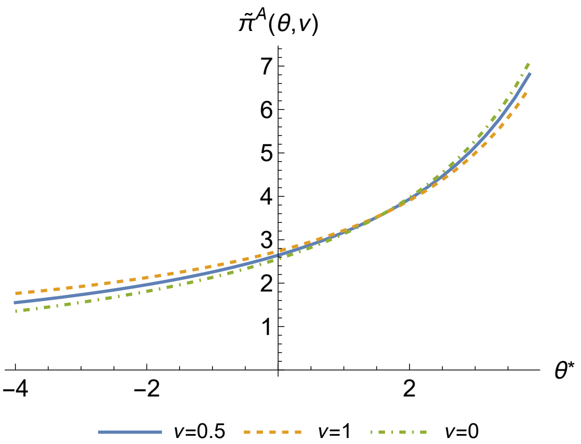
First, we note that, although agents' optimal consumption plans closely resemble those under logarithmic preferences, their optimal portfolios, $\hat{\pi}^A(\theta,\nu)$ and $\hat{\pi}^B(\theta,\nu)$, are *non-myopic*. By comparing Panels (a) and (b) with Panels (c) and (d), we observe that $\hat{\pi}^A(\theta,\nu)$ and $\hat{\pi}^B(\theta,\nu)$ significantly differ from the optimal mean-variance (myopic) portfolios, $\tilde{\pi}^A(\theta^*,\nu)$ and $\tilde{\pi}^B(\theta^*,\nu)$. Second, the optimal portfolios show significantly less variations with respect to sentiment than the myopic portfolios. Third, the optimal portfolios are more sensitive to changes in consumption share than myopic portfolio, which is due to the market clearing condition. For example, when $\nu_t = 0$, Agent B must invest his total wealth in the stock, therefore $\hat{\pi}^B(\theta_t, 0) = 1$ for $\theta^* \in (-\infty,\infty)$.



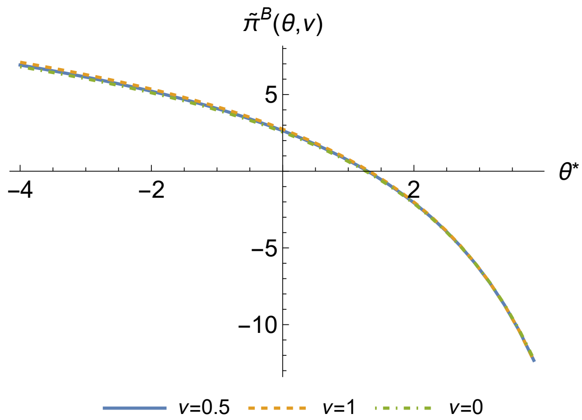
(a) Agent A's optimal portfolio



(b) Agent B's optimal portfolio



(c) Agent A's myopic portfolio



(d) Agent B's myopic portfolio

FIGURE 4. Agents' optimal portfolios, $\hat{\pi}^A(\theta,\nu)$ and $\hat{\pi}^B(\theta,\nu)$ in Panels (a) and (b), and myopic (optimal mean-variance) portfolios, $\tilde{\pi}^A(\theta,\nu)$ and $\tilde{\pi}^B(\theta,\nu)$ in Panels (c) and (d), as functions of the state variables θ^* and ν, with $\gamma = 10$, $\rho = 0.02$, $\bar{\theta} = -0.5$, $\zeta_\theta = 0.4$, $\delta_\theta = 0.2$, $\mu_Y = 0.018$, $\sigma_Y = 0.032$.

4. GENERAL KUJ PREFERENCES

In this section, we generalize the KUJ preferences such that the weights placed on beliefs, p and $1-p$, and the effective risk aversion are determined by two different parameters, similar to Galí (1994). More specifically, Agents A and B's instantaneous utilities in (H2) are now given by

$$U_t^A(c_t^A, c_t^B) \equiv \frac{e^{-\rho t}}{1-\gamma} \left(c_t^A\right)^{1-\gamma} \left(c_t^B\right)^{\varphi\gamma}$$
$$U_t^B(c_t^A, c_t^B) \equiv \frac{e^{-\rho t}}{1-\gamma} \left(c_t^B\right)^{1-\gamma} \left(c_t^A\right)^{\varphi\gamma}. \tag{40}$$

We assume the parameter $\varphi \in (0,1)$, so that each Agents A and B's marginal utilities,

$$\frac{\partial U_t^A}{\partial c_t^A} = e^{-\rho t}(c_t^A)^{-\gamma}(c_t^B)^{\varphi\gamma}, \qquad \frac{\partial U_t^B}{\partial c_t^B} = e^{-\rho t}(c_t^B)^{-\gamma}(c_t^A)^{\varphi\gamma},$$

are increasing in c_t^B and c_t^A, respectively.[8]

4.1. **Optimal Consumption.** We solve for agents' optimal consumption plans in the next proposition.

Proposition 4. *The optimal consumption plans for Agents A and B are given by*

$$\hat{c}_t^A = \left[e^{-\rho t} M_t^{1-p}/(\bar{y}^A \xi_t)\right]^{1/\tilde{\gamma}}, \tag{41}$$

and

$$\hat{c}_t^B = \left[e^{-\rho t} M_t^{p}/(\bar{y}^B \xi_t)\right]^{1/\tilde{\gamma}}, \tag{42}$$

respectively, where $\tilde{\gamma} = \gamma(1-\varphi)$, $p = \varphi/(1+\varphi)$, $\bar{y}^A = (y^A)^{1-p}(y^B)^p$, $\bar{y}^B = (y^B)^{1-p}(y^A)^p$, y^A and y^B are the Lagrange multipliers corresponding to Agents A and B's budget constraints, respectively.

Proposition 4 shows that Agents A and B's optimal consumption resemble those of CRRA agents with a relative risk aversion, $\tilde{\gamma} = \gamma(1-\varphi)$, and characteristics, M_t^{1-p} and M_t^p, where the weights, p and $1-p$, are determined solely by the parameter φ. Note that the baseline relative consumption model can be recovered by setting $\tilde{\gamma} = 1$, or equivalently, $\varphi = (\gamma-1)/\gamma$.

Next, from (41) and (42), we can derive the dynamics of Agent A' consumption share, $\nu_t = \hat{c}_t^A/(\hat{c}_t^A + \hat{c}_t^B)$ as

$$\frac{d\nu_t}{\nu_t} = \mu_{\nu,t}dt + \sigma_{\nu,t}d\omega_t, \tag{43}$$

where

$$\sigma_{\nu,t} = z(1-\nu_t)\theta_t, \qquad \mu_{\nu,t} = (1-\nu_t)\theta_t^2\left[z^2(1-\nu_t) - \frac{1}{2}z(1+z)\right], \qquad z = \frac{1-2p}{\tilde{\gamma}}.$$

4.2. **Equilibrium.** By the market clearing conditions in (20), we can derive the SPD process,

$$\xi_t = e^{-\rho t}\bar{M}_t Y_t^{-\tilde{\gamma}}, \qquad \bar{M}_t = \left[(1-\nu_0)M_t^{p/\tilde{\gamma}} + \nu_0 M_t^{(1-p)/\tilde{\gamma}}\right]^{\tilde{\gamma}},$$

[8]Note that, compared with the baseline model, now we no longer require the relative risk aversion $\gamma > 1$.

where $\nu_0 = (\bar{y}^A)^{-1/\tilde{\gamma}}/((\bar{y}^A)^{-1/\tilde{\gamma}} + (\bar{y}^B)^{-1/\tilde{\gamma}})$ is Agent A's initial consumption share. Thus, we can obtain that

$$\frac{\xi_u}{\xi_t} = e^{-\rho(u-t)} \left(\frac{Y_u}{Y_t}\right)^{-\tilde{\gamma}} \left[(1-\nu_t)\left(\frac{M_u}{M_t}\right)^{p/\tilde{\gamma}} + \nu_t \left(\frac{M_u}{M_t}\right)^{(1-p)/\tilde{\gamma}}\right]^{\tilde{\gamma}}. \tag{44}$$

Therefore, from (44) we can compute the P/D ratio,

$$\Psi(\theta_t, \nu_t) = \mathbb{E}\left\{\int_t^\infty e^{-\rho(u-t)} \left(\frac{Y_u}{Y_t}\right)^{1-\tilde{\gamma}} \left[(1-\nu_t)\left(\frac{M_u}{M_t}\right)^{p/\tilde{\gamma}} + \nu_t \left(\frac{M_u}{M_t}\right)^{(1-p)/\tilde{\gamma}}\right]^{\tilde{\gamma}} du \middle| \mathcal{F}_t\right\}.$$

The P/D ratio does not have a closed-form solution unless we assume $\tilde{\gamma}$ is a natural number. In order to compare with result of the baseline model ($\tilde{\gamma} = 1$), we will take the limit as $\nu_t \to 0$, which is the long-run outcome in the economy as Agent A gradually loses consumption share to Agent B over time. Also, similar to the baseline case, when the weight, p, is close to a half, the state variable, ν_t, has only minimal effect on the P/D ratio, much of the cyclical variation comes from sentiment, θ_t. In the limit as $\nu_t \to 0$, based on Proposition 3, the P/D ratio and stock volatility can be obtained by

$$\Psi(\theta_t) = \int_t^\infty e^{-\rho(u-t)} H(\theta_t, u-t, p, 1-\tilde{\gamma}) du, \tag{45}$$

and

$$\sigma_S(\theta_t) = \sigma_S(\theta_t, \nu_t) = \sigma_Y + \frac{1}{\Psi(\theta_t)} \Psi'(\theta_t) \delta_\theta. \tag{46}$$

Moreover, we can also compute the market price risk,

$$\kappa(\theta_t) = \tilde{\gamma}\sigma_Y - p\theta_t. \tag{47}$$

and the zero-coupon bond prices,

$$B(\theta_t, \tau) = e^{-\rho\tau} H(\theta_t, \tau; p, -\tilde{\gamma}). \tag{48}$$

In the following, we compare equilibrium for different values of $\tilde{\gamma}$, we fix p to be the same as in the baseline model by setting the KUJ parameter, $\varphi = 0.9$.

4.3. **Comparison with the baseline model.** In Figure 5, we examine the effect of risk aversion $\tilde{\gamma}$ on market equilibrium, when risk aversion deviates from $\tilde{\gamma} = 1$. Panels (a) and (b) show that a lower relative risk aversion increases the overall levels of P/D ratio and stock volatility.

Intuitively, more risk tolerant agents are willing to pay more for the risky stock. Moreover, the P/D ratio also becomes more sensitive to changes in sentiment with lower risk aversion, because sentiment has a greater effect on agents' propensity to consume. However, the effect on stock volatility is relatively small.

Panels (c) shows that risk aversion has little or no effect on the equity risk premium. This is due to the fact that although a lower risk aversion increases stock volatility, it reduces market price of risk, see (47), thus the two effects offset each other.

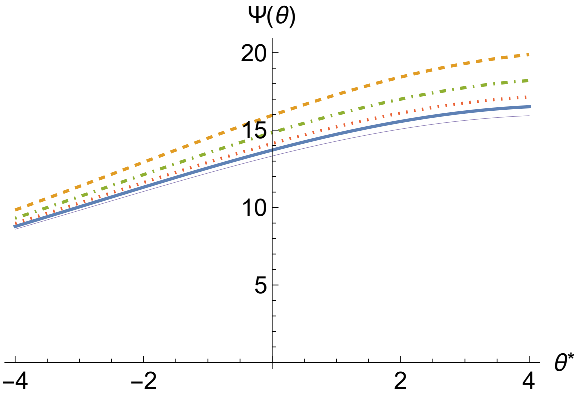
Lastly, Panel (d) shows the effect of $\tilde{\gamma}$ on the unconditional mean of the yield curve. We observe that a lower risk aversion leads to parallel downward shift of the yield curve, however, it is not low enough to match the average real rates in the U.S., which is between 1 to 2%. Note that, when $\nu_t \to 0$, the risk-free rate is given by

$$r(\theta_t) = \underbrace{\rho + \tilde{\gamma}\mu_Y - \frac{1}{2}\tilde{\gamma}(1+\tilde{\gamma})\sigma_Y^2}_{\text{standard risk-free rate}} + \underbrace{\frac{1}{2}p(1-p)\theta_t^2 + \sigma_Y p\theta_t}_{\text{KUJ component}}.$$

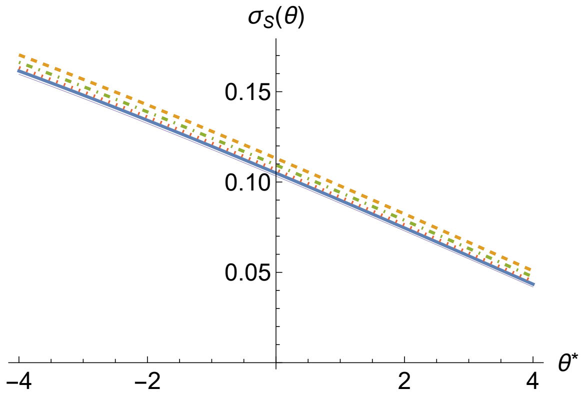
Therefore, even though a lower risk aversion reduces the standard risk-free rate, i.e., the risk-free rate in a CRRA representative agent economy, it does not diminish the KUJ component. The KUJ component is driven by a propensity-to-consume term, i.e., $1/2p(1-p)\theta_t^2$, average pessimism term, $\sigma_Y p\theta_t$. The first tends to dominate the second term, as a result, KUJ component tends to increase the risk-free rate.

5. Conclusion

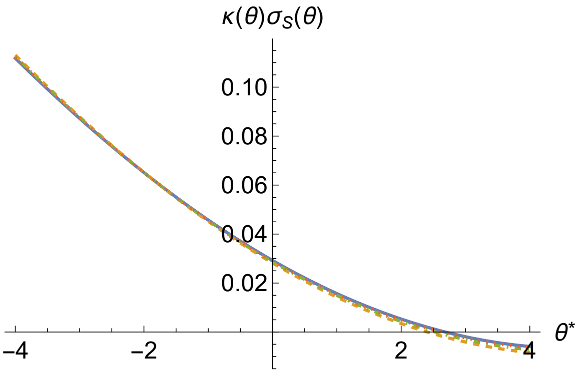
In this paper, we incorporate KUJ preference and stochastic sentiment in a two-agent dynamic equilibrium model. We show the model is able to produce a procyclical and concave P/D ratio, a countercyclical and excess stock volatility, and a sizeable and countercyclical equity premium. Although consumption share and sentiment are both state variable, we find that much of the cyclical variation in equilibrium quantities is due to sentiment. A key insight is that, due to KUJ preference, even though the irrational agent driven by sentiment does not survive in the long run, market equilibrium can behave very differently to that of a standard rational expectation representative-agent economy.



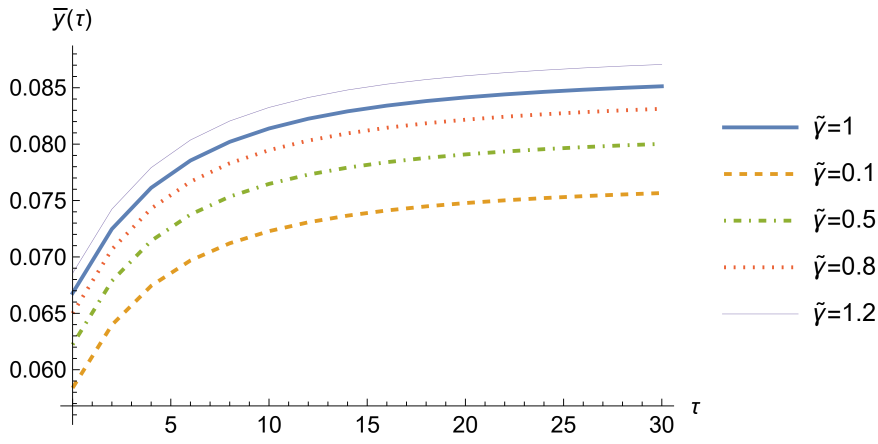
(a) Price-Dividend Ratio



(b) Stock Volatility



(c) Equity Premium



(d) Unconditional Yield Curve

FIGURE 5. Panels (a)-(c) show equilibrium quantities as a function of sentiment, θ^*, in the limit as $\nu_t \to 0$, for different relative risk aversion, $\tilde{\gamma}$. Panel (d) shows the unconditional average yield curve for different $\tilde{\gamma}$. Other parameters are $\varphi = 0.9$, $\rho = 0.02$, $\bar{\theta} = -0.5$, $\zeta_\theta = 0.4$, $\delta_\theta = 0.2$, $\mu_Y = 0.018$, $\sigma_Y = 0.032$.

APPENDIX A. PROOFS

A.1. **Proof of Proposition 1.** Given the optimization problem in (14) and the budget constraint in (12), by the first order condition, Agents A and B's optimal consumption plans satisfy

$$\hat{c}_t^A = \left(\frac{y^A e^{\rho t}\xi_t}{M_t}\right)^{-\frac{1}{\gamma}} (\hat{c}_t^B)^{\frac{\gamma-1}{\gamma}},$$
$$\hat{c}_t^B = (y^B e^{\rho t}\xi_t)^{-\frac{1}{\gamma}} (\hat{c}_t^A)^{\frac{\gamma-1}{\gamma}},$$

where y^A and y^B are the Lagrange multipliers corresponding to Agents A and B's budget constraints, respectively. Thus, the log optimal consumption plans can be written as the following set of simultaneous equations,

$$\begin{pmatrix} 1 & (1-\gamma)/\gamma \\ (1-\gamma)/\gamma & 1 \end{pmatrix} \begin{pmatrix} \ln(\hat{c}_t^A) \\ \ln(\hat{c}_t^B) \end{pmatrix} = \frac{1}{\gamma} \begin{pmatrix} -\rho t - \ln(\xi_t) + \ln(M_t/y^A) \\ -\rho t - \ln(\xi_t) - \ln(y^B) \end{pmatrix},$$

which has the solution

$$\begin{pmatrix} \ln(\hat{c}_t^A) \\ \ln(\hat{c}_t^B) \end{pmatrix} = \frac{1}{\gamma} \begin{pmatrix} 1 & (1-\gamma)/\gamma \\ (1-\gamma)/\gamma & 1 \end{pmatrix}^{-1} \begin{pmatrix} -\rho t - \ln(\xi_t) + \ln(M_t/y^A) \\ -\rho t - \ln(\xi_t) - \ln(y^B) \end{pmatrix}.$$

Hence, let $p = (\gamma-1)/(2\gamma-1)$, $\bar{y}^A = (y^B)^p (y^A)^{1-p}$, and $\bar{y}^B = (y^A)^p (y^B)^{1-p}$, the optimal consumption plans of Agent A and Agent B are given by (15) and (16), respectively. □

A.2. **Proof of Corollary 2.** The market clearing condition requires the consumption market to clear, that is $c_t^{\mathcal{R}} + c_t^{\mathcal{S}} = Y_t$. From (21) we have

$$\frac{d\xi_t}{\xi_t} = -\left\{\rho + \mu_Y - \sigma_Y^2 + \sigma_Y[p(1-\nu_t) + (1-p)\nu_t]\theta_t + \frac{1}{2}p(1-p)\theta^2\right\} dt$$
$$- \{\sigma_Y - [p(1-\nu_t) + (1-p)\nu_t]\theta_t\} d\omega_t, \qquad p = \frac{\gamma-1}{2\gamma-1}.$$

By matching the drift and diffusion coefficients with (9) we obtain the expressions for the market prices of risk in (23) and the risk-free rate in (24). □

A.3. **Proof of Proposition 3.** We recall the dividend, subjective belief, and sentiment processes under the objective probability measure $\mathbb{P}$,

$$\frac{dY_t}{Y_t} = \mu_Y dt + \sigma_Y d\omega_t,$$
$$\frac{dM_t}{M_t} = \theta_t d\omega_t,$$
$$d\theta_t = -\zeta_\theta(\theta_t - \bar{\theta})dt + \delta_\theta d\omega_t.$$

Next, we want to compute the function

$$F(M_t, Y_t, \theta_t, \tau; \alpha, \beta) \equiv \mathbb{E}\left[M_u^\alpha Y_u^\beta | \mathcal{F}_t\right], \qquad \tau = u - t.$$

We conjecture that

$$F(M_t, Y_t, \theta_t, \tau; \alpha, \beta) = M_t^\alpha Y_t^\beta H(\theta_t, \tau; \alpha, \beta).$$

For notational convenience, we drop the time t subscript for the rest of the proof. Using the Feyman-Kac formula, $F(M, Y, \theta, \tau; \alpha, \beta)$ satisfies the following partial differential equation,

$$\begin{aligned}\frac{\partial F}{\partial \tau} &= \frac{\partial F}{\partial \theta}\left(-\zeta_\theta(\theta-\bar{\theta})\right) + \frac{\partial F}{\partial Y}\mu_Y Y + \frac{1}{2}\frac{\partial^2 F}{\partial \theta^2}\delta_\theta^2 + \frac{1}{2}\frac{\partial^2 F}{\partial M^2}\theta^2 M^2 + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2}\sigma_Y^2 Y^2 \\ &\quad + \frac{\partial^2 F}{\partial M \partial Y}(\theta M)(\sigma_Y Y) + \frac{\partial^2 F}{\partial \theta \partial Y}\delta_\theta(\sigma_Y Y) + \frac{\partial^2 F}{\partial \theta \partial M}\delta_\theta \theta M,\end{aligned}$$

where

$$\begin{aligned}&\frac{\partial F}{\partial \tau} = M^\alpha Y^\beta \frac{\partial H}{\partial \tau},\ \frac{\partial F}{\partial \theta} = M^\alpha Y^\beta \frac{\partial H}{\partial \theta},\ \frac{\partial F}{\partial Y} = M^\alpha \beta Y^{\beta-1}H,\ \frac{\partial^2 F}{\partial \theta^2} = M^\alpha Y^\beta \frac{\partial^2 H}{\partial \theta^2}, \\ &\frac{\partial^2 F}{\partial M^2} = \alpha(\alpha-1)M^{\alpha-2}Y^\beta H,\ \frac{\partial^2 F}{\partial Y^2} = \beta(\beta-1)M^\alpha Y^{\beta-2}H,\ \frac{\partial^2 F}{\partial \theta \partial M} = \alpha M^{\alpha-1}Y^\beta \frac{\partial H}{\partial \theta}, \\ &\frac{\partial^2 F}{\partial M \partial Y} = \alpha\beta M^{\alpha-1}Y^{\beta-1}H,\ \frac{\partial^2 F}{\partial \theta \partial Y} = \beta M^\alpha Y^{\beta-1}\frac{\partial H}{\partial \theta},\end{aligned}$$

which leads to the following partial differential equation for $H(\theta, \tau; \alpha, \beta)$,

$$\begin{aligned}\frac{\partial H}{\partial \tau} &= \frac{1}{2}\delta_\theta^2 \frac{\partial^2 H}{\partial \theta^2} + \frac{\partial H}{\partial \theta}\left[(\alpha\delta_\theta - \zeta_\theta)\theta + \left(\beta\delta_\theta\sigma_Y + \zeta_\theta\bar{\theta}\right)\right] \\ &\quad + \left[\left(\beta\mu_Y + \frac{1}{2}\beta(\beta-1)\sigma_Y^2\right) + \alpha\beta\sigma_Y\theta + \frac{1}{2}\alpha(\alpha-1)\theta^2\right]H.\end{aligned}$$

To solve for $H(\theta, \tau; \alpha, \beta)$, we conjecture the following solution,

$$H(\theta, \tau; \alpha, \beta) = \exp\left\{\lambda_0(\tau) + \lambda_1(\tau)\theta + \frac{1}{2}\lambda_2(\tau)\theta^2\right\},$$

which has the following partial derivatives,

$$\begin{aligned}\frac{\partial H}{\partial \tau} &= \left(\lambda_0'(\tau) + \lambda_1'(\tau)\theta + \frac{1}{2}\lambda_2'(\tau)\theta^2\right)H, \qquad \frac{\partial H}{\partial \theta} = (\lambda_1(\tau) + \lambda_2(\tau)\theta)H \\ \frac{\partial^2 H}{\partial \theta^2} &= \left[(\lambda_1(\tau)^2 + \lambda_2(\tau)) + 2\lambda_1(\tau)\lambda_2(\tau)\theta + \lambda_2(\tau)^2\theta^2\right]H.\end{aligned}$$

Therefore, we have from the partial differential equation for $H(\theta, \tau; \alpha, \beta)$ that

$$\begin{aligned}\lambda_0'(\tau) + \lambda_1'(\tau)\theta + \frac{1}{2}\lambda_2'(\tau)\theta^2 &= \frac{1}{2}\delta_\theta^2\left[(\lambda_1(\tau)^2 + \lambda_2(\tau)) + 2\lambda_1(\tau)\lambda_2(\tau)\theta + \lambda_2(\tau)^2\theta^2\right] \\ &\quad + \left[(\alpha\delta_\theta - \zeta_\theta)\theta + \left(\beta\delta_\theta\sigma_Y + \zeta_\theta\bar{\theta}\right)\right](\lambda_1(\tau) + \lambda_2(\tau)\theta) \\ &\quad + \left[\left(\beta\mu_Y + \frac{1}{2}\beta(\beta-1)\sigma_Y^2\right) + \alpha\beta\sigma_Y\theta + \frac{1}{2}\alpha(\alpha-1)\theta^2\right].\end{aligned}$$

Thus, by matching the coefficients for θ^2, θ and the remainder term, we obtain the following system of ordinary differential equations,

$$\begin{aligned}\lambda_2'(\tau) &= a\lambda_2^2(\tau) - 2b\lambda_2(\tau) + c, \\ \lambda_1'(\tau) &= -[b - a\lambda_2(\tau)]\lambda_1(\tau) + k_1 + k_2\lambda_2(\tau), \\ \lambda_0'(\tau) &= \left(\beta\mu_Y + \frac{1}{2}\beta(\beta-1)\sigma_Y^2\right) + \frac{1}{2}a(\lambda_1^2(\tau) + \lambda_2(\tau)) + k_2\lambda_1(\tau),\end{aligned}$$

where $a = \delta_\theta^2$, $b = \zeta_\theta - \alpha\delta_\theta$, $c = \alpha(\alpha-1)$, $k_1 = \alpha\beta\sigma_Y$, $k_2 = \beta\delta_\theta\sigma_Y + \zeta_\theta\bar{\theta}$.

We first solve for $\lambda_2(\tau)$, which is given by

$$\lambda_2(\tau) = \frac{c(1-e^{-2q\tau})}{q+b+(q-b)e^{-2q\tau}}, \qquad q = \sqrt{b^2-ac}.$$

Next we solve for $\lambda_1(\tau)$. Let $\lambda_1(\tau) = u(\tau)v(\tau)$, so that $\lambda_1'(\tau) = u(\tau)v'(\tau) + u'(\tau)v(\tau)$, and thus we have

$$u(\tau)v'(\tau) + u'(\tau)v(\tau) = -m(\tau)\lambda_1(\tau) + n(\tau) = -m(\tau)[u(\tau)v(\tau)] + n(\tau),$$

where $m(\tau) = b - a\lambda_2(\tau)$ and $n(\tau) = k_1 + k_2\lambda_2(\tau)$. Hence, we have

$$[u'(\tau) + m(\tau)u(\tau)]v(\tau) + u(\tau)v'(\tau) = n(\tau).$$

To match the LHS and the RHS of the above equation we require

$$u'(\tau) + m(\tau)u(\tau) = 0, \qquad \text{and} \qquad u(\tau)v'(\tau) = n(\tau).$$

Therefore,

$$\int_0^\tau \frac{du(z)}{u(z)} = -\int_0^\tau m(z)dz = q\tau + \ln\left[\frac{2q}{(b+q)e^{2q\tau} - b + q}\right],$$

hence

$$u(\tau) = \frac{2qe^{q\tau}}{(b+q)e^{2q\tau} - b + q}.$$

Also, we have

$$v(\tau) = \int_0^\tau \frac{n(z)}{u(z)}dz = k_1\int_0^\tau \frac{1}{u(z)}dz + k_2\int_0^\tau \frac{\lambda_2(z)}{u(z)}dz = k_1\mathcal{D}_1(\tau) + k_2\mathcal{D}_2(\tau),$$

where

$$\mathcal{D}_1(\tau) = \frac{b(\cosh(q\tau)-1) + q\sinh(q\tau)}{q^2}, \qquad \mathcal{D}_2(\tau) = \frac{c(\cosh(q\tau)-1)}{q^2}.$$

Putting it altogether and simplify we obtain

$$\lambda_1(\tau) = u(\tau)v(\tau) = \frac{(\cosh(q\tau)-1)(b\,k_1 + c\,k_2) + k_1 q\sinh(q\tau)}{q(b\sinh(q\tau) + q\cosh(q\tau))}.$$

Lastly, for $\lambda_0(\tau)$ we have

$$\begin{aligned}\lambda_0(\tau) &= \left(\beta\mu_Y + \frac{1}{2}\beta(\beta-1)\sigma_Y^2\right)\tau + \frac{a}{2}\int_0^\tau \lambda_2(z)dz + \int_0^\tau k_2\lambda_1(z) + \frac{a}{2}\lambda_1^2(z)dz \\ &= \left(\beta\mu_Y + \frac{1}{2}\beta(\beta-1)\sigma_Y^2\right)\tau + \frac{1}{2}\left(\tau(b+q) - \ln\left((b+q)e^{2q\tau} - b + q\right) + \ln(2q)\right) \\ &+ \int_0^\tau k_2\lambda_1(z) + \frac{a}{2}\lambda_1^2(z)dz.\end{aligned}$$

□

A.4. **Proof of Proposition 4.** Given the general KUJ preferences in (40), Agents A and B's optimization problems are given by

$$\max_{c_t^A} \mathbb{E}\left[\int_0^\infty M_t \frac{e^{-\rho t}}{1-\gamma}\left(c_t^A\right)^{1-\gamma}\left(c_t^B\right)^{\varphi\gamma}\middle|\mathcal{F}_t\right]$$

and

$$\max_{c_t^B} \mathbb{E}\left[\int_0^\infty \frac{e^{-\rho t}}{1-\gamma}\left(c_t^A\right)^{1-\gamma}\left(c_t^B\right)^{\varphi\gamma}\middle|\mathcal{F}_t\right],$$

respectively, subjected to budget constraint in (12). Then, the first order conditions are given by

$$\hat{c}_t^A = \left(\frac{y^A e^{\rho t}\xi_t}{M_t}\right)^{-\frac{1}{\gamma}}\left(\hat{c}_t^B\right)^{\varphi},$$

$$\hat{c}_t^B = \left(y^B e^{\rho t}\xi_t\right)^{-\frac{1}{\gamma}}\left(\hat{c}_t^A\right)^{\varphi},$$

where y^A and y^B are the Lagrange multipliers corresponding to Agents A and B's budget constraints, respectively. The rest of the proof is similar to Proposition 1. □

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