Pricing European Options on Deferred Annuities

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Abstract

This paper considers the pricing of European call options written on pure endowment and deferred life annuity contracts, also known as guaranteed annuity options. These contracts provide a guaranteed value at the maturity of the option. The contract valuation is dependent on the stochastic interest rate and mortality processes. We assume single-factor stochastic square-root processes for both the interest rate and mortality intensity, with mortality being a time-inhomogeneous process. We then derive the pricing partial differential equation (PDE) and the corresponding transition density PDE for options written on the pure endowment and deferred annuity contracts. The general solution of the pricing PDE is derived as a function of the transition density function. We solve the transition density PDE by first transforming it to a system of characteristic PDEs using Laplace transform techniques and then applying the method of characteristics. Once an explicit expression for the density function is found, we then use sparse grid quadrature techniques to generate European call option prices on the pure endowment and deferred annuity contracts. This approach can easily be generalised to other contracts which are driven by similar stochastic processes presented in this paper. We test the sensitivity of the option prices by varying independent parameters in our model. As option maturity increases, the corresponding option prices significantly increase. The effect of mispricing the guaranteed annuity value is analysed, as is the benefit of replacing the whole-life annuity with a term annuity to remove volatility of the old age population.

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1. Introduction

In this paper we derive techniques for pricing deferred annuity and pure endowment options that can be used for managing longevity risk in life insurance and annuity providers' portfolios. Analytical techniques for deriving the joint interest and mortality rate probability density function are drawn from Chiarella and Ziveyi [10], where the dynamics of the underlying security evolve under the influence of stochastic volatility. Given our time-inhomogeneous mortality process, we use sparse grid quadrature methods to solve option prices under the risk-neutral measure.

Insurance companies and annuity providers are increasingly exposed to the risk of ever improving mortality trends across all ages, with a greater portion of survivors living beyond 100 years (Carriere [8], Currie et al. [12] and CMI(2005) among others). Such mortality improvements, coupled with the unavailability of suitable hedging instruments, pose significant challenges to annuity providers seizing from the risk of longer periods of annuity payments than initially expected. At present mortality risk is non-tradable (Blake et al. [6]) and there is no market to hedge these risks other than reinsurance. Blake and Burrows [5] highlight that annuity providers have been trying to hedge mortality risk using costly means such as the construction of hedged portfolios of long-term bonds (with no mortality risk). The hedging of both the interest rate and longevity risks is important to annuity providers, and the inability to purchase long-term bonds also hinders the annuity providers ability to hedge the interest rate risk, although there are a greater range of interest rate hedging methods and instruments than those for longevity risk. In this paper we focus on longevity risk. Due to the long-term nature of life annuity contracts, the development of an active market where longevity risk can easily be priced, traded and hedged, requires more advanced pricing and hedging methods. A number of securities that can make up this market have already been proposed in the literature and these include longevity bonds, mortality derivatives, securitised products among others (Bauer [2] and Blake et al. [6]).

Blake et al. [6], Bauer [1], and Bauer [2] demonstrate that, if mortality risk can be traded through securities such as longevity bonds and swaps, then the techniques developed in financial markets can be adapted and implemented for mortality risk. A number of papers have also developed models for pricing guaranteed annuity options. Milevsky and Promislow [23] develop algorithms for valuing mortality contingent claims by taking the underlying securities as defaultable coupon paying bonds with the time of death as a stopping time. Boyle and Hardy [7] use the numeraire approach to value options written on guaranteed annuities. They detail the challenges experienced in the UK where long-dated and low guaranteed rates were provided relative to the high prevailing interest rates in the 1970s. Interest rates fell significantly in the 1990s, leading to sharp increases in the value of guaranteed contracts, and this had a significant impact on annuity providers' profitability.

A significant number of empirical studies have been presented showing that mortality trends are generally improving and the future development of mortality rates are uncertain and require stochastic models. Proposed stochastic models include Milevsky and Promislow [23], Dahl et al. [13], Biffis [4]. A number of stochastic mortality models have been motivated by interest rate term-structure modelling literature (Cox et al. [11], Dahl et al. [13], Litterman and Scheinkman [20]) as well as stochastic volatility models such as that proposed in Heston [18].

The main aim of this paper is to devise a novel numerical approach for the pricing and hedging of deferred mortality contingent claims with special emphasis on pure endowment options and deferred immediate annuity options. We start off by devising techniques for pricing deferred insurance contracts, which are the underlying assets. Analytical solutions can be derived for pure endowment contracts using the forward measure approach; however, this is not possible for deferred immediate annuities where analytical approximation techniques have mostly been used as in Singleton and Umantsev [25] when valuing options on coupon paying bonds. Having devised models for the underlying securities, we then present techniques for valuing European style options written on these contracts.

We assume that the interest rate dynamics is driven by a single-factor stochastic square-root process, while the time-inhomogeneous mortality dynamics is a one-factor version of the model proposed in Biffis [4]. The long-term mean reversion level of the mortality process is a time-varying function following the Weibull mortality law. This provides a reference mortality level for each age in a cohort. The model definition guarantees positive mortality rates. Although the mortality intensity can fall below our reference rate, careful selection of the parameters limits this occurring.

We use hedging arguments and Ito's Lemma to derive a partial differential equation (PDE) for options written on the deferred insurance contracts. We also present the backward Kolmogorov PDE satisfied by the two stochastic processes under consideration. We present the general solution of the pricing PDE by using Duhamel's principle. The solution is a function of the joint probability density function, which is also the solution of the Kolmogorov PDE. We solve the transition density PDE with the aid of Laplace transform techniques, thereby obtaining an explicit expression for the joint transition density function. Using the explicit density function, we then use sparse grid quadrature methods to price options on deferred insurance contracts.

While Monte Carlo simulation techniques are effective for option pricing, the analytic solution of the joint density function provides valuable insight into longevity risk. From the analytic solution one can easily derive expressions for the hedge ratios, such as the option price sensitivity with respect to the interest rate and mortality processes. The framework can be extended to multiple interest and mortality risk factors, with only a small increase in computation requirements.

The remainder of this paper is organised as follows: Section 2 presents the modelling framework for the interest rate and mortality rate processes. We then provide the option pricing framework in Section 3. It is in this section where we derive the option pricing PDE and the corresponding backward Kolmogorov PDE for the density function. With the general solution of the pricing PDE presented, we then outline a step-by-step approach for solving the transition density PDE using Laplace transform techniques. The explicit expressions for the deferred pure endowment and deferred annuity contracts together with their corresponding option prices are presented in Section 4. All numerical results are presented in Section 5. Section 6 concludes the paper. Where appropriate, the derivations and proofs are included to the appendices.

2. Modelling Framework

The intensity-based modelling of credit risky securities has a number of parallels with mortality modelling (Lando [19] and Biffis [4]). We are interested in the first stopping time, τ, of the intensity process $\mu(t;x)$, for a person aged x at time zero. Starting with a filtered probability space (Ω, $\mathcal{F}$, $\mathbb{F}$, $\mathbb{P}$), where $\mathbb{P}$ is the real-world probability measure. The information at time-t is given by $\mathbb{F} = \mathbb{G} \vee \mathbb{H}$. The sub-filtration $\mathbb{G}$ contains all financial and actuarial information except the actual time of death.

There are two $\mathbb{G}$-adapted short-rate processes, $r(t)$ and $\mu(t;x)$, representing the instantaneous interest and mortality processes respectively. The sub-filtration $\mathbb{H}$ is the σ-algebra with death information. Let $N(t) := \mathbb{1}_{\tau \leq t}$ be an indicator function, if the compensator $A(t) = \int_0^t \mu(s;x)ds$ is a predictable process of $N(t)$ then $dM(t) = dN(t) - dA(t)$ is a $\mathbb{P}$-martingale, where $dA(t) = \mu(t;x)dt$.

There also exists another measure where $dM(t)$ is a $\mathbb{Q}$-martingale, under which the compensator becomes $dA(t) = \mu^Q(t;x)dt$, with $\mu^Q(t;x) = (1+\phi(t))\mu(t;x)$ and $\phi(t) \geq -1$. The predictable process $\phi(t)$ represents a market price of idiosyncratic, or individual, mortality risk of the insurance contract. The idiosyncratic risk is important to the annuity provider, but for the market this risk can be diversified, and we assume $\phi(t) = 0$ in this paper. In the absence of market data, we make the usual simplifying assumption that under the $\mathbb{Q}$-measure interest and mortality rates are independent; we do not assume a change in the economic conditions to affect the risk premium in longevity securities.

Proposition 2.1. *In the absence of arbitrage opportunities there exists an equivalent martingale measure $\mathbb{Q}$ where $C(t,T,r,\mu;x)$ is the t-value of an option contract with a pay-off function, $P(T,r,\mu;x)$, at time-T. The payment of $P(T,r,\mu;x)$, which is a $\mathbb{G}$-adapted process, is conditional on survival to the start of the period T, otherwise the value is zero. The time t-value of the option can be represented as*

$$\begin{aligned} C(t,T,r,\mu;x) =& \mathbb{E}^Q\left[e^{-\int_t^T r(u)du} P(T,r,\mu;x)\mathbb{1}_{\tau>T}|\mathcal{F}_t\right] \\ =& 1_{\tau>t}\mathbb{E}^Q\left[e^{-\int_t^T [r(u)+\mu(u;x)]du} P(T,r,\mu;x)|\mathcal{G}_t\right] \end{aligned} \tag{2.1}$$

Proof: The law of iterated expectations can be used to show this; see Bielecki and Rutkowski

[3] or Biffis [4] for detail. □

In our framework, time-T is always the option maturity age. If $P(T, r, \mu; x) \equiv 1$ then our contract resembles a credit risky zero coupon bond. In actuarial terms, this is a pure endowment contract written at time-t that receives 1 at time T if the holder is still alive. If $P(T, r, \mu; x)$ is the value of a stream of payments starting at T, conditional on survival, then we are pricing a deferred immediate annuity. The contract value, $P(T, r, \mu; x)$, can also take the form of an option pay-off. In this scenario, the strike price, K, represents a guaranteed value at time-T on an endowment or annuity contract.

One approach to solving equation (2.1) is to use a forward measure approach. If we use $P(T, r, \mu; x)$ as numeraire, we can rewrite (2.1) as

$$C(t, T, r, \mu; x) = 1_{\tau > t} P(t, r, \mu; x) \mathbb{E}^{Q^P} \left[P(T, r, \mu; x) | \mathcal{F}_t \right] \tag{2.2}$$

where $\mathbb{E}^{Q^P}$ is our new forward probability measure. When $P(T, r, \mu; x)$ is the value of a general payment stream, closed form solutions to this problem do not exist. One solution is to use Monte Carlo simulation or numerical approximation methods. These approximation methods are derived by Singleton and Umantsev [25] for coupon bearing bond options in a general affine framework, while Schrager [24] proposes a numerical approximation method for pricing guaranteed annuity options in a Gaussian affine framework. For a guaranteed annuity option, this requires using the deferred annuity as a numeraire as outlined in Boyle and Hardy [7].

In comparison, our approach derives a closed-form joint density function for interest and mortality rates. This method uses Laplace transform techniques allowing us to directly solve equation (2.1) under the risk neutral measure. With a closed-form density function we use numerical integration to price various types of insurance contracts. We use sparse grids quadrature techniques to evaluate double integral expressions.

2.1. Interest Rate Model

In our filtered probability space (Ω, $\mathcal{F}$, $\mathbb{F}$, $\mathbb{P}$), defined above, the $\mathbb{G}$-adapted short-rate process r represents the instantaneous interest rate on risk-free securities. There exists an equivalent martingale measure $\mathbb{Q}$ where the arbitrage-free price of a risk-free zero coupon bond is given by

$$B(t, T) = \mathbb{E}^Q \left[e^{-\int_t^T r(u) du} | \mathcal{G}_t \right], \tag{2.3}$$

where T is the maturity of the bond and $r(u)$ is the short-rate process.

We model the interest rate as a single factor affine process (Duffie and Kan [16] and Dai and Singleton [15]). The short-rate is modelled as a Cox-Ingersoll-Ross (CIR) process with the risk-neutral dynamics defined as

$$dr(t) = \kappa_r(\theta_r - r(t))dt + \sigma_r\sqrt{r(t)}dW^Q, \tag{2.4}$$

where κ_r is the speed of mean reversion of r with θ_r being the corresponding long-run mean. The volatility of the process is denoted by σ_r. For the process (2.4) to be guaranteed positive, Cox et al. [11] show that the parameters must satisfy the following condition $\frac{2\kappa_r}{\theta_r} > \sigma_r^2$.

The explicit solution of equation (2.4) can be represented as

$$B(t,T) = e^{\alpha_r(t,T)-\beta_r(t,T)r(t)} \tag{2.5}$$

where α_r and β_r are expressions of the form

$$\begin{aligned}
\alpha_r(t,T) =& \frac{2\kappa_r\theta_r}{\sigma_r^2}\texttt{log}\left[\frac{2\gamma_r e^{\frac{(\kappa_r+\gamma_r)(T-t)}{2}}}{(\gamma_r+\kappa_r)(e^{\gamma_r(T-t)}-1)+2\gamma_r}\right]\\
\beta_r(t,T) =& \frac{2(e^{\gamma_r(T-t)}-1)}{(\gamma_r+\kappa_r)(e^{\gamma_r(T-t)}-1)+2\gamma_r}\\
\gamma_r =& \sqrt{\kappa_r^2+2\sigma_r^2}
\end{aligned}$$

Our adoption of the CIR model for interest rates is based on option pricing papers such as Grzelak and Oosterlee [17] and Chiarella and Kang [9]. The CIR interest rate model has a closed-form solution to the Bond pricing equation and it is compatible with the Laplace transform techniques used in the paper. There is extensive research on the class of affine interest rate models, and Dai and Singleton [15] provide a general classification of these models. Multiple factor affine and stochastic volatility interest rate models can be implemented in this framework. This model assumption ensure that it is feasible to derive explicit integral expressions for both contract values and associated option prices. Other numerical techniques for handling more complex interest rate models include simulation techniques such as Monte Carlo. Our aim is to propose and present a numerical technique that can be efficiently implemented with the affine model.

2.2. Mortality Model

Mortality is modelled as an affine process. Luciano and Vigna [22] show that a time-homogeneous model like equation (2.4) does not capture mortality rates very well. Another approach is to use time-inhomogeneous models as proposed in Biffis [4] and Dahl and Moller [14]. In this paper we use a 1-factor version of the Biffis [4] 2-factor square-root diffusion model. The mortality intensity process is modelled as

$$d\mu(t;x) = \kappa_\mu(m(t) - \mu(t;x))dt + \sigma_\mu\sqrt{\mu(t;x)}dW(t) \tag{2.6}$$

where

$$\sigma_\mu = \Sigma_\mu \sqrt{m(t)}$$

By using similar arguments as in Cox et al. [11], the mortality process is positive definite if parameters are chosen such that $2\kappa_\mu > \Sigma_\mu^2$. Biffis [4] chooses $m(t)$ to be a deterministic function given by

$$m(x+t) = \frac{c}{\theta^c}(x+t)^{c-1} \tag{2.7}$$

which is the Weibull mortality law; we adopt this functional form as it fits well to observed mortality trends.

Using the same filtered probability space $(\Omega,\ \mathcal{F},\ \mathbb{F},\ \mathbb{P})$, the $\mathbb{G}$-adapted process μ represents the instantaneous mortality rate. There exists an equivalent martingale measure $\mathbb{Q}$ such that the survival probability can be represented as

$$S(t,T;x) = \mathbb{E}^Q\left[e^{-\int_t^T \mu(u)du}|\mathcal{G}_t\right] \tag{2.8}$$

The general solution of equation (2.8) can be shown to be

$$S(t,T;x) = e^{\alpha_\mu(t,T;x)-\beta_\mu(t,T;x)\mu(x,t)} \tag{2.9}$$

where $\alpha_\mu(t,T;x)$ and $\beta_\mu(t,T;x)$ are the solutions to the following ordinary differential equations

$$\frac{\partial}{\partial t}\beta_\mu(t,T;x) = 1 - \kappa_\mu(t;x)\beta_\mu(t,T;x) - \frac{1}{2}\left(\Sigma_\mu(x,t)\right)^2\left(\beta_\mu(t,T;x)\right)^2 m(t), \tag{2.10}$$

$$\frac{\partial}{\partial t}\alpha_\mu(t,T;x) = -\kappa_\mu(t;x)m(t)(x,t)\beta_\mu(t,T;x), \tag{2.11}$$

with $\beta_\mu(T,T;x) = 0$ and $\alpha_\mu(T,T;x) = 0$; see Duffie and Kan [16]. A closed form solution of $\beta_\mu(t,T;x)$ and $\alpha_\mu(t,T;x)$ does not exist for this model but can be solved via numerical techniques such as the Runge-Kutta methods.

3. The Option Pricing Framework

By using hedging arguments and Ito's Lemma the time-t value of an option, $C(t,T,r,\mu;x)$, written on an insurance contract with value $P(T,r,\mu;x)$ at time-T, is the solution to the partial differential equation (PDE)

$$\frac{\partial C}{\partial t}(t,T,r,\mu;x) = \mathcal{D}C(t,T,r,\mu;x) - r_x C(t,T,r,\mu;x), \tag{3.1}$$

which is solved subject to the initial condition $C(T,T,r,\mu;x) = 0$, where

$$\mathcal{D} = \kappa_r(\theta_r - r)\frac{\partial}{\partial r} + \kappa_\mu(m(t)-\mu)\frac{\partial}{\partial \mu} + \frac{1}{2}\sigma_r^2 r\frac{\partial^2}{\partial r^2} + \frac{1}{2}\sigma_\mu^2\mu\frac{\partial^2}{\partial \mu^2}, \tag{3.2}$$

is the Dynkin operator and $0 < r, \mu < \infty$, $r_x = r + \mu$ and $t < T$. Detailed discussion on deriving PDEs like (3.1) can be found in Chiarella and Ziveyi [10]. Equation (3.1) is solved subject to a boundary condition, which is the payoff of the option at the time it matures. Also associated with the system of SDEs (2.4) and (2.6) is the transition density function, which we denote here as $G(\psi, r, \mu; r_T, \mu_T, x)$ with $\psi = T - t$ being the time-to-maturity. Chiarella and Ziveyi (2011) show that SDEs like (2.4) and (2.6) satisfy the associated Kolmogorov backward PDE

$$\frac{\partial G}{\partial \psi}(\psi, r, \mu; r_0, \mu_0, x) = \mathcal{L}G(\psi, r, \mu; r_0, \mu_0, x). \tag{3.3}$$

Equation (3.3) is solved subject to the boundary condition

$$G(0, r, \mu; r_0, \mu_0, x) = \delta(r - r_0)\delta(\mu - \mu_0) \tag{3.4}$$

where $\delta(\cdot)$ is the Dirac Delta function, r_0 and μ_0 are the terminal values of the interest and mortality rates respectively.

3.1. General Solution of the Option Pricing Problem

By using the law of iterated expectation, as detailed in Biffis [4], and the independence assumption between r and μ, equation (2.1) can be re-expressed as

$$C(t, T, r, \mu; x) = 1_{\tau > t}\mathbb{E}^Q\left[e^{-\int_t^T [r(u) + \mu(u;x)]du}\mathbb{E}^Q\left[P(T, r, \mu; x)|\mathcal{G}_T\right]|\mathcal{G}_t\right]. \tag{3.5}$$

The problem then becomes that of finding the contract value, $P(T, r, \mu; x)$, at time-T discounted to time-t.

Given the inhomogeneous PDE in equation (3.1) we can use Duhamel's principle to solve the option price $C(t, T, r, u; x)$. See Logan [21] for a discussion on Duhamel's principle. In solving the inner expectation in equation (3.5) with filtration $\mathcal{G}_T$, we present the general solution as a function of the associated transition density function $G(\psi, r, \mu; r_0, \mu_0, x)$, which is a solution of the PDE (3.3). The general solution of the option price is presented in the proposition below.

Proposition 3.1. *Duhamel's principle allows us to represent the general solution of equation* (3.5) *as*

$$\begin{aligned} C(t, T, r, \mu; x) =& 1_{\tau > t}\mathbb{E}^Q\left[e^{-\int_t^T [r(u) + \mu(u;x)]du}|\mathcal{G}_t\right]\int_0^\infty\int_0^\infty P(T, r, \mu; x)G(\psi, r, \mu; w_1, w_2)dw_1dw_2 \\ =& 1_{\tau > t}e^{\alpha_r(t,T) - \beta_r(t,T)r(t)} \times e^{\alpha_\mu(t,T;x) - \beta_\mu(t,T;x)\mu(x,t)} \\ &\times \int_0^\infty\int_0^\infty P(T, r, \mu; x)G(\psi, r, \mu; w_1, w_2)dw_1dw_2, \end{aligned} \tag{3.6}$$

where $G(\psi, r, \mu; w_1, w_2)$ is the solution of the transition density PDE (3.3)*.*

Proof: A detailed proof of Duhamel's principle is presented in Chiarella and Ziveyi [10]. $\square$

Now that we have managed to present the general solution of our pricing function, we only need to solve (3.3) for the density function which is the only unknown term in equation (3.6). We accomplish this in the next section.

3.2. Applying the Laplace Transform

In solving equation (3.3) we make use of Laplace transform techniques to transform the PDE to a corresponding system of characteristic PDEs. The Laplace transform has the following definition

$$\mathcal{L}[G] = \int_0^\infty \int_0^\infty e^{-s_r r - s_\mu \mu} G(\psi, r, \mu) dr d\mu \equiv \widetilde{G}(\psi, s_r, s_\mu), \tag{3.7}$$

which we use in the following proposition.

Proposition 3.2. *Laplace transform of equation* (3.3) *can be represented as*

$$\begin{aligned} &\frac{\partial \widetilde{G}}{\partial \psi} + \left\{ \frac{1}{2}\sigma_r^2 s_r^2 - \kappa_r s_r \right\} \frac{\partial \widetilde{G}}{\partial s_r} + \left\{ \frac{1}{2}\sigma_\mu^2 s_\mu^2 - \kappa_\mu s_\mu \right\} \frac{\partial \widetilde{G}}{\partial s_\mu} \\ &= \Big[(\kappa_r \theta_r - \sigma_r^2) s_r + (\kappa_\mu m(t) - \sigma_\mu^2) s_\mu + \kappa_r + \kappa_\mu \Big] \widetilde{G} + f_1(\psi, s_\mu) + f_2(\psi, s_r) \end{aligned} \tag{3.8}$$

where

$$f_1(\psi, s_\mu) = \left(\frac{1}{2}\sigma_r^2 - \kappa_r \theta_r \right) \widetilde{G}(\psi, 0, s_\mu) \tag{3.9}$$

$$f_2(\psi, s_r) = \left(\frac{1}{2}\sigma_\mu^2 - \kappa_\mu m(t) \right) \widetilde{G}(\psi, s_r, 0). \tag{3.10}$$

Equation (3.8) *is to be solved subject to the initial condition*

$$\tilde{G}(0, r, \mu; r_0, \mu_0, x) = e^{-s_r r_0 - s_\mu \mu_0} \tag{3.11}$$

Proof: Refer to Appendix 1. $\square$

Proposition 3.3. *The solution to equation* (3.3) *can be represented as*

$$
\begin{aligned}
\widetilde{G}(\psi, s_r, s_\mu; r_0, \mu_0, x) = &\left(\frac{2\kappa_r}{\sigma_r^2 s_r(e^{\kappa_r\psi}-1)+2\kappa_r}\right)^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}} \times \left(\frac{2\kappa_\mu}{\sigma_\mu^2 s_\mu(e^{\kappa_\mu\psi}-1)+2\kappa_\mu}\right)^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}}\\
&\exp\Big[\{\kappa_r+\kappa_\mu\}\psi\Big] \times \exp\left\{-\frac{2s_r\kappa_r e^{\kappa_r\psi}}{\sigma_r^2 s(e^{\kappa_r\psi}-1)+2\kappa_r} r_0 - \frac{2s_\mu\kappa_\mu e^{\kappa_\mu\psi}}{\sigma_\mu^2 s_\mu(e^{\kappa_\mu\psi}-1)+2\kappa_\mu}\mu_0\right\}\\
&\times \left[\Gamma\left(\frac{2\Phi_r}{\sigma_r^2}-1; \frac{4\kappa_r^2 e^{\kappa_r\psi}}{(e^{\kappa_r\psi}-1)(\sigma_r^2 s(e^{\kappa_r\psi}-1)+2\kappa_r)} r_0\right)\right.\\
&\left.+\Gamma\left(\frac{2\Phi_\mu}{\sigma_\mu^2}-1; \frac{4\kappa_\mu^2 e^{\kappa_\mu\psi}}{(e^{\kappa_\mu\psi}-1)(\sigma_\mu^2 s_\mu(e^{\kappa_\mu\psi}-1)+2\kappa_\mu)}\mu_0\right)-1\right]
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
\Phi_r &= \kappa_r\theta_r\\
\Phi_\mu &= \kappa_\mu m(t)
\end{aligned}
$$

Proof: Refer to Appendix 2. □

Proposition 3.4. *The inverse Laplace transform of equation* (3.12) *can be represented as*

$$
\begin{aligned}
G(\psi, r, \mu; r_0, \mu_0, x) = &\exp\left\{-\frac{2\kappa_r}{\sigma_r^2(e^{\kappa_r\psi}-1)}(r_0 e^{\kappa_r\psi}+r) - \frac{2\kappa_\mu}{\sigma_\mu^2(e^{\kappa_\mu\psi}-1)}(\mu_0 e^{\kappa_\mu\psi}+\mu)\right\}\\
&\times\left(\frac{r_0 e^{\kappa_r\psi}}{r}\right)^{\frac{\Phi_r}{\sigma_r^2}-\frac{1}{2}} \times \left(\frac{\mu_0 e^{\kappa_\mu\psi}}{\mu}\right)^{\frac{\Phi_\mu}{\sigma_\mu^2}-\frac{1}{2}} I_{\frac{2\Phi_r}{\sigma_r^2}-1}\left(\frac{4\kappa_r}{\sigma_r^2(e^{\kappa_r\psi}-1)}(r \times r_0 e^{\kappa_r\psi})^{\frac{1}{2}}\right)\\
&\times I_{\frac{2\Phi_\mu}{\sigma_\mu^2}-1}\left(\frac{4\kappa_\mu}{\sigma_\mu^2(e^{\kappa_\mu\psi}-1)}(\mu \times \mu_0 e^{\kappa_\mu\psi})^{\frac{1}{2}}\right) \times \frac{2\kappa_r e^{\kappa_r\psi}}{\sigma_r^2(e^{\kappa_r\psi}-1)}\frac{2\kappa_\mu e^{\kappa_\mu\psi}}{\sigma_\mu^2(e^{\kappa_\mu\psi}-1)}
\end{aligned}
\tag{3.13}
$$

Proof: Refer to Appendix 3. □

4. Deferred Insurance Contracts

When pricing deferred insurance contracts, we first define general functions to denote such contracts. The value of a pure endowment contract at option maturity is given by

$$
F_E(T, r, \mu; x) = \mathbb{E}^Q\left[e^{-\int_T^{T_m}[r(u)+\mu(u)]du}\Big|\mathcal{G}_T\right],
\tag{4.1}
$$

where T_m is the time when the pure endowment contract matures and T is the option maturity date. Similarly the payoff for a deferred immediate annuity, when the annuity starts at the option maturity date, T, can be represented as

$$F_A(T, r, \mu; x) = \sum_{i=T}^{\omega} \mathbb{E}^Q \left[e^{-\int_T^i [r(u)+\mu(u)]du} \middle| \mathcal{G}_T \right]. \tag{4.2}$$

More complicated structures can arise when an option on a deferred immediate annuity expires before the annuity start time such that

$$F_A(T, r, \mu; x) = \mathbb{E}^Q \left[e^{-\int_T^{T+h} [r(u)+\mu(u)]du} \left[\sum_{i=T+h}^{\omega} \mathbb{E}^Q \left[e^{-\int_{T+h}^i [r(u)+\mu(u)]du} \middle| \mathcal{G}_{T+h} \right] \right] \middle| \mathcal{G}_T \right], \tag{4.3}$$

where the filtration, $\mathcal{G}_T$, defines the option maturity date, ω is the maximum age in the cohort for a whole life immediate annuity and h is the time between option maturity and annuity start age. Equations like (4.3) naturally lead to the pricing of American style options on life insurance contracts; we have left this for future research. In this paper, we present results for functional forms in equations (4.1) and (4.2).

Given our affine definition of the short-rate interest rate and mortality processes we have an explicit solution for the expectation in equation (4.1) given by

$$F_E(T, r, \mu; \varphi_1, \varphi_2, x) = e^{\alpha_r(T_m - T) - \beta_r(T_m - T)\varphi_1} \times e^{\alpha_\mu(T_m - T; x) - \beta_\mu(T_m - T; x)\varphi_2}, \tag{4.4}$$

and for equation (4.2) we have

$$F_A(T, r, \mu; \varphi_1, \varphi_2, x) = \sum_{i=T}^{\omega} \left[e^{\alpha_r(i-T) - \beta_r(i-T)\varphi_1} \times e^{\alpha_\mu(i-T;x) - \beta_\mu(i-T;x)\varphi_2} \right]. \tag{4.5}$$

For a pure endowment contract, when $T = T_m$ the option and the contract matures at the same time such that

$$F_E(T, r, \mu; \varphi_1, \varphi_2, x) = 1, \tag{4.6}$$

and by definition of a probability density function we obtain

$$\int_0^{\infty} \int_0^{\infty} G(\psi, r, \mu; \varphi_1, \varphi_2, x) d\varphi_1 d\varphi_2 = 1, \tag{4.7}$$

implying that the price of a pure endowment option is simply

$$C(t, T, r, \mu; x) = 1_{\tau > t} e^{\alpha_r(T-t) - \beta_r(T-t)r(t)} e^{\alpha_\mu(T-t;x) - \beta_\mu(T-t;x)\mu(t;x)}, \tag{4.8}$$

which is known at time-t.

4.1. Options on Endowment and Deferred Annuity Contracts

We take the perspective of the insurer and focus on European call options written on insurance contracts defined in equations (4.1) and (4.2). An option on a pure endowment contract has a pay-off at option maturity which we represent here as

$$P(T, r, \mu; x) = \max \left[0, \mathbb{E}^Q \left[e^{-\int_T^{T_m} [r(u)+\mu(u)]du} \middle| \mathcal{G}_T \right] - K \right], \tag{4.9}$$

where T_m is the time when the pure endowment contract matures, T is the option maturity date and K is the guaranteed value. Similarly the pay-off of a deferred immediate annuity, when the annuity starts at the same time as the option expires, time-T, can be represented as

$$P(T, r, \mu; x) = \max \left[0, \sum_{i=T}^{\omega} \mathbb{E}^Q \left[e^{-\int_T^{i} [r(u)+\mu(u)]du} \middle| \mathcal{G}_T \right] - K \right], \tag{4.10}$$

where K is the guaranteed annuity value at time-T.

By substituting equations (4.9) and (4.10) into equation (3.6) we obtain the option price of a pure endowment and deferred immediate annuity, respectively. Such option prices can be represented as

$$\begin{aligned} C(t, T, r, \mu; x) &= 1_{\tau > t} e^{\alpha_r(T-t) - \beta_r(T-t) r(t)} e^{\alpha_\mu(T-t;x) - \beta_\mu(T-t;x)\mu(t;x)} \\ &\times \int_0^\infty \int_0^\infty \max \left[0, e^{\alpha_r(T_m-T) - \beta_r(T_m-T)\varphi_1} \times e^{\alpha_\mu(T_m-T;x) - \beta_\mu(T-T;x)\varphi_2} - K\right] G(\psi, r, \mu; \varphi_1, \varphi_2, x) d\varphi_1 d\varphi_2. \end{aligned} \tag{4.11}$$

and

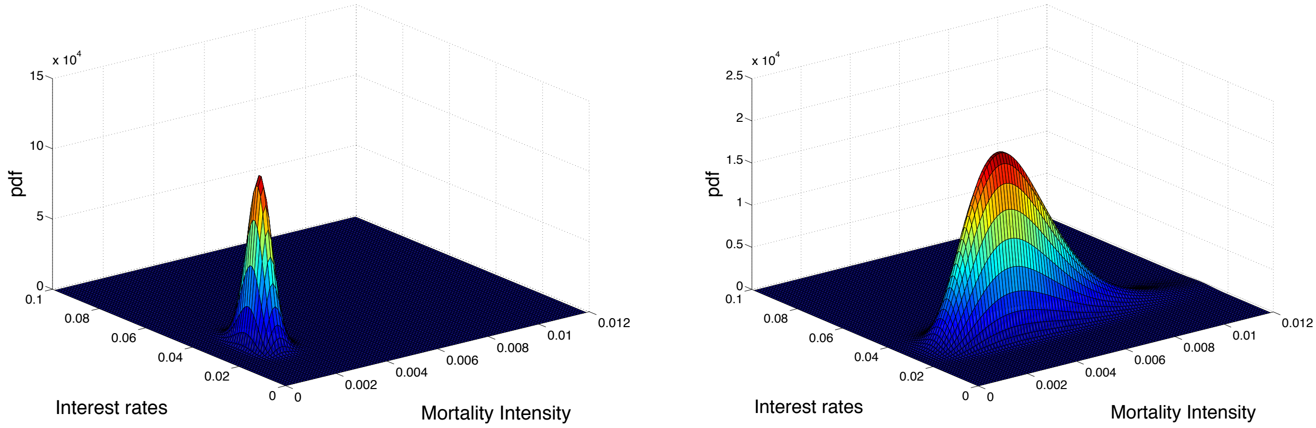
$$\begin{aligned} C(t, T, r, \mu; x) &= 1_{\tau > t} e^{\alpha_r(T-t) - \beta_r(T-t) r(t)} e^{\alpha_\mu(T-t;x) - \beta_\mu(T-t;x)\mu(t;x)} \\ &\times \int_0^\infty \int_0^\infty \max \left[0, \sum_{i=T}^{\omega} e^{\alpha_r(i-T) - \beta_r(i-T)\varphi_1} \times e^{\alpha_\mu(i-T;x) - \beta_\mu(i-T;x)\varphi_2} - K\right] G(\psi, r, \mu; \varphi_1, \varphi_2, x) d\varphi_1 d\varphi_2. \end{aligned} \tag{4.12}$$

5. Numerical Results

5.1. Model Parameters

Figure 5.1 shows the joint density function for varying values of κ_μ and Σ_μ for a cohort aged 65. The interest rate parameters are fixed; typical parameter values are given in 1. We note different shapes of the joint density function when we vary the speed of reversion parameter, κ_μ and volatility Σ_μ in the mortality process, equation 2.6.

The effect of increasing κ_μ increases the peak mortality density, hence a lower survival probability. Increasing the volatility, Σ_μ, significantly increases the mortality intensity dispersion. Practically, the integral limits to infinity are not required in equations (4.11) and (4.12). We can see in figure 5.1 that the interest rate and mortality intensity processes decay to zero; we will use these observed limits when performing numerical integration.



(a) $\kappa_\mu = 0.015$, $\Sigma_\mu = 0.03$ (b) $\kappa_\mu = 0.065$, $\Sigma_\mu = 0.15$

Figure 5.1: Probability Density Function - Age = 65

Figure 5.2 shows the mortality intensity probability density function at various ages for a volatility fixed at $\Sigma_\mu = 0.15$; in this figure we have integrated the interest rate process. The figure is truncated at a mortality intensity of 0.12 to observe detail at the younger ages, all the intensity distributions decay to zero and integrate to one to indicate a proper probability distribution. As the cohort's age increases, i.e. time to maturity, the peak of the mortality intensity also increases. The dispersion of mortality intensity is increasing with age, giving a high level of uncertainty of mortality intensity as our cohort ages.

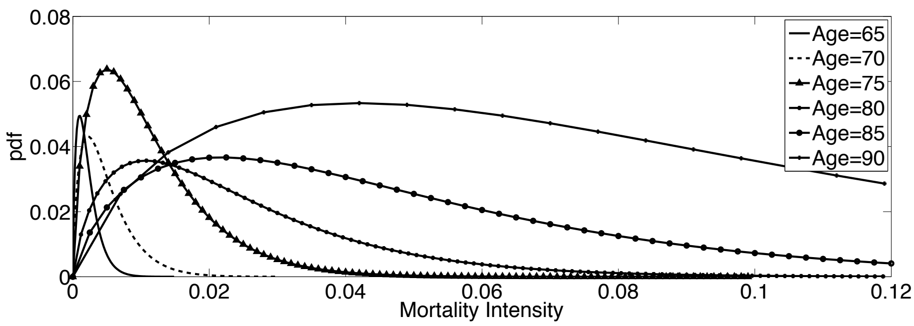


Figure 5.2: Mortality Intensity for Different Ages

We price guaranteed pure endowment and deferred annuity options in the framework derived in section 3. Table 1 shows the parameters of the Weibull function used in this paper as a base level of our cohort at age 50, equation (2.7). These are the initial values used by Biffis [4]. Table 1 also contains the parameter values of the interest rate and mortality stochastic processes derived in sections 2.1 and 2.2 respectively. Unless otherwise specified these values are used in the analysis that follows.

In this analysis we model a cohort aged 50 at time zero and pricing options maturing at time T. The time-to-maturity is defined as $\psi = T - t$.

κ_r	θ_r	σ_r	Σ_μ	c	θ	r_0	μ_0
0.05	0.05	0.01	0.15	10.841	86.165	0.03	$m(0)$

Table 1: Stochastic process and Weibull parameters

To derive the survival probabilities at time zero we solve the system of ordinary differential equations given in equation (2.10). The shape of the survival curves can be controlled by changing the κ_μ and Σ_μ parameters of the mortality stochastic process. Figures 5.3a and 5.3c shows the survival curves at time zero for various values of κ_μ and Σ_μ respectively. Lower values of κ_μ, corresponding to a slower speed of mean-reversion, produce survival curves with higher survival probabilities for a fixed Σ_μ. Similarly, we fix κ_μ and observe the survival curves as Σ_μ varies. Higher values of Σ_μ increase the survival probability in the older ages.

The force of mortality for our cohort aged 50 at time zero is defined as

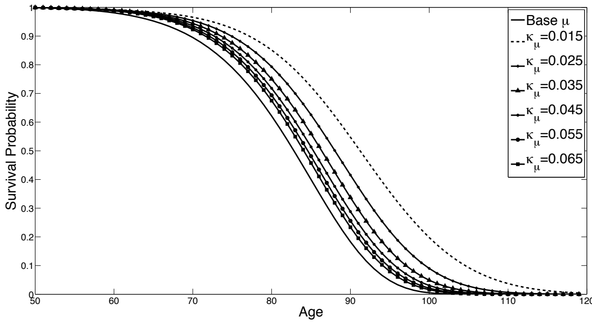
$$\begin{aligned}\mu(t,T;x) &= -\frac{d}{dT}\log\left[\mathbb{E}^Q\left[e^{-\int_t^T \mu(u)du}\middle|\mathcal{G}_t\right]\right]\\ &= -\frac{d}{dT}\left[\alpha_\mu(T-t;x)-\beta_\mu(T-t;x)\mu(t;x)\right].\end{aligned} \tag{5.1}$$

The force of mortality for varying levels of κ_μ and Σ_μ is shown in Figures 5.3b and 5.3d respectively. The scale of each figure changes for ages below and above age 90. In Figure 5.3b changing values of κ_μ produces a large variety of curves below age 90; for a fixed Σ_μ, above age 90 the force of mortality for each κ_μ does not diverge significantly. The opposite can be seen in Figure 5.3d: for a fixed κ_μ, varying Σ_μ produces a divergence in the force of mortality over age 90. By varying these two parameters, we have a flexible mortality model that can cover a large variety of survival curves.

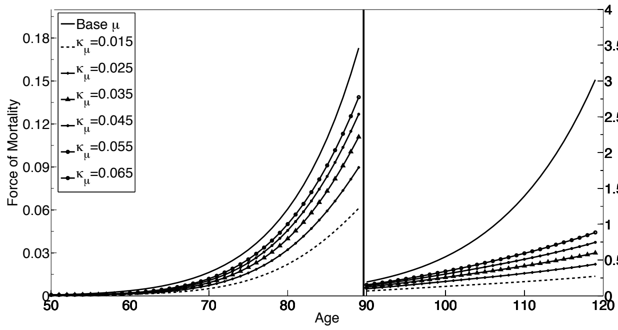
We also wish to observe the probability density function of the mortality process at some time-T in the future. This can be done by integrating the joint density function, equation (3.13), with respect to r_0. Figure 5.4 shows the mortality probability density function with $T = 65$, or $\psi = 15$, for varying values of κ_μ and Σ_μ; interest rate process parameters are given in Table 1. The peak in the density function in Figure 5.4a corresponds to the lowest value of κ_μ. The peak density decreases and mortality intensity dispersion increases as κ_μ increases. Figure 5.4b shows the mortality probability density function as Σ_μ varies. The peak in the density function is greatest at the lowest volatility level. Increasing Σ_μ decreases (and then increases) the density level while the mortality intensity dispersion increases.

5.2. Option Pricing

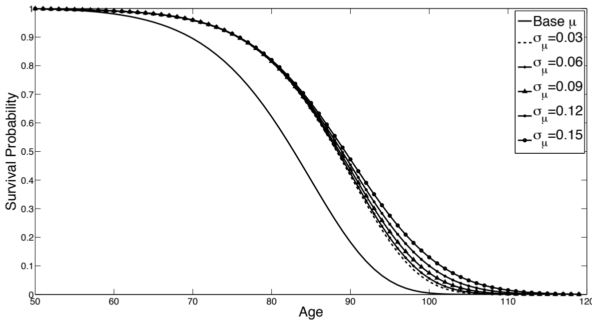
Using the parameters from Table 1 we analyse the effect of a changing κ_μ on pure endowment and deferred annuity contract valuations and option prices. For the pure endowment contract, a payment of one is made to the contract holder at the maturity of contract if they are alive. Similarly for the deferred annuity contract holder, a periodic payment of one is made to the contract holder while they are still alive; the annuity is a whole-life immediate type. The



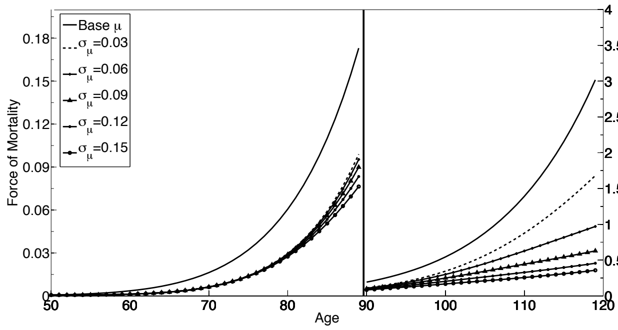
(a) Survival Probability ($\Sigma_\mu = 0.15$)



(b) Force of Mortality ($\Sigma_\mu = 0.15$)

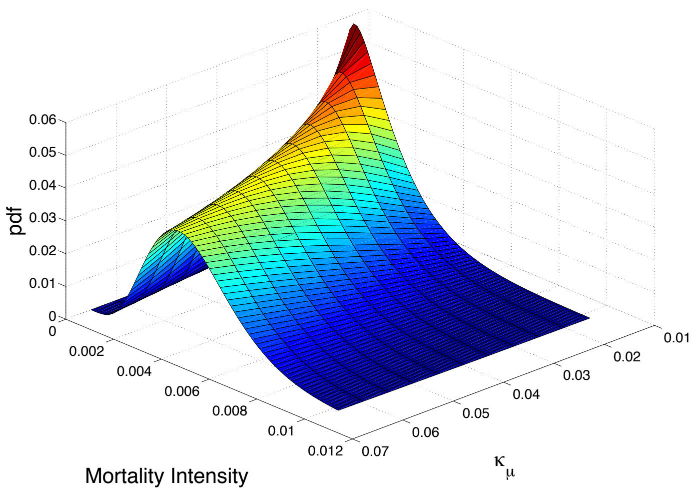


(c) Survival Probability ($\kappa_\mu = 0.015$)

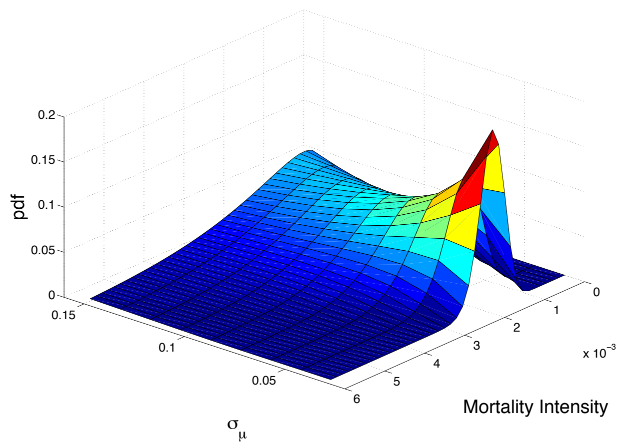


(d) Force of Mortality ($\kappa_\mu = 0.015$)

Figure 5.3: Survival Probability and Force of Mortality



(a) Probability Density Function ($\Sigma_\mu = 0.15$)



(b) Probability Density Function ($\kappa_\mu = 0.015$)

Figure 5.4: Probability Density Function

guaranteed value in the options, K, will typically be given as the model's contract valuation at time-T. All the options are European call options based on a cohort aged 50 at time zero.

5.2.1. Pure Endowment Pricing

Figure 5.5 shows results for a pure endowment with an option maturing at $T = 65$ and the contract maturity varying from age 65 to 100. When the option maturity and contract maturity are both 65, the contract valuation at time-T is 1 and the option price is zero, similar to payoff of an at-the-money option. As the maturity of the pure endowment contract increases to 100 the effect of varying the κ_μ parameter can be seen in Figure 5.5a. The contracts with lower κ_μ's have higher values at time-T. For the option price, the guaranteed value is the model market value of the pure endowment contract valued at time-T. The pure endowment contract value at time-t is given in Figure 5.5b.

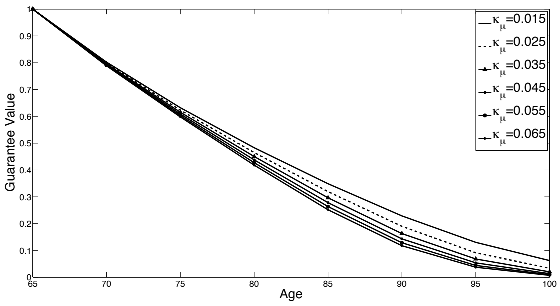
Figure 5.5c shows the option prices for the pure endowment with the guaranteed value given in Figure 5.5a. Even though the contract values are always decreasing with age, the option prices are increasing until age 75 or 80. Then the prices of the options decrease with age; this is due to the shape of the survival curves in Figure 5.3a. The guaranteed values are dependent on the model's market value of the pure endowment contract at time T. In reality, an insurer may prefer to analyse option prices for a fixed guaranteed value and varying assumptions in κ_μ, and we perform this mispricing analysis below.

The option prices relative to the face-value of the pure endowment valuation at time-t are shown in Figure 5.5d. These percentages are increasing in age but flattens out after 90. This shows that the longer the contract maturity date relative to the option maturity date, the higher the cost of the hedge. The option price as a percentage of the contract face value is relatively insensitive to changes in κ_μ in this example.

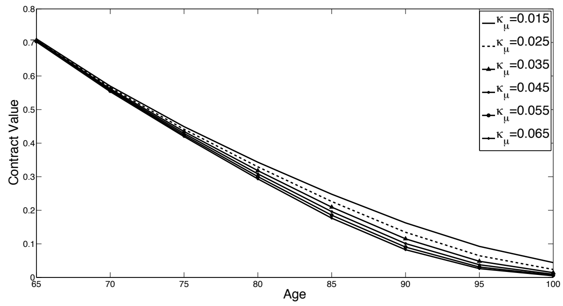
Age \ κ_μ	0.015	0.025	0.035	0.045	0.055	0.065	$\bar{K}$
70	0.8022	0.7986	0.7955	0.7928	0.7905	0.7885	0.7947
75	0.6321	0.6231	0.6153	0.6086	0.6029	0.5979	0.6133
80	0.4830	0.4650	0.4501	0.4376	0.4271	0.4181	0.4468
85	0.3487	0.3193	0.2963	0.2781	0.2634	0.2516	0.2929
90	0.2287	0.1899	0.1627	0.1431	0.1285	0.1175	0.1617
95	0.1301	0.0912	0.0681	0.0536	0.0440	0.0374	0.0707
100	0.0622	0.0334	0.0201	0.0132	0.0093	0.0070	0.0242

Table 2: Guaranteed Value for different κ_μ

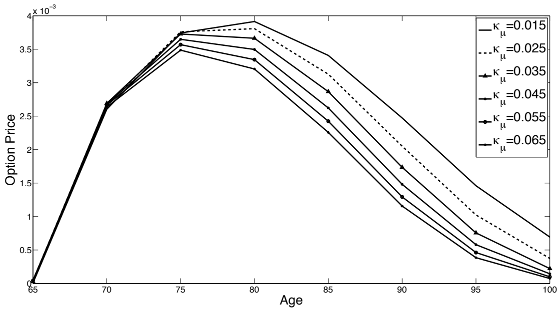
The guaranteed values at time-T are dependent on our choice of κ_μ. Since we cannot know the correct κ_μ, we analyse mispricing of the options by fixing guaranteed values. This time we choose a guaranteed value that is the mean of the contract values at time-T, denoted by $\bar{K}$; see Table 2. As the contract maturity increases, there are larger changes in model's market value of the contracts. Table 3 shows the option prices and the percentages of face-value for guaranteed values of $\bar{K}$ given in Table 2. Under these assumptions the option prices and



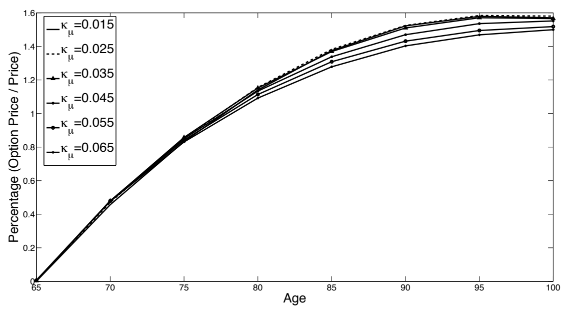
(a) Contract Valuation at $T = 65$



(b) Contract Valuation at Age=50



(c) Option Price



(d) Option Price as Percentage of Face Value

Figure 5.5: Pure Endowment Options

Age \ κ_μ	0.015	0.025	0.035	0.045	0.055	0.065
70	0.0062	0.0043	0.0030	0.0020	0.0014	0.0009
75	0.0139	0.0083	0.0045	0.0022	0.0009	0.0004
80	0.0258	0.0135	0.0050	0.0011	0.0001	0.0000
85	0.0397	0.0188	0.0043	0.0001	0.0000	0.0000
90	0.0476	0.0200	0.0021	0.0000	0.0000	0.0000
95	0.0422	0.0145	0.0001	0.0000	0.0000	0.0000
100	0.0270	0.0065	0.0000	0.0000	0.0000	0.0000

(a) Option Price

Age \ κ_μ	0.015	0.025	0.035	0.045	0.055	0.065
70	1.10	0.77	0.53	0.37	0.25	0.17
75	3.19	1.92	1.04	0.50	0.21	0.08
80	8.11	4.25	1.59	0.34	0.03	0.00
85	19.06	9.04	2.06	0.05	0.00	0.00
90	41.39	17.42	1.85	0.00	0.00	0.00
95	84.00	28.98	0.29	0.00	0.00	0.00
100	156.86	38.15	0.00	0.00	0.00	0.00

(b) Percentage

Table 3: Pure Endowment Options

percentages of face-value are highly dependent on κ_μ. This shows that in reality offering guaranteed pure endowment contracts that mature over age 80 is extremely expensive.

5.2.2. Deferred Immediate Annuity Pricing

The pricing of deferred immediate annuity options is similar to the previous sections. From equation (4.2) we can see that the annuity payment is the sum of pure endowment contracts as shown in equation (4.1). In this section the annuity contract is a whole life deferred immediate annuity, with the annuity contract deferred to time-T.

Figure 5.6 shows our analysis for a changing option maturity date, T, varying from 55 to 100. Figure 5.6a shows the guaranteed value or model market value of the deferred immediate annuity at time-T. As expected, the lower values of κ_μ have higher guaranteed values. The contract values at time-t for different annuity start years, time-T, are shown in Figure 5.6b, with the corresponding option prices in Figure 5.6c. We can see a relatively low percentage of option price to contract face-value when the deferment period is short; see Figure 5.6d. This shows a relative increase in the cost of options with a deferment period, there is a quite large difference in the percentage for varying κ_μ at the longer deferment periods. Unlike pure endowment option prices, the relative prices of the options on deferred immediate annuities are sensitive to the value of κ_μ.

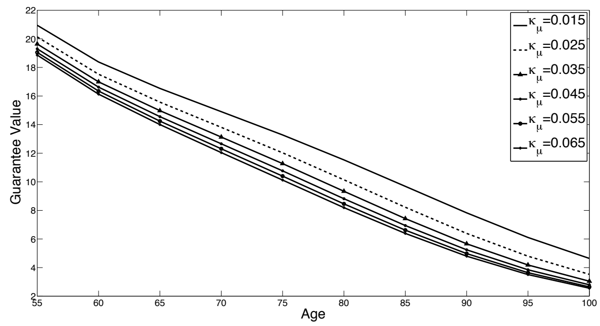
Age \ κ_μ	0.015	0.025	0.035	0.045	0.055	0.065	$\bar{K}$
55	20.9465	20.1327	19.6371	19.2976	19.0496	18.8612	19.6541
60	18.3791	17.5232	16.9917	16.6207	16.3445	16.1307	16.9983
65	16.5308	15.5718	14.9750	14.5567	14.2447	14.0029	14.9803
70	14.9051	13.8099	13.1300	12.6581	12.3098	12.0425	13.1426
75	13.2778	12.0286	11.2739	10.7643	10.3973	10.1214	11.3106
80	11.5325	10.1404	9.3374	8.8174	8.4571	8.1954	9.4134
85	9.6840	8.2205	7.4280	6.9410	6.6177	6.3906	7.5470
90	7.8290	6.3911	5.6667	5.2460	4.9790	4.7983	5.8183
95	6.1091	4.7963	4.1838	3.8483	3.6450	3.5126	4.3492
100	4.6438	3.5298	3.0478	2.7977	2.6523	2.5606	3.2053

Table 4: Guaranteed Value for different κ_μ

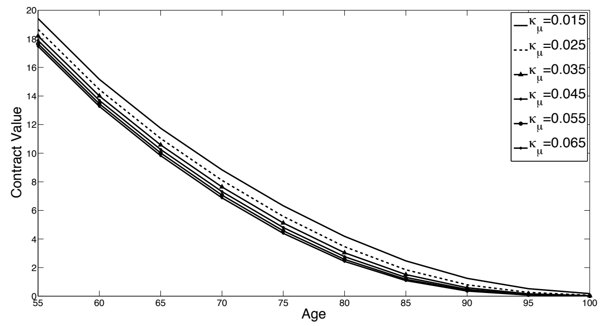
Similar to the previous section, we also test the sensitivity of the option prices for fixed guaranteed values, rather than the model's market values. The last column of Table 4 gives the average 'fair-values' of the annuity contracts at time-T. Table 5 shows the option prices and percentage of option price to contract value. Since these contract valuations depend on the remaining life of the contract holder, we see higher sensitivity to the guaranteed value than pure endowment contracts.

5.3. Volatilities of Interest and Mortality Processes

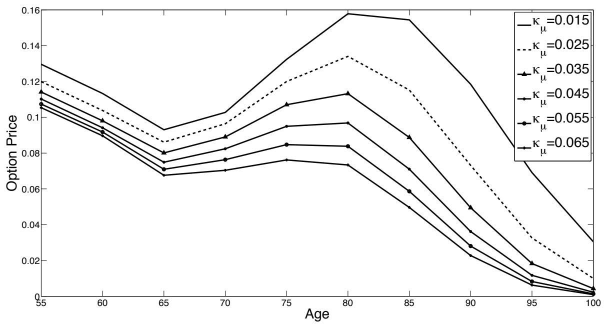
We have two stochastic processes affecting the price of our deferred annuity options. Interest rate processes are generally known as highly volatile in the short term, with the volatility of



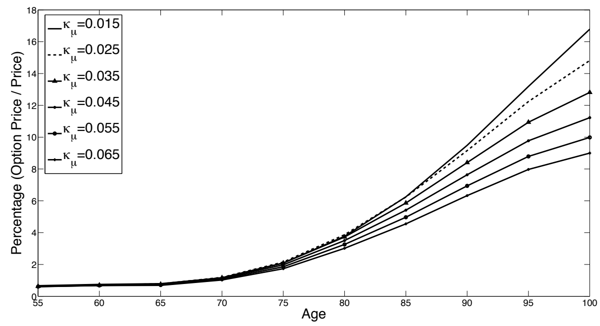
(a) Contract Valuation at time-T



(b) Contract Valuation at Age=50



(c) Option Price



(d) Option Price as Percentage of Face Value

Figure 5.6: Deferred Immediate Annuity Options

Age \ κ_μ	0.015	0.025	0.035	0.045	0.055	0.065
55	1.1978	0.4545	0.1068	0.0139	0.0012	0.0001
60	1.1419	0.4420	0.0954	0.0081	0.0003	0.0000
65	1.0991	0.4236	0.0782	0.0029	0.0000	0.0000
70	1.0466	0.4052	0.0853	0.0035	0.0000	0.0000
75	0.9425	0.3650	0.0978	0.0113	0.0002	0.0000
80	0.7848	0.3041	0.1000	0.0236	0.0031	0.0001
85	0.5696	0.2112	0.0764	0.0249	0.0068	0.0014
90	0.3455	0.1151	0.0418	0.0156	0.0058	0.0020
95	0.1680	0.0460	0.0153	0.0057	0.0023	0.0010
100	0.0643	0.0130	0.0036	0.0012	0.0005	0.0002

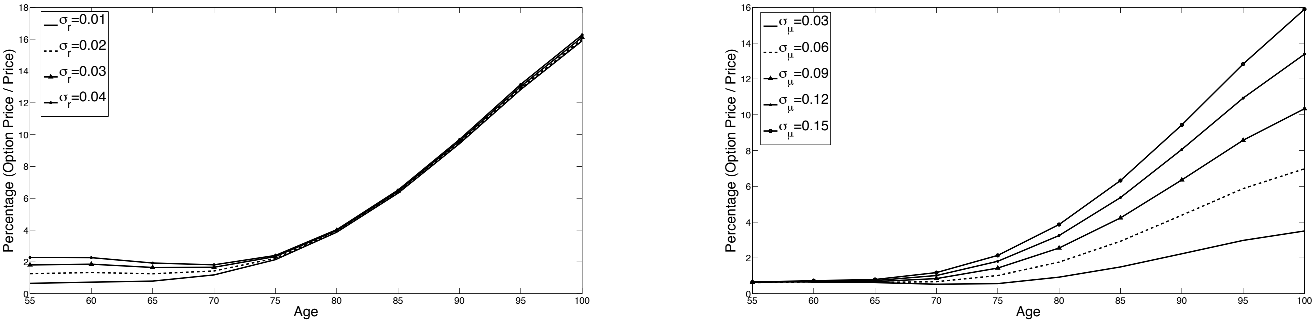
(a) Option Price

Age \ κ_μ	0.015	0.025	0.035	0.045	0.055	0.065
55	6.58	2.50	0.59	0.08	0.01	0.00
60	8.14	3.15	0.68	0.06	0.00	0.00
65	10.32	3.99	0.74	0.03	0.00	0.00
70	13.43	5.25	1.11	0.05	0.00	0.00
75	17.50	6.96	1.91	0.22	0.00	0.00
80	22.97	9.46	3.26	0.80	0.11	0.01
85	29.58	12.45	4.97	1.75	0.51	0.11
90	37.23	15.83	6.93	2.98	1.22	0.46
95	45.02	19.05	8.80	4.24	2.08	1.01
100	51.34	21.34	10.13	5.21	2.81	1.56

(b) Percentage

Table 5: Deferred Whole-Life Annuity

prices decreasing with time as the process reverts to its long-term mean value. The effect of changing interest rate volatility is presented in Figure 5.7a. We can see the difference in relative price in the contracts with the shortest deferment period. This difference is due to interest rate volatility converging around the age of 75 for our selected parameters.

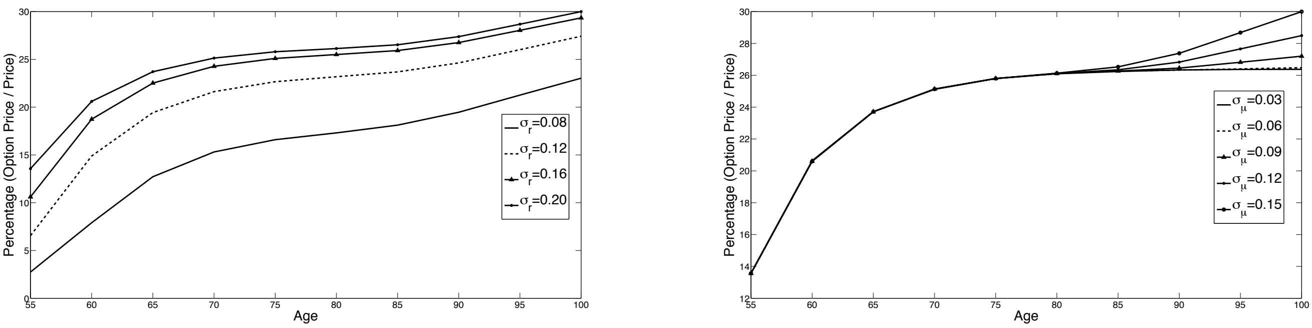


(a) Annuity - Changing Interest Rate Volatility

(b) Annuity - Changing Mortality Volatility

Figure 5.7: Changing Volatility

The situation is reversed for mortality volatility. The different levels of volatility significantly change the mortality rates at the older ages; see Figure 5.3d. We present the effect of varying volatility on relative option prices in Figure 5.7b. These effects are small at the younger ages, but as the deferment age increases the relative price of the options starts to diverge. Prolonged deferment periods may not be practical; a realistic deferment period for a person aged 50 may be between 15 and 30 years. At a 15 year deferment the effect of volatility is small on the relative option price, which varies between 1 and 4 percent.



(a) Annuity - Changing Interest Rate Volatility

(b) Annuity - Changing Mortality Volatility

Figure 5.8: High Interest Rate Volatility

We test deferred annuity option prices in an extremely volatile interest rate environment. In Figure 5.8a, we set $\kappa_r = 0.15$ and $\theta_r = 0.15$ and vary interest rate volatility, σ_r, between 0.08 and 0.20. In this figure the relative option price is significantly higher than compared to the standard case presented in Figure 5.7a. Likewise, by fixing $\sigma_r = 0.20$ and varying the mortality volatility between 0.03 and 0.15, in Figure 5.8b, the effect of mortality volatility is only apparent in the very old ages.

5.4. Term Annuities

In this analysis we compare the differences in the option price for a deferred whole life annuity and a deferred term annuity contract. The deferment period for both contracts is 15 years, corresponding to a person currently aged 50 with a guaranteed annuity value at aged 65. The guaranteed value is the model's market value of the annuity price at age 65. The term annuity contracts matures at age 80 and removes the risk associated with the old age uncertainty in the cohort.

Table 6 shows the guaranteed value of each contract with varying values of Σ_μ and κ_μ. As expected, deferred whole life annuity contracts are more expensive compared to the corresponding deferred term annuity contracts. These findings also hold for the corresponding option prices which are higher for whole life contracts as shown in Table 7. The percentage of option price to contract value is shown in Table 8. We can see that the percentage for whole-life annuities show fairly large changes in κ_μ for a given Σ_μ, whereas for the term annuity options, the percentage is relatively insensitive to κ_μ for a given Σ_μ.

Σ_μ \ κ_μ	0.015	0.025	0.035	0.045	0.055	0.065
0.03	11.1076	11.0146	10.9335	10.8601	10.7896	10.7562
0.06	11.1066	11.0098	10.9278	10.8576	10.7975	10.7454
0.09	11.1071	11.0104	10.9284	10.8583	10.7981	10.7461
0.12	11.1077	11.0112	10.9293	10.8593	10.7991	10.7470
0.15	11.1111	11.0120	10.9300	10.8600	10.7997	10.7476

(a) Deferred Term Annuity Contract Price

Σ_μ \ κ_μ	0.015	0.025	0.035	0.045	0.055	0.065
0.03	16.1282	15.3085	14.7803	14.3988	14.1015	13.9045
0.06	16.1811	15.3370	14.7984	14.4158	14.1285	13.9046
0.09	16.2698	15.3954	14.8418	14.4504	14.1569	13.9286
0.12	16.3875	15.4747	14.9013	14.4978	14.1961	13.9618
0.15	16.5308	15.5718	14.9750	14.5567	14.2447	14.0029

(b) Deferred Whole-Life Annuity Contract Price

Table 6: Contract Prices

Σ_μ \ κ_μ	0.015	0.025	0.035	0.045	0.055	0.065
0.03	0.0314	0.0296	0.0287	0.0296	0.0330	0.0260
0.06	0.0333	0.0330	0.0325	0.0320	0.0315	0.0312
0.09	0.0355	0.0352	0.0349	0.0342	0.0337	0.0332
0.12	0.0381	0.0381	0.0378	0.0370	0.0364	0.0357
0.15	0.0400	0.0414	0.0412	0.0404	0.0397	0.0389

(a) Deferred Term Annuity Option Price

Σ_μ \ κ_μ	0.015	0.025	0.035	0.045	0.055	0.065
0.03	0.0692	0.0606	0.0559	0.0548	0.0575	0.0470
0.06	0.0742	0.0668	0.0622	0.0588	0.0562	0.0545
0.09	0.0803	0.0723	0.0672	0.0631	0.0601	0.0579
0.12	0.0873	0.0789	0.0733	0.0686	0.0651	0.0623
0.15	0.0930	0.0861	0.0801	0.0749	0.0710	0.0676

(b) Deferred Whole-Life Annuity Option Price

Table 7: Option Prices

5.5. Variable Strike

In this section we analyse how the option price changes when we vary the strike price. As the strike goes to zero the price of the option converges to the market value of the contract. Figure 5.9 shows the option price as the strike price increases to the market value. We vary the strike for a whole life deferred annuity contract with option maturities at age 65 and 80.

Σ_μ \ κ_μ	0.015	0.025	0.035	0.045	0.055	0.065
0.03	0.3983	0.3796	0.3719	0.3870	0.4352	0.3439
0.06	0.4221	0.4226	0.4213	0.4177	0.4150	0.4133
0.09	0.4494	0.4519	0.4516	0.4466	0.4433	0.4400
0.12	0.4829	0.4884	0.4891	0.4837	0.4789	0.4735
0.15	0.5064	0.5304	0.5331	0.5282	0.5227	0.5158

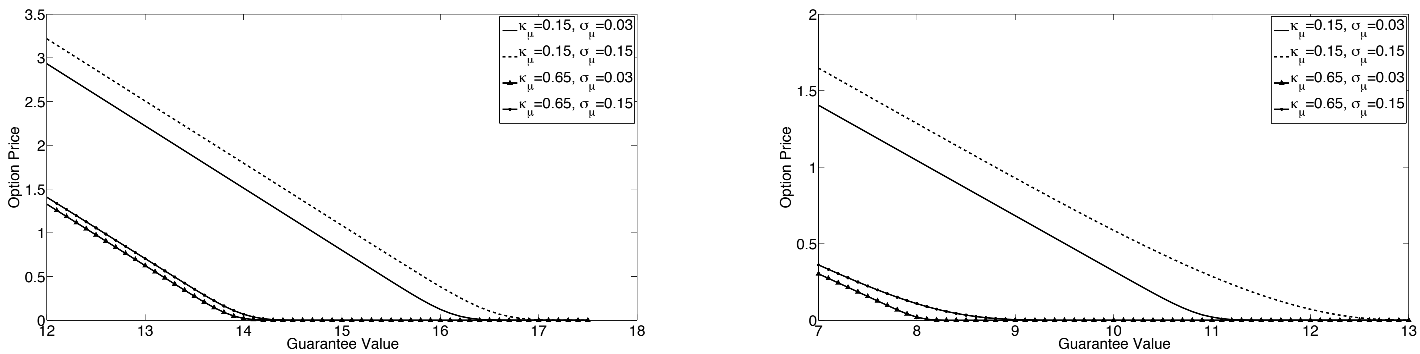
(a) Deferred Term Annuity Option Percentage

Σ_μ \ κ_μ	0.015	0.025	0.035	0.045	0.055	0.065
0.03	0.6035	0.5584	0.5356	0.5401	0.5794	0.4816
0.06	0.6451	0.6152	0.5951	0.5786	0.5657	0.5582
0.09	0.6944	0.6632	0.6409	0.6199	0.6040	0.5928
0.12	0.7499	0.7199	0.6960	0.6712	0.6517	0.6358
0.15	0.7920	0.7804	0.7571	0.7301	0.7083	0.6877

(b) Deferred Whole-Life Annuity Percentage

Table 8: Percentage (Option Price / Price)

Figures 5.9a and 5.9b shows large shifts in the option price when κ_μ varies. Smaller shifts in the option price occur when Σ_μ is varied.



(a) Option Price Age 65

(b) Option Price Age 80

Figure 5.9: Deferred Whole Life Annuity Option Price

6. Conclusion

We have presented a framework for pricing deferred pure endowment and annuity contracts and options written on such contracts. The approach involves the derivation of the pricing partial differential equation (PDE) for options written on deferred insurance products and the corresponding Kolmogorov PDE for the joint transition density function. The general solution of the pricing PDE has been presented with the aid of Duhamel's principle and this is a function of the joint transition density, where the transition density is a solution of the Kolmogorov PDE for the two stochastic processes under consideration.

We have outlined a systematic approach for solving the transition density PDE with the aid of Laplace transform techniques. The application of Laplace transforms to the PDE transforms it to a corresponding system of characteristic PDEs which can then be solved by the method of characteristics. This yields an explicit closed-form expression for the bivariate transition density function. The closed-form expression allows us to price option contracts on deferred insurance products without the need to perform Monte Carlo simulations or other numerical approximation techniques. Guided by the shape of the joint transition density function, we establish finite upper bounds for the integrals that appear in the pricing expressions. We then use sparse grids quadrature techniques to solve option pricing functions.

Numerical analysis has been provided for the contract values, option prices and sensitivity analysis for varying volatility and interest rates. We have analysed the effects of changing the speed of mean-reversion, κ_μ, for the mortality process on contracts and option prices where we noted that lower speeds of mean-reversion will always yield higher contract prices and corresponding higher option prices. We have shown that the guaranteed value is dependent on κ_μ and the problem of mispricing contracts can lead to very high option prices relative to contract value.

For the purposes of understanding the key risks and sensitivities in longevity risk we have assumed a single factor affine interest rate model. This has allowed us to provide key insights into pricing and risks for mortality linked contracts including options. We provide an extensive sensitivity analysis to the model assumptions. The methodology developed and presented can be extended to multiple factors.

7. Acknowledgement

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Appendix 1. Proof of Proposition 3.2

By using the Laplace transform, defined in equation (3.7), we can derive the following transforms

$$\begin{aligned}
\mathcal{L}\left[\frac{\partial G}{\partial r}\right] &= -\widetilde{G}(\psi, 0, s_\mu) + s\widetilde{G}(\psi, s_r, s_\mu),\\
\mathcal{L}\left[\frac{\partial G}{\partial \mu}\right] &= -\widetilde{G}(\psi, s_r, 0) + s_\mu\widetilde{G}(\psi, s_r, s_{mu}),\\
\mathcal{L}\left[r\frac{\partial G}{\partial r}\right] &= -\widetilde{G}(\psi, s_r, s_{mu}) - s_r\frac{\partial \widetilde{G}}{\partial s_r}(\psi, s_r, s_\mu),\\
\mathcal{L}\left[\mu\frac{\partial G}{\partial \mu}\right] &= -\widetilde{G}(\psi, s_r, s_\mu) - s_\mu\frac{\partial \widetilde{G}}{\partial s_\mu}(\psi, s_r, s_\mu),\\
\mathcal{L}\left[r\frac{\partial^2 G}{\partial r^2}\right] &= \widetilde{G}(\psi, 0, s_\mu) - 2s_r\widetilde{G}(\psi, s_r, s_\mu) - s_r^2\frac{\partial \widetilde{G}}{\partial s_r}(\psi, s_r, s_\mu),\\
\mathcal{L}\left[\mu\frac{\partial^2 G}{\partial \mu^2}\right] &= \widetilde{G}(\psi, s_r, 0) - 2s_\mu\widetilde{G}(\psi, s_r, s_\mu) - s_\mu^2\frac{\partial \widetilde{G}}{\partial s_\mu}(\psi, s_r, s_\mu).
\end{aligned}$$

Applying these transforms to equation (3.3) we obtain the results in equation (3.8) of Proposition 3.2.

Appendix 2. Proof of Proposition 3.3

Using the method of characteristics to solve (3.8) with initial condition (3.11). Equation (3.8) can be re-expressed in characteristic form as

$$\begin{aligned}
d\psi =& \frac{ds_r}{\frac{1}{2}\sigma_r^2 s_r^2 - \kappa_r s_r} = \frac{ds_\mu}{\frac{1}{2}\sigma_\mu^2 s_\mu^2 - \kappa_\mu s_\mu} = d\widetilde{G}\Big/\Big[\left\{(\kappa_r\theta_r - \sigma_r^2)s + \left(\kappa_\mu m(t) - \sigma_\mu^2\right)s_\mu + \kappa_r + \kappa_\mu\right\}\widetilde{G}\\
&+ f_1(\psi, s_\mu) + f_2(\psi, s_r)\Big] \qquad \text{(A2.1)}
\end{aligned}$$

We solve the first characteristic pair of (A2.1) by integrating to obtain

$$\int d\psi = \frac{2}{\sigma_r^2}\int \frac{ds_r}{s\left(s_r - \frac{2\kappa_r}{\sigma_r^2}\right)}$$

This implies that

$$\psi + c_1 = \frac{1}{\kappa_r}\int\left[\frac{1}{s_r - \frac{2\kappa_r}{\sigma_r^2}} - \frac{1}{s_r}\right]ds_r$$

integrating the RHS gives

$$\kappa_r\psi + c_2 = \mathtt{ln}\left[\frac{s_r - \frac{2\kappa_r}{\sigma_r^2}}{s_r}\right]$$

which implies

$$e^{\kappa_r\psi+c_2} = \frac{s_r - \frac{2\kappa_r}{\sigma_r^2}}{s_r}$$

hence

$$c_3 = e^{c_2} = \frac{(s_r - \frac{2\kappa_r}{\sigma_r^2})e^{-\kappa_r\psi}}{s_r} = \frac{(\sigma_r^2 s - 2\kappa_r)e^{-\kappa_r\psi}}{\sigma_r^2 s_r}$$

This can be rearranged to solve for s_r

$$\begin{aligned}
c_3\sigma_r^2 s &= (\sigma_r^2 s - 2\kappa_r)e^{-\kappa_r\psi} \\
c_3\sigma_r^2 s_r - \sigma_r^2 s_r e^{-\kappa_r\psi} &= -2\kappa_r e^{-\kappa_r\psi} \\
s_r &= \frac{-2\kappa_r e^{-\kappa_r\psi}}{\sigma_r^2(c_3 - e^{-\kappa_r\psi})}
\end{aligned}$$

Similarly the second characteristic pair of equation (A2.1) can be given by

$$d_3 = \frac{(s_\mu - \frac{2\kappa_\mu}{\sigma_\mu^2})e^{-\kappa_\mu\psi}}{s_\mu} = \frac{(\sigma_\mu^2 s_\mu - 2\kappa_\mu)e^{-\kappa_\mu\psi}}{\sigma_\mu^2 s_\mu}$$

with

$$s_\mu = \frac{-2\kappa_\mu e^{-\kappa_\mu\psi}}{\sigma_\mu^2(d_3 - e^{-\kappa_\mu\psi})}$$

The last characteristic pair can be represented as

$$\frac{d\widetilde{G}}{d\psi} + \left\{(\kappa_r\theta_r - \sigma_r^2)s + \left(\kappa_\mu m(t) - \sigma_\mu^2\right)s_\mu + \kappa_r + \kappa_\mu\right\}\widetilde{G} \quad = f_1(\psi, s_\mu) + f_2(\psi, s_r) \qquad \text{(A2.2)}$$

The integrating factor of equation (A2.2) is

$$R(\psi) = \mathtt{exp}\left[\int \left\{(\sigma_r^2 - \kappa_r\theta_r)s_r + \left(\sigma_\mu^2 - \kappa_\mu m(t)\right)s_\mu - \kappa_r - \kappa_\mu\right\}d\psi\right] \qquad \text{(A2.3)}$$

simplifying

$$\begin{aligned}
&\int\left((\kappa_r\theta_r - \sigma_r^2)\frac{2\kappa_r e^{-\kappa_r\psi}}{\sigma_r^2(c_3 - e^{-\kappa_r\psi})} + (\kappa_\mu m(t) - \sigma_\mu^2)\frac{2\kappa_\mu e^{-\kappa_\mu\psi}}{\sigma_\mu^2(d_3 - e^{-\kappa_\mu\psi})} - \kappa_r - \kappa_\mu\right)d\psi \\
&= \frac{2(\kappa_r\theta_r - \sigma_r^2)}{\sigma_r^2}\mathtt{ln}|c_3 - e^{-\kappa_r\psi}| + \frac{2(\kappa_\mu m(t) - \sigma_\mu^2)}{\sigma_\mu^2}\mathtt{ln}|d_3 - e^{-\kappa_\mu\psi}| - \{\kappa_r + \kappa_\mu\}\psi \qquad \text{(A2.4)}
\end{aligned}$$

The integrating factor can be represented as

$$R(\psi) =\left|c_3 - e^{-\kappa_r\psi}\right|^{\frac{2(\kappa_r\theta_r-\sigma_r^2)}{\sigma_r^2}} \times \left|d_3 - e^{-\kappa_\mu\psi}\right|^{\frac{2(\kappa_\mu m(t)-\sigma_\mu^2)}{\sigma_\mu^2}} \times \texttt{exp}\Big[-\{\kappa_r+\kappa_\mu\}\psi\Big] \qquad \text{(A2.5)}$$

Simplify some of the terms, we let

$$\Phi_r =\kappa_r\theta_r, \qquad \Phi_\mu = \kappa_\mu m(t)$$

and rewrite equation (A2.5) as

$$R(\psi) =\left|c_3 - e^{-\kappa_r\psi}\right|^{\frac{2(\Phi_r-\sigma_r^2)}{\sigma_r^2}} \times \left|d_3 - e^{-\kappa_\mu\psi}\right|^{\frac{2(\Phi_\mu-\sigma_\mu^2)}{\sigma_\mu^2}} \times \texttt{exp}\Big[-\{\kappa_r+\kappa_\mu\}\psi\Big] \qquad \text{(A2.6)}$$

We can solve equation (A2.2) by writing it as

$$\frac{d}{d\psi}\Big(R(\psi)\widetilde{G}\Big) =R(\psi)\Big[f_1(\psi,s_\mu)+f_2(\psi,s_r)\Big]$$

integrating and rearranging we obtain an the following expression,

$$\widetilde{G} =\frac{1}{R(\psi)}\left\{\int_0^\psi R(t)\Big[f_1(t,s_\mu)+f_2(t,s_r)\Big]dt + c_4\right\}$$

This can be represented as

$$\begin{aligned}\widetilde{G} =&\left|c_3 - e^{-\kappa_r\psi}\right|^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}} \times \left|d_3 - e^{-\kappa_\mu\psi}\right|^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}} \times \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big] \times \left\{\int_0^\psi \Big[f_1(t,s_\mu)+f_2(t,s_r)\Big]\right.\\ &\left.\times \left|c_3 - e^{-\kappa_r t}\right|^{\frac{2(\Phi_r-\sigma_r^2)}{\sigma_r^2}} \times \left|d_3 - e^{-\kappa_\mu t}\right|^{\frac{2(\Phi_\mu-\sigma_\mu^2)}{\sigma_\mu^2}} \times \texttt{exp}\Big[-\{\kappa_r+\kappa_\mu\}t\Big]dt + c_4\right\} \qquad \text{(A2.7)}\end{aligned}$$

Here c_4 is a constant and can be determined when $\psi = 0$, equation (A2.7) can then be expressed as

$$\widetilde{G}(0,s_r,s_\mu) =\left|c_3 - 1\right|^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}} \times \left|d_3 - 1\right|^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}} \times c_4$$

Rearranging in terms of c_4 and writing c_4 as a function of c_3 and d_3 we have

$$A(c_3,d_3) = \left|c_3 - 1\right|^{\frac{2(\Phi_r-\sigma_r^2)}{\sigma_r^2}} \times \left|d_3 - 1\right|^{\frac{2(\Phi_\mu-\sigma_\mu^2)}{\sigma_\mu^2}} \times \widetilde{G}\left(0, \frac{-2\kappa_r}{\sigma_r^2(c_3-1)}, \frac{-2\kappa_\mu}{\sigma_\mu^2(d_3-1)}\right)$$

In terms of the constant from equation (A2.7) we can write

$$
\begin{aligned}
&|c_3 - e^{-\kappa_r\psi}|^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}} \times \left|d_3 - e^{-\kappa_\mu\psi}\right|^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}} \times \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big] \times A(c_3,d_3)\\
=&\left|\frac{c_3 - e^{-\kappa_r\psi}}{c_3-1}\right|^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}} \times \left|\frac{d_3 - e^{-\kappa_\mu\psi}}{d_3-1}\right|^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}} \times \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big] \times \widetilde{G}\left(0, \frac{-2\kappa_r}{\sigma_r^2(c_3-1)}, \frac{-2\kappa_\mu}{\sigma_\mu^2(d_3-1)}\right)\\
=&\left|\frac{2\kappa_r e^{-\kappa_r\psi}}{\sigma_r^2 s_r(1-e^{-\kappa_r\psi})+2\kappa_r e^{-\kappa_r\psi}}\right|^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}} \times \left|\frac{2\kappa_\mu e^{-\kappa_\mu\psi}}{\sigma_\mu^2 s_\mu(1-e^{-\kappa_\mu\psi})+2\kappa_\mu e^{-\kappa_\mu\psi}}\right|^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}} \times \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big]\\
&\times \widetilde{G}\left(0, \frac{-2\kappa_r}{\sigma_r^2(c_3-1)}, \frac{-2\kappa_\mu}{\sigma_\mu^2(d_3-1)}\right)
\end{aligned}
$$

Equation (A2.7) can be simplified to

$$
\begin{aligned}
\widetilde{G} =&\left|\frac{2\kappa_r e^{-\kappa_r\psi}}{\sigma_r^2 s(1-e^{-\kappa_r\psi})+2\kappa_r e^{-\kappa_r\psi}}\right|^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}} \times \left|\frac{2\kappa_\mu e^{-\kappa_\mu\psi}}{\sigma_\mu^2 s_\mu(1-e^{-\kappa_\mu\psi})+2\kappa_\mu e^{-\kappa_\mu\psi}}\right|^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}} \times \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big]\\
&\times \widetilde{G}\left(0, \frac{-2\kappa_r}{\sigma_r^2(c_3-1)}, \frac{-2\kappa_\mu}{\sigma_\mu^2(d_3-1)}\right) + \left\{\int_0^\psi \Big[f_1(t,s_\mu)+f_2(t,s_r)\Big] \times \left|\frac{c_3 - e^{-\kappa_r\psi}}{c_3 - e^{-\kappa_r t}}\right|^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}}\right.\\
&\left.\times \left|\frac{d_3 - e^{-\kappa_\mu\psi}}{d_3 - e^{-\kappa_\mu t}}\right|^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}} \times \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}(\psi-t)\Big]dt\right\}
\end{aligned}
$$

This reduces to

$$
\begin{aligned}
\widetilde{G} =&\left(\frac{2\kappa_r e^{-\kappa_r\psi}}{\sigma_r^2 s_r(1-e^{-\kappa_r\psi})+2\kappa_r e^{-\kappa_r\psi}}\right)^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}} \times \left(\frac{2\kappa_\mu e^{-\kappa_\mu\psi}}{\sigma_\mu^2 s_\mu(1-e^{-\kappa_\mu\psi})+2\kappa_\mu e^{-\kappa_\mu\psi}}\right)^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}}\\
&\times \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big] \times \widetilde{G}\left(0, \frac{2s\kappa_r}{\sigma_r^2 s_r(1-e^{-\kappa_r\psi})+2\kappa_r e^{-\kappa_r\psi}}, \frac{2s_\mu\kappa_\mu}{\sigma_\mu^2 s_\mu(1-e^{-\kappa_\mu\psi})+2\kappa_\mu e^{-\kappa_\mu\psi}}\right)\\
&+\left\{\int_0^\psi \Big[f_1(t,s_\mu)+f_2(t,s_r)\Big] \times \left(\frac{2\kappa_r e^{-\kappa_r\psi}}{\sigma_r^2 s(e^{-\kappa_r t}-e^{-\kappa_r\psi})+2\kappa_r e^{-\kappa_r\psi}}\right)^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}}\right.\\
&\left.\times \left(\frac{2\kappa_\mu e^{-\kappa_\mu\psi}}{\sigma_\mu^2 s_\mu(e^{-\kappa_\mu t}-e^{-\kappa_\mu\psi})+2\kappa_\mu e^{-\kappa_\mu\psi}}\right)^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}} \times \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}(\psi-t)\Big]dt\right\} \qquad \text{(A2.8)}
\end{aligned}
$$

We can solve $f_1(t, s_\mu)$ as $s_r \to \infty$ equation (A2.8) simplifies to

$$
\begin{aligned}
0 =& \left(\frac{2\kappa_r e^{-\kappa_r \psi}}{\sigma_r^2 (1 - e^{-\kappa_r \psi})} \right)^{\frac{2(\sigma_r^2 - \Phi_r)}{\sigma_r^2}} \times \left(\frac{2\kappa_\mu e^{-\kappa_\mu \psi}}{\sigma_\mu^2 s_\mu (1 - e^{-\kappa_\mu \psi}) + 2\kappa_\mu e^{-\kappa_\mu \psi}} \right)^{\frac{2(\sigma_\mu^2 - \Phi_\mu)}{\sigma_\mu^2}} \times \texttt{exp}\Big[\{\kappa_r + \kappa_\mu\}\psi \Big] \\
& \times \widetilde{G} \left(0, \frac{2\kappa_r}{\sigma_r^2 (1 - e^{-\kappa_r \psi})}, \frac{2 s_\mu \kappa_\mu}{\sigma_\mu^2 s_\mu (1 - e^{-\kappa_\mu \psi}) + 2\kappa_\mu e^{-\kappa_\mu \psi}} \right) \\
& + \left\{ \int_0^\psi \Big[f_1(t, s_\mu) \Big] \times \left(\frac{2\kappa_r e^{-\kappa_r \psi}}{\sigma_r^2 (e^{-\kappa_r t} - e^{-\kappa_r \psi})} \right)^{\frac{2(\sigma_r^2 - \Phi_r)}{\sigma_r^2}} \times \left(\frac{2\kappa_\mu e^{-\kappa_\mu \psi}}{\sigma_\mu^2 s_\mu (e^{-\kappa_\mu t} - e^{-\kappa_\mu \psi}) + 2\kappa_\mu e^{-\kappa_\mu \psi}} \right)^{\frac{2(\sigma_\mu^2 - \Phi_\mu)}{\sigma_\mu^2}} \right. \\
& \left. \times \texttt{exp}\Big[\{\kappa_r + \kappa_\mu\}(\psi - t) \Big] dt \right\}
\end{aligned}
$$

This can be rearranged as

$$
\begin{aligned}
& -\widetilde{G} \left(0, \frac{2\kappa_r}{\sigma_r^2 (1 - e^{-\kappa_r \psi})}, \frac{2 s_\mu \kappa_\mu}{\sigma_\mu^2 s_\mu (1 - e^{-\kappa_\mu \psi}) + 2\kappa_\mu e^{-\kappa_\mu \psi}} \right) \\
& = \left\{ \int_0^\psi f_1(t, s_\mu) \times \left(\frac{1 - e^{-\kappa_r \psi}}{e^{-\kappa_r t} - e^{-\kappa_r \psi}} \right)^{\frac{2(\sigma_r^2 - \Phi_r)}{\sigma_r^2}} \times \left(\frac{\sigma_\mu^2 s_\mu (1 - e^{-\kappa_\mu \psi}) + 2\kappa_\mu e^{-\kappa_\mu \psi}}{\sigma_\mu^2 s_\mu (e^{-\kappa_\mu t} - e^{-\kappa_\mu \psi}) + 2\kappa_\mu e^{-\kappa_\mu \psi}} \right)^{\frac{2(\sigma_\mu^2 - \Phi_\mu)}{\sigma_\mu^2}} \right. \\
& \left. \times \texttt{exp}\Big[-\{\kappa_r + \kappa_\mu\} t \Big] dt \right\}
\end{aligned}
$$

let

$$
\zeta_r^{-1} = 1 - e^{-\kappa_r t}, \qquad z_r^{-1} = 1 - e^{-\kappa_r \psi} \tag{A2.9}
$$

$$
\zeta_\mu^{-1} = 1 - e^{-\kappa_\mu t}, \qquad z_\mu^{-1} = 1 - e^{-\kappa_\mu \psi} \tag{A2.10}
$$

define

$$
g(\zeta_r) = f_1(t, s_\mu) \times \frac{\zeta_r^{\frac{2(\sigma_r^2 - \Phi_r)}{\sigma_r^2}}}{\zeta_r(\zeta_r)} \times \left(\frac{\zeta_\mu [\sigma_\mu^2 s_\mu + 2\kappa_\mu (z_\mu - 1)]}{\sigma_\mu^2 s_\mu (\zeta_\mu - z_1) + 2\kappa_\mu \zeta_\mu (z_\mu - 1)} \right)^{\frac{2(\sigma_\mu^2 - \Phi_\mu)}{\sigma_\mu^2}} \texttt{exp}\Big[-\{\kappa_r + \kappa_\mu\} t \Big] \tag{A2.11}
$$

substituting with a change of integral variable and limits

$$
\int_{z_r}^\infty g(\zeta_r)(\zeta_r - z_r)^{-\frac{2(\sigma_r^2 - \Phi_r)}{\sigma_r^2}} d\zeta_r = -\kappa_r \widetilde{G} \left(0, \frac{2\kappa_r z_r}{\sigma_r^2}, \frac{2 s_\mu \alpha_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu (z_\mu - 1)} \right)
$$

By the definition of Laplace transform

$$\widetilde{G}\left(0, \frac{2\kappa_r z_r}{\sigma_r^2}, \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\right)$$
$$= \int_0^\infty \int_0^\infty \texttt{exp}\Bigg\{ - \frac{2\kappa_r z_r}{\sigma_r^2} r - \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\mu \Bigg\} G(0, r, \mu) dr d\mu$$

We introduce the Gamma function such that

$$\widetilde{G}\left(0, \frac{2\kappa_r z_r}{\sigma_r^2}, \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\right)$$
$$= \frac{\Gamma(\frac{2\Phi_r}{\sigma_r^2} - 1)}{\Gamma(\frac{2\Phi_r}{\sigma_r^2} - 1)} \times \int_0^\infty \int_0^\infty \texttt{exp}\Bigg\{ - \frac{2\kappa_r z_r}{\sigma_r^2} r - \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\mu \Bigg\} \times G(0, r, \mu) dr d\mu$$

expanding the Gamma function

$$\widetilde{G}\left(0, \frac{2\kappa_r z_r}{\sigma_r^2}, \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\right) = \frac{1}{\Gamma(\frac{2\Phi_r}{\sigma_r^2} - 1)} \times \int_0^\infty \int_0^\infty \int_0^\infty e^{-a} a^{\frac{2\Phi_r}{\sigma_r^2} - 2}$$
$$\times \texttt{exp}\Bigg\{ - \frac{2\kappa_r z_r}{\sigma_r^2} r - \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\mu \Bigg\} \times G(0, r, \mu) da dr d\mu$$

Make the substitution $a = (\frac{2\kappa_r r}{\sigma_r^2})y$ we obtain

$$\widetilde{G}\left(0, \frac{2\kappa_r z_r}{\sigma_r^2}, \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\right) = \frac{1}{\Gamma(\frac{2\Phi_r}{\sigma_r^2} - 1)} \times \int_0^\infty \int_0^\infty \int_0^\infty \texttt{exp}\left\{ - \frac{2\kappa_r r}{\sigma_r^2} y \right\}$$
$$\left(\frac{2\kappa_r r}{\sigma_r^2} y\right)^{\frac{2\Phi_r}{\sigma_r^2} - 2} \times \left(\frac{2\kappa_r r}{\sigma_r^2}\right) \times \texttt{exp}\Bigg\{ - \frac{2\kappa_r z_r}{\sigma_r^2} r - \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\mu \Bigg\} \times G(0, r, \mu) dy dr d\mu$$

this can be rearranged to

$$\widetilde{G}\left(0, \frac{2\kappa_r z_r}{\sigma_r^2}, \frac{2s_\mu \alpha_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\right) = \frac{1}{\Gamma(\frac{2\Phi_r}{\sigma_r^2} - 1)} \times \int_0^\infty \int_0^\infty G(0, r, \mu) \left(\frac{2\kappa_r r}{\sigma_r^2}\right)^{\frac{2\Phi_r}{\sigma_r^2} - 1}$$
$$\times \texttt{exp}\Bigg\{ - \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\mu \Bigg\} \left[\int_0^\infty \texttt{exp}\left\{ - \left(\frac{2\kappa_r r}{\sigma_r^2}\right)(y + z_r) \right\} y^{\left(\frac{2\Phi_r}{\sigma_r^2} - 2\right)} dy \right] dr d\mu \tag{A2.12}$$

let $\varrho = y + z_r$ and substituting into equation (A2.12)

$$\widetilde{G}\left(0, \frac{2\kappa_r z_r}{\sigma_r^2}, \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\right) = \frac{1}{\Gamma(\frac{2\Phi_r}{\sigma_r^2} - 1)} \times \int_0^\infty \int_0^\infty G(0, r, \mu) \left(\frac{2\kappa_r r}{\sigma_r^2}\right)^{\frac{2\Phi_r}{\sigma_r^2} - 1}$$
$$\times \texttt{exp}\Bigg\{ - \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\mu \Bigg\} \times \left[\int_{z_r}^\infty \texttt{exp}\left\{ - \left(\frac{2\kappa_r r}{\sigma^2}\right) \varrho \right\} (\varrho - z_r)^{-\frac{2(\sigma_r^2 - \Phi_r)}{\sigma^2}} d\varrho \right] dr d\mu \tag{A2.13}$$

rearranging gives

$$\widetilde{G}\left(0, \frac{2\kappa_r z_r}{\sigma_r^2}, \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\right) = \int_{z_r}^{\infty} (\varrho - z_r)^{-\frac{2(\sigma_r^2 - \Phi_r)}{\sigma_r^2}} \times \Bigg[\int_0^\infty \int_0^\infty \frac{G(0, r, \mu)}{\Gamma(\frac{2\Phi_r}{\sigma_r^2} - 1)}$$
$$\times \left(\frac{2\kappa_r r}{\sigma_r^2}\right)^{\frac{2\Phi_r}{\sigma_r^2} - 1} \mathtt{exp}\Bigg\{ - \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\mu \Bigg\} \mathtt{exp}\left\{ - \left(\frac{2\kappa_r \varrho}{\sigma_r^2}\right) r \right\} dr d\mu \Bigg] d\varrho \qquad \text{(A2.14)}$$

Thus we have shown that

$$g(\zeta_r) = -\, \kappa_r \int_0^\infty \int_0^\infty \frac{G(0, r, \mu)}{\Gamma(\frac{2\Phi_r}{\sigma_r^2} - 1)} \left(\frac{2\kappa_r r}{\sigma_r^2}\right)^{\frac{2\Phi_r}{\sigma_r^2} - 1} \times \mathtt{exp}\Bigg\{ - \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\mu \Bigg\}$$
$$\mathtt{exp}\left\{ - \left(\frac{2\kappa_r \varrho}{\sigma_r^2}\right) r \right\} dr d\mu \qquad \text{(A2.15)}$$

the initial condition from equation (3.4) is

$$G(0, r, \mu; r_0, \mu_0) = \delta(r - r_0)\delta(\mu - \mu_0),$$

substituting the initial condition into equation (A2.15)

$$g(\zeta_r) = -\, \kappa_r \int_0^\infty \int_0^\infty \frac{\delta(r - r_0)\delta(\mu - \mu_0)}{\Gamma(\frac{2\Phi_r}{\sigma_r^2} - 1)} \left(\frac{2\kappa_r r}{\sigma_r^2}\right)^{\frac{2\Phi_r}{\sigma_r^2} - 1} \times \mathtt{exp}\Bigg\{ - \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\alpha_\mu(z_\mu - 1)}\mu \Bigg\}$$
$$\mathtt{exp}\left\{ - \left(\frac{2\kappa_r \varrho}{\sigma_r^2}\right) r \right\} dr d\mu \qquad \text{(A2.16)}$$

using the properties of the Dirac Delta function, we can simplify,

$$g(\zeta_r) = \frac{-\kappa_r}{\Gamma(\frac{2\Phi_r}{\sigma_r^2} - 1)} \left(\frac{2\kappa_r r_0}{\sigma_r^2}\right)^{\frac{2\Phi_r}{\sigma_r^2} - 1} \times \mathtt{exp}\Bigg\{ - \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\mu \Bigg\} \mathtt{exp}\left\{ - \left(\frac{2\kappa_r \zeta_r}{\sigma_r^2}\right) r_0 \right\} \qquad \text{(A2.17)}$$

We can then substitute our results into equation (A2.11) to get an explicit form of $f_1(t, s_\mu)$

$$f_1(t, s_\mu) = \frac{-\kappa_r}{\Gamma(\frac{2\Phi_r}{\sigma_r^2} - 1)} \left(\frac{2\kappa_r r_0}{\sigma_r^2}\right)^{\frac{2\Phi_r}{\sigma_r^2} - 1} \times \mathtt{exp}\Bigg\{ - \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\mu_0 \Bigg\} \times \mathtt{exp}\left\{ - \left(\frac{2\kappa_r \zeta_r}{\sigma_r^2}\right) r_0 \right\}$$
$$\times \frac{\zeta_r(\zeta_r - 1)}{\zeta_r^{\frac{2(\sigma_r^2 - \Phi_r)}{\sigma_r^2}}} \times \left(\frac{\zeta_\mu[\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)]}{\sigma_\mu^2 s_\mu(\zeta_\mu - z_\mu) + 2\kappa_\mu \zeta_\mu(z_\mu - 1)}\right)^{\frac{2(\Phi_\mu - \sigma_\mu^2)}{\sigma_\mu^2}} \times \mathtt{exp}\Big[\{\kappa_r + \kappa_\mu\}t\Big] \qquad \text{(A2.18)}$$

By a similar operation we can show

$$f_2(t,s_r) = \frac{-\kappa_\mu}{\Gamma(\frac{2\Phi_\mu}{\sigma_\mu^2}-1)}\left(\frac{2\kappa_\mu\mu_0}{\sigma_\mu^2}\right)^{\frac{2\Phi_\mu}{\sigma_\mu^2}-1} \times \texttt{exp}\left\{-\frac{2s_r\kappa_r z_r}{\sigma_r^2 s_r + 2\kappa_r(z_r-1)}r_0\right\} \times \texttt{exp}\left\{-\left(\frac{2\kappa_\mu\zeta_\mu}{\sigma_\mu^2}\right)\mu_0\right\}$$
$$\times \frac{\zeta_\mu(\zeta_\mu-1)}{\zeta_\mu^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}}} \times \left(\frac{\zeta_r[\sigma_r^2 s_r + 2\kappa_r(z_r-1)]}{\sigma_r^2 s_r(\zeta_r - z_r) + 2\kappa_r\zeta_r(z_r-1)}\right)^{\frac{2(\Phi_r-\sigma_r^2)}{\sigma_r^2}} \times \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}t\Big] \tag{A2.19}$$

The first component on the RHS of equantion (A2.8) can be written as

$$J_1 = \left(\frac{2\kappa_r e^{-\kappa_r\psi}}{\sigma_r^2 s_r(1-e^{-\kappa_r\psi}) + 2\kappa_r e^{-\kappa_r\psi}}\right)^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}} \times \left(\frac{2\kappa_\mu e^{-\kappa_\mu\psi}}{\sigma_\mu^2 s_\mu(1-e^{-\kappa_\mu\psi}) + 2\kappa_\mu e^{-\kappa_\mu\psi}}\right)^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}}$$
$$\times \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big] \times \widetilde{G}\left(0, \frac{2s_r\kappa_r}{\sigma_r^2 s_r(1-e^{-\kappa_r\psi}) + 2\kappa_r e^{-\kappa_r\psi}}, \frac{2s_\mu\kappa_\mu}{\sigma_\mu^2 s_\mu(1-e^{-\kappa_\mu\psi}) + 2\kappa_\mu e^{-\kappa_\mu\psi}}\right)$$

Making use of equation (A2.9), we can write

$$J_1 = \left(\frac{2\kappa_r(z_r-1)}{\sigma_r^2 s + 2\kappa_r(z_r-1)}\right)^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}} \times \left(\frac{2\alpha_\mu(z_\mu-1)}{\sigma_\mu^2 s + 2\kappa_\mu(z_\mu-1)}\right)^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}}$$
$$\texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big] \times \widetilde{G}\left(0, \frac{2s_r\kappa_r z_r}{\sigma_r^2 s + 2\kappa_r(z_r-1)}, \frac{2s_\mu\kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu-1)}\right)$$

Substituting the initial conditions equation (3.11) yields

$$J_1 = \left(\frac{2\kappa_r(z_r-1)}{\sigma_r^2 s + 2\kappa_r(z_r-1)}\right)^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}} \times \left(\frac{2\kappa_\mu(z_\mu-1)}{\sigma_\mu^2 s + 2\kappa_\mu(z_\mu-1)}\right)^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}}$$
$$\texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big] \times \texttt{exp}\left\{-\frac{2s_r\kappa_r z_r}{\sigma_r^2 s + 2\kappa_r(z_r-1)}r_0 - \frac{2s_\mu\kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu-1)}\mu_0\right\}$$

The second component on the RHS of equation (A2.8) can be written as

$$J_2 = \int_{z_r}^{\infty}\Big[f_1(t,s_\mu))\Big] \times \left(\frac{2\kappa_r\zeta_r(z_r-1)}{\sigma_r^2 s(\zeta_r-z_r) + 2\kappa_r\zeta_r(z_r-1)}\right)^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}}$$
$$\times \left(\frac{2\kappa_\mu\zeta_\mu(z_\mu-1)}{\sigma_\mu^2 s_\mu(\zeta_\mu-z_\mu) + 2\kappa_\mu\zeta_\mu(z_\mu-1)}\right)^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}} \times \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}(\psi-t)\Big]\frac{d\zeta_r}{\kappa_r\zeta_r(\zeta_r-1)}$$

Substituting equation (A2.18) yields

$$J_2 = \frac{-1}{\Gamma(\frac{2\Phi_r}{\sigma_r^2}-1)} \int_{z_r}^{\infty} \left(\frac{2\kappa_r r_0}{\sigma_r^2}\right)^{\frac{2\Phi_r}{\sigma_r^2}-1} \times \texttt{exp}\Bigg\{ - \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\mu_0\Bigg\} \times \texttt{exp}\left\{-\left(\frac{2\kappa_r \zeta_r}{\sigma_r^2}\right) r_0\right\}$$
$$\times \left(\frac{2\kappa_\mu(z_\mu-1)}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu-1)}\right)^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}} \times \left(\frac{2\kappa_r(z_r-1)}{\sigma_r^2 s(\zeta_r - z_r) + 2\kappa_r\zeta_r(z_r-1)}\right)^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}} \times \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big] d\zeta_r$$

Rearranging gives

$$\begin{aligned} J_2 =& \frac{-[2\kappa_r(z_r-1)]^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}}}{\Gamma(\frac{2\Phi_r}{\sigma_r^2}-1)} \left(\frac{2\kappa_r r_0}{\sigma_r^2}\right)^{\frac{2\Phi_r}{\sigma_r^2}-1} \times \texttt{exp}\Bigg\{ - \frac{2s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu - 1)}\mu_0\Bigg\} \\ & \times \left(\frac{2\kappa_\mu(z_\mu-1)}{\sigma_\mu^2 s_1 + 2\kappa_\mu(z_\mu-1)}\right)^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}} \times \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big] G_1(r_0) \end{aligned} \tag{A2.20}$$

where

$$G_1(r) = \int_{z_r}^{\infty} \texttt{exp}\left\{-\left(\frac{2\kappa_r r}{\sigma_r^2}\right)\zeta_r\right\} \Big[\sigma_r^2 s_r(\zeta_r - z_r) + 2\kappa_r\zeta_r(z_r-1)\Big]^{\frac{2(\Phi_r-\sigma_r^2)}{\sigma_r^2}} d\zeta_r \tag{A2.21}$$

Let $y = \sigma_r^2 s(\zeta_r - z_r) + 2\kappa_r\zeta_r(z_r-1)$, so that

$$d\zeta_r = \frac{dy}{\sigma_r^2 s_r + 2\kappa_r(z_r-1)}$$

substituting into equation (A2.21) we obtain

$$\begin{aligned} G_1(r) =& \frac{1}{\sigma_r^2 s_r + 2\kappa_r(z_r-1)} \texttt{exp}\left\{-\left(\frac{2\kappa_r r}{\sigma_r^2}\right)\left(\frac{\sigma_r^2 s_r z_r}{\sigma_r^2 s + 2\kappa_r(z_r-1)}\right)\right\} \\ & \int_{2\kappa_r z_r(z_r-1)}^{\infty} \texttt{exp}\left\{-\left(\frac{2\kappa_r r}{\sigma_r^2}\right)\left(\frac{y}{\sigma_r^2 s_r + 2\kappa_r(z_r-1)}\right)\right\} y^{\frac{2(\Phi_r-\sigma_r^2)}{\sigma_r^2}} dy \end{aligned} \tag{A2.22}$$

Now let

$$\xi = \left(\frac{2\kappa_r r}{\sigma_r^2}\right)\left(\frac{y}{\sigma_r^2 s + 2\kappa_r(z_r-1)}\right) = \left(\frac{2\kappa_r r y}{\sigma_r^2\left[\sigma_r^2 s + 2\kappa_r(z_r-1)\right]}\right)$$

This implies

$$dy = \frac{\sigma_r^2[\sigma_r^2 s_r + 2\kappa_r(z_r-1)]}{2\kappa_r r} d\xi$$

substituting into equation (A2.22) yields

$$G_1(r) = \frac{\sigma_r^2}{2\kappa_r r}\texttt{exp}\left\{-\left(\frac{2\kappa_r r}{\sigma_r^2}\right)\left(\frac{\sigma_r^2 s_r z_r}{\sigma_r^2 s + 2\kappa_r(z_r-1)}\right)\right\}$$
$$\left[\frac{\sigma_r^2[\sigma_r^2 s + 2\kappa_r(z_r-1)]}{2\kappa_r r}\right]^{\left(\frac{2\Phi_r}{\sigma_r^2}-2\right)}\int_{\frac{4\kappa_r^2 r z_r(z_r-1)}{\sigma_r^2 s + 2\kappa_r(z_r-1)}}^{\infty}\texttt{exp}\{-\xi\}\,\xi^{\left(\frac{2\Phi_r}{\sigma_r^2}-1\right)-1}d\xi$$

rearranging and inserting the Gamma function

$$G_1(r) = \left(\sigma_r^2 s + 2\kappa_r(z_r-1)\right)^{\left(\frac{2\Phi_r}{\sigma_r^2}-2\right)}\left(\frac{\sigma_r^2}{2\kappa_r r}\right)^{\left(\frac{2\Phi_r}{\sigma_r^2}-1\right)}\texttt{exp}\left\{-\left(\frac{2\kappa_r r}{\sigma_r^2}\right)\left(\frac{\sigma_r^2 s_r z_r}{\sigma_r^2 s + 2\kappa_r(z_r-1)}\right)\right\}$$
$$\left[\Gamma\left(\frac{2\Phi_r}{\sigma_r^2}-1\right) - \int_0^{\frac{4\kappa_r^2 r z_r(z_r-1)}{\sigma_r^2 s_r + 2\kappa_r(z_r-1)}}\texttt{exp}\{-\xi\}\,\xi^{\left(\frac{2\Phi_r}{\sigma_r^2}-1\right)-1}d\xi\right]$$

substituting back into equation (A2.20) gives

$$J_2 = \frac{-[2\kappa_r(z_r-1)]^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}}}{\Gamma(\frac{2\Phi_r}{\sigma_r^2}-1)}\left(\frac{2\kappa_r r_0}{\sigma_r^2}\right)^{\frac{2\Phi_r}{\sigma_r^2}-1}\times\texttt{exp}\left\{-\frac{2s_\mu\kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu-1)}\mu_0\right\}$$
$$\times\left(\frac{2\kappa_\mu(z_\mu-1)}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu-1)}\right)^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}}\times\texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big]\left(\sigma_r^2 s + 2\kappa_r(z_r-1)\right)^{\left(\frac{2\Phi_r}{\sigma_r^2}-2\right)}\left(\frac{\sigma_r^2}{2\kappa_r r_0}\right)^{\left(\frac{2\Phi_r}{\sigma_r^2}-1\right)}$$
$$\texttt{exp}\left\{-\left(\frac{2\kappa_r r_0}{\sigma_r^2}\right)\left(\frac{\sigma_r^2 s z}{\sigma_r^2 s + 2\kappa_r(z_r-1)}\right)\right\}\left[\Gamma\left(\frac{2\Phi_r}{\sigma_r^2}-1\right) - \int_0^{\frac{4\kappa_r^2 r z_r(z_r-1)}{\sigma_r^2 s_r + 2\kappa_r(z_r-1)}}\texttt{exp}\{-\xi\}\,\xi^{\left(\frac{2\Phi_r}{\sigma_r^2}-1\right)-1}d\xi\right] \tag{A2.23}$$

this reduces to

$$J_2 = \frac{-1}{\Gamma(\frac{2\Phi_r}{\sigma_r^2}-1)}\times\texttt{exp}\left\{-\frac{2\kappa_r s_r z_r}{\sigma_r^2 s_r + 2\kappa_r(z_r-1)}r_0 - \frac{2s_\mu\kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu-1)}\mu_0\right\}$$
$$\times\left(\frac{2\kappa_r(z_r-1)}{\sigma_r^2 s + 2\kappa_r(z_r-1)}\right)^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}}\times\left(\frac{2\kappa_\mu(z_\mu-1)}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu-1)}\right)^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}}$$
$$\times\texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big]\times\Gamma\left(\frac{2\Phi_r}{\sigma_r^2}-1\right)\left[1-\Gamma\left(\frac{2\Phi_r}{\sigma_r^2}-1;\frac{4\kappa_r^2 z_r(z_r-1)}{\sigma_r^2 s + 2\kappa_r(z_r-1)}r_0\right)\right] \tag{A2.24}$$

Similarly the third component on the RHS of equation (A2.8) can be written as

$$
\begin{aligned}
J_3 =& \frac{-1}{\Gamma(\frac{2\Phi_\mu}{\sigma_\mu^2}-1)} \times \texttt{exp}\Bigg\{ -\frac{2\kappa_r s_r z_r}{\sigma_r^2 s_r + 2\kappa_r(z_r-1)} r_0 - \frac{2 s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu-1)}\mu_0 \Bigg\} \\
& \times \left(\frac{2\kappa_r(z_r-1)}{\sigma_r^2 s + 2\kappa_r(z_r-1)} \right)^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}} \times \left(\frac{2\alpha_\mu(z_\mu-1)}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu-1)} \right)^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}} \\
& \times \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big] \times \Gamma\left(\frac{2\Phi_\mu}{\sigma_\mu^2}-1\right)\left[1-\Gamma\left(\frac{2\Phi_\mu}{\sigma_\mu^2}-1; \frac{4\alpha_\mu^2 z_\mu(z_\mu-1)}{\sigma_\mu^2 s + 2\kappa_\mu(z_\mu-1)}\mu_0\right)\right]
\end{aligned}
\tag{A2.25}
$$

By combining $J1$, $J2$ and $J3$ equation (A2.8) becomes

$$
\begin{aligned}
\widetilde{G} =& \left(\frac{2\kappa_r(z_r-1)}{\sigma_r^2 s + 2\kappa_r(z_r-1)} \right)^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}} \times \left(\frac{2\kappa_\mu(z_\mu-1)}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu-1)} \right)^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}} \\
& \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big] \times \texttt{exp}\Bigg\{ -\frac{2 s_r \kappa_r z_r}{\sigma_r^2 s_r + 2\kappa_r(z_r-1)} r_0 - \frac{2 s_\mu \kappa_\mu z_\mu}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu-1)}\mu_0 \Bigg\} \\
& \times \left[\Gamma\left(\frac{2\Phi_r}{\sigma_r^2}-1; \frac{4\kappa_r^2 z_r(z_r-1)}{\sigma_r^2 s_r + 2\kappa_r(z_r-1)} r_0\right) + \Gamma\left(\frac{2\Phi_\mu}{\sigma_\mu^2}-1; \frac{4\kappa_\mu^2 z_\mu(z_\mu-1)}{\sigma_\mu^2 s_\mu + 2\kappa_\mu(z_\mu-1)}\mu_0\right) - 1 \right]
\end{aligned}
$$

and finally our solution is,

$$
\begin{aligned}
\widetilde{G} =& \left(\frac{2\kappa_r}{\sigma_r^2 s(e^{\kappa_r\psi}-1) + 2\kappa_r} \right)^{\frac{2(\sigma_r^2-\Phi_r)}{\sigma_r^2}} \times \left(\frac{2\kappa_\mu}{\sigma_\mu^2 s_\mu(e^{\kappa_\mu\psi}-1) + 2\kappa_\mu} \right)^{\frac{2(\sigma_\mu^2-\Phi_\mu)}{\sigma_\mu^2}} \\
& \texttt{exp}\Big[\{\kappa_r+\kappa_\mu\}\psi\Big] \times \texttt{exp}\Bigg\{ -\frac{2 s_r \kappa_r e^{\kappa_r\psi}}{\sigma_r^2 s(e^{\kappa_r\psi}-1) + 2\kappa_r} r_0 - \frac{2 s_\mu \kappa_\mu e^{\kappa_\mu\psi}}{\sigma_\mu^2 s_\mu(e^{\kappa_\mu\psi}-1) + 2\kappa_\mu}\mu_0 \Bigg\} \\
& \times \Bigg[\Gamma\left(\frac{2\Phi_r}{\sigma_r^2}-1; \frac{4\kappa_r^2 e^{\kappa_r\psi}}{(e^{\kappa_r\psi}-1)(\sigma_r^2 s(e^{\kappa_r\psi}-1)+2\kappa_r)} r_0\right) \\
& + \Gamma\left(\frac{2\Phi_\mu}{\sigma_\mu^2}-1; \frac{4\kappa_\mu^2 e^{\kappa_\mu\psi}}{(e^{\kappa_\mu\psi}-1)(\sigma_\mu^2 s_\mu(e^{\kappa_\mu\psi}-1)+2\kappa_\mu)}\mu_0\right) - 1 \Bigg]
\end{aligned}
\tag{A2.26}
$$

Appendix 3. Proof of Proposition 3.4

Start by making the following transforms

$$
\begin{aligned}
A_r =& \frac{2\kappa_r r_0}{\sigma_r^2(1-e^{-\kappa_r\psi})}, \qquad A_\mu = \frac{2\kappa_\mu \mu_0}{\sigma_\mu^2(1-e^{-\kappa_\mu\psi})} \\
z_r =& \frac{\sigma_r^2 s(e^{\kappa_r\psi}-1) + 2\kappa_r}{2\kappa_r}, \qquad z_\mu = \frac{\sigma_\mu^2 s_\mu(e^{\kappa_\mu\psi}-1) + 2\kappa_\mu}{2\kappa_\mu}
\end{aligned}
\tag{A3.1}
$$

and

$$h = \texttt{exp}\Big[\{\kappa_r + \kappa_\mu\}\psi\Big] \tag{A3.2}$$

We can rewrite equation (A2.26) as

$$\begin{aligned}\widetilde{G} =& h \times z_r^{\frac{2\Phi_r}{\sigma_r^2}-2} \times z_\mu^{\frac{2\Phi_\mu}{\sigma_\mu^2}-2} \texttt{exp}\left\{-\frac{A_r}{z_r}(z_r-1)\right\} \times \texttt{exp}\left\{-\frac{A_\mu}{z_\mu}(z_\mu-1)\right\} \\ &\times \left[\Gamma\left(\frac{2\Phi_r}{\sigma_r^2}-1; \frac{A_r}{z_r}\right) + \Gamma\left(\frac{2\Phi_\mu}{\sigma_\mu^2}-1; \frac{A_\mu}{z_\mu}\right) - 1\right] \\ =& h \times z_r^{\frac{2\Phi_r}{\sigma_r^2}-2} \times z_\mu^{\frac{2\Phi_\mu}{\sigma_\mu^2}-2} \texttt{exp}\left\{-\frac{A_r}{z_r}(z_r-1)\right\} \times \texttt{exp}\left\{-\frac{A_\mu}{z_\mu}(z_\mu-1)\right\} \\ &\times \left[\frac{1}{\Gamma\left(\frac{2\Phi_r}{\sigma_r^2}-1\right)} \int_0^{\frac{A_r}{z_r}} e^{-\beta_r} \beta_r^{\frac{2\Phi_r}{\sigma_r^2}-2} d\beta_r + \frac{1}{\Gamma\left(\frac{2\Phi_\mu}{\sigma_\mu^2}-1\right)} \int_0^{\frac{A_\mu}{z_\mu}} e^{-\beta_\mu} \beta_\mu^{\frac{2\Phi_\mu}{\sigma_\mu^2}-2} d\beta_\mu - 1\right]\end{aligned}$$

Separate $\widetilde{G}$ into 3 parts

$$\widetilde{F}_1 = h \times z_r^{\frac{2\Phi_r}{\sigma_r^2}-2} \times z_\mu^{\frac{2\Phi_\mu}{\sigma_\mu^2}-2} \texttt{exp}\left\{-\frac{A_r}{z_r}(z_r-1)\right\} \times \texttt{exp}\left\{-\frac{A_\mu}{z_\mu}(z_\mu-1)\right\} \times \frac{1}{\Gamma\left(\frac{2\Phi_r}{\sigma_r^2}-1\right)} \int_0^{\frac{A_r}{z_r}} e^{-\beta_r} \beta_r^{\frac{2\Phi_r}{\sigma_r^2}-2} d\beta_r$$

$$\widetilde{F}_2 = h \times z_r^{\frac{2\Phi_r}{\sigma_r^2}-2} \times z_\mu^{\frac{2\Phi_\mu}{\sigma_\mu^2}-2} \texttt{exp}\left\{-\frac{A_r}{z_r}(z_r-1)\right\} \times \texttt{exp}\left\{-\frac{A_\mu}{z_\mu}(z_\mu-1)\right\} \times \frac{1}{\Gamma\left(\frac{2\Phi_\mu}{\sigma_\mu^2}-1\right)} \int_0^{\frac{A_\mu}{z_\mu}} e^{-\beta_\mu} \beta_\mu^{\frac{2\Phi_\mu}{\sigma_\mu^2}-2} d\beta_\mu$$

$$\widetilde{F}_3 = -h \times z_r^{\frac{2\Phi_r}{\sigma_r^2}-2} \times z_\mu^{\frac{2\Phi_\mu}{\sigma_\mu^2}-2} \texttt{exp}\left\{-\frac{A_r}{z_r}(z_r-1)\right\} \times \texttt{exp}\left\{-\frac{A_\mu}{z_\mu}(z_\mu-1)\right\}$$

Starting with $\widetilde{F}_1$, let $\xi = 1 - \frac{z_r}{A_r}\beta_r$ then

$$\widetilde{F}_1 = h \times e^{-(A_r+A_\mu)} \times \frac{A_r^{\frac{2\Phi_r}{\sigma_r^2}-1}}{\Gamma\left(\frac{2\Phi_r}{\sigma_r^2}-1\right)} e^{\frac{A_\mu}{z_\mu}} z_\mu^{\frac{2\Phi_\mu}{\sigma_\mu^2}-2} \times \int_0^{e^{\frac{A_r}{z_r}}} (1-\xi)^{\frac{2\Phi_r}{\sigma_r^2}-2} z_r^{-1} e^{-\frac{A_r}{z_r}\xi} d\xi \tag{A3.3}$$

from equation (A3.1) we can express the variable s_r and s_μ as

$$s_r = \frac{2\kappa_r(z_r-1)}{\sigma_r^2(e^{\kappa_r\psi}-1)}, \quad s_\mu = \frac{2\alpha_\mu(z_\mu-1)}{\sigma_\mu^2(e^{\kappa_\mu\psi}-1)}$$

representing the Laplace transform as

$$\mathcal{L}\{\hat{F}_1\} = \int_0^\infty \int_0^\infty \texttt{exp}\left\{-\frac{2\kappa_r(z_r-1)}{\sigma_r^2(e^{\kappa_r\psi}-1)} r\right\} \times \texttt{exp}\left\{-\frac{2\kappa_\mu(z_\mu-1)}{\sigma_\mu^2(e^{\kappa_\mu\psi}-1)}\mu\right\} \hat{F}_1 dr d\mu$$

substituting

$$y_r = \frac{2\kappa_r r}{\sigma_r^2(e^{\kappa_r\psi}-1)}, \quad y_\mu = \frac{2\kappa_\mu \mu}{\sigma_\mu^2(e^{\kappa_\mu\psi}-1)} \tag{A3.4}$$

then

$$\mathcal{L}\{\hat{F}_1\} = \frac{\sigma_r^2(e^{\kappa_r\psi}-1)}{2\kappa_r}\frac{\sigma_\mu^2(e^{\kappa_\mu\psi}-1)}{2\kappa_\mu}\int_0^\infty\int_0^\infty e^{-y_r z_r}e^{-y_\mu z_\mu}\times e^{y_r}e^{y_\mu}\hat{F}_1 dy_r$$
$$= \frac{\sigma_r^2(e^{\kappa_r\psi}-1)}{2\kappa_r}\frac{\sigma_\mu^2(e^{\kappa_\mu\psi}-1)}{2\kappa_\mu}\mathcal{L}\{e^{y_r}e^{y_\mu}\hat{H}\}$$

and in terms of the inverse Laplace transform

$$\mathcal{L}^{-1}\{\widetilde{F}_1\} = \frac{2\kappa_r}{\sigma_r^2(e^{\kappa_r\psi}-1)}\frac{2\kappa_\mu}{\sigma_\mu^2(e^{\kappa_\mu\psi}-1)}e^{-y_r}e^{-y_\mu}\mathcal{L}^{-1}\{\widetilde{F}_1\}$$

Applying the inverse transform to equation (A3.3) we obtain

$$\hat{F}_1 = h\times e^{-(A_r+A_\mu)}\times\frac{A_r^{\frac{2\Phi_r}{\sigma_r^2}-1}}{\Gamma\left(\frac{2\Phi_r}{\sigma_r^2}-1\right)}\times\frac{2\kappa_r}{\sigma_r^2(e^{\kappa_r\psi}-1)}\frac{2\kappa_\mu}{\sigma_\mu^2(e^{\kappa_\mu\psi}-1)}e^{-y_r}e^{-y_\mu}$$
$$\times\int_0^{e^{\frac{A_r}{z_r}}}(1-\xi)^{\frac{2\Phi_r}{\sigma_r^2}-2}\mathcal{L}^{-1}\left\{e^{\frac{A_\mu}{z_\mu}}z_\mu^{\frac{2\Phi_\mu}{\sigma_\mu^2}-2}\times e^{\frac{A_r}{z_r}\xi}z^{-1}\right\}d\xi$$

The inverse Laplace can be solved as

$$\hat{F}_1 = h\times e^{-(A_r+A_\mu)}\times\frac{A_r^{\frac{2\Phi_r}{\sigma_r^2}-1}}{\Gamma\left(\frac{2\Phi_r}{\sigma_r^2}-1\right)}\times\frac{2\kappa_r}{\sigma_r^2(e^{\kappa_r\psi}-1)}\frac{2\kappa_\mu}{\sigma_\mu^2(e^{\kappa_\mu\psi}-1)}e^{-y_r}e^{-y_\mu}$$
$$\times\left(\frac{y_\mu}{A_\mu}\right)^{\frac{1}{2}-\frac{\Phi_\mu}{\sigma_\mu^2}}I_{1-\frac{2\Phi_\mu}{\sigma_\mu^2}}(2\sqrt{A_\mu y_\mu})\int_0^1(1-\xi)^{\frac{2\Phi_r}{\sigma_r^2}-2}I_0(2\sqrt{A_r y_r\xi})d\xi$$

We can simplify

$$\int_0^1(1-\xi)^{\frac{2\Phi_r}{\sigma_r^2}-2}I_0(2\sqrt{A_r y_r\xi})d\xi = \Gamma\left(\frac{2\Phi_r}{\sigma_r^2}-1\right)(A_r y_r)^{\frac{1}{2}-\frac{\Phi_r}{\sigma_r^2}}I_{\frac{2\Phi_r}{\sigma_r^2}-1}(2\sqrt{A_r y_r})$$

on substitution

$$\hat{F}_1 = h\times e^{-(A_r+A_\mu)}\times\frac{2\kappa_r}{\sigma_r^2(e^{\kappa_r\psi}-1)}\frac{2\kappa_\mu}{\sigma_\mu^2(e^{\kappa_\mu\psi}-1)}e^{-y_r}e^{-y_\mu}\left(\frac{y_r}{A_r}\right)^{\frac{1}{2}-\frac{\Phi_r}{\sigma_r^2}}\times I_{1-\frac{2\Phi_r}{\sigma_r^2}}(2\sqrt{A_r y_r})$$
$$\times\left(\frac{y_\mu}{A_\mu}\right)^{\frac{1}{2}-\frac{\Phi_\mu}{\sigma_\mu^2}}I_{1-\frac{2\Phi_\mu}{\sigma_\mu^2}}(2\sqrt{A_\mu y_\mu})$$

similarly we can show $\hat{F}_2 = \hat{F}_1$ and $\hat{F}_3 = -\hat{F}_1$. This implies $\hat{G} = \hat{F}_1$. Substituting for (A3.1), (A3.2) and (A3.4) we obtain our result,

$$\begin{aligned}\hat{G} =& \mathtt{exp}\left\{ - \frac{2\kappa_r}{\sigma_r^2(e^{\kappa_r\psi}-1)}(r_0 e^{\kappa_r\psi}+r) - \frac{2\kappa_\mu}{\sigma_\mu^2(e^{\kappa_\mu\psi}-1)}(\mu_0 e^{\kappa_\mu\psi}+\mu)\right\} \\ &\times \left(\frac{r_0 e^{\kappa_r\psi}}{r}\right)^{\frac{\Phi_r}{\sigma_r^2}-\frac{1}{2}} \times \left(\frac{\mu_0 e^{\kappa_\mu\psi}}{\mu}\right)^{\frac{\Phi_\mu}{\sigma_\mu^2}-\frac{1}{2}} I_{\frac{2\Phi_r}{\sigma_r^2}-1}\left(\frac{4\kappa_r}{\sigma_r^2(e^{\kappa_r\psi}-1}(r\times r_0 e^{\kappa_r\psi})^{\frac{1}{2}}\right) \\ &\times \times I_{\frac{2\Phi_\mu}{\sigma_\mu^2}-1}\left(\frac{4\kappa_\mu}{\sigma_\mu^2(e^{\kappa_\mu\psi}-1}(\mu\times\mu_0 e^{\kappa_\mu\psi})^{\frac{1}{2}}\right) \times \frac{2\kappa_r e^{\kappa_r\psi}}{\sigma_r^2(e^{\kappa_r\psi}-1)}\frac{2\kappa_\mu e^{\kappa_\mu\psi}}{\sigma_\mu^2(e^{\kappa_\mu\psi}-1)}\end{aligned} \quad \text{(A3.5)}$$

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