When factors don't span their basis portfolios

Konark Saxena*

October 2014

ABSTRACT

It is common practice to construct factors by combining basis portfolios using prespecified simple weights, such as differencing the returns of two characteristic-sorted portfolios. I provide necessary conditions for such factors to produce unbiased alpha estimates: the rank of the basis portfolios' covariance matrix should be equal to the number of factors constructed. As this condition is rarely satisfied, I offer a method to combine basis portfolios into adjusted factors that correct for this bias. This adjustment significantly improves the root-mean-squared-error of anomalous trading strategy alphas, and cross-sectional tests yield estimates that are closer to theoretically predicted values. In addition, performance estimates of the aggregate mutual fund industry are sensitive to the adjustment.

Keywords: Factor Models, Abnormal Returns, Biased Estimators, Orthogonal Mean Variance Efficient Portfolios, HML, SMB

JEL Classification: C58, G12, G14

*Saxena is at the University of New South Wales, Sydney.

The idea that all systematic factors influencing asset prices can be captured by a small set of traded portfolios is appealing and used in many finance and economic applications. When economists want to measure the effect of a phenomenon or event on the value of firms, factor models often help them control for the effect of systematic variations. In such applications, it is of interest to understand the econometric properties of estimates of abnormal returns obtained from factor models. Recent papers by Lewellen, Nagel, and Shanken (2010), Kan, Robotti, and Shanken (2013), Harvey, Liu, and Zhu (2014), and Pukthuanthong and Roll (2014) propose methods to improve empirical tests when evaluating such models. Kogan and Tian (2012) find that it is difficult to evaluate factor models based solely on their ex-post performance and highlight the need to evaluate the theoretical plausibility and empirical evidence in favor or against their main theoretical mechanisms.

I study the econometric restrictions imposed by a widely-used factor construction method. In an seminal study, Fama and French (1993) (hereafter FF) introduced a method to construct mimicking portfolios for underlying risk factors related to characteristics of interest. In this method, basis portfolios are first constructed using portfolios of individual securities ranked according to characteristics of interest. These basis portfolios are then combined into a smaller number of factors using prespecified simple weights. This intuitive method has been widely adopted by researchers attempting to control for expected returns in various dimensions. Carhart (1997) (hereafter FFC), Chen, Novy-Marx, and Zhang (2010), Fama and French (2013), and Novy-Marx (2013) are examples of alternate factors constructed using a similar construction method.

Fama (1996) and Fama and French (1996) provide conditions when using *any* weights to combine basis portfolios into factors yields unbiased estimates: if the set of basis portfolios considered span all priced factors in the economy *and* all the basis portfolios are just linearly dependent portfolios of these underlying priced factors. Under these assumptions, any set of linearly independent portfolios can proxy for the correct priced factors in the economy as long as the correct number of linearly independent factors is chosen.

This paper provides a testable restriction to evaluate whether a factor model constructed using the Fama (1996) framework is likely to produce unbiased estimates: the rank of the covariance matrix of the basis portfolios should be equal to the number of factors constructed. When this necessary condition is not met, factor models constructed using most prespecified combinations of basis portfolios are likely to exclude a component required to span the mean-variance efficient (MVE) portfolio (an omitted-variable-bias) and to leave an unpriced component in the factors (an confounded-variable-bias). This is because the prespecified weights tend to impose invalid restrictions on the factor structure. These restrictions prevent the basis portfolios from spanning the MVE portfolio even when the unconstrained basis portfolios, and an adjusted factor model with the same number of factors, do span the MVE portfolio. Estimates of expected and abnormal returns from a factor model that does not satisfy this rank condition tend to be biased. For cases when this condition is not satisfied, which is usually the case, I propose an adjustment to the construction method that significantly reduces the likelihood that estimates are subject to these biases.

There are at least three scenarios in which this bias is likely to be present: (i) when there are unpriced factors in the economy, (ii) when there are unpriced state variables in the economy, or (iii) when the set of chosen basis portfolios are exposed to unpriced sources of variation.[1] In all these scenarios, the rank of the covariance matrix of basis portfolios is likely

[1]Covariance with any portfolio that has a zero-beta with respect to the MVE portfolio will have a zero risk premium. This is also true for the orthogonal portfolio on the efficient frontier of risky assets, which is likely to be well-diversified. Fama (1998) refers to any unpriced state variable as one whose mimicking

to be greater than the number of factors constructed. Using prespecified factor construction weights here is likely to result in a factor model producing biased estimates.

The influence of potential biases can be illustrated through a simple example: consider an economy with one priced state variable s_p and two basis portfolio processes r_l and r_s. Let the processes of these basis portfolios be $r_l = \mu_l + b_l s_p + s_u$ and $r_s = \mu_s + s_p$, where μ represents expected excess returns and s_u is a mean-zero unpriced state variable. In this setting, it is possible to construct a priced state variable mimicking portfolio r_p, which can be used to estimate abnormal returns. Any such factor should have a process of the form $r_p = \mu_p + k s_p$, where k is nonzero.[2] However, the use of simple prespecified weights, +1 and -1, to construct the state variable hedge portfolio will result in a mimicking portfolio r_{ls} whose process is $r_{ls} = \mu_l - \mu_s + (b_l - 1)s_p + s_u$. If r_{ls} is assumed to proxy for r_p then estimates of β_i for any security i obtained using this proxy will be subject to an error-in-variables downward bias due to the s_u component of r_{ls}. If s_p has a positive risk premium, this procedure will induce an upward bias in estimates of α_i. If security i is also positively correlated with s_u then this procedure will induce an confounded-variable-bias (with s_u as the confounding variable) that produces upward biased estimates of β_i and downward biased estimates of α_i. If $b_l = 1$, then the omission of s_p will introduce a bias, the direction of which will depend on the sign of the risk premium of s_p.

The misspecification in the example could have been detected by estimating the rank of the covariance matrix of r_l and r_s (which is two) and observing that it is not equal to the number of factors constructed (which is one). When this rank condition is not met, I

portfolio has an expected risk premium that is completely explained by the market premium and a set of mimicking portfolios of the priced state variables.

[2]Huberman, Kandel, and Stambaugh (1987) refer to traded portfolios that can be used in place of the original state variables for pricing assets as mimicking portfolios.

propose an adjustment to identify optimal weights to construct factors. In this example, optimal weights of 0 and +1 could be estimated and imposed to obtain a factor that is not exposed to s_u and thereby provides consistent estimates of abnormal returns.[3] Evidence from simulations suggests that the estimator proposed works as well as the unadjusted estimator when the prespecified simple factor construction weights are assumed to be optimal. However, when the adjusted weights are assumed to be optimal, factors constructed using prespecified weights produce biased estimates.

Welch (2008) finds that the benchmark FF factors don't span their basis portfolios. This implies that the rank condition is not satisfied, which further predicts a bias in estimates from the factors due to an omitted spanning component and an included unpriced component. Gerakos and Linnainmaa (2014) find evidence for such a bias in the FF factors—they detect an unpriced component in the HML factor that distorts inferences. Cremers, Petajisto, and Zitzewitz (2013) find that one of the main causes of estimates of such nonzero alphas of money managers are the weights used to construct the factors. They suggest that this is because three factors are not enough to span the returns across the basis portfolios of the FF factors. I characterize the misspecification when three factors *are* enough to span the basis portfolios, and the misspecification is only due to the choice of weights, not due to more priced state variables than considered by the factor model. I provide expressions for the potential bias due to prespecified simple weights and identify alternate weights to combine the same basis portfolios into factors that are not subject to this risk.

[3] Note that the weights do not need to sum to zero as the processes represent excess returns. If these were returns, then the remaining appropriate amount of the riskless asset could be subtracted to ensure that the factor is zero-investment portfolio.

I. Characterization of the bias

Consider an economy with N traded securities, with excess returns of security i denoted by $r_{i,t}$ and S uncorrelated state variables that influence security returns. A riskless security is assumed to exist. Investors require a nonzero risk premium over the riskless rate to hold securities exposed to a *priced state variable*, defined as one whose state-variable mimicking portfolio or multifactor minimum-variance (MMV) portfolio has a nonzero risk premium.[4] Let the number of priced state variable be $S_{\mathtt{P}}$ and the number of state variables whose MMV portfolios have zero-beta with respect to the tangency MVE portfolio be $S_{\mathtt{Z}} = S - S_{\mathtt{P}}$. The realization of these state variables at time t is denoted by $\boldsymbol{s}_{\mathtt{P}}$ and $\boldsymbol{s}_{\mathtt{Z}}$.

Security excess returns are described by the following process:

$$r_{i,t} = \mu_i + \beta_{i,\mathrm{m}}(r_{m,t} - \mu_m) + \boldsymbol{\beta}_{i,\mathtt{P}}\boldsymbol{s}_{\mathtt{P},t} + \boldsymbol{\beta}_{i,\mathtt{Z}}\boldsymbol{s}_{\mathtt{Z},t} + \eta_{i,t}. \tag{1}$$

where μ_i is the expected excess return of security i, $\mathbf{r}_m$ is the excess market return, and $\beta_{i,j}$ is the sensitivity of the return of security i to variable j. Variables in bold denote vectors or matrices of convenient dimensions. Without loss of generality, the state variables are assumed to mean zero and identity covariance matrix. This process requires that $\mathbf{r}_m$ is also an MMV portfolio.[5]

The process in Equation 1 can also be expressed in terms of the MMV portfolios $\mathbf{f}_{\mathtt{P},t}$,

[4]Fama (1996) refers to portfolios that have minimum variance given their sensitivity to the state variables as MMV portfolios. Any portfolio f that minimizes σ_f^2, given μ_f, $\boldsymbol{\beta}_{f,\mathtt{P}}$, and $\boldsymbol{\beta}_{f,\mathtt{Z}}$ is refereed to as an He defines an MMV portfolio. The two MMV portfolios with $\boldsymbol{\beta}_{f,\mathtt{P}} = 0$ and $\boldsymbol{\beta}_{f,\mathtt{Z}} = 0$ are the MVE portfolio or the global minimum variance (GMV) portfolio.

[5]Fama (1996) assumes market clearing to motivate this assumption in a discrete-time version of the Merton (1973) ICAPM. There also exists an equivalent Ross (1976) *APT* representation where $\mathbf{r}_m$, $\boldsymbol{s}_{\mathtt{P}}$, and $\boldsymbol{s}_{\mathtt{Z}}$ are the common factors in the economy.

a $S_P \times 1$ vector of linearly independent portfolios that have minimum variance given their nonzero sensitivity to one state variable in $\boldsymbol{s}_{\mathtt{P},t}$ and zero to the others:

$$r_{i,t} = \mu_i + \beta_{i,\mathrm{m}}(r_{m,t} - \mu_m) + \boldsymbol{\beta}_{i,\mathtt{P}}(\mathbf{f}_{\mathtt{P},t} - \boldsymbol{\lambda}_{\mathtt{P}}) + \epsilon_{i,t}, \tag{2}$$

where $\epsilon_{\mathtt{i},t}$ denotes the cumulative returns of the zero-beta state variable exposures ($\boldsymbol{\beta}_{i,\mathtt{Z}}\boldsymbol{s}_{\mathtt{Z},t}$) and specific returns ($\eta_{i,t}$) of securities i. $\boldsymbol{\lambda}_{\mathtt{P}} = \mathrm{E}[\mathbf{f}_{\mathtt{P},t}]$ is the risk premium vector ($S_{\mathtt{P}} \times 1$) for the priced state variables.

In this framework, the risk return relation of any security or portfolio i is given by:

$$\mu_i = \beta_{i,\mathrm{m}}\mu_m + \boldsymbol{\beta}_{i,\mathtt{P}}\boldsymbol{\lambda}_{\mathtt{P}} = \beta_{i,\mathrm{m}}\mathrm{E}\left[r_{m,t}\right] + \boldsymbol{\beta}_{i,\mathtt{P}}\mathrm{E}\left[\mathbf{f}_{\mathtt{P},t}\right], \tag{3}$$

where the MVE portfolio of risky assets is spanned by the $\mathbf{r}_{m,t}$ and $\mathbf{f}_{\mathtt{P},t}$[6].

Note that the set of factors $\mathbf{f}_{\mathtt{P}}$ consistent with Equation 3 are unique only up to an orthogonal transformation. It is convenient to denote any portfolio of $\mathbf{f}_{\mathtt{P}}$ and r_m as $r_{\mathtt{p}}$, which is referred to as a priced MMV (PMMV) portfolio. Further, denote any linearly independent set of $K(= S_{\mathtt{P}} + 1)$ PMMV portfolios as $\mathbf{r}_{\mathtt{p}}$, where:

$$\mathbf{r}_{\mathtt{p}} = \mathbf{W}'[r_{m,t}\ \mathbf{f}_{\mathtt{P},t}], \tag{4}$$

and $\mathbf{W}$ is a $K \times K$ invertible matrix.

[6]The MMV portfolios that only load on the state variables $\boldsymbol{s}_{\mathtt{Z},t}$ with minimum variance are, by definition, zero beta portfolios with respect to the MVE portfolio spanned by $\mathbf{r}_{m,t}$ and $\mathbf{f}_{\mathtt{P},t}$. These portfolios are orthogonal to the tangency portfolio from the riskless asset on the mean standard deviation plane and are located on the line parallel to the standard-deviation axis, which intersects the riskless rate. The minimum-variance portfolio of risky assets, which has the same return as the riskless rate, is an example of an efficient (and probably well-diversified) risky portfolio that is a potential candidate for $\mathbf{f}_{\mathtt{Z}}$

A. Estimation of expected returns using a set of basis portfolios

Denote the difference between the true expected return and its average predicted value using a set of factors $\mathbf{f}_{\mathtt{x},t}$ as $\Delta\mu_{i,\mathtt{x}} \equiv \mu_i - E[\hat{\mu}_{i,\mathtt{x}}]$ and the difference between the true unpriced variance and its average predicted value as $\Delta\sigma_{\epsilon_i,\mathtt{x}} \equiv \sigma_{\epsilon_i} - E[\hat{\sigma}_{\epsilon_i,\mathtt{x}}]$. The goal is to understand the properties of $\Delta\mu_{i,\mathtt{x}}$ and $\Delta\sigma_{\epsilon_i,\mathtt{x}}$, when the factors are constructed using the following commonly used method:

Step 1: A vector of firm-level characteristics $\mathbf{d}_i$ is identified that produces a dispersion in average returns[7]. Individual securities are ranked based on $\mathbf{d}_i$, as these characteristics are suspected to be correlated (imperfectly) with $\boldsymbol{\beta}_{i,\mathtt{P}}$. On the basis of these ranks, the securities are assigned to $C-1$ groups.

Step 2: A value-weighted (or equally weighted) portfolio of securities in each of the $C-1$ characteristic-sorted groups is constructed. Let $\mathbf{r}_{\mathtt{C},t}$ be the $C\times 1$ vector of excess returns of these $C-1$ portfolios and the market portfolio.

Step 3: The number of factors constructed (K) is chosen so that $K = S_{\mathtt{P}} + 1 \leq C$. Use a nonsingular $K \times C$ weight matrix $\mathbf{A}_{\mathtt{x}}$ to combine $\mathbf{r}_{\mathtt{C},t}$ into a $K \times 1$ vector of factor returns $\mathbf{f}_{\mathtt{x},t} = \mathbf{A}_{\mathtt{x}}\mathbf{r}_{\mathtt{C},t}$, where the subscript $\mathtt{x}$ corresponds to matrix used to construct the factors.

Step 4: Use factor returns $\mathbf{f}_{\mathtt{x},t}$ and asset returns $r_{i,t}$ to estimate expected excess returns $\hat{\mu}_{i,x} = \widehat{\mathbf{B}}_{i,x}\overline{\mathbf{f}}_{\mathtt{x}}$ and average abnormal returns $\hat{\alpha}_{i,x} = \overline{r}_i - \hat{\mu}_{i,x}$, where the overline denotes the

[7] For example, Fama (1998) explains that the FF factors are formed to produce a large spread in average returns.

sample time-series average $\overline{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$, and the $1 \times K$ beta vector $\widehat{\mathbf{B}}_{i,\mathtt{x}}$ is estimated by:

$$\widehat{\mathbf{B}}_{i,\mathtt{x}} = \left[\frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{f}_{\mathtt{x},t} - \overline{\mathbf{f}}_{\mathtt{x},t}\right)\left(r_{i,t} - \overline{r}_i\right)\right]' \left[\frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{f}_{\mathtt{x},t} - \overline{\mathbf{f}}_{\mathtt{x},t}\right)\left(\mathbf{f}_{\mathtt{x},t} - \overline{\mathbf{f}}_{\mathtt{x},t}\right)'\right]^{-1} = \widehat{\mathbf{\Sigma}}_{i,\mathtt{x}}\widehat{\mathbf{\Sigma}}_{\mathtt{x}}^{-1}, \qquad (5)$$

The covariance matrix of the a vector of portfolios $\mathtt{x}$ is denoted as $\mathbf{\Sigma}_{\mathtt{x}}$ and the covariance between a security i and $\mathtt{x}$ as $\widehat{\mathbf{\Sigma}}_{i,\mathtt{x}}$. I first state assumptions under which any nonsingular weight matrix $\mathbf{A}_{\mathtt{x}}$ produces unbiased estimates ($\Delta\mu_{i,\mathtt{x}} = 0$, $\Delta\sigma_{\epsilon_i,\mathtt{x}} = 0$).

A.1. The rank condition that predicts the bias

The first assumption requires the chosen C portfolios span the excess returns of K linearly independent PMMV portfolios. For this reason, I refer to these portfolios as *basis portfolios.*

Assumption I: $\exists$ a full rank $K \times C$ matrix $\mathbf{A}_{\mathtt{P}}$ such that $[r_{m,t}\ \mathbf{f}_{\mathtt{P},t}]' = \mathbf{A}_{\mathtt{P}}\mathbf{r}_{\mathtt{C},t}$, where $K = S_{\mathtt{P}} + 1$, $K < C$, and the portfolios $\mathbf{f}_{\mathtt{P},t}$ and $r_{m,t}$ are linearly independent.

This assumption further implies that the chosen C portfolios span the MVE portfolio.

Further, since K is known, Assumption I also implies that $S_{\mathtt{P}}$ is known. Note that this assumption only requires the number of priced state variables ($S_{\mathtt{P}}$) to be known, it does not require the state variables to be identified or their state-variable mimicking portfolios to be known. Also note that this assumption does not imply that step 3 can be used to obtain factors that satisfy Equation 3. For any $K \times C$ full rank arbitrary matrix $\mathbf{A}_{\mathtt{x}}$ to produce the relevant priced factor mimicking portfolios, a stronger assumption is required. Fama (1996) assumes that *all* the C basis portfolios are PMMV. With this assumption, any linearly independent portfolio of these C basis portfolios is also PMMV.[8] I assume that the

[8] See Fama (1996) for proof.

basis portfolios under consideration do not satisfy this property:

Assumption II: $\exists$ a full rank $K \times C$ matrix $\mathbf{A}_{\mathtt{x}}$ such that $[r_{m,t}\ \mathbf{f}_{\mathtt{P},t}]' \neq \mathbf{Z}_{\mathtt{x}}\mathbf{A}_{\mathtt{x}}\mathbf{r}_{\mathtt{C},t}$ $\forall$ non-singular $K \times K$ matrix $\mathbf{Z}_{\mathtt{x}}$.

I refer to the factors constructed using any such arbitrary matrix $\mathbf{A}_{\mathtt{x}}$ as *unadjusted factors*, and denote them using a subscript $\mathtt{x}$: $\mathbf{f}_{\mathtt{x}}$. Assumption II is equivalent to assuming that the unadjusted factors $\mathbf{f}_{\mathtt{x},t}$ do not span the PMMV portfolios, and consequently do not span the space spanned by these portfolios.

Proposition 1. *If Assumption I holds and:*

1. *Assumption II does not hold, then:*
   (a) $\forall$ *arbitrary full rank* $K \times C$ *matrix* $\mathbf{A}_{\mathtt{x}}$:
      i. *factors* $\mathbf{f}_{\mathtt{x},t} = \mathbf{A}_{\mathtt{x}}\mathbf{r}_{\mathtt{C},t}$ *span the basis portfolios* $\mathbf{r}_{\mathtt{C},t}$,
      ii. $\forall$ *security* i, $\Delta\mu_{i,\mathtt{x}} = 0$ and $\Delta\sigma_{\epsilon_i,\mathtt{x}} = 0$.
   (b) *Rank of* $\mathbf{\Sigma}_{\mathtt{C}} = K$,
2. *Assumption II holds, then*:
   (a) $\exists$ *a full rank* $K \times C$ *matrix* $\mathbf{A}_{\mathtt{x}}$ *such that*:
      i. *factors* $\mathbf{f}_{\mathtt{x},t} = \mathbf{A}_{\mathtt{x}}\mathbf{r}_{\mathtt{C},t}$ *do not span at least one basis portfolio in* $\mathbf{r}_{\mathtt{C},t}$,
      ii. $\exists$ *security* i, $\Delta\mu_{i,\mathtt{x}} \neq 0$ and $\Delta\sigma_{\epsilon_i,\mathtt{x}} \neq 0$.
   (b) *Rank of* $\mathbf{\Sigma}_{\mathtt{C}} > K$,

Proof: Appendix A contains the proof of Proposition 1. □

Assumptions I and II together imply that all the basis portfolios are not all PMMV and at least one portfolio has an unpriced component. This proposition provides a testable condition for any arbitrary weight matrix to produce PMMV portfolios that span $[r_{m,t}\ \mathbf{f}_{\mathtt{P},t}]$: rank of $\mathbf{\Sigma}_{\mathtt{C}} = K$. When this condition is satisfied, estimates from $\mathbf{f}_{\mathtt{x}}$ will satisfy $\Delta\mu_{i,\mathtt{x}} = 0$.

Considering the case when Assumption I holds ensures that the bias is not introduced in the first two steps of the construction method. When Assumption II does not hold, any nonsingular *arbitrary* weight matrix $\mathbf{A}_{\mathtt{x}}$ can be used to construct an unbiased factor model and estimate μ_i for any asset i. As our interest is in understanding whether imposing arbitrary restriction matrices $\mathbf{A}_{\mathtt{x}}$ is likely to produce biased estimates ($\Delta\mu_{i,\mathtt{x}} \neq 0$), I only consider the possibility that Assumptions I and II hold.

Given these assumptions, I focus on modeling how excess return estimates obtained in step 4 are influenced by the choice of arbitrary weight matrix $\mathbf{A}_{\mathtt{x}}$ in step 3. This analysis will also help understand the properties of alternate matrices $\mathbf{A}_{\psi}$ that can be used to replace matrix $\mathbf{A}_{\mathtt{x}}$ in step 3 so that the factors constructed $\mathbf{f}_{\psi,t} = \mathbf{A}_{\psi}\mathbf{r}_{\mathtt{C},t}$ from the same set of basis portfolios $\mathbf{r}_{\mathtt{C},t}$ produce unbiased estimates of expected returns ($\Delta\mu_{i,\psi} = 0$).

Proposition 1.1 provides a testable restriction and a necessary condition to support the use of arbitrary weights in constructing factors: if an prespecified simple K factor model is being constructed from C basis portfolios, then the rank of the covariance matrix of the C portfolios must equal the number of factors being constructed (K). Further, Assumption I requires that K is equal to $S_{\mathtt{P}} + 1$.

Proposition 1.2 considers the case when this testable restriction is not satisfied for the set of basis portfolios under consideration. Further, if, for a set of C basis portfolios, econometricians find that $K < Q$, then they should not construct factors using arbitrary weights

$\mathbf{A}_{\mathtt{x}}$ as they are likely to produce biased estimates.

Appendix B shows that the expression for this bias can be decomposed as $\Delta\mu_{i,\mathtt{x}} = \Delta\mu^{\mathtt{P}}_{i,\mathtt{x}} - \Delta\mu^{\mathtt{Z}}_{i,\mathtt{x}}$, where:[9]

$$\Delta\mu^{\mathtt{P}}_{i,\mathtt{x}} = \mathbf{B}_{i,\mathtt{P}}\boldsymbol{\alpha}_{\mathtt{P}}; \quad \Delta\mu^{\mathtt{Z}}_{i,\mathtt{x}} = \mathbf{b}_{\mathtt{i},\mathtt{Z}}\boldsymbol{\mu}_{\mathtt{x}}. \tag{6}$$

$\Delta\mu^{\mathtt{Z}}_{i,\mathtt{x}}$ is the bias due to the *included unpriced component*, which is a zero-beta portfolio of the unpriced components in the basis portfolios $r_{\mathtt{Z},\mathtt{x}} = \boldsymbol{\epsilon}_{\mathtt{x},t}\boldsymbol{\Sigma}^{-1}_{\mathtt{x}}\boldsymbol{\mu}_{\mathtt{x}}$. This bias depends on $\mathbf{b}_{\mathtt{i},\mathtt{Z}}$, which is defined as the covariance of the residual $\epsilon_{\mathtt{i}}$ of portfolio i and the included unpriced component of the unadjusted factor model $r_{\mathtt{Z},\mathtt{x}}$. This is essentially an error-in-variables problem that depends on the covariation of residuals of the independent and dependent variables.

$\Delta\mu^{\mathtt{P}}_{i,\mathtt{x}}$, is the bias due to the *omitted spanning component*. It is expressed as a function of the $K \times 1$ vector $\boldsymbol{\alpha}_{\mathtt{P}} = \Delta\boldsymbol{\mu}_{\mathtt{P},\mathtt{x}}$, where $\mathtt{P}$ represents PMMV portfolios (see Equation 4). This component of bias depends on the expected value of $\boldsymbol{\alpha}_{\mathtt{P}}$, the omitted spanning components, and the covariation of a security with those omitted components $\mathbf{B}_{i,\mathtt{P}}$. It is well-known that $\boldsymbol{\alpha}_{\mathtt{P}} \neq \mathbf{0}$ when the factors do not span the MVE portfolio of the basis portfolios.[10] The exposure of any security to these portfolio is denoted as $\mathbf{B}_{i,\mathtt{P}}$.[11]

[9] Lehmann (1987), MacKinlay (1987), MacKinlay (1995), and Diacogiannis and Feldman (2013) find expressions relating the abnormal return of an orthogonal portfolio and abnormal return estimates of a security, similar to the one for $\Delta\mu^{\mathtt{P}}_{i,\mathtt{x}}$ in Equation 6. The difference here is that $\Delta\mu^{\mathtt{P}}_{i,\mathtt{x}}$ relates to the spanning of the basis portfolios whereas the orthogonal portfolios relate to the spanning of all assets. These two are the same under Assumption I.

[10] See e.g. Roll (1977).

[11] The vectors $\boldsymbol{\alpha}_{\mathtt{P}}$ and $\mathbf{B}_{i,\mathtt{P}}$ will depend on the arbitrary constraint matrix $\mathbf{A}_{\mathtt{x}}$. This dependence is suppressed for notational convenience

II. Adjustments assuming common state variables

The goal is to construct an adjusted $K \times 1$ vector of factor returns $\mathbf{f}_{\psi,t}$ from the same set of basis portfolios $\mathbf{r}_{\mathtt{C},t}$ that produce unbiased estimates of expected returns ($\Delta\mu_{i,\psi} = 0$) under Assumption I. In this section, I propose alternate methods to estimate a factor construction matrix (FCM) $\mathbf{A}_\psi$ for step 3 of the factor construction process. The factors constructed using these matrices are PMMV even when Assumption II holds. The methods discussed in this section are based on weaker assumptions than when Assumption II does not hold and are likely to more robust in a setting when the rank of $\mathbf{\Sigma}_{\mathtt{C}}$ is more than K.

To better understand the source of the bias described in the previous section, I discuss the potential method of using the single-factor MVE portfolio representation in Section II.A. I show that using a single-factor representation of the basis portfolios will result in biased standard errors for time-series alphas. In Section II.B, I discuss alternative methods to create multi-factor models that are not subject to this bias (ignoring estimation error). These methods are based on a decomposition of the covariance matrix into common components and specific components. These methods take advantage of this structure to identify priced state variable exposures under certain assumptions. I recommend the use of the least restrictive of these methods to estimate abnormal returns.

A. Single-factor model

Denote the portfolio obtained using the MVE weights $\mathbf{w}_{\mathtt{C}^*} = \mathbf{\Sigma}_{\mathtt{C}}^{-1}\boldsymbol{\mu}_{\mathtt{C}}$ to combine the basis portfolios into a portfolio $\mathtt{C}^*$ with returns $r_{\mathtt{C}^*,t} = \mathbf{w}'_{\mathtt{C}^*}\mathbf{r}_{\mathtt{C},t}$. The results of Grinblatt and Titman (1987) and Huberman and Kandel (1987) suggest that one method to construct an unbiased factor model is to use $r_{\mathtt{C}^*,t}$ as a single factor with $\mathbf{w}_{\mathtt{C}^*}$ representing the matrix $\mathbf{A}_\psi$, even though the number of factors constructed using this method ($K = 1$) is less than $S_{\mathtt{P}} + 1$:

Proposition 2. *If $\boldsymbol{\Sigma}_{\mathtt{C}}$ and $\boldsymbol{\mu}_{\mathtt{C}}$ are known and Assumption I holds then, $\forall$ i*

1. $\Delta\mu_{i,\mathtt{C}^*} = \mathrm{E}[\hat{\alpha}_{i,\mathtt{C}^*}] = 0$,

2. $\Delta\sigma_{\epsilon_i,\mathtt{C}^*} \geq 0$, *with equality if* $K = 1$,

3. $\mathrm{Var}[\widehat{\boldsymbol{\alpha}}_{i,\mathtt{C}^*}] \geq \mathrm{Var}[\widehat{\boldsymbol{\alpha}}_{i,\mathtt{P}}]$, *with equality if* $K = 1$.

Proof: Appendix C contains the proof of Proposition 2. □

The first result of this proposition is a special case of a well-understood—a single-factor MVE combination of the basis portfolios in a one factor model will provide unbiased estimates of expected and abnormal returns. The second and third result indicate that, under Assumption I, this method of estimating expected abnormal returns will produce upward biased estimates of standard errors and consequently downward biased estimates of t-statistics. This is because the additional residual variance of priced state variable in the multi-factor model, which remains in the residuals of the single-factor model, will make the t-statistics larger than those from the multi-factor model, even though both models will produce identical estimates of $\Delta\mu_{i,\mathtt{C}^*}$.

Since the ICAPM predicts that the market portfolio is PMMV, it can be included along with $r_{\mathtt{C}^*,t}$ in the regression to control for independent variation related to one priced state variable. This is sufficient when there is only one priced state variable and therefore only two factors need to be constructed ($K = 2$). However, when $K > 2$, then $K - 2$ additional PMMV portfolios are required to get unbiased standard error estimates of abnormal returns. In this case optimal matrices, under some assumptions, can be used to construct factors that produce better estimates.

B. Multi-factor model

To identify the appropriate FCM, it is helpful to specify the expected process of the basis portfolios in accordance with the implicit motivation in earlier steps of the factor construction process. For example, one reason to choose a characteristic $\mathbf{d}_i$ in step 1 of the factor construction process is to capture a dispersion in exposure to common state variables across securities such that $\mathbf{d}_i \propto \boldsymbol{\beta}_{i,\mathtt{P}}$.[12] If this property of $\mathbf{d}_i$ is satisfied, then more than one basis portfolio $r_{\mathtt{C}_j}$ should have nonzero exposure to the priced state variables.

Denote M state variables that are common to more than one basis portfolio in $\mathbf{r}_{\mathtt{C},t}$ as the $M \times 1$ vector of *common* state variables $\boldsymbol{\phi}_t$. The process in Equation 1 can be written in terms of $\boldsymbol{\phi}_t$ as:

$$\mathbf{r}_{\mathtt{C},t} = \boldsymbol{\mu}_{\mathtt{C}} + \mathbf{b}_{\mathtt{C},\mathrm{m}}(r_{m,t} - \mu_m) + \mathbf{B}_{\mathtt{C},\phi}\boldsymbol{\phi}_t + \boldsymbol{\zeta}_{\mathtt{C},t}, \tag{7}$$

where $\mathbf{b}_{\mathtt{C},\mathrm{m}} \equiv \mathrm{Cov}[\mathbf{r}_{\mathtt{C},t}, r_{m,t}]/\sigma_m^2$ is a $C \times 1$ vector of univariate beta coefficients of $\mathbf{r}_{\mathtt{C},t}$ on the market portfolio, $\mathbf{B}_{\mathtt{C},\phi}$ is the factor loadings of the C basis portfolios on the M common factors $\boldsymbol{\phi}_t$, and $\boldsymbol{\zeta}_{\mathtt{C},t}$ denotes the $C \times 1$ vector of *specific* components, defined as those that are not common to more than one basis portfolio. By definition of common state variables, each column of the matrix $\mathbf{B}_{\mathtt{C},\phi}$ contains two or more nonzero elements. The common state variables $\boldsymbol{\phi}_t$ are normalized to have mean zero and identity covariance matrix. With this normalization, $\mathbf{B}_{\phi}\mathbf{B}'_{\phi}$ represents the variance of the C basis portfolios due to M common state variables. The diagonal matrix consisting of the variance of $\boldsymbol{\zeta}_{\mathtt{C},t}$ is denoted as $\boldsymbol{\Psi}_{\mathtt{C}}$.

This representation of the basis portfolio processes implies the following decomposition

[12] For example, Berk, Green, and Naik (1999) model betas as functions of characteristics such as size and book-to-market.

of the covariance matrix of the basis portfolios:[13]

$$\boldsymbol{\Sigma}_{\mathtt{C}} = \mathbf{b}_{\mathtt{C},\mathrm{m}}\mathbf{b}'_{\mathtt{C},\mathrm{m}}\sigma_m^2 + \mathbf{B}_{\mathtt{C},\phi}\mathbf{B}'_{\mathtt{C},\phi} + \boldsymbol{\Psi}_{\mathtt{C}}, \tag{8}$$

where σ_m^2 is the variance of the market portfolio. Next, I outline alternate methods and their corresponding assumptions to use this decomposition to obtain $\mathbf{f}_{\phi,t}$, a set of PMMV portfolio returns that can replace the unadjusted factor returns $\mathbf{f}_{\mathtt{x},t}$ to obtain unbiased estimates such that $\Delta\mu_{i,\phi} = 0$ and $\Delta\sigma_{i,\phi} \equiv \text{Var}[\hat{\alpha}_{i,\phi}] - \text{Var}[\hat{\alpha}_{i,\mathtt{P}}] = 0$. Note that in this section, estimation error in $\hat{\mathbf{f}}_{\phi,t}$ will be ignored. In the next section, I show using simulations that the suggested method outperforms despite estimation error.

We know that $r_{\mathtt{C}^*}$ is a PMMV portfolio from Assumption I. Further, $\Delta\mu_{i,\mathtt{C}^*} = 0$ and $\Delta\sigma_{i,\mathtt{C}^*} \geq 0$ from Lemma 2. If we use $r_{\mathtt{C}^*}$ in the factor model, perhaps after orthogonalization to other factors, then only $K-2$ additional PMMV portfolios are required to construct the factor model. To identify these $K-2$ portfolios such that both $\Delta\mu_{i,\phi} = 0$ and $\Delta\sigma_{i,\phi} = 0$, I first make the following assumption:

Assumption III: $\boldsymbol{s}^{\phi}_{\mathtt{P},t} = \mathbf{G}\boldsymbol{\phi}_t$, where $\mathbf{G}$ is an $M \times M$ vector that transforms the common factors $\boldsymbol{\phi}_t$ to $M = K-2 = S_P - 1$ state variables $\boldsymbol{s}^{\phi}_{\mathtt{P},t}$, which are both common and priced.

This assumption describes a set of basis portfolios whose common variance is *only* due to their common exposures to the market portfolio or $K-2$ priced state variables. For each priced state variables $\phi_{j,t}$, there are at least two basis portfolios with the corresponding elements b_{k,ϕ_j} of $\mathbf{B}_{\mathtt{C},\phi}$ such that $b_{k,\phi_j} \neq 0$. The specific variances of the well-diversified basis

[13]This decomposition is similar to but distinct from the decomposition in various implementations of APT such as Roll and Ross (1980). Here, $\boldsymbol{\Psi}_{\mathtt{C}}$ does not contain the idiosyncratic variances of individual securities; it contains the specific variance of a spanning portfolio composed of a large number of securities. The possibility that this component is related to priced or unpriced state variables is not ruled out and depends on the process used to construct the basis portfolios.

portfolios represent undiversifiable risks that investors probably demand compensation for in expected returns.[14]

If Assumption III holds, a maximum-likelihood factor analysis decomposition of the covariance matrix can be used to obtain consistent estimates of $\mathbf{B}_{\mathtt{C},\phi}$ and $\mathbf{\Psi}_{\mathtt{C}}$ with a large number of time-series observations T, even when the number of basis portfolios is small.[15] These estimates can further be used to construct factor returns $\boldsymbol{f}_{\phi,t}$ that are PMMV portfolios, with minimum variance exposures to the common priced state variables $\boldsymbol{\phi}_t$. Ignoring estimation issues, consider the case when $\mathbf{B}_{\mathtt{C},\phi}$ and $\mathbf{\Psi}_{\mathtt{C}}$ are known.

Proposition 3. *If Assumptions I and III hold, then:*

1. *The $M \times 1$ weight vector $\mathbf{w}_{\phi}$ with elements $\mathbf{w}_{\phi_j}$ can be used to construct M PMMV portfolios $\boldsymbol{f}_{\phi,t}$:*

$$\mathbf{w}_{\phi_j} = \frac{\mathbf{\Phi}^{\dagger}_{\mathtt{C},j}\boldsymbol{\iota}}{\boldsymbol{\iota}'\mathbf{\Phi}^{\dagger}_{\mathtt{C},j}\boldsymbol{\iota}}, \tag{9}$$

where $\mathbf{\Phi}_{\mathtt{C},j} = \mathbf{\Sigma}_{\mathtt{C}} - \mathbf{b}_{\mathtt{C},m}\mathbf{b}'_{\mathtt{C},m}\sigma^2_m - \mathbf{b}_{\mathtt{C},\phi_j}\mathbf{b}'_{\mathtt{C},\phi_j}$ denotes the j^{th} partial covariance matrix for common factor ϕ_j; $\boldsymbol{\iota}$, a vector of ones of convenient dimensions; and the superscript $\dagger$, the generalized inverse.

2. *The cross-sectional mean squared-error (MSE) of expected abnormal returns estimates produced by factors $\boldsymbol{f}_{\mathtt{x},t}$ is greater than that produced by $\boldsymbol{f}_{\phi,t}$: $\sum_i \mathrm{E}[\alpha_{i,\mathtt{x}}]^2 \geq \sum_i \mathrm{E}[\alpha_{i,\phi}]^2$.*

Proof: Appendix D contains the proof of Proposition 3. □

The remaining PMMV portfolio hedges variation in expected returns that is independent of loadings on the common priced state variables and the market portfolio. The weights of

[14]See e.g. Fama (1996).
[15]See e.g. Grinblatt and Titman (1987).

this portfolio of the basis portfolios minimize exposure to the specific variance of the basis portfolios for a given level of expected returns and zero exposure to common state variables and the market portfolio.[16] I solve for a weight vector of $\mathbf{w}_{\phi^*}$ that minimizes the variance of a portfolio of the specific residuals $\boldsymbol{\zeta}_{\mathtt{C},t}$ with the level of expected excess returns constrained to be $\mu_{\phi^*} = \boldsymbol{\alpha}'_{\mathtt{C},\phi}\boldsymbol{\Psi}^{\dagger}_{\mathtt{C}}\boldsymbol{\alpha}_{\mathtt{C},\phi}$, where $E[\boldsymbol{\zeta}_{\mathtt{C},t}] = \boldsymbol{\alpha}_{\mathtt{C},\phi}$.[17] The solution to this problem is:

$$\mathbf{w}_{\phi^*} = \boldsymbol{\Psi}^{\dagger}_{\mathtt{C}}\boldsymbol{\alpha}_{\mathtt{C},\phi}, \tag{10}$$

It is well known that including the optimal orthogonal portfolio in the set of chosen PMMV portfolios will span $r_{\mathtt{C}^*,t}$.[18] This is consistent with the earlier discussion that the OMVE portfolio is a valid candidate for the remaining PMMV portfolio.

The weights $\mathbf{w}_{\phi}$ and $\mathbf{w}_{\phi^*}$ of these $K-1$ portfolios are combined with a row consisting of a one for the market portfolio and zeros for the other basis portfolios. I obtain a $K \times C$ matrix $\mathbf{A}_{\phi}$ such that $\boldsymbol{f}_{\phi,t} = \mathbf{A}_{\phi}\mathbf{r}_{\mathtt{C},t}$. Under Assumptions I and III, this alternate matrix $\mathbf{A}_{\phi}$ will provide unbiased estimates of abnormal returns such that $\Delta\mu_{i,\phi} = 0$ and $\Delta\sigma_{i,\phi} = 0$.

[16] $r_{\mathtt{C}^*}$ can also be used as the last PMMV portfolio. However, given the MVE weight vector contains substantial estimation error, I recommend using $r_{\phi^*,t}$, the optimal orthogonal portfolio (OMVE). This will avoid the estimation error in MVE from impacting the estimated weights $\mathbf{w}_{\phi^*}$ of the remaining PMMV portfolios. See e.g. Kan and Smith (2008) for distribution of sample MVE portfolio.

[17] This choice of μ_{ϕ^*} is innocuous and used to eliminate terms and simplify notation.

[18] MacKinlay (1995) defines a related problem with respect to spanning in the space of all assets, in contrast to spanning in the space of the basis portfolios considered here. The solution of weights is then multiplied by a different constant than the solutions of MacKinlay (1995) as the constraint used differs. As various optimal orthogonal portfolios trace a straight line in mean-standard deviation space, the Sharpe ratio is not affected by the choice of expected return (see Lehmann (1987))

B.1. Variations of the suggested factor construction matrix

Variations in Assumption III, under a similar broad theme, will imply variations in the matrix that should be used to construct factors, given Assumptions I and II. For example, consider the following alternate assumption:

Assumption III^a: $\boldsymbol{\phi}_t = \mathbf{G}\boldsymbol{s}_{\mathtt{P},t}$, where $\mathbf{G}$ is a $S_{\mathtt{P}} \times S_{\mathtt{P}}$ rotation matrix such that $\mathbf{G}\mathbf{G}' = \mathbf{I}_{S_{\mathtt{P}}}$.

If this assumption holds, then $K-1$ common factors, not just the $K-2$ considered earlier, are also the priced state variables. In this scenario, the optimal orthogonal portfolio does not need to be added to the set of PMMV portfolios. This assumption requires that the common variance across the basis portfolios is *only* due to common exposures to priced state variable.[19] Also, it implies that $M = S_P$ and only specific unpriced variance is captured by $\boldsymbol{\Psi}_{\mathtt{C}}$. Consider a $K \times C$ matrix $\mathbf{A}_{\phi^a}$ with $K-1$ columns as $1 \times C$ vectors $\mathbf{w}_\phi$ and one column as a weight vector with one for the market portfolio and zero for other basis portfolios. Assumption III^a implies that $\mathbf{A}_{\phi^a}$ can be used in step 3 to construct the factors $\boldsymbol{f}_{\phi^a,t} = \mathbf{A}_{\phi^a}\mathbf{r}_{\mathtt{C},t}$.

This assumption is also appropriate when the number of time-series observations is relatively small and estimation error in weights of $r_{\phi^*,t}$ may dominate the information contained in the mean abnormal returns of the basis portfolios. Absent estimation error, it will yield unbiased estimates of abnormal returns and their standard deviations under the Assumption III^a.

Similarly, principal component analysis can also be used to obtain factors from the basis portfolios. This method is known to produce consistent weights with a large cross-section of

[19] $\mathbf{G}$ relates the orthogonal components of the common state variables to the actual common state variables The rotation condition ensures that the covariance of both sets of state variables is an identity matrix.

assets.[20] However, with a small cross-section of basis portfolios (such as six basis portfolios in the case of the FF factors) this is not necessarily the case and additional assumptions are needed.[21] This is because the lower eigenvalue principal components of the basis portfolios are not necessarily diversifiable and unpriced. In the scenario when this possibility is ruled out, using the principal components of the basis portfolios with the $K-1$ largest eigenvalues can identify the priced factors:

Assumption III^b: $[r_{m,t}\ \mathbf{f}_{\mathtt{P},t}]' = \mathbf{\Gamma}_{\mathtt{K}}'\mathbf{r}_{\mathtt{C},t}$, where $\mathbf{\Gamma}_{\mathtt{M}}$ contains the K eigenvectors of the covariance matrix $\mathbf{\Sigma}_{\mathtt{C}}$ with the largest eigenvalues.

If the basis portfolios are constructed using a sort that maximizes the spread in exposure to priced state variables, then the largest principal components of the covariance matrix will pick up this dispersion. The remaining variation in returns captured by eigenvectors with smaller eigenvalues should be due to the unpriced state variables. With this assumption, a $K \times C$ matrix $\mathbf{A}_{\phi^b} = \mathbf{\Gamma}_{\mathtt{K}}'$ can be used in step 3 to construct the factors $\boldsymbol{f}_{\phi^b,t} = \mathbf{A}_{\phi^b}\mathbf{r}_{\mathtt{C},t}$.

The choice between $\boldsymbol{f}_{\phi,t}$, $\boldsymbol{f}_{\phi^a,t}$, and $\boldsymbol{f}_{\phi^b,t}$ depends on steps 1 and 2 of the factor construction process. Of the three, $\boldsymbol{f}_{\phi,t}$ is the safest choice when little is known about the process of the basis portfolios and there is enough time-series data to estimate $\boldsymbol{f}_{\phi,t}$ relatively precisely. This is because $\Delta\mu_{i,\phi}$ will asymptotically be zero, even if Assumption III does not hold.

[20] See Chamberlain (1983) for proof. Connor and Korajczyk (1986) discuss the benefits of using an alternative principal component estimation technique when the number of securities is large compared to the number of time periods.

[21] In finite samples, Lehmann and Modest (2005) find that maximum likelihood factor analysis coupled with minimum idiosyncratic risk portfolio formation yields economically and statistically superior basis portfolios compared with those derived from asymptotic principal components.

III. Comparison of construction methods using simulations

If the true underlying process is known, then corresponding asymptoticly optimal PMMV portfolio construction method can often be derived analytically in a manner similar to the one described in the previous section. However, the finite-sample properties of these estimators and the out-of-sample performance, taking into consideration estimation error, are usually difficult to derive analytically. Also, we typically do not know the true underlying process and the factor construction method used may be sensitive to assumptions regarding this process. To understand the finite sample properties and robustness of various factor construction methods proposed earlier, I simulate data under various assumptions of the return-generating process and compare the performance of factors constructed using the various methods under consideration.

To make the simulations realistic and allow comparison of the simulation results with actual parameter estimates, I calibrate the parameters of various data-generating processes (DGPs) to the corresponding sample estimates of the basis portfolios. I use the basis portfolios employed to construct the factors in the pioneering Fama and French (1993) study. The six portfolios are formed on size and book-to-market (BTM) sorts.[22] These FF portfolios are used in step 3 of the factor construction method with $C = 7$ and $K = 3$. Under Assumption I, these basis portfolios are expected to span the priced factors in the economy and consequently the MVE portfolio of all assets. I enforce this restriction in the simulations.

I calibrate the parameters by using the monthly returns of the six FF size and BTM portfolios from July 1963 to December 2011. The DGP for the returns of the three factors is multivariate normal with mean $\boldsymbol{\mu}_f$ and variance $\boldsymbol{\Sigma}_f$ equal to the corresponding FF factors

[22] See Fama and French (1993) for details on how these portfolios are constructed.

sample estimates and that of their residuals. The riskless rate is assumed to be zero. The market portfolio has $\beta_{\mathrm{m}} = 1$ and other $\beta_{m,\mathtt{P}}$ as zero. The $\boldsymbol{\beta}^{j}_{\mathtt{P}}$ vector for the remaining six portfolios varies according to the assumptions regarding the DGP. For example, the construction matrix $\mathbf{A}_{\phi}$ is obtained using estimates of $\mathbf{B}_{\mathtt{C},\phi}$, $\boldsymbol{\Psi}_{\mathtt{C}}$, and $\mathbf{w}_{\phi}$ applying data from the sample period. For each such DGP, the true $\boldsymbol{\beta}^{j}_{\mathtt{P}}$ is then obtained from sample time-series regressions of the basis portfolios on the estimated factors given the FCM $\hat{\mathbf{A}}_{j}$ is appropriate for that DGP. The basis portfolios are obtained using their $\boldsymbol{\beta}$ exposures to the three simulated factors and a set of multivariate normal residuals with mean zero and covariance matrix the same as that of the time-series regression residuals.

Given the model parameters, returns of the factors and basis portfolios can be simulated for any sample size T. In the simulations that follow, I draw 10,000 sample data sets for each scenario considered. The alternative construction matrices are estimated using a simulated "training" sample. This is treated as the estimate of the FCM that econometricians can obtain given observations from T periods. Weights estimated from this experiment are used in all subsequent simulated data sets to construct the various alternate factor models. The subsequent simulations measure the distribution of estimates and statistics obtained using these alternate factor models, given econometricians are applying weights estimated using a particular "training" sample.

A. True weight matrices for various data generating processes

I consider four DGP corresponding to Section II.B. DGP 1 assumes that the true PMMV portfolios underlying the basis portfolios can be obtained using the weight matrix $\mathbf{A}^{\mathtt{FF}}_{\mathtt{x}}$, which is used to construct the FF factors. DGP 2 assumes that the true PMMV portfolios can be constructed using the weight matrix $\mathbf{A}_{\phi}^{\texttt{FF}}$, estimated under Assumption III. Similarly, DGP 3 and DGP 4 assume that matrices $\mathbf{A}_{\phi^a}^{\texttt{FF}}$ and $\mathbf{A}_{\phi^b}^{\texttt{FF}}$ can be used to identify the true PMMV portfolios.

A.1. Simple prespecified constraint parameters are true parameters (DGP 1)

The prespecified simple weights used by FF to construct the FF factors can be represented in matrix form as:

$$\mathbf{A}_{\texttt{x}}^{\texttt{FF}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & -1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}, \tag{11}$$

where the columns represents the market portfolio and the six FF basis portfolios.[23] To see how the matrix is formed, it is to be noted that the FF factors are constructed as SMB (small minus big) and HML (high minus low), where the small, big, value, and growth portfolios are combinations of their six basis portfolios as SMB ($\equiv SL + SN + SH$ - BL - BN- BH) and HML ($\equiv -SL + SH$ - $BL + BH$). These weights are assumed to be an instance of the potentially suboptimal arbitrary weight matrix $\mathbf{A}_{\texttt{x}}$ in the simulations.

A.2. Common state variables as priced state variables (DGP 2 and DGP 3)

Data on market returns (MKT), treasury bill rate (RF), FFC factors, and the various characteristic-sorted portfolios are used to estimate the covariance matrix $\Sigma_{\texttt{C}}^{\texttt{FF}}$.[24] Maximum likelihood factor analysis (without rotation) is used to decompose $\mathbf{\Sigma}_{\texttt{C}}^{\texttt{FF}}$ as per Equation 8. Table I, Panel A shows estimates of $\mathbf{B}_{\texttt{C},\phi}$, $\mathbf{\Psi}_{\texttt{C}}$, and $\mathbf{w}_{\phi}$ for the six FF basis portfolios.

[23] The columns in the matrix 11 are ordered in the following sequence: SL, SN, SH, BL, BN, and BH, where the letter B denotes Big; S, Small; H, High BTM; N, Neutral BTM; and L, Low BTM.

[24] Return data are downloaded from Kenneth French's website: `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`

Estimates of $\mathbf{\Psi_C}$ indicate that the specific variance of the small cap, neutral (SN) portfolio is nearly zero. Under Assumption III, it is nearly a PMMV portfolio with exposure only to a priced state variables. This is true for the full sample as well as the Pre-1992 sample.[25] This zero specific variance of the SN portfolio also reflects in the solution to MMV weights problem, which is a weight that is approximately one on SN and zero on all other portfolios.

The literature on MVE weight estimation suggests that a shrinkage estimator is a natural candidate to obtain $\mathbf{w}_{\phi^*}$, the weights for the OMVE portfolio. This has been shown to reduce the mean-squared error in the MVE weight estimates[26]. I use the Jorion (1986) shrinkage estimator in the estimation of OMVE weights. Various estimates of $\mathbf{w}_{\phi^*}$ are shown in Table I, Panel B. The weights look relatively stable across the subsamples and methods.

A.3. Principal components as PMMV portfolios (DGP 4)

I consider a a factor construction method based on principal components with the highest eigenvalues. Table I, Panel C presents the estimated principal component of the basis portfolios. The principal component are numbered from Γ_1 to Γ_6 in order of decreasing eigenvalues. Interestingly, Γ_1, Γ_2, and Γ_3, resemble the weights used by FF and have the highest correlation with the FF factors (Panel D). The market portfolio has a correlation of 0.95 with Γ_1 and has positive weights on all the FF basis portfolios. The SMB factor has

[25] I use January 1992 as the sample breaking period as that was end of the sample used in the classical Fama and French (1993) study, which pioneered the unadjusted factor construction method. This will help check whether this bias arises due to the benefit of hindsight in estimating MVE weights.

[26] In an influential study, James and Stein (1961) show that the shrinkage estimator is admissible. Jorion (1986) argues that this estimator should always improve on the classical sample mean. He derives a weight estimator for a Bayesian MVE portfolio with the average excess returns of the sample global minimum-variance portfolio as the shrinkage target. Using simulations, he shows that the out-of-sample performance of this portfolio dominates that of the sample MVE portfolio. More recently, Kan and Zhou (2007) derive a related shrinkage estimator motivated from the perspective of an investor who faces a parameter uncertainty that can also be interpreted as having the sample global minimum-variance portfolio as the shrinkage target.

a correlation of -0.77 with Γ_2, and has the same sign on portfolios belonging to the same size category. The HML factor has a correlation of -0.84 with Γ_3 and has the same signs on portfolios belonging to extreme BTM categories. Weights under various other assumptions can be estimated in a manner similar to the ones described in Table I.

B. Test statistics

I study the ability of the two set of factors produced by the constraint matrices to price the basis portfolios that they were constructed from under various DGP assumptions. I focus on the hypothesis that the factors generate zero bias when pricing the basis portfolios that they are constructed from: $\Delta\mu_{\mathtt{C,x}} = 0$. This is because, under Assumption I, the expression for the bias in estimate of any securities abnormal return $\Delta\mu_{\mathtt{i,x}}$ is closely related to $\Delta\mu_{\mathtt{C,x}}$ (see Equation 6). If the factors span the basis portfolios, they will also span the MVE portfolio and consequently not bias estimates from any security.

To test this hypothesis, I use the GRS (Gibbons, Ross, and Shanken (1989)) test statistic:

$$\hat{t}_{\mathtt{x}}^2 = \left[\frac{\widehat{\Delta\mu}_{\mathtt{C}^*,\mathtt{x}}}{\widehat{SD}(\widehat{\Delta\mu}_{\mathtt{C}^*,\mathtt{x}})}\right]^2 = \frac{T(\hat{s}_{\mathtt{C}}^2 - \hat{s}_{\mathtt{x}}^2)}{(1+\hat{s}_{\mathtt{x}}^2)}, \tag{12}$$

where $s_{\mathtt{G}}$ denotes the Sharpe ratio of the MVE portfolio of $r_{\mathtt{G},t}$. Note that this statistic has a close relation to the omitted spanning component bias $\Delta\mu_i^{\mathtt{P}}$, which can be expressed as:

$$\Delta\mu_{i,\mathtt{x}}^{\mathtt{P}} = \mathbf{B}_{i,\mathtt{P}}\mathbf{B}_{\mathtt{P},\mathtt{xC}^*}\Delta\mu_{\mathtt{C}^*,\mathtt{x}} = \mathbf{B}_{i,\mathtt{P}}\mathbf{B}_{\mathtt{P},\mathtt{xC}^*}s_{\mathtt{xC}}^2 = \mathbf{B}_{i,\mathtt{P}}\mathbf{B}_{\mathtt{P},\mathtt{xC}^*}(s_{\mathtt{C}}^2 - s_{\mathtt{x}}^2) = \mathbf{B}_{i,\mathtt{P}}\mathbf{B}_{\mathtt{P},\mathtt{xC}^*}\frac{(1+s_{\mathtt{x}}^2)t_{\mathtt{x}}^2}{T}, \tag{13}$$

where the parameters calculated with respect to the optimal orthogonal portfolio $r_{\mathtt{xC}^*,t}$ with expected return of the unadjusted factors are denoted using the subscript $\mathtt{xC}^*$.

Table I
Adjusted Factor Weight Estimates for FF Basis Portfolios

Panel A presents the weights used to form the MMV portfolio from the base assets of the FF factors. The weights of the MMV portfolio are calculated from the partial covariance matrix between the six base assets (six size and BTM portfolios), controlling for exposures to the market portfolio. Panel B presents the OMVE weights ($\mathbf{w}_{\phi^*}$) of the specific components of the basis portfolios. These weights are estimated using the maximum likelihood (ML) estimator, and with the Jorian shrinkage (JS) estimator. Panel C presents the estimates of the principal component (Γ) weights of the factors estimated using principal component decomposition of the FF basis portfolios. In the panel C and D, Γ_1 denotes the component with the highest eigenvalue and Γ_6, the lowest. Panel D presents the correlation of returns of the FF factors (denoted as MKT, SMB, and HML) and the principal components. ***, **, and * indicate statistical significance at the 1%, 5%, and 10% levels, respectively. The full sample period is from July 1963 to December 2011. The Pre-92 sample period is from July 1963 to December 1991.

Panel A: PMMV Portfolio Parameters

	BTM Rank			BTM Rank		
Size Rank	Low	Medium	High	Low	Medium	High
	Full Sample			Pre 92 Sample		
			$\mathbf{B}_{\mathtt{C},\phi}$			
Small	0.75	1.00	0.90	0.79	1.00	0.91
Big	-0.48	0.03	0.26	-0.44	-0.07	0.33
			$\mathbf{\Psi}_{\mathtt{C}}$			
Small	5.03	0.00	1.85	3.23	0.00	1.49
Big	1.06	2.66	5.09	1.19	1.67	3.28
			$\mathbf{w}_{\phi}$			
Small	0.00	1.00	0.00	0.00	1.00	0.00
Big	0.00	0.00	0.00	0.00	0.00	0.00

Panel B: OMVE Portfolio Parameters ($\mathbf{w}_{\phi^*}$)

	BTM Rank			BTM Rank		
Size Rank	Low	Medium	High	Low	Medium	High
	Full Sample (JS)			Post 92 Sample (JS)		
Small	-0.12	-0.00	0.09	-0.15	-0.00	0.06
Big	0.15	-0.01	-0.04	0.19	-0.04	-0.07
	Pre-92 Sample(ML)			Pre-92 Sample(JS)		
Small	-0.13	-0.00	0.08	-0.11	-0.00	0.08
Big	0.23	0.06	0.03	0.20	0.05	0.03

PANEL C: Principal Component Parameters

	Full Sample						Pre 92 Sample					
	Small Portfolios			Big Portfolios			Small Portfolios			Big Portfolios		
BTM Rank	Low	Medium	High	Low	Medium	High	Low	Medium	High	Low	Medium	High
Γ_1	0.54	0.44	0.44	0.34	0.32	0.34	0.53	0.44	0.44	0.35	0.31	0.34
Γ_2	0.58	0.19	0.08	-0.22	-0.51	-0.56	-0.28	-0.30	-0.40	0.60	0.50	0.26
Γ_3	0.28	-0.24	-0.53	0.71	0.11	-0.25	-0.54	0.05	0.35	-0.44	0.23	0.58
Γ_4	0.02	0.20	-0.15	-0.34	0.77	-0.48	0.23	0.09	-0.27	-0.50	0.73	-0.29
Γ_5	0.50	-0.19	-0.49	-0.46	0.06	0.51	0.49	-0.27	-0.42	-0.27	-0.22	0.62
Γ_6	-0.23	0.80	-0.50	0.06	-0.20	0.11	-0.22	0.80	-0.53	0.02	-0.16	0.12

PANEL D: Correlation of Principal Components with FF Factor Returns

	Full Sample						Pre 92 Sample					
Factor	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
MKT	0.95***	−0.18***	0.24***	−0.01	−0.03	−0.02	0.96***	0.27***	−0.04	−0.01	−0.03	−0.00
SMB	0.55***	0.77***	−0.31***	0.02	−0.04	0.01	0.60***	−0.78***	−0.15**	0.02	−0.04	0.00
HML	−0.20***	−0.49***	−0.84***	−0.07	−0.01	−0.03	−0.27***	−0.26***	0.92***	−0.07	−0.01	−0.02

The expected return of the optimal orthogonal portfolio is $\Delta\mu_{\texttt{C}^*,\texttt{x}}$. The relation follows from the properties of the optimal orthogonal portfolio[27]. The GRS statistic also tests the hypothesis that the square of the Sharpe ratio achieved by the set of unadjusted factors is the same as that from the unrestricted set of basis portfolios. Equation 13 shows that when the hypothesis that $s^2_{\texttt{C}} > s^2_{\texttt{x}}$ cannot be rejected, then it is likely that $\Delta\mu^{\texttt{P}}_{i,\texttt{x}} > 0$. In simulations, I focus on spanning tests of the basis portfolios by factors constructed using various methods. I estimate the mean and standard-deviation of the GRS test statistic, p-value of the GRS test statistic, the bias multipler $\Delta\mu_{\texttt{C}^*,\texttt{x}}$ (Equation 13), and root-mean-square-error (RMSE), in finites samples of various sizes.

C. Simulation results

The simulation results are reported in Table II, Panel A. I find that both the unadjusted factor method and the adjusted factor method are able, on average, to price the simulated basis portfolios. Both sets of factors are close to the ex-post frontier in the simulations and the hypothesis that that the factors are ex-ante efficient cannot be rejected using the GRS statistic. They also have similar RMSE of estimates of alphas of the simulated spanning portfolio. The principal component-based factors produce similar estimates for smaller samples, but when the sample size increases, the hypothesis that they are ex-ante efficient can also be rejected.

The simulations in Table II, Panel B generate data under the assumption that the systematic constraint restrictions identify the true factors. The DGP for the basis portfolios is a three-factor model with the factors calibrated to the market portfolio, the MMV portfolio,

[27] See e.g. MacKinlay (1995).

and the OMVE portfolio.[28]

As expected, the ability of factors constructed using $\mathbf{A}_{\phi}^{\mathtt{FF}}$ is robust to the underlying DGP. It also has a lower RMSE of spanning portfolio alphas than other methods, except the one corresponding to the true DGP. According to the GRS statistic, the other FCMs are only able to price their basis portfolios when the underlying DGP is in accordance with the assumptions of the corresponding FCM. The bias multiplier $\Delta\mu_{\mathtt{C}^*,\mathtt{x}}$ for $\mathbf{A}_{\phi}^{\mathtt{FF}}$ is generally comparable to that of the DGP related FCM and mostly several times smaller than that of other FCMs.

IV. Adjusted and unadjusted estimates: a sensitivity analysis

This section empirically investigates the extent to which biases identified in previous sections can influence our conclusions in applications of influential benchmark factor models which assume a basis portfolio covariance matrix rank of K to construct factors. Specifically, I analyze the impact of this bias on estimates of two widely used and influential benchmark factor models: the FF three-factor model and the FFC four-factor model.

Though the economic rationale for these factors is widely debated in the literature, this factor model has become a benchmark in finance research. For this reason, it is important to understand whether estimates from the model could be significantly impacted by prespecified simple constraints used in its construction?

I first test whether the hypothesis that FF factors price their basis portfolios can be rejected with a significant probability. I find that it can be. According to Proposition 1 and

[28]The full sample (shrinkage) weights from Table I are used to generate the factors. The basis portfolios are calibrated to have the same betas as the six FF size and BTM portfolios, and multivariate normal residuals with mean and covariance matrix the same as the residuals of the FF basis portfolios with respect to the adjusted factors.

Table II
Basis Portfolio Spanning Tests Using Simulated Data

The table reports the average GRS statistic, p-value, $\Delta\mu_{\mathtt{C}^*,\mathtt{x}}$, and RMSE over 10,000 simulated data sets. Standard deviations of parameters estimates across samples are in brackets. These statistics are estimated for each simulated sample using four versions of factors constructed from the same six simulated portfolios. The betas and residuals of these portfolios are calibrated to betas of the sample estimates of the six FF basis portfolios with respect to factors constructed using the matrices (FCM) $\mathbf{A}^{\mathtt{FF}}_{\mathtt{x}}$, $\mathbf{A}^{\mathtt{FF}}_{\phi}$, $\mathbf{A}^{\mathtt{FF}}_{\phi^a}$, $\mathbf{A}^{\mathtt{FF}}_{\phi^b}$ described in section III.A are used to simulate basis portfolio returns in Panels A, B, C, and D respectively.

	Unadjusted Factors ($\mathbf{A}^{\mathtt{FF}}_{\mathtt{x}}$)				*Common Factors w/ OMVE* ($\mathbf{A}^{\mathtt{FF}}_{\phi}$)				*Common Factors* ($\mathbf{A}^{\mathtt{FF}}_{\phi^a}$)				*Principal Components* ($\mathbf{A}^{\mathtt{FF}}_{\phi^b}$)			
Sample Size	60	300	600	6000	60	300	600	6000	60	300	600	6000	60	300	600	6000
Panel A (*DGP*1): *Priced Factors Simulated Using Simple Prespecified FCM*($\mathbf{A}^{\mathtt{FF}}_{\mathtt{x}}$)																
GRS	0.39	0.39	0.40	0.39	0.53	0.73	0.71	1.31	1.33	2.19	3.02	33.66	1.08	0.68	0.71	6.07
p(GRS)	0.87	0.86	0.86	0.86	0.78	0.66	0.68	0.38	0.37	0.14	0.05	0.00	0.48	0.70	0.67	0.00
	(0.19)	(0.20)	(0.20)	(0.20)	(0.24)	(0.29)	(0.29)	(0.31)	(0.31)	(0.21)	(0.11)	(0.00)	(0.32)	(0.28)	(0.29)	(0.01)
$12\times\Delta\mu_{\mathtt{C}^*,\mathtt{x}}$	0.12	0.06	0.04	0.00	0.16	0.11	0.07	0.01	0.39	0.32	0.29	0.24	0.32	0.10	0.07	0.04
	(0.09)	(0.05)	(0.03)	(0.00)	(0.11)	(0.07)	(0.05)	(0.01)	(0.23)	(0.15)	(0.12)	(0.03)	(0.20)	(0.07)	(0.05)	(0.01)
$\sqrt{12}\times\Delta\sigma_{\epsilon_i,\mathtt{x}}$	-0.00	0.00	0.00	0.00	-0.10	-0.40	-0.16	-0.05	-1.09	-1.19	-1.29	-1.22	-2.38	-1.92	-2.00	-1.89
	(0.01)	(0.00)	(0.00)	(0.00)	(0.04)	(0.04)	(0.02)	(0.01)	(0.10)	(0.08)	(0.06)	(0.02)	(0.15)	(0.10)	(0.08)	(0.02)
$\sqrt{12}\times$ RMSE	0.41	0.29	0.24	0.07	0.48	0.49	0.33	0.12	1.38	1.42	1.47	1.41	1.60	0.55	0.48	0.37
	(0.16)	(0.12)	(0.10)	(0.03)	(0.19)	(0.22)	(0.13)	(0.04)	(0.51)	(0.40)	(0.34)	(0.09)	(0.73)	(0.28)	(0.25)	(0.07)
Panel B (*DGP*2): *Priced Factors Simulated Using Common Factor Analysis w/ OMVE FCM* ($\mathbf{A}^{\mathtt{FF}}_{\phi}$)																
GRS	2.61	4.92	7.22	92.65	0.62	0.83	1.79	0.61	3.65	6.88	9.96	127.39	0.53	1.95	4.61	23.89
p(GRS)	0.09	0.00	0.00	0.00	0.73	0.60	0.21	0.73	0.02	0.00	0.00	0.00	0.78	0.18	0.00	0.00
	(0.15)	(0.02)	(0.00)	(0.00)	(0.27)	(0.30)	(0.24)	(0.26)	(0.07)	(0.00)	(0.00)	(0.00)	(0.24)	(0.23)	(0.02)	(0.00)
$12\times\Delta\mu_{\mathtt{C}^*,\mathtt{x}}$	0.78	0.73	0.71	0.68	0.20	0.13	0.19	0.00	1.08	1.01	0.97	0.92	0.17	0.30	0.46	0.18
	(0.34)	(0.24)	(0.19)	(0.05)	(0.14)	(0.08)	(0.09)	(0.00)	(0.41)	(0.28)	(0.23)	(0.06)	(0.12)	(0.14)	(0.15)	(0.03)
$\sqrt{12}\times\Delta\sigma_{\epsilon_i,\mathtt{x}}$	0.97	0.97	0.97	0.97	0.02	-0.07	-0.08	0.02	-0.31	-0.34	-0.25	-0.24	-1.59	-0.87	-0.66	-0.98
	(0.10)	(0.07)	(0.06)	(0.02)	(0.02)	(0.02)	(0.02)	(0.00)	(0.07)	(0.05)	(0.04)	(0.01)	(0.18)	(0.07)	(0.07)	(0.02)
$\sqrt{12}\times$ RMSE	1.03	1.00	0.99	0.97	0.64	0.55	0.69	0.10	1.85	1.73	1.71	1.67	0.79	1.03	1.45	0.92
	(0.23)	(0.17)	(0.14)	(0.04)	(0.28)	(0.23)	(0.22)	(0.05)	(0.48)	(0.34)	(0.27)	(0.08)	(0.44)	(0.38)	(0.34)	(0.10)
Panel C (*DGP*3): *Priced Factors Simulated Using Common Factors FCM* ($\mathbf{A}^{\mathtt{FF}}_{\phi^a}$)																
GRS	0.66	0.94	1.21	11.27	0.54	0.62	0.66	1.21	0.53	0.54	0.54	0.54	1.19	1.10	1.47	12.43
p(GRS)	0.70	0.55	0.43	0.00	0.77	0.73	0.70	0.42	0.79	0.78	0.78	0.78	0.43	0.47	0.32	0.00
	(0.29)	(0.32)	(0.32)	(0.00)	(0.24)	(0.27)	(0.28)	(0.31)	(0.23)	(0.24)	(0.24)	(0.24)	(0.32)	(0.32)	(0.30)	(0.00)
$12\times\Delta\mu_{\mathtt{C}^*,\mathtt{x}}$	0.19	0.14	0.12	0.08	0.16	0.09	0.06	0.01	0.16	0.08	0.05	0.00	0.35	0.16	0.14	0.09
	(0.14)	(0.09)	(0.07)	(0.02)	(0.11)	(0.06)	(0.05)	(0.01)	(0.11)	(0.06)	(0.04)	(0.00)	(0.21)	(0.10)	(0.08)	(0.02)
$\sqrt{12}\times\Delta\sigma_{\epsilon_i,\mathtt{x}}$	1.23	1.23	1.23	1.23	-0.13	-0.13	-0.09	-0.13	-0.07	-0.02	0.03	-0.00	-1.33	-0.72	-0.84	-0.86
	(0.11)	(0.08)	(0.06)	(0.02)	(0.04)	(0.04)	(0.03)	(0.01)	(0.03)	(0.01)	(0.01)	(0.00)	(0.20)	(0.09)	(0.08)	(0.02)
$\sqrt{12}\times$ RMSE	0.53	0.44	0.41	0.35	0.70	0.53	0.44	0.21	0.65	0.45	0.37	0.10	1.72	0.74	0.76	0.65
	(0.20)	(0.16)	(0.13)	(0.04)	(0.33)	(0.26)	(0.21)	(0.08)	(0.32)	(0.22)	(0.17)	(0.05)	(0.78)	(0.34)	(0.31)	(0.10)
Panel D (*DGP*4): *Priced Factors Simulated Using Principal Components FCM* ($\mathbf{A}^{\mathtt{FF}}_{\phi^b}$)																
GRS	1.24	2.13	3.05	36.10	1.20	0.99	1.10	1.58	3.40	6.28	9.40	119.76	0.82	0.62	1.18	0.92
p(GRS)	0.41	0.15	0.05	0.00	0.42	0.52	0.47	0.27	0.03	0.00	0.00	0.00	0.61	0.73	0.42	0.56
	(0.31)	(0.21)	(0.11)	(0.00)	(0.31)	(0.31)	(0.31)	(0.27)	(0.08)	(0.00)	(0.00)	(0.00)	(0.30)	(0.26)	(0.31)	(0.31)
$12\times\Delta\mu_{\mathtt{C}^*,\mathtt{x}}$	0.38	0.32	0.30	0.27	0.37	0.15	0.11	0.01	1.00	0.91	0.90	0.85	0.26	0.10	0.12	0.01
	(0.22)	(0.15)	(0.12)	(0.03)	(0.22)	(0.09)	(0.07)	(0.01)	(0.39)	(0.27)	(0.22)	(0.06)	(0.17)	(0.07)	(0.07)	(0.00)
$\sqrt{12}\times\Delta\sigma_{\epsilon_i,\mathtt{x}}$	1.97	1.97	1.97	1.97	0.91	0.97	1.20	1.38	0.73	0.66	0.75	0.74	0.28	-0.11	0.27	-0.07
	(0.12)	(0.09)	(0.07)	(0.02)	(0.11)	(0.07)	(0.06)	(0.02)	(0.13)	(0.09)	(0.08)	(0.02)	(0.04)	(0.01)	(0.02)	(0.00)
$\sqrt{12}\times$ RMSE	0.81	0.77	0.76	0.74	1.17	0.74	0.49	0.15	2.24	2.31	2.23	2.23	0.80	0.55	0.54	0.15
	(0.25)	(0.18)	(0.15)	(0.04)	(0.47)	(0.32)	(0.18)	(0.05)	(0.55)	(0.41)	(0.32)	(0.09)	(0.35)	(0.25)	(0.19)	(0.06)

Equation 6, this further suggests that inferences from these factors are likely to be biased. I test for the presence of such a bias in various applications of benchmark factor models.

A. Can unadjusted benchmark factors price their basis portfolios?

Table III shows the abnormal returns from time-series regressions of the actual returns of the basis portfolios on the benchmark factors. The results in Panel A are consistent with those reported by Fama and French (1996): the returns on the different legs of the long and short portfolios of SMB and HML are priced by the three-factor model. This suggests that the FF three-factor model does span the MVE portfolio of the market portfolio and the small, big, value, and growth portfolios. However, these legs of the factors are themselves weighted combinations of a larger set of the basis portfolios. The results in Panel B indicate that five of the seven FF basis portfolios are not spanned by the unadjusted three-factor model.[29] This could result in a significant bias in estimates from these factors as compared to the adjusted factors that span these portfolios. Similar results are obtained when the six additional basis portfolios of the FFC factors are used to test spanning by the four-factor model.

This table suggests that estimates obtained using the FF factors are likely to be biased, as per proposition 1. The GRS test indicates that the multiplier of the bias $\Delta\mu_{\mathtt{C^*,x}}$ is highly significant. The magnitude of the statistic is similar to the corresponding magnitude estimated in the simulations. The bias $\Delta\mu_{\mathtt{P,x}}$ is significant in the full sample period as well as the post-estimation (Post-92) sample period.

[29]Welch (2008) finds similar results.

Table III
Basis Portfolio Spanning Tests

This table presents the time-series regression abnormal returns of various FF basis portfolios with respect to the FF factors, and the momentum basis portfolios with respect to the FFC factors. Robust Newey and West (1987) t-statistics with nine lags are reported in brackets. GRS is the F-statistic of Gibbons, Ross, and Shanken (1989), testing the hypothesis that all the regression intercepts for a set of test portfolios are all 0. The p-value of GRS is in brackets. $\Delta\mu_{\mathtt{C}^*,\mathtt{x}}$ denotes the bias in abnormal return of the MVE portfolio measured by the unadjusted factors and is a multiplier in the expression of the omitted spanning component bias. ***, **, and * indicate statistical significance at the 1%, 5%, and 10% levels, respectively. The sample period is from July 1963 to December 2011.

PANEL A: Long-Short Portfolios

	$\Delta\mu_{\mathtt{P},\mathtt{x}}$	$t(\Delta\mu_{\mathtt{P},\mathtt{x}})$	GRS	$\Delta\mu_{\mathtt{C}^*,\mathtt{x}}$	RMSE
FF factors					
Small/Big	-0.02	(-0.81)	1.03	0.01	0.05
Value/Growth	-0.03	(-1.30)	(0.39)		
FFC factors					
Winners/Losers	0.07	(1.85)	2.10**	0.01	0.09
		(0.12)	(0.05)		

PANEL C: Omitted Priced Components

	Full Sample			Post-92 Sample		
	$\Delta\mu_{\mathtt{P},\mathtt{x}}$	$t(\Delta\mu_{\mathtt{P},\mathtt{x}})$	$\bar{R}^2$	$\Delta\mu_{\mathtt{P},\mathtt{x}}$	$t(\Delta\mu_{\mathtt{P},\mathtt{x}})$	$\bar{R}^2$
$r_{\mathtt{1},\mathtt{P}}$	0.07***	(2.18)	0.98	0.10	(1.59)	0.97
$r_{\mathtt{P}^*}$	0.05***	(5.21)	0.96	0.06***	(4.15)	0.96

Panel B: Size and Value, and Size and Momentum Portfolios

	FF factors			*FFC factors*		
Size Rank	BTM Rank			MOM Rank		
	Low	Medium	High	Low	Medium	High
Small	-0.19***	0.07**	0.07***	-0.12*	0.09*	0.18***
	(-4.23)	(2.17)	(2.80)	(-1.74)	(1.86)	(3.55)
Big	0.13***	-0.05	-0.13***	0.26***	-0.01	-0.04
	(4.27)	(-1.01)	(-2.81)	(3.72)	(-0.12)	(-0.85)
GRS		5.54***			4.91***	
$p(GRS)$		(0.00)			(0.00)	
$\Delta\mu_{\mathtt{C}^*,\mathtt{x}}$		0.06			0.06	
RMSE		0.28			0.32	

The results of these tests predict that estimates of abnormal returns in applications that use this benchmark model are likely to be sensitive to weight assumptions. Adjustments to these benchmark models can help researchers differentiate between hypotheses of returns driven by covariance with respect to an omitted priced or included unpriced component of the basis portfolios verses abnormal returns that are not explained by the basis portfolios associated with the set of characteristics under consideration.

This testing approach can also be used to diagnose whether other factor models constructed using the construction method outlined in Section I are likely to be subject to this bias. The result of these tests will predict whether estimates using such factor models may be influenced by this bias.

B. Sensitivity analysis of anomalous return estimates

Can construction methods and the choice of FCMs influence inferences regarding the existence of anomalous returns of trading strategies? I test fourteen trading strategies that have been reported as producing anomalous returns in the literature.

I find that abnormal returns on five of these strategies are no longer significant after controlling for the weight-related bias in the FFC factors. The results in table IV indicate that FCMs influence inferences regarding the existence of anomalous abnormal returns of trading strategies. When adjusted factors are used to estimate abnormal returns, they correct for this bias.

Bootstrapped confidence intervals of time-series α and RMSE are also reported in the table to account for the estimation in the weights used to construct the factors. The samples are drawn with replacement and the time-series regression is re-estimated for each sample. The bias corrected and accelerated percentile (BCA) method is used to compute the confidence intervals of the bootstrapped statistic.

After the adjustment, the cross-sectional RMSE is reduced. This is consistent with the prediction of proposition 3 that using the adjustment will provide less noisy out-of-sample estimates of abnormal returns. For non-basis portfolio strategies, this reduction is significant at the 95% level.

Table IV
Sensitivity Analysis of Abnormal Return Estimates of Trading Strategies

This table reports the average excess returns (E[r^e])to strategies formed by sorting on the variables used in factor construction: market capitalization, BTM, performance (returns) over previous 11 months, gross profits-to-assets demeaned by industry and hedged for industry exposure, return on assets, default risk (Ohlson O-score), net stock issuance, and total accruals. Strategies are long-short extreme deciles from a sort on the corresponding variable, using NYSE breaks, and returns are value-weighted. Momentum, return on assets, and default probability strategies are rebalanced monthly, whereas other strategies are rebalanced annually, at the end of June. Strategies based on variables scaled by assets exclude financial firms. The table also reports abnormal returns relative to alternative models. ***, **, and * indicate statistical significance at the 1%, 5%, and 10% levels, respectively. t-statistics in round brackets are calculated using Newey and West standard errors, with nine lags. L95 and U95 denote the bootstrapped 95% confidence interval; RMSE, root-mean-squared error; ΔRMSE, the difference between RMSE of unadjusted and alternative estimators. The sample covers July 1973 through December 2011, and is determined by the availability of quarterly earnings data.

	Excess Returns		FF		FF (adjusted)				FFC		FFC (adjusted)			
	E[r^e]	σ	α	σ_ϵ	α	[L95	U95]	σ_ϵ	α	σ_ϵ	α	[L95	U95]	σ_ϵ
Basis Characteristics														
Market Equity	-0.29	4.60	0.08	2.33	-0.01	[-0.28	0.23]	2.70	0.09	2.33	-0.03	[-0.29	0.21]	2.44
	(-1.34)		(0.72)		(-0.10)				(0.75)		(-0.20)			
Book-to-Market	0.49***	3.92	-0.00	2.78	0.15	[-0.24	0.46]	3.44	0.01	2.78	0.19	[-0.12	0.53]	3.52
	(2.70)		(-0.02)		(0.82)				(0.10)		(1.07)			
Prior Performance	1.36***	6.66	1.67***	6.50	1.56***	[0.83	2.19]	6.62	0.49***	2.67	0.57*	[-0.14	1.17]	5.59
	(4.40)		(6.63)		(4.68)				(4.33)		(1.71)			
RMSE	0.85		0.96		0.90				0.29		0.35			
ΔRMSE					0.06	[-0.07	0.18]				-0.06	[-0.32	0.25]	
Other Characteristics														
Profitability (adj)	0.21**	1.98	0.33***	1.87	0.18**	[0.02	0.39]	1.82	0.32***	1.87	0.20***	[0.04	0.38]	1.81
	(2.29)		(3.91)		(2.07)				(4.51)		(2.67)			
Return on Assets	0.59***	4.77	1.02***	3.84	0.62***	[0.29	1.05]	3.77	0.74***	3.59	0.35	[-0.00	0.77]	3.60
	(2.66)		(4.88)		(2.76)				(3.70)		(1.50)			
Return on Equity	1.08***	4.95	1.19***	4.41	0.99***	[0.50	1.41]	4.45	0.87***	4.11	0.79***	[0.23	1.27]	4.40
	(4.70)		(5.15)		(4.08)				(3.97)		(2.84)			
Asset Turnover	0.47***	3.68	0.68***	3.47	0.40*	[0.07	0.83]	3.66	0.60***	3.45	0.37	[-0.04	0.75]	3.67
	(2.73)		(3.58)		(1.92)				(3.21)		(1.46)			
Gross Margins	-0.07	2.92	0.26*	2.41	0.10	[-0.16	0.33]	2.52	0.29**	2.40	0.14	[-0.11	0.47]	2.56
	(-0.50)		(1.90)		(0.65)				(2.11)		(0.86)			
SUE	0.84***	4.78	1.05***	4.64	0.91***	[0.48	1.29]	4.63	0.82***	4.49	0.68***	[0.20	1.21]	4.56
	(3.79)		(5.00)		(4.24)				(3.84)		(2.75)			
O-score	-0.24	4.45	-0.77***	2.73	-0.53***	[-0.77	-0.25]	2.75	-0.55***	2.50	-0.36**	[-0.64	-0.07]	2.67
	(-1.18)		(-4.87)		(-3.53)				(-3.61)		(-2.23)			
Net Stock Issuance	-0.77***	2.87	-0.81***	2.78	-0.65***	[-0.93	-0.40]	2.71	-0.70***	2.73	-0.65***	[-0.93	-0.35]	2.75
	(-5.76)		(-5.63)		(-4.72)				(-4.87)		(-3.87)			
Accruals	-0.32**	3.32	-0.34**	3.18	-0.41***	[-0.71	-0.09]	3.22	-0.31**	3.18	-0.49**	[-0.93	-0.16]	3.22
	(-2.05)		(-2.53)		(-2.63)				(-2.02)		(-2.36)			
Asset Growth	-0.63***	3.42	-0.38***	2.85	-0.34**	[-0.63	0.01]	3.04	-0.27*	2.79	-0.32*	[-0.64	0.04]	3.13
	(-3.99)		(-2.77)		(-2.14)				(-1.83)		(-1.82)			
Organizational Capital	0.42**	4.14	0.33**	2.81	0.34*	[-0.00	0.71]	3.36	0.30**	2.80	0.25	[-0.07	0.61]	3.22
	(2.19)		(2.49)		(1.74)				(2.30)		(1.39)			
RMSE	0.59		0.73		0.56				0.57		0.46			
ΔRMSE					0.17**	[0.10	0.26]				0.11**	[0.02	0.23]	
RMSE (*All*)	0.65		0.78		0.65				0.52		0.44			
ΔRMSE (*All*)					0.13**	[0.07	0.22]				0.08	[-0.02	0.22]	

C. Sensitivity analysis of cross-sectional asset pricing test estimates

The tests mentioned in the previous sections suggest that a priced component in the basis portfolios may be excluded from, or an unpriced component may be included in, the factors constructed using the matrix $\mathbf{A}_{\mathtt{x}}^{\mathtt{FF}}$. I test whether these restrictions bias inferences from cross-sectional regressions on the standard 25 FF test portfolios. I estimate these cross-sectional regressions using a common variant of the FM (Fama and MacBeth (1973)) procedure.[30] In the first stage, I calculate betas using a full-sample time-series regression. In the second stage, I regress the cross section of monthly returns on the estimated beta. The time-series average of the monthly estimated coefficients is the reported estimates of the risk premiums ($\hat{\gamma}$) and intercept ($\hat{\alpha}$). I report the Shanken (1992) error-in-variables-bias-adjusted t-statistic. For specifications in which the adjusted factors are used, I also report bootstrap t-statistics that take into account estimation error in the adjusted FCM.

I test for the presence of the bias described in Proposition 1 by including the optimal orthogonal component of the FF factors. It is obtained as a regression residual of the returns of the estimated MVE portfolio on the returns of the FF factors. The risk premium of this orthogonal component, γ^{FF^*}, is significant. Further, including it reduces the bias in the risk premium estimates. For example, the estimated market risk premium is nearer to the expected value of 0.44% per month, the average return in the sample.

[30] perform this test using betas estimated from rolling windows. Following, Jagannathan and Wang (1996), I use the full-sample to estimate betas.

Table V
Sensitivity Tests for Cross Sectional Regression Estimates

Panels A and C show the results of an FM regression. The test assets are the 25 FF test portfolios. SIZE is the log of the average firm size and BTM is the value-weighted average of the ratio of book to market value of equity; γ represents the risk premium estimated from the FM procedure. FF^* denotes the optimal orthogonal portfolio of the FF factors, which spans the FF basis portfolios. MKT, PMMV, and OMMV are the adjusted FF factors. $E[\hat{\gamma}]$ represents the estimated expected value of estimate $\hat{\gamma}$ in the sample (from theory or the time-series average). R^2 follows Jagannathan and Wang (1996). $\bar{R}^2$ adjusts for degrees of freedom. SH denotes Shanken (1992) error-in-variables-bias-adjusted t-statistics; BS, bootstrapped t-statistics. Panels B shows the results of a time-series regression using the time series of the intercept (r_0^{FF}) of the cross-sectional regressions from Panel A. γ^{xy} represents the risk premium on the FF basis portfolios, where x is S or B representing small or big firms; and y is L, N, or H, representing low, neutral, or high BTM values. ***, **, and * indicate statistical significance at the 1%, 5%, and 10% levels, respectively. The sample period is from July 1963 to December 2011.

PANEL A: Cross-sectional Regression with Unadjusted FF Factors

	$E[r_z - r_f]$	γ^{MKT}	γ^{SMB}	γ^{HML}	γ^{FF^*}	γ^{SIZE}	γ^{BTM}	R^2	$\bar{R}^2$
$E[\hat{\gamma}]$	$\leq$ 0.17	0.44	0.26	0.38	1.31	0	0	1	1
	1.13***	-0.66*	0.19	0.42***				0.72	0.68
(SH)	(3.87)	(-1.88)	(1.39)	(3.40)					
	1.29***	-0.49	-0.31	0.15		-0.14***	0.16*	0.76	0.72
(SH)	(3.86)	(-1.20)	(-1.19)	(0.93)		(-2.19)	(1.85)		
	0.38	0.10	0.23*	0.35***	1.30**			0.88	0.86
(SH)	(1.05)	(0.23)	(1.75)	(2.81)	(4.16)				
	0.64	-0.13	0.27	0.17	1.33***	0.02	0.16*	0.90	0.89
(SH)	(1.56)	(-0.29)	(1.08)	(1.02)	(4.15)	(0.28)	(1.72)		

PANEL B: Betas of the "Zero Beta" Portfolio

	α_0	β^{SL}	β^{SN}	β^{SH}	β^{BL}	β^{BN}	β^{BH}	R^2	$\bar{R}^2$
$E[r_z - r_f]$	0.38	-1.56***	1.79***	0.40	0.79***	0.60***	-1.75***	0.29	0.28
	(1.40)	(-7.13)	(5.53)	(1.62)	(4.21)	(2.41)	(-6.76)		
	α_0	β^{MKT}	β^{PMMV}	β^{OMVE}				R^2	$\bar{R}^2$
$E[r_z - r_f]$	0.38	-0.20*	-0.42	0.44***				0.15	0.16
	(1.22)	(-1.91)	(-1.60)	(4.86)					

PANEL C: Cross-sectional Regression with Adjusted FF Factors

	$E[r_z - r_f]$	γ^{MKT}	γ^{PMMV}	γ^{OMVE}		γ^{SIZE}	γ^{BTM}	R^2	$\bar{R}^2$
$E[\hat{\gamma}]$	$\leq$ 0.17	0.44	0.31	2.18		0	0	1	1
	0.22	0.24	0.30***	1.91***				0.89	0.86
(SH)	(0.62)	(0.60)	(2.81)	(3.95)					
(BS)	(0.74)	(0.68)	(3.05)	(2.53)					
	0.31	0.27	0.15	1.68***		-0.05	0.04	0.88	0.86
(SH)	(0.80)	(0.66)	(0.72)	(2.73)		(-0.67)	(0.53)		
(BS)	(0.90)	(0.75)	(0.81)	(2.21)		(-0.83)	(0.85)		

Jagannathan, Skoulakis, and Wang (2009) prescribe including the time-average of characteristics associated with average returns to the cross-sectional regression to detect potential misspecification.[31] I find that when the time-average of size and BTM are added to the regression, the risk premium coefficients of the original FF factors become insignificant, whereas the characteristics remain significant. In contrast, the significance of the risk premium related to FF* is robust to including characteristics (though it is not sufficient to make BTM insignificant). This evidence supports the hypothesis that the exogenous constraints imposed on the FF factors induce a bias in expected return estimates as per Proposition 1.

A small and insignificant estimated intercept in cross-sectional tests is another important prediction of any asset pricing model. These models imply that the zero-beta rate should either be equal the risk-free rate, or when the borrowing and lending rates differ, be slightly greater than zero, about 2% per annum or approximately 0.17% month.[32] The unadjusted FF specification indicates a large and significant zero-beta rate. This is significantly higher than the difference between borrowing and lending rates, which might lead to nonzero intercepts.

I analyze the time-series of monthly estimated intercepts from second-stage cross-sectional

[31] Jagannathan and Wang (1998) show that, when the factor model is correctly specified, the cross-sectional estimator is consistent under very general conditions. However, if the null hypothesis model is misspecified, the estimator will be asymptotically biased and the factor risk premium in tests can converge to infinity (under certain conditions) even when the true factor risk premium is zero. They show that this misspecification can be detected by using firm characteristics, which could be significant in cross sectional regressions because the linear-beta pricing model is misspecified, not because the firm characteristics are priced factors. Jagannathan et al. (2009) extend this analysis and find that characteristics included in the cross-sectional should not be time-varying, otherwise they can induce a bias and make the estimator inconsistent. They recommend time-averaging of firm characteristics.

[32] See e.g. Brennan (1971) and Lewellen et al. (2010). Kan et al. (2013) note that the estimated zero-beta rates from most models in the literature are far too high to be consistent with plausible spreads between borrowing and lending rates, as required by theory. Lewellen et al. (2010) investigate many asset pricing models and find that for almost all models the alphas in cross sectional Fama-Macbeth tests of the size and book-to-market portfolios are large and significantly greater than zero. Because an important testable implication of a well-specified asset pricing model is a small and insignificant alpha, these models do not seem to satisfactorily pass this test for the 25 FF portfolios.

regressions. A significant average intercept can also be due to nonzero average returns of the zero-beta portfolio. Panel B shows time-series regressions of this estimated zero-beta portfolio on the basis portfolios and on the adjusted FF factors. I find that the estimated "zero-beta" portfolio does not have zero beta with respect to the FF basis portfolios or the adjusted factors.[33] Further, the estimated average return of the zero-beta portfolio is smaller and insignificant, after controlling for exposures to the basis portfolios and the adjusted factors. This is consistent with the hypothesis that the large and significant intercept estimated using the unadjusted factors is not due to true zero-beta risk, but due to bias related with the dimensionality of the space spanned by the basis portfolios.

In Panel C, I test the hypothesis that the adjusted FF model is well specified for this set of the test portfolios. I find that this hypothesis cannot be rejected by the tests: the risk premiums on the factors are close to their theoretical values, the estimated zero-beta rate is low and insignificantly different from zero, and characteristics become insignificant when included in the cross-sectional regressions. This suggests that the FF basis portfolios do span the MVE portfolio of the test assets, but the FF factors fail to do so because of the restrictions imposed at the time of factor construction. After correcting for this bias, the basis portfolios chosen by FF seem to describe this cross section of test portfolio returns even better than in their unadjusted specification.

D. Sensitivity analysis of mutual fund performance estimates

I test the effect of the unadjusted factor construction method on estimates of mutual fund performance evaluation—another influential application of the FF and FFC factor models. These factors have often been used in studies to control for common variation in mutual fund

[33] The betas with respect to the unadjusted FF factors are zero by construction.

returns. I report time-series regression estimates of the returns of the aggregate portfolio of the mutual funds (weighted based on their assets) on unadjusted and adjusted factors. I use the monthly mutual fund data provided by Center for Research in Security Prices (CRSP) between January 1984 and December 2011 to estimate fund alphas.[34]

Table VI
Sensitivity Analysis of Mutual Fund Performance Estimates

This table presents the results of the regression of the value-weighted average of mutual fund returns on various specifications of the FF and FFC factors. The adjustment for FF factors is a three-factor model that includes the market portfolio, an estimated PMMV portfolio (the small-neutral portfolio), and their corresponding OMVE portfolios. ***, **, and * indicate statistical significance at the 1%, 5%, and 10% levels, respectively. The sample period is from January 1984 to December 2011.

PANEL A: Fama, French, and Carhart Factors					
$12 \times \alpha$	EMKT	SMB	HML	MOM	RSQ
-1.11**	0.92***	0.09***	0.02		0.98
(-2.45)	(107.01)	(6.92)	(1.50)		
-1.01**	0.92***	0.09***	0.02	-0.01	0.98
(-2.19)	(103.90)	(6.97)	(1.25)	(-1.23)	

PANEL B: Adjustment for Fama and French factors				
$12 \times \alpha$	MKT	PMMV	OMVE	RSQ
-0.52	0.95***	0.14***	-0.03***	0.98
(-0.87)	(43.55)	(5.61)	(-4.45)	

The results in Table VI indicate that the estimates using FFC factors are biased downwards. The adjustment for bias significantly impacts inference in this application. In Panel B, I find that the average mutual fund performance estimate α reduces by more than 50 % and becomes insignificant after correcting for this bias.[35]

[34]The period before 1984 is not included due to selection bias issues in reported monthly returns. See Fama and French (2010) and Elton, Gruber, and Blake (2001). Following Barras, Scaillet, and Wermers (2010), I select only funds having at least 60 monthly return observations in the sample and match the CRSP database with the Thomson/CDA database using the MFLINKs product of Wharton Research Data Services.

[35]A bootstrapped confidence interval (to account for estimation error due to estimation of PMMV portfolio weights) also contains zero.

V. Conclusion

I find that when the basis portfolios used to construct factors have a rank greater than the number of factors constructed, the use of the prespecified simple weights to construct factors will induce biases in estimates from the resultant factors. These biases can be characterized as having two components: the omitted spanning component bias and the included unpriced component bias. The former relates to the component of the MVE portfolio omitted due to binding weight constraints. The latter is an confounded-variable-bias that influences estimates when the residuals of securities covary with the unpriced components of the unadjusted factors.

I propose an adjustment to correct for the identified bias. This adjustment is based on optimal weights of the basis portfolios under the assumption that more than one basis portfolio has exposure to each priced state variables. I find that this adjustment to the remarkably successful Fama and French (1993) factor construction method helps reduce the bias and MSE in estimates of expected and abnormal returns. Specification tests using simulated data indicate that factors adjusted using this method seem to be less biased than those constructed using prespecified weights for realistic sample sizes. Also, FF factors, adjusted using this method, describe observed data even better than the original specification.

Although these empirical results focus on the benchmark FF and FFC factors, the biases and adjustments identified in this paper are relevant to factors constructed by combining any set of portfolios using prespecified simple weights. The methodology proposed in this paper will contribute to improving estimates in asset pricing and corporate finance applications that require the use of such factor models.

Appendix

Appendix A. Proof of Proposition 1

1. (a) i. If Assumption II does not holds then $\forall$ full rank $K \times C$ matrix $\mathbf{A}_{\mathtt{x}}$, there exists a nonsingular $K \times K$ matrix $\mathbf{Z}_{\mathtt{x}}$ such that $[r_{m,t}\ \mathbf{f}_{\mathtt{P},t}]' = \mathbf{Z}_{\mathtt{x}}\mathbf{A}_{\mathtt{x}}\mathbf{r}_{\mathtt{C},t}$. Since this must be true for all choices of $\mathbf{A}_{\mathtt{x}}$, consider a K $\times$ K identity matrix bordered with zeros in the remaining C - K rows. This $K \times C$ matrix $\mathbf{A}_{\mathtt{I}}$ must also satisfy the condition that there exists a non-singular $\mathbf{Z}_{\mathtt{I}}$ such that: $[r_{m,t}\ \mathbf{f}_{\mathtt{P},t}]' = \mathbf{Z}_{\mathtt{I}}\mathbf{A}_{\mathtt{I}}\mathbf{r}_{\mathtt{C},t}$. This implies that all C basis portfolios $\mathbf{r}_{\mathtt{C}}$ are spanned by the K PMMV portfolios. Further, when Assumption II is not satisfied, all random vectors $\mathbf{f}_{\mathtt{x},t} = \mathbf{A}_{\mathtt{x}}\mathbf{r}_{\mathtt{C},t}$ span the PMMV portfolios. This implies all $\mathbf{f}_{\mathtt{x}}$ span $\mathbf{r}_{\mathtt{C}}$.

   ii. From Assumption I, there exist K linearly independent combinations of $\mathbf{r}_{\mathtt{C}}$ that are the PMMV portfolios. By assumption, these PMMV portfolios produce unbiased estimates for all securities.

   From standard OLS results, two sets of regressors, where one set is a linear combinations of the other, will produce identical estimates of intercept and residual variance. Since any K factors will span the PMMV portfolios, therefore the assumption for identical OLS estimates of intercept and residual variance is satisfied. Since, by definition, the PMMV portfolios produce unbiased estimates of the intercept and residual variance: $\Delta\mu_{i,\mathtt{x}} = \Delta\mu_{i,\mathtt{P}} = 0$ and $\Delta\sigma_{i,\mathtt{x}} = \Delta\sigma_{i,\mathtt{P}} = 0$.

   (b) Since the space spanned by $\mathbf{r}_{\mathtt{C}}$ can be spanned by a basis of K elements $\mathbf{r}_{\mathtt{P}}$, the rank of the covariance matrix, which is the rank of the rank of $\mathrm{E}[\mathbf{r}_{\mathtt{C},t}\mathbf{r}'_{\mathtt{C},t}]$, is K.

2. (a) i. From Assumption II, there exists a matrix $\mathbf{A}_{\mathtt{x}}$ such that $\mathbf{f}_{\mathtt{x}} \neq \mathbf{Z}_{\mathtt{x}}^{-1}\mathbf{A}_{\mathtt{P}}\mathbf{r}_{\mathtt{C}}$ for any matrix $\mathbf{Z}_{\mathtt{x}}$. This implies that factors $\mathbf{f}_{\mathtt{x}}$ do not span $\mathbf{r}_{\mathtt{C}}$.

ii. Let $\xi_{P,t}$ denote the residuals of a regression of $r_{P,t}$ on $\mathbf{f}_{\mathtt{x},t}$. This residual vector can be expressed as:

$$\begin{aligned}\xi_{\mathtt{P},t} &= \mathbf{r}_{\mathtt{P},t} - \mathbf{Z}_{\mathtt{x}}\mathbf{f}_{\mathtt{x},t} = \mathbf{A}_{\mathtt{P}}\mathbf{r}_{\mathtt{C},t} - \mathbf{Z}_{\mathtt{x}}\mathbf{A}_{\mathtt{x}}\mathbf{r}_{\mathtt{C},t} \\ &= (\mathbf{A}_{\mathtt{P}} - \mathbf{Z}_{\mathtt{x}}\mathbf{A}_{\mathtt{x}})\mathbf{r}_{\mathtt{C},t}.\end{aligned}$$

Under Assumptions I and II, there does not exist a nonsingular $K \times K$ matrix $\mathbf{Z}_{\mathtt{x}}$ such that $\mathbf{Z}_{\mathtt{x}}\mathbf{A}_{\mathtt{x}} = \mathbf{A}_{\mathtt{P}}$, therefore $[r_{m,t}\ \mathbf{f}_{\mathtt{P},t}] = \mathbf{A}_{\mathtt{P}}\mathbf{r}_{\mathtt{C},t} \neq \mathbf{Z}_{\mathtt{x}}\mathbf{A}_{\mathtt{x}}\mathbf{r}_{\mathtt{C},t} = \mathbf{Z}_{\mathtt{x}}\mathbf{f}_{\mathtt{x}}$ for any non-singular matrix $\mathbf{Z}_{\mathtt{x}}$. This implies that $\mathrm{Var}[\xi_{\mathtt{P},t}] \neq 0$ for at least one PMMV portfolio. Therefore, for some PMMV portfolio p, $\Delta\sigma_{\epsilon_p,\mathtt{x}} > 0$.

Next, consider the residual ϵ_p of this portfolio. If $\mathrm{E}[\epsilon_p] = 0$, then p is not a minimum variance portfolio with respect to a priced state variable—the same priced exposure and lower variance could be achieved using $\mathbf{f}_{\mathtt{x}}$. This is because $\mathrm{Var}[r_{\mathtt{p},t}] = \mathbf{Z}_{\mathtt{x}}\mathrm{Var}[f_{\mathtt{x},t}]\mathbf{Z}_{\mathtt{x}}' + \mathrm{Var}[\xi_{\mathtt{P},t}]$. This contradicts the assumption that $r_{\mathtt{p},t}$ is PMMV. Therefore $\mathrm{E}[\epsilon_p] \neq 0$ and the factors $\mathbf{f}_{\mathtt{x}}$ will estimate a nonzero abnormal returns for this portfolio: $\Delta\mu_{p,\mathtt{x}} \neq 0$.

(b) If $\mathbf{f}_{\mathtt{x}}$ does not span $\mathbf{r}_{\mathtt{C}}$ then the space spanned by $\mathbf{f}_{\mathtt{x}}$ is a subspace of the space spanned by $\mathbf{r}_{\mathtt{C}}$. Since the rank of the space spanned by $\mathbf{f}_{\mathtt{x}}$ is K, the rank of $\mathbf{\Sigma}_{\mathtt{C}} > K$.

□

Appendix B. Expression for Bias in Equation 6

First I express $\mathrm{E}[\hat{\mu}_{i,\mathtt{x}}]$ as functions of the underlying parameters:

$$
\begin{aligned}
E[\hat{\mu}_{i,\mathtt{x}}] &= E[\widehat{\mathbf{B}}_{i,x}\overline{\mathbf{f}}_{\mathtt{x}}] = \mathbf{B}_{i,x}\boldsymbol{\mu}_{\mathtt{x}} = \boldsymbol{\Sigma}_{i,\mathtt{x}}\boldsymbol{\Sigma}_{\mathtt{x}}^{-1}\boldsymbol{\mu}_{\mathtt{x}}, && \text{(from definitions and standard results)}\\
&= \mathrm{Cov}(\mathbf{B}_{i,\mathtt{P}}\mathbf{r}_{\mathtt{P},t} + \epsilon_{i,t}, \mathbf{B}_{\mathtt{x},\mathtt{P}}\mathbf{r}_{\mathtt{P},t} + \boldsymbol{\epsilon}_{\mathtt{x},t})\boldsymbol{\Sigma}_{\mathtt{x}}^{-1}\boldsymbol{\mu}_{\mathtt{x}}, && \text{(from equation 2)}\\
&= \mathbf{B}_{i,\mathtt{P}}\boldsymbol{\Sigma}_{\mathtt{P}}\mathbf{B}'_{\mathtt{x},\mathtt{P}}\boldsymbol{\Sigma}_{\mathtt{x}}^{-1}\boldsymbol{\mu}_{\mathtt{x}} + \mathrm{Cov}(\epsilon_i, \boldsymbol{\epsilon}_{\mathtt{x},t})\boldsymbol{\Sigma}_{\mathtt{x}}^{-1}\boldsymbol{\mu}_{\mathtt{x}}, &&\\
&= \mathbf{B}_{i,\mathtt{P}}\boldsymbol{\Sigma}_{\mathtt{P}}\boldsymbol{\Sigma}_{\mathtt{P}}^{-1}\boldsymbol{\Sigma}'_{\mathtt{x},\mathtt{P}}\boldsymbol{\Sigma}_{\mathtt{x}}^{-1}\boldsymbol{\mu}_{\mathtt{x}} + \mathbf{b}_{\mathtt{i},\mathtt{Z}}\boldsymbol{\mu}_{\mathtt{x}}, && \text{(from definition of } \mathbf{B}_{\mathtt{x},\mathtt{P}} \text{ and } \mathbf{b}_{\mathtt{i},\mathtt{Z}})\\
&= \mathbf{B}_{i,\mathtt{P}}\mathbf{B}_{\mathtt{P},\mathtt{x}}\boldsymbol{\mu}_{\mathtt{x}} + \mathbf{b}_{\mathtt{i},\mathtt{Z}}\boldsymbol{\mu}_{\mathtt{x}}, && \text{(from definition of } \mathbf{B}_{\mathtt{P},\mathtt{x}})\\
&= \mathbf{B}_{i,\mathtt{P}}\mathbf{B}_{\mathtt{P},\mathtt{x}}\boldsymbol{\mu}_{\mathtt{x}} + \Delta\mu_{i,\mathtt{x}}^{\mathtt{Z}}. &&
\end{aligned}
$$

Therefore, $\Delta\mu_{i,\mathtt{x}}$ can be expressed as:

$$
\begin{aligned}
\Delta\mu_{i,\mathtt{x}} &= \mu_i - E[\hat{\mu}_{i,\mathtt{x}}] = \mu_i - \mathbf{B}_{i,\mathtt{P}}\mathbf{B}_{\mathtt{P},\mathtt{x}}\boldsymbol{\mu}_{\mathtt{x}} - \Delta\mu_{i,\mathtt{x}}^{\mathtt{Z}}, && \text{(from definitions and result above)}\\
&= \mathbf{B}_{i,\mathtt{P}}\left(\boldsymbol{\mu}_{\mathtt{P}} - \mathbf{B}_{\mathtt{P},\mathtt{x}}\boldsymbol{\mu}_{\mathtt{x}}\right) - \Delta\mu_{i,\mathtt{x}}^{\mathtt{Z}}, && \text{(from equations 3 and 4)}\\
&= \mathbf{B}_{i,\mathtt{P}}\Delta\boldsymbol{\mu}_{\mathtt{P},\mathtt{x}} - \Delta\mu_{i,\mathtt{x}}^{\mathtt{Z}} = \mathbf{B}_{i,\mathtt{P}}\boldsymbol{\alpha}_{\mathtt{P}} - \Delta\mu_{i,\mathtt{x}}^{\mathtt{Z}}, && \text{(from definition of } \Delta\boldsymbol{\mu}_{\mathtt{P},\mathtt{x}})\\
&= \Delta\mu_{i,\mathtt{x}}^{\mathtt{P}} - \Delta\mu_{i,\mathtt{x}}^{\mathtt{Z}} &&
\end{aligned}
$$

Appendix C. Proof of Proposition 2

1. If Assumption I holds, then for any $S_P + 1$ weight vector $\mathbf{w}_{\mathtt{P}}$ such that $r_{\mathtt{P}^*,t} = \mathbf{w}'_{\mathtt{P}}r_{\mathtt{P},t}$, there exists a matrix $\mathbf{A}_{\mathtt{P}^*} = \mathbf{w}'_{\mathtt{P}}\mathbf{A}_{\mathtt{P}}$ such that $r_{\mathtt{P}^*,t} = \mathbf{A}_{\mathtt{P}^*}\mathbf{r}_{\mathtt{C},t}$. This is also true for the weight vector that produces the MVE portfolio. Therefore, applying Huberman and Kandel (1987) Proposition 3 (when the riskless asset exists), exact linear pricing holds with respect to $\mathbf{r}_{\mathtt{C},t}$: that is $\Delta\mu_{i,\mathtt{C}} = 0$. Again applying the same theorem to MVE

portfolio of the basis portfolios $\mathbf{r}_{\mathtt{C}^*,t}$: exact linear pricing also holds with respect to $\mathbf{r}_{\mathtt{C}^*,t}$ and $E[\boldsymbol{\alpha}_{i,\mathtt{C}^*}] = E[\widehat{\boldsymbol{\alpha}}_{i,\mathtt{C}^*}] = 0$. Therefore, $\Delta\mu_{i,\mathtt{C}^*} = 0$.

2. Let $\xi_{i,t}$ denote the residuals of a regression of $r_{i,t}$ on $\mathbf{r}_{\mathtt{C}^*,t}$, and $\epsilon_{i,t}$ denote that on $\mathbf{r}_{\mathtt{P},t}$. Note that $\epsilon_{i,t}$ captures the cumulative returns of the unpriced state variable exposures ($\boldsymbol{\beta}_{i,\mathtt{Z}}\boldsymbol{s}_{\mathtt{Z},t}$) and specific returns ($\eta_{i,t}$) of securities i. $\xi_{i,t}$ can be expressed as:

$$\begin{aligned}\xi_{i,t} &= r_{i,t} - \beta_{i,\mathtt{C}^*} r_{\mathtt{C}^*,t} = \mathbf{B}_{i,\mathtt{P}}\mathbf{r}_{\mathtt{P},t} + \epsilon_{i,t} - \beta_{i,\mathtt{C}^*}\boldsymbol{\Sigma}_{\mathtt{C}^*\mathtt{P}}\mathbf{r}_{\mathtt{P},t} \\ &= (\mathbf{B}_{i,\mathtt{P}} - \beta_{i,\mathtt{C}^*}\boldsymbol{\Sigma}_{\mathtt{C}^*\mathtt{P}})\mathbf{r}_{\mathtt{P},t} + \epsilon_{i,t}. \qquad (1)\end{aligned}$$

This further implies that:

$$\begin{aligned}\mathrm{Var}[\xi_{i,t}] &= \mathrm{Var}[\epsilon_{i,t}] + (\mathbf{B}_{i,\mathtt{P}} - \beta_{i,\mathtt{C}^*}\mathbf{B}_{\mathtt{C}^*\mathtt{P}})\boldsymbol{\Sigma}_{\mathtt{P}}(\mathbf{B}'_{i,\mathtt{P}} - \beta_{i,\mathtt{C}^*}\mathbf{B}'_{\mathtt{C}^*\mathtt{P}}), \\ &\geq \mathrm{Var}[\epsilon_{i,t}]. \qquad (2)\end{aligned}$$

When $K = 1$, then the term $(\mathbf{B}_{i,\mathtt{P}} - \beta_{i,\mathtt{C}^*}\mathbf{B}'_{\mathtt{C}^*\mathtt{P}})$ will be zero. If this is not the case, then exact linear pricing will not hold with respect to $\mathtt{C}^*$, a contradiction. However, $(\mathbf{B}_{i,\mathtt{P}} - \beta_{i,\mathtt{C}^*}\mathbf{B}'_{\mathtt{C}^*\mathtt{P}})$ will not be zero in general, when $K > 1$. For instance, for at least one PMMV portfolio p, it is greater than zero (see proof of Proposition 1 [2 (c) ii]).

3. From standard *OLS* results, we know that the variance of the intercept increases in the variance of the regression residuals. Since $\mathrm{Var}[\xi_{i,t}] \geq \mathrm{Var}[\epsilon_{i,t}]$, therefore $\mathrm{Var}[\widehat{\boldsymbol{\alpha}}_{i,\mathtt{C}^*}] \geq \mathrm{Var}[\widehat{\boldsymbol{\alpha}}_{i,\mathtt{P}}]$, with equality when $K = 1$. □

Appendix D. Proof of Proposition 3

1. Consider a $C \times 1$ weight vector $\mathbf{w}_j$ that combines the basis portfolios with minimum residual variance controlling for the market portfolio return and $\phi_{j,t}$. The residual variance ($\upsilon_{\phi_j,t}$) of this portfolio is:

$$\sigma^2(\upsilon_{\phi_j,t}) = E[((\mathbf{r}_{\mathtt{C},t} - \boldsymbol{\mu}_{\mathtt{C}} - \mathbf{b}_m(r_{m,t} - \mu_m) - \mathbf{b}_{\phi_j}\phi_{j,t}))'\mathbf{w}_j)^2], \tag{3}$$

   The weight vector $\mathbf{w}_{\phi_j}$ that minimizes Equation 3 must satisfy the first-order condition: $\boldsymbol{\Phi}_{\mathtt{C},j}\mathbf{w}_{\phi_j} - \lambda_j\boldsymbol{\iota} = 0$. This implies that $\mathbf{w}_{\phi_j} = \lambda_j\boldsymbol{\Phi}^{\dagger}_{\mathtt{C},j}\boldsymbol{\iota}$. Imposing the constraint $\boldsymbol{\iota}'\mathbf{a}_{\phi_j} = 1$ gives the expression for minimum residual variance portfolio in Equation 9. If Assumption III^a holds then these portfolios are also a PMMV portfolio of the basis portfolios. The PMMV portfolios of the basis portfolios are also PMMV for all securities (Assumption I). □

2. The cross-sectional expected MSE of $\hat{\alpha}_{i,\mathtt{x}}$, given a large time-series, is:

$$\sum_i E[\hat{\alpha}_{i,\mathtt{x}}]^2 = \sum_i (\mathbf{B}_{i,\mathtt{P}}\boldsymbol{\alpha}_{\mathtt{P}} - \mathbf{b}_{\mathtt{i},\mathtt{Z}}\boldsymbol{\mu}_{\mathtt{x}})^2. \tag{4}$$

   From 1, $\mathrm{E}[\alpha_{i,\phi}] = 0$ for all i. Also, from Proposition 1, the bias is nonzero for at least one asset i. This implies $\sum_i \mathrm{E}[\hat{\alpha}_{i,\mathtt{x}}]^2 > \sum_i \mathrm{E}[\hat{\alpha}^2_{i,\phi}] = 0$. □

REFERENCES

Barras, Laurent, Olivier Scaillet, and Russ Wermers, 2010, False discoveries in mutual fund performance: Measuring luck in estimated alphas, *The Journal of Finance* 65, 179–216.

Berk, Jonathan B., Richard C. Green, and Vasant Naik, 1999, Optimal investment, growth options, and security returns, *The Journal of Finance* 54, 1553–1607.

Brennan, M. J., 1971, Capital market equilibrium with divergent borrowing and lending rates, *The Journal of Financial and Quantitative Analysis* 6, 1197–1205.

Carhart, Mark M., 1997, On persistence in mutual fund performance, *The Journal of Finance* 52, 57–82.

Chamberlain, Gary, 1983, Funds, factors, and diversification in arbitrage pricing models, *Econometrica* 51, 1305–1323.

Chen, L., R. Novy-Marx, and L. Zhang, 2010, An alternative three-factor model, *Working paper* .

Connor, Gregory, and Robert A. Korajczyk, 1986, Performance measurement with the arbitrage pricing theory: A new framework for analysis, *Journal of Financial Economics* 15, 373 – 394.

Cremers, Martijn, Antti Petajisto, and Eric Zitzewitz, 2013, Should benchmark indices have alpha? revisiting performance evaluation., *Critical Finance Review* 2, 001–048.

Diacogiannis, George, and David Feldman, 2013, Linear beta pricing with inefficient benchmarks, *Quarterly Journal of Finance* 3, 1–35.

Elton, Edwin J., Martin J. Gruber, and Christopher R. Blake, 2001, A first look at the accuracy of the crsp mutual fund database and a comparison of the crsp and morningstar mutual fund databases, *The Journal of Finance* 56, 2415–2430.

Fama, Eugene F., 1996, Multifactor portfolio efficiency and multifactor asset pricing, *Journal of Financial and Quantitative Analysis* 31, 441–465.

Fama, Eugene F., 1998, Determining the number of priced state variables in the ICAPM, *Journal of Financial and Quantitative Analysis* 33, 217–231.

Fama, Eugene F., and Kenneth R. French, 1993, Common risk factors in the returns on stocks and bonds, *Journal of Financial Economics* 33, 3 – 56.

Fama, Eugene F., and Kenneth R. French, 1996, Multifactor explanations of asset pricing anomalies, *The Journal of Finance* 55–84.

Fama, Eugene F., and Kenneth R. French, 2010, Luck versus skill in the cross-section of mutual fund returns, *The Journal of Finance* 65, 1915–1947.

Fama, Eugene F, and Kenneth R French, 2013, A five-factor asset pricing model, *Fama-Miller working paper, University of Chicago, Dartmouth College and NBER* .

Fama, Eugene F., and James D. MacBeth, 1973, Risk, return, and equilibrium: Empirical tests, *The Journal of Political Economy* 81, 607–636.

Gerakos, Joseph, and Juhani T. Linnainmaa, 2014, Dissecting factors, *Chicago Booth Research Paper* .

Gibbons, Michael R., Stephen A. Ross, and Jay Shanken, 1989, A test of the efficiency of a given portfolio, *Econometrica* 57, 1121–1152.

Grinblatt, Mark, and Sheridan Titman, 1987, The relation between mean-variance efficiency and arbitrage pricing, *The Journal of Business* 60, pp. 97–112.

Harvey, Campbell, Yan Liu, and Heqing Zhu, 2014, âĂę and the cross-section of expected returns, *Working Paper* .

Huberman, Gur, and Shmuel Kandel, 1987, Mean-variance spanning, *The Journal of Finance* 42, 873–888.

Huberman, Gur, Shmuel Kandel, and Robert F. Stambaugh, 1987, Mimicking portfolios and exact arbitrage pricing, *The Journal of Finance* 42, pp. 1–9.

Jagannathan, Ravi, Georgios Skoulakis, and Zhenyu Wang, 2009, The analysis of the cross-section of security returns, *Handbook of Financial Econometrics, Vol 2: Applications* 2, 73–134.

Jagannathan, Ravi, and Zhenyu Wang, 1996, The conditional capm and the cross-section of expected returns, *Journal of Finance* 51, 3–53.

Jagannathan, Ravi, and Zhenyu Wang, 1998, An asymptotic theory for estimating beta-pricing models using cross-sectional regression, *The Journal of Finance* 53, 1285–1309.

James, W., and C. Stein, 1961, Estimation with quadratic loss., *Proceedings of the 4th Berkeley Symposium on Probability and Statistics I. Berkeley: Univ. of Calif. Press* 361–379.

Jorion, Philippe, 1986, Bayes-Stein estimation for portfolio analysis, *Journal of Financial and Quantitative Analysis* 21, 279–292.

Kan, Raymond, Cesare Robotti, and Jay Shanken, 2013, Pricing model performance and the two-pass cross-sectional regression methodology, *The Journal of Finance* .

Kan, Raymond, and Daniel R. Smith, 2008, The distribution of the sample minimum-variance frontier, *Management Science* 54, 1364–1380.

Kan, Raymond, and Guofu Zhou, 2007, Optimal portfolio choice with parameter uncertainty, *Journal of Financial and Quantitative Analysis* 42, 621–656.

Kogan, Leonid, and Mary Tian, 2012, Firm characteristics and empirical factor models: a data-mining experiment, *International Finance Discussion Papers, Board of Governors of the Federal Reserve System (U.S.)* 1070.

Lehmann, Bruce N., 1987, Orthogonal frontiers and alternative mean-variance efficiency tests, *The Journal of Finance* 42, 601–619.

Lehmann, Bruce N., and David M. Modest, 2005, Diversification and the optimal construction of basis portfolios, *Management Science* 51, 581–598.

Lewellen, Jonathan, Stefan Nagel, and Jay Shanken, 2010, A skeptical appraisal of asset pricing tests, *Journal of Financial Economics* 96, 175 – 194.

MacKinlay, A.Craig, 1987, On multivariate tests of the {CAPM}, *Journal of Financial Economics* 18, 341 – 371.

MacKinlay, A.Craig, 1995, Multifactor models do not explain deviations from the CAPM, *Journal of Financial Economics* 38, 3 – 28.

Merton, Robert C., 1973, An intertemporal capital asset pricing model, *Econometrica* 41, 867–887.

Newey, Whitney K, and Kenneth D West, 1987, A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix, *Econometrica* 55, 703–708.

Novy-Marx, Robert, 2013, The other side of value: The gross profitability premium, *Journal of Financial Economics* 108, 1 – 28.

Pukthuanthong, Kuntara, and Richard Roll, 2014, A protocol for factor identification, *Working Paper, UCLA* .

Roll, Richard, 1977, A critique of the asset pricing theory's tests part i: On past and potential testability of the theory, *Journal of Financial Economics* 4, 129–176.

Roll, Richard, and Stephen A. Ross, 1980, An empirical investigation of the arbitrage pricing theory, *The Journal of Finance* 35, pp. 1073–1103.

Ross, Stephen A, 1976, The arbitrage theory of capital asset pricing, *Journal of Economic Theory* 13, 341–360.

Shanken, Jay, 1992, On the estimation of beta-pricing models, *Review of Financial Studies* 5, 1–33.

Welch, Ivo, 2008, The link between Fama-French time-series tests and Fama-MacBeth cross-sectional tests, *UCLA Working Paper* .