The Conditional Expected Return and Autocorrelation from the Derivatives

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Abstract

We express conditional expected future returns and stock market autocorrelations with publicly available derivatives data. Our approach is robust to pricing kernel process choice, and provides a real-time conditional point of view. We demonstrate a moderate short-term reversal of market returns with this approach. Furthermore, our approach implies comparable autocorrelation by statistical inference model with a gradually fading memory feature. We construct a reversal signal based on this approach and show that the corresponding market timing strategy outperforms the buy-and-hold strategy overall. Finally, we demonstrate that the term structure of one-month future returns is pro-cyclical.

Keywords: Autocorrelation, conditional expected return, derivatives, market spanning, recovery

JEL Classification: G1, G12, G13

1 Introduction

Researchers have devoted themselves to examining the stock market efficiency since Keynes's (1936) *Animal Spirits* pronouncement. The majority of these studies have focused on the forecastability of stock market returns. Can past returns forecast future returns? Is the return of the stock market index autocorrelated? How to estimate the unconditional autocorrelation coefficients of market returns? Despite extensive research about autocorrelation with historical stock market data (see, for instance, Lo and MacKinlay, 1988; Fama and French, 1988; Poterba and Summers, 1988; Moskowitz, Ooi, and Pedersen, 2012; Campbell, 2017), how to express the autocorrelation using derivatives is mainly missing in the literature.[1]

In this paper, we present a Q-approach to express the stock market index autocorrelation with publicly available *derivative price* data.[2] These derivatives include index options, VIX futures, and VIX options. Our approach is real-time, and provides a time-varying conditional point of view about the return predictability in the stock market. Moreover, we demonstrate the economic value of this approach and derive its new implications to the stock market.

Conceptually, to compute autocorrelation coefficient, it suffices to compute *conditional expected spot return*, *conditional expected future return*, and other statistical quantities that involves spot return and future return.[3] Because the derivatives market offers a forward-looking perspective about the underlying stock market, many authors have studied the conditional expected spot return by the Q-approach (see Martin, 2017; Martin and Wagner, 2019; Kadan and Tang, 2020; Chabi-Yo and Loudis, 2020, etc.). Therefore, we first present a theoretical study about the conditional expected future return, among other things, by the Q-approach.

In our theoretical contribution, we reformulate the conditional expected future return as a *futures price* of a new index, a *power VIX index (PVIX)*, given a power-specification of stochastic discount factor process (SDF). Precisely, a PVIX measures risk-neutral moments of a spot market return that all market prices of index options can span, similar to the CBOE VIX index. All PVIX indices together characterize a spot return's conditional distribution. The futures contract on PVIX is an innovation of the VIX derivatives market. Building on the relation between VIX and PVIX, we obtain a no-arbitrage futures price of PVIX in terms of VIX derivatives

[1] Martin (2021), and Chabi-Yo (2019) are two notable exceptions in which they study the autocorrelation coefficient from the derivatives market information. We later compare our paper with theirs and highlight the difference.

[2] Following standard terminology, we name it a Q-approach because the derivative price is computed by a risk-neutral (Q) measure.

[3] Here, the spot return from the time t perspective is $R_{t\to T}$, the growth (rate of) asset return over the period from t to T. Similarly, $R_{T_1\to T_2}$ is a future return from time t perspective for any $T_2 > T_1 > t$. By a conditional expected spot return we mean $\mathbb{E}_t[R_{t\to T}]$, and $\mathbb{E}_t[R_{T_1\to T_2}]$ is a conditional expected future return for $T_2 > T_1 > t$. We also let $R_{f,T_1\to T_2}$ denote the gross risk-free return over the period from T_1 to T_2.

(VIX futures and VIX options) and stock market index options. Therefore, we can express autocorrelation with publicly available derivatives prices. For a robustness purpose, we also demonstrate that the Q-approach applies to general specifications of the stochastic discount factor process.

We then implement our theoretical results to the S&P 500 index with market data of S&P 500 index options, VIX futures, and VIX options. Our empirical results document significantly negative autocorrelation on the S&P 500 index. For instance, the conditional autocorrelation between two consecutive monthly returns is on average -20.90% with a t-stat of -18.10 (Table 3). The negative sign holds robustly for different risk-aversion parameters in the power-specified SDFs, and different lengths of the two consecutive periods. Notably, the autocorrelation coefficients by the Q-approach are around -20% to -40%, comparable in magnitude to the findings in Martin (2021) and Chabi-Yo (2019). These empirical results reveal a persistent and moderate short-term reversal of monthly returns from a forward-looking perspective.

Nevertheless, since Lo and MacKinlay (1999), it has been well documented that the unconditional market (S&P 500 index) autocorrelation using historical monthly return data is either a very small positive value or virtually zero, suggesting a weak or no return predictability. For simplicity and comparison, we call it a "P-approach" to highlight that it uses historical stock market index data. The Q- and P-approaches imply significant differences in both sign and magnitude of the autocorrelation of market returns. So, how to explain such a discrepancy of the autocorrelation by these two approaches?

We next argue that the choice of statistical inference model leads to a noteworthy difference between the P- and the Q-approaches. Remarkably, a statistical inference model with a gradually fading memory feature (a fading memory methodology in Nagel and Xu (2021)) generates autocorrelation comparable to the Q-approach (Figure 4), despite significant difference between the P- and Q-approaches. In this regard, the fading memory methodology is appealing because it combines both historical and forward-looking perspectives. Therefore, our Q-approach provides supportive evidence to investors' learning with gradually fading memory.

To further understand the empirical implications of the Q-approach, we investigate its economic value. Precisely, because of the moderate short-term reversal of monthly market return by the Q-approach, we construct a reversal signal to trade the market, and the reversal signal relies on the autocorrelation identified by the Q-approach. We empirically show that this reversal signal predicts the market downturn well, particularly when the market declines significantly in the next month. Furthermore, we show that the market timing strategy, using this reversal signal, is conservative and delivers higher Sharpe ratios than the buy-to-hold benchmark strategy. Moreover, the economic value of our market-timing strategy can be substantial in a downward period. For example, we show that investors are willing to pay as high as 11% per annum to switch from the buy-and-hold benchmark to the market timing strategy between January 2008 and June 2009 (Table 7).

Finally, we examine the term structure of one-month future returns, $f_{t,T} = \mathbb{E}_t\left[R_{t+T\to t+T+1}\right], T = 1, 2, \cdots$, and equity risk premiums, $f_{t,T} - \mathbb{E}_t[R_{f,t+T\to t+T}]$, from the time t-perspective. Based on our analytical formula of conditional expected future returns with derivatives data only, we show that the term structure of one-month future returns (and equity risk premiums) is mainly upward on average. Moreover, the term structures of one-month future returns and equity risk premiums are *pro-cyclical*—they are downward sloping in bad times, but upward sloping in good times (Figure 9).

Related Literature. Martin (2021) is the most relevant paper on the market autocorrelation coefficient derived by the derivative data. For a log-specified SDF, Martin (2021) reduces the conditional expected future return to a no-arbitrage price of a "forward-start option". Since the forward-start option is not publicly tradable, Martin (2021) relies on quoted prices from a sophisticated investment bank. From a no-arbitrage perspective, Chabi-Yo (2019) derives lower and upper bounds on the market autocorrelation coefficient with index options. In contrast, we use VIX derivatives (VIX futures and VIX options) in addition to index options and derive a real-time market autocorrelation coefficient at any trading date. More importantly, we investigate new asset pricing implications of the Q-approach to the stock market.

Since both Martin (2021) and this paper rely on derivatives' no-arbitrage prices, it is plausible to use the option spanning theory with basket options (Ross, 1976; Carr and Laurence, 2011; Tian, 2014). In this paper, our approach does not rely on those basket options, instead on publicly VIX derivative data to price PVIX futures. In this respect, our approach is also different from the no-arbitrage model approach of VIX derivatives (See e.g. Mencia and Sentana, 2013). Moreover, we show that the approximation error in the (almost) pricing formula is sufficiently tight for empirical applications.

In some important recent studies, the derivatives market plays an essential role in revealing the underlying stock market information (Ross, 2015; Schneider and Trojani, 2019; Jensen, Lando, and Pedersen, 2019). Many authors including Bakshi and Madan (2000); Bakshi, Kapadia, and Madan (2003); Martin (2017); Bakshi, Crosby, Gao Bakshi, and Zhou (2019a); Bakshi, Gao Bakshi, and Xue (2019b) have utilized derivatives to study the conditional expected spot return. Others investigate individual stock variance, disaster probability, and currency rates by using risk-neutral measure and derivatives data (see, for instance, Backus, Chernov, and Martin, 2011; Martin and Wagner, 2019; Kremens and Martin, 2019; Kadan, Liu, and Tang, 2019). Our contribution in this area is to express the conditional expected future return and other relevant statistical measures with derivatives data. This approach allows us to derive a *bivariate distribution* of two spot returns under the physical probability from derivatives, whereas previous studies only focus on the *marginal distribution* of a spot return.

This paper is also related to a strand of recent studies that investors learn asset prices with a gradually fading memory (Collin-Dufresne, Johannes, and Lochstoer, 2017; Bordalo, Gennaioli, Porta, and Shleifer, 2019; Malmendier and Nagel, 2016; Nagel and Xu, 2021). Unlike those studies, we use the Q-approach as a benchmark to compare several statistical inference methods. Our empirical results support the gradually fading memory methodology suggested in these articles.

Lastly, our results shed new light on the recent equity term structure studies in the literature. Binsbergen, Brandt, and Koije (2012) document that the equity term structure is downward sloping on average. Gormsen (2021) further shows that the term structure is downward sloping in good times, but upward sloping in bad times (counter-cyclical). These authors discuss the hypothetical one-period returns on dividend-claims, or the conditional expected annual return to long-maturity minus the annual return to short maturity equity. In contrast, we study the expected one-month future return; thus, our finding of the term structure of equity risk premiums is a new stylized fact. Similar to Binsbergen, Brandt, and Koije (2012) and Gormsen (2021) in which they use derivatives, such as stock market index (call and put) options and dividend futures, we also use derivatives data (VIX derivatives) to document the shape of the term structure. Remarkably, we show that the term structures of one-month future returns and expected equity risk premiums are pro-cyclical.

The rest of the paper is organized as follows. Section 2 introduces the Q-approach. Section 3 presents the theoretical results, followed by the empirical results with publicly available derivative data in Section 4. Section 5 explains the difference between the P- and Q-approaches. Section 6 demonstrates the economic value of the Q-approach. Section 7 investigates the term structure of equity risk premiums. Finally, Section 8 concludes. The details of the theory are presented in the Appendixes. More empirical supportive results are available in the Online Appendix.

2 The Q-approach Framework

This section presents the Q-approach framework to investigate the serial dependence coefficients between stock market returns over two consecutive periods.

2.1 Serial dependence of the market return

Let S_t denote the time-t price of the stock index. $R_{t\to t+1} = \frac{S_{t+1}}{S_t}$ is the gross market return over the time period from t to the next time $t+1$, and $R_{f,t\to t+1}$ is the gross risk-free return over the same time period. The information set at time t is $\mathscr{F}_t$. Similarly, we define $R_{t\to t+T} = \frac{S_T}{S_t}$ and $R_{f,t\to t+T}$ for any $T = 1, 2, \cdots$ in a dynamic setting. The length of the period can be arbitrary.

Let $X = R_{t\to t+T_1}, Y = R_{t+T_1\to t+T_1+T_2}$ be stock market returns over two consecutive periods. Conditional on the time-t perspective, the serial dependence between X and Y is captured by the conditional correlation coefficient,

$$corr_t = \frac{Cov_t(X,Y)}{\sqrt{Var_t(X)} \times \sqrt{Var_t(Y)}}. \tag{1}$$

In general, we can investigate the serial dependence closely by using the following standard regression equation

$$Y = \alpha_{t\to t+T} + \beta_{t\to t+T} X + \varepsilon, \tag{2}$$

where $Cov(X,\varepsilon) = 0$. We omit the script "T" for brevity if the meaning is evident. The beta coefficient in Equation (2) is given by

$$\beta_t = \frac{Cov_t(X,Y)}{Var_t(X)}, \tag{3}$$

and the intercept term is

$$\alpha_t = \frac{Cov_t(XY,X) - \beta_t Cov_t(X^2,X)}{Var_t(X)}, \tag{4}$$

assuming that $Cov(X^2,\varepsilon) = 0$.[4]

With numerous statistical inference models, econometricians estimate serial dependence coefficients $\{corr_t, \beta_t, \alpha_t\}$ using historical return data over a specific horizon (before time t). We name this statistical inference approach a "P-approach" as it relies on the (real-world) physical probability and historical stock market data. In contrast, we name an approach to derive the serial dependence coefficients with derivatives market data a "Q-approach", as the derivatives market reveals information under the risk-neutral Q measure from the derivative pricing theory.

To compute the above coefficients in Equations (1) - (4) in the Q-approach, it suffices to compute the following terms with available derivatives data. (1) The conditional covariance of a *spot return* $R_{t\to t+T_1}$ and a *future return* $R_{t+T_1\to t+T_1+T_2}$. (2) The conditional covariance between two spot returns, $R_{t\to t+T_1}$ and $R_{t\to t+T_1+T_2}$. (3) The conditional covariance between the spot return and its square, and (4) the conditional first two moments of the spot return and the future return.

[4] Multiplying X on both sides of Equation (2), we obtain $XY = \alpha_t X + \beta_t X^2 + \varepsilon X$, here $XY = R_{t\to t+T_1+T_2}$. Since $Cov_t(\varepsilon, X^2) = 0$, $Cov_t(\varepsilon_t X, X) = \mathbb{E}_t[\varepsilon_t X^2] - \mathbb{E}_t[\varepsilon_t X]\mathbb{E}_t[X] = \mathbb{E}_t[\varepsilon_t](\mathbb{E}_t[X^2] - \mathbb{E}_t[X]^2) = 0$. Therefore, $Cov_t(XY,X) = \alpha_t Var_t(X) + \beta_t Cov_t(X^2,X)$, yielding Equation (4).

2.2 Conditional expected return and PVIX index

We start with the calculation of the conditional expected spot return. Other remaining terms in the Q-approach are calculated in Section 3.

Let (M_t) be a pricing kernel process and $m_{t,t+T} = \frac{M_{t+T}}{M_t}$ be the stochastic discount factor (SDF) over the period from t to $t+T$. Equivalently, the risk-neutral probability measure Q is given by

$$\frac{dQ}{dP}|_{\mathscr{F}_{t+T}} = R_{f,t\to t+T} m_{t,t+T}.$$

Then, for any $f(S_{t+T}) \in \mathscr{F}_{t+T}$ with suitable integrable condition, its conditional expectation under the real-world probability measure P is

$$\mathbb{E}_t[f(S_{t+T})] = \mathbb{E}_t^Q\left[\frac{dP}{dQ} f(S_{t+T})\right] = \frac{1}{R_{f,t\to t+T}} \mathbb{E}_t^Q\left[\frac{f(S_{t+T})}{m_{t,t+T}}\right]. \tag{5}$$

This equation states that a conditional expectation of $f(S_{t+T})$ under the real-world probability measure is the no-arbitrage time-t price of a contingent claim with payoff $\frac{f(S_{t+T})}{m_{t,t+T}}$ at time $t+T$.[5]

We consider a power-specification of the stochastic discount factor for now, and we discuss a general specification of the stochastic discount factor in Section 3.4. Let

$$m_{t,t+T} = \left(\frac{S_t}{S_{t+T}}\right)^\gamma, \quad \gamma > 0,$$

be a stochastic discount factor for a representative CRRA-type agent with a coefficient of constant relative risk aversion γ. We define a power VIX (PVIX) index over the period $[t, t+T]$ with exponent n by

$$PVIX_{t\to t+T}^{(n)} \equiv \frac{1}{T} \mathbb{E}_t^Q\left[\left(\frac{S_{t+T}}{S_t}\right)^n\right], n \geqslant 1. \tag{6}$$

Given the power-specification of the pricing kernel process, the conditional moment of the spot return over the period $[t+T_1, t+T_1+T_2]$ is equivalent to a PVIX,

$$\mathbb{E}_t[R_{t\to t+T}^n] = \frac{T}{R_{f,t\to t+T}} PVIX_{t\to t+T}^{(\gamma+n)}. \tag{7}$$

As is well-studied in the literature (see, e.g. Bakshi, Kapadia, and Madan, 2003; Bakshi and Madan, 2000; Carr and Madan, 1999; Carr, Ellis, and Gupta, 1998), we can synthesize the risk-neutral conditional moments

[5] Equation (5) is known as the *inverting Girsanov theorem* in Bakshi, Gao Bakshi, and Xue (2019b).

of the spot return $R_{t\to t+T}$ in terms of market prices of index call options $C_{t\to t+T}(K)$. Precisely,

$$PVIX_{t\to t+T}^{(\gamma+1)} = \frac{(\gamma+1)\gamma R_{f,t\to t+T}}{TS_t^{\gamma+1}} \int_0^\infty K^{\gamma-1} C_{t\to t+T}(K) dK. \tag{8}$$

By its definition, PVIX measures the risk-neutral (conditional) moments of a spot return. However, PVIX is not tradable in the market, whereas the publicly-traded CBOE volatility index product is VIX. Specifically, VIX measures the risk-neutral entropy and it is given formally by

$$VIX_{t\to t+T}^2 = \frac{2}{T} L_t^Q \left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}} \right), \tag{9}$$

where $L_t^Q(X) \equiv log\mathbb{E}_t^Q X - \mathbb{E}_t^Q logX$. The relation between PVIX and VIX plays a crucial role in our theoretical results of the next section.

3 Theoretical Results

This section presents our theoretical results. The first subsection introduces PVIX futures. The second subsection presents two assumptions behind our method. The third subsection presents a no-arbitrage method to compute the PVIX futures price. Then we use these results to derive the serial dependence coefficients of the market returns in the fourth subsection. Finally, we discuss the robustness of our method for a general specification of the pricing kernel process.

3.1 PVIX futures

The challenge in computing the conditional correlation and other serial dependence coefficients is to calculate the conditional moments of *future returns*. For this purpose, we first introduce a new VIX-derivative, a forward contract on the PVIX index.

Like VIX futures written on the VIX index, a forward contract is written on the PVIX index. A forward contract on PVIX is an agreement between two parties to "buy" or "sell" a PVIX at a specific future time $t+T_1$ (maturity) for a fixed price, FPVIX, and the underlying PVIX is calculated over the time period $t+T_1$ to $t+T_1+T_2$. The fixed price FPVIX is the PVIX forward price. Those forward contracts can be standardized by trading in the Exchange, and if so, we name them PVIX futures contracts.

3.2 Assumptions

We make the following two assumptions for the subsequent theoretical results.

Assumption 3.1. *The risk-free rate $R_{f,t\to t+T}$ is deterministic for any $T>0$.*

Assumption 3.1 assumes a deterministic risk-free rate. Since we focus on the serial dependence of the market return, we ignore the effect of the interest rate risk to the market return. By Assumption 3.1, the no-arbitrage futures price is the same as the forward price. Moreover, by the derivative pricing theory, the PVIX futures price at time t for the time period $[t+T_1, t+T_1+T_2]$ satisfies

$$FPVIX^{(\gamma+1)}_{t,t+T_1\to t+T_1+T_2} = \mathbb{E}^Q_t\left[PVIX^{(\gamma+1)}_{t+T_1\to t+T_1+T_2}\right].$$

By the law of iterated expectation and the definition of PVIX, the PVIX futures price at time t for the time period $[t+T_1, t+T_1+T_2]$ is a risk-neutral conditional moments of future return as follows.

$$FPVIX^{(\gamma+1)}_{t,t+T_1\to t+T_1+T_2} = \frac{1}{T_2}\mathbb{E}^Q_t\left[\left(\frac{S_{t+T_1+T_2}}{S_{t+T_1}}\right)^{\gamma+1}\right] = \frac{1}{T_2}\mathbb{E}^Q_t\left[R^{\gamma+1}_{t+T_1\to t+T_1+T_2}\right]. \tag{10}$$

Assumption 3.2. *Stein's Lemma holds for $X = R_{t\to t+T_1}$ and $Y = R_{t+T_1\to t+T_1+T_2}$ under the risk-neutral measure. That is, $Cov^Q_t(g(X),Y) = \mathbb{E}^Q_t[g'(X)]Cov^Q_t(X,Y)$ and $Cov^Q_t(X,g(Y)) = \mathbb{E}^Q_t[g'(Y)]Cov^Q_t(X,Y)$.*

Stein's lemma is about a first-order approximation of the covariances for a bivariate distribution. To see it, we write the first-order approximation as $g(X)\sim g(X_0)+(X-X_0)g'(X_0)$ and $X_0=\mathbb{E}_t[X]$ under a general probability measure. Then

$$\begin{aligned} Cov_t(g(X),Y) &\sim Cov_t(g(X_0)+g'(X_0)(X-X_0),Y) \\ &= Cov_t(g'(X_0)(X-X_0),Y) = g'(X_0)Cov_t(X,Y). \end{aligned}$$

Similarly, $Cov_t(X,g(Y))\sim g'(Y_0)Cov_t(X,Y)$. Stein's lemma states that the first-order approximation is accurate.

As shown in Vanduffel and Yao (2017); Adcock, Landsman, and Shushi (2019), Stein's lemma holds for a general class of bivariate distribution, including elliptic distribution, student distribution and hyperbolic distributions. Assumption 3.2 essentially states that the first-order approximation is sufficiently accurate under the risk-neutral measure.[6]

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[6] As shown in Camara (2003); Schroder (2004), the risk-neutral distribution shapes remain similar by changing only the location parameters through a simple restriction on the pricing kernel process.

3.3 No-arbitrage price of PVIX futures

The following result presents a no-arbitrage (almost) pricing formula of the PVIX futures under Assumption 3.1. Building on the relation between VIX and PVIX, we use VIX derivatives to price the PVIX futures in a static-replication approach.

Therefore, we do not need model assumptions or parameter estimations.[7]

Proposition 3.1. *Assume Assumption 3.1, and for simplicity, let* $R = R_{t+T_1 \to t+T_1+T_2}$, $R_f = R_{f,t+T_1 \to t+T_1+T_2}$. *Let* $F_t = FVIX_{t,t+T_1 \to t+T_1+T_2}$ *be the futures price on VIX index with the time to maturity of* T_1, *and* σ_t *be the implied volatility of at-the-money options on VIX index with the time to maturity of* T_1, *then* $\mathbb{E}_t^Q[R^n]$ *can be obtained recursively for* $n = 2, 3, 4, \cdots$ *via the following approximation formulae,*

$$F_t^2(1+\sigma_t^2 T_1) \sim \frac{1}{T_2}\left(\mathbb{E}_t^Q\left[\left(\frac{R}{R_f}\right)^2\right] - 1\right), \tag{11}$$

$$\frac{T_2}{2}F_t^2\left(1+\sigma_t^2 T_1\right) \sim \sum_{i=1}^{n}(-1)^i \frac{1}{i}\mathbb{E}_t^Q\left[\left(\frac{R}{R_f}-1\right)^i\right], n \geqslant 3. \tag{12}$$

Proof. See Appendix B. ☐

Proposition 3.1 offers a recursive no-arbitrage pricing formula of FPVIX, and it only depends on the available real-time CBOE VIX index and VIX derivatives data. Although it is given in an approximative version, the approximation error is very small for the empirical application (See the proof in Appendix B).

It is useful to explain the central idea behind Proposition 3.1 using the second-moment of the future return as an example. We compute the conditional second moment of a future VIX because of the relation between the second moment and the entropy. The second moment of a future VIX is the sum of a square of a VIX futures price and a conditional variance of a future VIX, and the latter is derived by the implied volatility (variance) of VIX options. Therefore, VIX derivatives are used in Proposition 3.1.

Since Proposition 3.1 calculates all risk-neutral moments of a future return, it is equivalent to a risk-neutral conditional cumulant generating function of a future return as follows.

Remark 3.1. *Following Martin (2013), the conditional cumulant-generating function* $K(\lambda)$ *of the relative future return* $\frac{R}{R_f}$ *is* $K(\lambda) = log\left(\mathbb{E}_t^Q\left[e^{\lambda R/R_f}\right]\right)$. *Notice that* $\mathbb{E}_t^Q\left[\frac{R}{R_f}\right] = \mathbb{E}_t^Q\left[\mathbb{E}_{t+T}^Q\left(\frac{R}{R_f}\right)\right] = 1$, *then*

$$K(\lambda) = log\left(1+\lambda+\sum_{n=2}^{\infty}\frac{1}{n!}\mathbb{E}_t^Q\left[\left(\frac{R}{R_f}\right)^n\right]\lambda^n\right).$$

[7]In contrast with the VIX derivative literature, for instance, Mencia and Sentana (2013), our pricing formula depends only on publicly-traded VIX derivatives data, without building a no-arbitrage model.

3.4 Autocorrelation and regression coefficients

By Proposition 3.1, we can quantify the futures price of PVIX. We next present the following proposition to calculate all remaining statistical terms in the Q-approach.

Proposition 3.2. *Given the power-specification of the pricing kernel process, and under Assumption 3.1 and Assumption 3.2, then*

1. *the conditional moment of the future return over the period $[t+T_1, t+T_1+T_2]$ is*

$$\mathbb{E}_t\left[R^n_{t+T_1\to t+T_1+T_2}\right] = \frac{T_2}{R_{f,t+T_1\to t+T_1+T_2}} FPVIX^{(\gamma+n)}_{t,t+T_1\to t+T_1+T_2}, \tag{13}$$

2. *the conditional expectation of the product of two spot returns over different periods is*

$$\mathbb{E}_t[R_{t\to t+T_1}R_{t\to t+T_1+T_2}] = \frac{T_1T_2}{R_{f,t\to t+T_1+T_2}} \times PVIX^{(\gamma+2)}_{t\to t+T_1} \times FPVIX^{(\gamma+1)}_{t,t+T_1\to t+T_1+T_2}. \tag{14}$$

Proof. See Appendix C. □

The novelty of Proposition 3.2 is that the conditional moments of the future return are computed directly from the market prices of VIX-related instruments.

3.5 Robustness of the pricing kernel process

In the last subsection, the expressions of serial dependence coefficients depend on a power specification of the pricing kernel process. We relax this restriction of the pricing kernel process in the Q-approach for robustness.

First, for the conditional expectation (or moments) of the spot return with a large class of SDFs, $\mathbb{E}_t[R_{t\to t+1}]$, many authors in such as Bakshi, Crosby, Gao Bakshi, and Zhou (2019a) have derived an analytical formula. Therefore, we focus on the conditional expectation (or moments) of the future return for a general specification of pricing kernel processes.

Assume the stochastic discount factor over the period $t+1$ to $t+T$ is given by $m_{t+1,t+T} = f(R_{t+1\to t+T}; t, T)$, for a function $f(x;t,T)$ of a component x with coefficients depending on time variables t and T. Under Assumption 3.2, it is straightforward to show that (See the proof in Appendix B),

$$\mathbb{E}_t[R_{t+1\to t+T}] \quad = \quad \frac{1}{R_{f,t+1\to +T}} \mathbb{E}^Q_t\left[\frac{R_{t+1\to t+T}}{m_{t+1,t+T}}\right].$$

For $\mathbb{E}_t^Q\left[\frac{R_{t+1\to t+T}}{m_{t+1,t+T}}\right]$, we use the Taylor series expansion of the function $\frac{x}{f(x;t,T)}$ and all higher moments of the future returns (or the cumulant generating function) in Proposition 3.1. Therefore, the conditional moments of the future return, and thus the coefficients in Equations (3) - (4), can be obtained by Q-approach with VIX derivatives and index options data for the this specification of the pricing kernel process, $m_{t+1,t+T}$.

We illustrate the Q-approach with several examples of the pricing kernel process.

Example 3.1. *Assume that $m_{t,t+1} = exp(a - \phi R_{t\to t+1}), a, \phi > 0, \forall t$, an analytical expression of the conditional expected spot return is available in Bakshi, Crosby, Gao Bakshi, and Zhou (2019a). Moreover, since*

$$\frac{R_{t+1\to t+2}}{m_{t+1,t+2}} = exp(-a) \times R_{t+1\to t+2} \times exp\left(\phi R_{t+1\to t+2}\right),$$

we obtain

$$\mathbb{E}_t^Q\left[\frac{R_{t+1\to t+2}}{m_{t+1,t+2}}\right] = e^{-a}\sum_{n=0}^{\infty}\frac{\phi^n}{n!}\mathbb{E}_t^Q\left[R_{t+1\to t+2}^{n+1}\right].$$

By Proposition 3.1, we obtain the autocorrelation coefficient between $R_{t\to t+1}$ and $R_{t+1\to t+2}$, and the regression coefficients $\{\alpha_{t,t+1}, \beta_{t,t+1}\}$ by the Q-approach.

Example 3.2. *Let $m_{t,t+1} = f(R_{t\to t+1}, t)$ for all t. For instance, let $f(\cdot)$ be a polynomial function as in Harvey and Siddique (2000) and Dittmar (2002), and $\frac{R_{t+1\to t+2}}{m_{t+1,t+2}}$ is a smooth function of $R_{t+1\to t+2}$, under Assumption 3.2, we can derive an analytical expression of the autocorrelation and the regression coefficients from the following equation,*

$$\mathbb{E}_t^Q\left[\frac{R_{t+1\to t+2}}{m_{t+1,t+2}}\right] = \sum_{n=0}^{\infty}\mathbb{E}_t^Q\left[R_{t+1\to t+2}^{n}\right]\frac{g^{(n)}(0)}{n!},$$

where $g(x) = \frac{x}{f(x)}$ is a smooth function.

Given a general pricing kernel process, we can project the stochastic discount factor $m_{t,t+1}$ onto the space generated by $\{R_{t\to t+1}, R_{t\to t+1}^2, \cdots, R_{t\to t+1}^n, \cdots\}$ (see, e.g. Chaudhuri and Schroder, 2015; Schneider and Trojani, 2019). Therefore, the Q-approach can be applied to derive autocorrelation for a general class of pricing kernel process.

Example 3.3. *Given a pricing kernel process $m_{t,t+1} = f(R_{t\to t+1}, t), \forall t$, we can derive a bivariate conditional distribution of $(R_{t\to t+1}, R_{t+1\to t+2})$ at time t. The bivariate distribution is characterized by the following moment* *generating function* $\mathbb{E}_t\left[e^{x_1 R_{t\to t+1}+x_2 R_{t+1\to t+2}}\right], (x_1,x_2)\in\mathscr{R}^2$,

$$\begin{aligned}\mathbb{E}_t\left[e^{x_1 R_{t\to t+1}+x_2 R_{t+1\to t+2}}\right] &= \frac{1}{R_{f,t\to t+2}}\mathbb{E}_t^Q\left[\frac{e^{x_1 R_{t\to t+1}+x_2 R_{t+1\to t+2}}}{m_{t\to t+2}}\right]\\ &= \frac{1}{R_{f,t\to t+2}}\mathbb{E}_t^Q\left[\frac{e^{x_1 R_{t\to t+1}}}{f(R_{t\to t+1},t)}\frac{e^{x_2 R_{t+1\to t+2}}}{f(R_{t+1\to t+2},t+1)}\right]\\ &= \frac{1}{R_{f,t\to t+2}}\mathbb{E}_t^Q\left[\frac{e^{x_1 R_{t\to t+1}}}{f(R_{t\to t+1},t)}\right]\mathbb{E}_t^Q\left[\frac{e^{x_2 R_{t+1\to t+2}}}{f(R_{t+1\to t+2},t+1)}\right], (\textit{ by Assumption 3.2})\end{aligned}$$

in which the first component is calculated similar to Equation (8), and the second component is derived explicitly in Examples 3.1 - 3.2.

4 Empirical Results

In this section, we implement the theoretical results of Section 3. Using publicly available derivatives data, we first demonstrate that the autocorrelation between two consecutive returns is significantly negative in the Q-approach. It means a moderate short-term market return reversal. We next explain that the long-run trend of the stock market is upward from a forward-looking perspective.

4.1 Data

We use three types of derivative instruments, namely, S&P 500 index options, CBOE VIX futures, and VIX options. We collect index options and VIX options data from the OptionMetrics, VIX index and futures data from the CBOE. After applying standard filters and merging data from different databases, we end up with a sample of daily observations from February 24, 2006, to December 31, 2019.

Following the same procedure in CBOE and in Martin (2017), at each day t, we compute $\text{PVIX}_{t\to t+T}$ for T = 1, 2, 3, 4, 6, and 9 months by Equation (8).[8] All results are annualized. We also follow Hu and Jacobs (2020) to use linear interpolation, on each trading day, to compute daily VIX futures prices, $\text{FVIX}_{t,t+T_1\to t+T_1+T_2}$, with constant maturities for T_1 = 1, 2, 3, 4, 6, and 9 month.[9] Since both VIX and VIX futures measure the forward-looking implied index volatility over 30 days, T_2 always represents one month.

[8] We use market prices of out-of-the-money call and put options and the same linear interpolation method as in CBOE and in Martin (2017) to calculate PVIX. For the log-utility-based SDF, $\gamma = 1$, the PVIX index is related to Martin's (2017) SVIX. In Appendix A we show that SVIX square essentially provides a lower bound for all PVIX indices. The detailed procedure is provided in the online Appendix.

[9] CFE may list futures for up to nine near-term serial months, as well as five months on the February quarterly cycle associated with the March quarterly cycle for options on S&P 500 (Mencia and Sentana, 2013). We thus choose the maximum constant maturity to be nine months. VIX futures expiration calendar can be found at `https://www.macroption.com/vix-expiration-calendar/`.

Figure 1 (a) displays the 1-month PVIX index for three different values of $\gamma = 1, 2, 3$. The PVIX index spiked around the 2008-2009 crisis period. For example, the average value of PVIX when $\gamma = 1$ is about 12.24, but the highest value is about 12.7. This spike feature is significant with a higher exponent. Moreover, Figure 1 (b) plots both the 1-month PVIX ($\gamma = 1$) and the CBOE VIX index. As shown, PVIX and VIX indices are highly correlated.

Table 1 reports the summary statistics for PVIX indices over different T_1 periods when $\gamma = 1$. For completeness, we consider two different sample periods. Panel A considers the period from January 4, 1996, the first available date in OptionMetrics. Panel B restricts the sample starting from February 24, 2006. In general, PVIX is increasing monotonically to γ, but decreasing monotonically to T_1 (See Appendix A for the proof).

We next compute the PVIX futures price by Proposition 3.1. Table 2 reports key statistics for PVIX futures price. We use the average value between the implied volatility of at-the-money put and call VIX options as a proxy for σ_t. On average, PVIX futures prices are higher than the PVIX index, displaying an upward term structure. It is intuitive because of the VIX futures' contango trap (Eraker and Wu, 2017), and the close relation between VIX and PVIX as displayed in Figure 1. For robustness, we further compute PVIX indices and PVIX futures price for other power-utility-based SDFs (when $\gamma = 2$ and 3) in the online Appendix. The patterns for different value of the parameter γ remain similar.

4.2 Autocorrelation by Q-approach

Table 3 reports the summary statistics for market autocorrelation under the Q-approach. We observe significantly *negative* coefficients across different T_1, from one to nine months.[10] Specifically, the market autocorrelation on S&P 500 index is around -20% to -40%. For instance, when T_1 is 1 month, $corr_t(R_{t\to t+1mo}, R_{t+1mo\to t+2mo})$ is on average -20.90%, with a t-stat of 18.10. Notably, the numbers in Table 3 are comparable to Chabi-Yo (2019) and Martin (2021). Using S&P 500 index options data, Chabi-Yo (2019) estimates that the no-arbitrage bounds of autocorrelation varies from -28% to -3%. Martin (2021) computes the implied autocorrelation of the S&P 500 index on 8 specific dates, with values also between -20% and -40%. Since Martin (2021) relies on market quotes of forward-start options from investment banks, Table 3 suggest that our no-arbitrage pricing of the PVIX futures, in Proposition 3.1, is consistent with the pricing of OTC derivatives in the market.

As a comparison, Table 3 also reports the market autocorrelation under the P-approach. As shown in Panel B of Table 3, the unconditional autocorrelation coefficients over the same sample period are statistically

[10]We conduct a two-tail t-test on the time-series of the aucotorrelation coefficients from the Q-approach to assess the statistical significance. The Newey-West estimators are used to adjust the standard errors.

insignificant across different value of T_1, and are much smaller in magnitudes compared to Panel A. For example, when $T_1 = 1$ month, the autocorrelation is 9.3% from the P-approach, whereas it is on average -21% from the Q-approach. The difference between these approaches is even more striking if we consider the correlation between the current one-month return and the past three-month return: from the Q-approach, it is about -36%; but, it is only 8% from the P-approach. The P-approach in Table 3 suggests that market returns are almost independent.

Using the same historical return data, we further study the *realized month-to-month autocorrelation*. Specifically, we compute the sample autocorrelation between two consecutive calendar months, including January/February, February/March, $\cdots$, December/January. Figure 2 plots the realized month-to-month autocorrelation over various periods with the data prior to January 1927 obtained from Robert Shiller's website. As displayed in Figure 2, the month-to-month autocorrelation based on historical data can be either positive or negative, depending on the sample of the data.[11] For example, the autocorrelation of March/April is around 10% over 1871-2019, but -20% over a recent time period 1990-2019. The month-to-month autocorrelation in Figure 2 also displays a moderate reversal between two consecutive calendar months in specific periods. For instance, the corresponding autocorrelations are significantly negative from February to March, March to April, May to June, November to December, and December to January. However, the autocorrelation coefficients of January/February, April/May, and October/November are significantly positive. In this regard, Figure 2 shares similar magnitudes of the autocorrelation by the Q-approach in Table 3.

Why is the difference between the P-approach and the Q-approach so substantial in Table 3? One plausible explanation is that the derivatives' prices are real-time and highly volatile. Take the expected return as an example. Martin (2017) shows that the expected excess return computed from the derivatives (index options) market is higher than that from Campbell and Thompson's (2008) variation ratio approach, as the risk-neutral variance contains real-time and forward-looking information about the index volatility. Due to the rich information embedded in the VIX derivatives market, the difference in autocorrelation is more striking than that in the expected return. However, the Q-approach implies prominent negative autocorrelation coefficients (-21% to -36%), regardless of the time, risk-aversion, or the frequency of the data (length of the period), whereas the P-approach yields a small positive value (3% to 11%). Therefore, the feature of the derivatives market alone cannot explain the significant difference between these two approaches. Moreover, it is also puzzling that the month-to-month autocorrelation with the P-approach might take considerable negative or positive values.

It is important to explore what causes such a discrepancy between the two approaches, and we delve into this issue in Section 5. Before doing so, we present more empirical results of the Q-approach.

[11] In Section 5, we will also compute the month-to-month autocorrelation from the Q-apprpach and make a comparison.

4.3 The market return dynamics by Q-approach

Table 4, Panel A reports the summary statistics for the beta coefficients of the Q-approach. Similarly, Table 5, Panel A reports the intercept coefficient of the Q-approach for the period over February 2006 to December 2019. In addition, we report the beta and the intercept coefficients from the P-approach in Panel B of each table for comparison purposes.

Similar to the market autocorrelation, we observe consistently negative conditional beta coefficients across columns. The beta coefficients from the Q-approach are much more prominent in magnitude and are statistically significant than the P-approach. In contrast, the betas from OLS are virtually zero (from 0.013 to 0.093 in absolute values) and are insignificant. Different from beta coefficients, the intercepts are primarily positive in both approaches. The mean values are all larger than 1.4. On the contrary, the OLS intercepts are around one.

To better understand the Q-approach, we decompose Equation (4) as

$$\alpha_t = \frac{Cov_t(X,XY)}{Var_t(X^2)} - \beta_t \frac{Cov_t(X^2,X)}{Var_t(X^2)},$$

where $X = R_{t\to t+T_1}$, $Y = R_{t+T_1\to t+T_1+T_2}$, and $XY = R_{t\to t+T_1+T_2}$. For a positive gross return X, the two variables X and X^2 move in the same direction, yielding $Cov_t(X^2,X) > 0$. Together with $\beta_t < 0$, the second component on the right hand of the last equation is positive. On the other hand, the first component on the right hand has the same sign as $Cov_t(X,XY)$, the conditional covariance between two spot returns $R_{t\to t+T_1}$ and $R_{t\to t+T_1+T_2}$.

By Proposition 3.2, the two autocovariances, $Cov_t(X,Y)$ and $Cov_t(X,XY)$, can be analytically computed from no-arbitrage prices of contingent claims. That is,

$$Cov_t\left(X,Y\right) = \frac{T_1+T_2}{R_{f,t\to t+T_1+T_2}} \times PVIX^{(2)}_{t\to t+T_1+T_2} - \frac{T_1T_2}{R_{f,t\to t+T_1+T_2}} \times PVIX^{(2)}_{t\to t+T_1} \times FPVIX^{(2)}_{t,t+T_1\to t+T_1+T_2}, \tag{15}$$

and

$$\begin{aligned} Cov_t\left(X,XY\right) = &\frac{T_1T_2}{R_{f,t\to t+T_1+T_2}} \times PVIX^{(3)}_{t\to t+T_1} \times FPVIX^{(2)}_{t,t+T_1\to t+T_1+T_2} \\ &- \frac{T_1(T_1+T_2)}{R_{f,t\to t+T_1}R_{f,t\to t+T_1+T_2}} \times PVIX^{(2)}_{t\to t+T_1} \times PVIX^{(2)}_{t\to t+T_1+T_2}. \end{aligned} \tag{16}$$

Given the last two equations, a negative $Cov_t(X,Y)$ and a positive $Cov_t(X,XY)$ are equivalent to a *lower bound* for the PVIX futures price as follows,

$$FPVIX^{(2)}_{t,t+T_1\to t+T_1+T_2} > \max\left\{ \frac{T_1+T_2}{T_1T_2}\frac{PVIX^{(2)}_{t\to t+T_1+T_2}}{PVIX^{(2)}_{t\to t+T_1}}, \frac{T_1+T_2}{T_2R_{f,t+T_1}}\frac{PVIX^{(2)}_{t\to t+T_1} \times PVIX^{(2)}_{t\to t+T_1+T_2}}{PVIX^{(3)}_{t\to t+T_1}} \right\}. \tag{17}$$

By Proposition 3.1, Equation (17) essentially becomes

$$\begin{aligned}
&FVIX^2_{t,t+T_1\to t+T_1+T_2}(1+\sigma_t^2 T_1)+\frac{1}{T_2}\\
&>\max\left\{\frac{T_1+T_2}{T_1T_2}\frac{PVIX^{(2)}_{t\to t+T_1+T_2}}{PVIX^{(2)}_{t\to t+T_1}},\frac{T_1+T_2}{T_2R_{f,t+T_1}}\frac{PVIX^{(2)}_{t\to t+T_1}\times PVIX^{(2)}_{t\to t+T_1+T_2}}{PVIX^{(3)}_{t\to t+T_1}}\right\}.
\end{aligned}\tag{18}$$

Equation (18) displays a relation among market prices of index options, VIX futures, and VIX options. In theory, a violation of the relation in Equation (18) does not suggests an arbitrage opportunity occurs, since its not necessarily true that $Cov_t(X,Y)<0$ and $Cov_t(X,XY)>0$ follow from a simple no-arbitrage condition. Equation (18) merely implies a short-term reversal and a long-run upward trend from a forward-looking perspective, and Table 6 empirically documents this relation. Specifically, Panel A reports the summary statistics for the negative autocovariance $Cov_t(X,Y)$ by the Q-approach, and Panel B reports the positivity of $Cov_t(X,XY)$. From a forward-looking perspective, these two autovariances display entirely different signs. In Panel A, $Cov_t(X,Y)$ are, on average, negative across columns, with a mean value around -1.5. Nevertheless, $Cov_t(X,Y)$ in Panel B are always positive, and the magnitude is around 3% on average.[12] Economically, these results show that the monthly return reversal is modest.

Finally, we rewrite Equation (2) as

$$Y-\frac{\alpha_t}{1-\beta_t}=\beta_t\left(X-\frac{\alpha_t}{1-\beta_t}\right)+\varepsilon.$$

A large number of $\frac{\alpha_t}{1-\beta_t}$ indicates an *upward trend* of the market return in the long-run. Figure 3 plots the time series of the conditional beta and the intercept coefficients by the Q-approach. As shown, the beta coefficients are mostly below zero over time, whereas the intercept terms are most above zero. We also plot the times series of long-run rate, $\frac{\alpha_t}{1-\beta_t}$, when T_1 and T_2 are 1 month. The average of $\frac{\alpha_t}{1-\beta_t}$ is around 1.34, indicating an upward trend of the market, in spite of the short-term reversal.

In summary, we have shown a substantial difference between the P-approach and Q-approach about the serial dependence of the market return. Remarkably, the Q-approach demonstrates significant negative dependence between two consecutive monthly market returns, whereas the classical P-approach provides virtually independent consecutive market returns. In the following two sections, we explain where the difference comes from and explore the economic value of the Q-approach.

[12] In the Online Appendix, we relax the log-utility assumption and assume that $\gamma=2$ or 3 for general power-utility-based SDFs. The sign properties of these two conditional covariances are robust regardless of the risk-aversion coefficients in the SDF specification.

5 The Difference Between the P- and Q-approaches

In this section, we demonstrate that the methodology of the P-approach itself leads to the difference of the market autocorrelation between Q- and P-approaches. To do so, we compare the Q-approach with alternative P-approach-based methodologies with the same historical stock market data. In particular, we find that the Q-approach is comparable to the Nagel and Xu's (2021) fading memory distribution, which will be explained in the next several paragraphs.

In general, given a variable X with historical data $X_1, \cdots, X_N$ from the beginning to the latest, X_t is the observation at time t. Suppose our objective is to compute statistical quantities, say, the expected value, of X at time N with those observations. We implement three methodologies below to compute the monthly stock market autocorrelation by the P-approach with the same sample of historical data.

First, we adopt a fading memory methodology to compute the market autocorrelation. Inspired by Equation (15) in Nagel and Xu (2021), we write

$$\mu_t = \mu_{t-1} + \nu(X_t - \mu_{t-1}), \tag{19}$$

where μ_t is the expected value of the variable X formed at t, and ν is a decay parameter. By using the above equation recursively, we obtain

$$\mu_t = \nu X_t + (1-\nu)\nu X_{t-1} + \cdots + (1-\nu)^{t-1}\nu X_1.$$

For simplicity, let $\mathbb{E}_t^f[\cdot]$ denote the expectation of X in the last equation. Economically, $\mathbb{E}_t^f[\cdot]$ reflects the empirical fact that investors may learn from the most recent personal experiences when forming expectations (Nagel and Xu, 2021; Bordalo, Gennaioli, Porta, and Shleifer, 2019; Collin-Dufresne, Johannes, and Lochstoer, 2017; Malmendier and Nagel, 2016). Given the expectation operator $\mathbb{E}_t^f[\cdot]$, we also compute the conditional covariance and correlation coefficients between two variables X and Y.[13] Since $\mathbb{E}_t^f[\cdot]$ is motivated by the fading memory specification, we label the autocorrelation by this methodology as the *fading-memory autocorrelation*.

In the second methodology, we replace the number ν in Equation (19) by $\frac{1}{t}$, yielding $\mu_t = \frac{X_1+\cdots+X_t}{t}$. This is the standard sample (empirical) mean of the random variable X. Likewise, we use $\mathbb{E}_t^s[\cdot]$ to represent this empirical expectation of X. Intuitively, at each time N, X has an equal probability, $\frac{1}{N}$, among its historical

[13] Specifically, $Var_t^f(X) = \mathbb{E}_t^f\left[(X - \mathbb{E}_t^f[X])^2\right]$, $Cov_t^f(X,Y) = \mathbb{E}_t^f[XY] - \mathbb{E}_t^f[X]\mathbb{E}_t^f[Y] = (\nu X_N Y_N + (1-\nu)\nu X_{N-1}Y_{N-1} + \cdots + (1-\nu)^{t-1}\nu X_1 Y_1) - \mathbb{E}_t^f[X]\mathbb{E}_t^f[Y]$. See Rougier (2016), page 32.

observations $\{X_1, \cdots, X_N\}$. Since the classical statistical inference method essentially relies on this *empirical distribution*, we name the autocorrelation under the *empirical autocorrelation*.

In the third methodology, we focus on the *month-to-month autocorrelation* that is discussed in Section 4. In this methodology, since we compute the autocorrelation between two consecutive calendar months, we view the corresponding month's return in the last calendar year as the most recent historical data. To compare with the empirical autocorrelation, we still impose equal weights on the historical data.

We start with the comparison of the first two methodologies. At the beginning of each month, we compute the monthly stock market autocorrelation on S&P 500 index "recursively" with an initial window of 10 years. Following Nagel and Xu (2021), we choose the decay parameter $\nu = 0.018$ in computing the fading-memory autocorrelation. The comparison results are presented in Figure 4. Consistent with Lo and MacKinlay (1999), the covariance between historical monthly returns is virtually zero; thus, the empirical autocorrelation is almost zero. However, the fading-memory autocorrelation is time-varying and significantly nonzero. For instance, most of the monthly autocorrelation coefficients during 2019 are larger than -20% in absolute values. Similar results are observed in Table 3 by Q-approach. Even though the Q-approach implies a more fluctuating and mostly negative market autocorrelation, the fading-memory autocorrelation is comparable to that by Q-approach, especially after 2018.

For robustness, we also use a "rolling method" in the first two methodologies, by fixing the number of historical sample data at each time. The results with the rolling method are displayed in Figure 5, and the patterns are similar to Figure 4. Overall, Figures 4 and 5 indicate that the derivated-based market autocorrelation reveals different market movements from the empirical distribution, but, the fading-memory distribution reconciles to some extent the Q-approach. While the empirical distribution implies virtually zero (insignificant) autocorrelation between two consecutive monthly returns, a different statistical inference method with the same data can lead to different and significantly nonzero autocorrelation.

Conceptually, it is not surprising that different statistical inference methods generate different market autocorrelation. What is remarkable is that some statistical inference methods can be comparable to the Q-approach. We can also confirm this fact by the third methodology to calculate the month-to-month autocorrelation of the S&P 500 index between two consecutive calendar months. To calculate the month-to-month autocorrelation by the Q-approach, we compute $corr_t(R_{t\to t+1mo}, R_{t+1mo\to t+2mo})$ on the first day of each month and take the simple average within each of 12 calendar months of the year. For consistency, we restrict the sample period to 2006$-$2019 and plot the autocorrelation in Figure 6. The month-to-month autocorrelations under the Q-approach (Figure 6) and the P-approach (Figure 2) display a relatively similar pattern, despite the Q-based autocorrelations are still more negative in magnitude. For example, both of them yield a similar autocorrelation coefficients of -35% between February and March, and -10% between December and January.

Between May and June, however, the Q-approach implies an autocorrrelation as large as -40%, while the P-approach suggests a value of -20%. Moreover, the monthly return displays a stronger reversal in specific periods than others (for instance, from February to March, May to June, July to August, and December to January) by both approaches.

The above findings have several important implications. First, since the Q-approach depends on real-time derivatives data, it provides a forward-looking conditional point of view. It is well-known that all option prices derive an *implied distribution* (see, for instance, Bakshi, Gao Bakshi, and Xue, 2019b). Specifically, the implied distribution can be derived by market prices of call options, put options, and digital options (Backus, Chernov, and Martin, 2011; Bakshi, Gao Bakshi, and Xue, 2019b). Therefore, we have shown that the autocorrelation derived from the "implied distribution" of the market index is somewhat close to the "fading-memory distribution" of the market index. In contrast, this implied distribution is significantly different from the "empirical distribution". In this regard, the Q-approach supports the learning with fading memory and vice versa, as the implied distribution and the fading-memory distribution generate comparable market autocorrelations.

Second, when we compute the month-to-month market autocorrelation, the corresponding consecutive month returns in different calendar years have a more substantial effect than those in the same calendar year. As an example, we consider the effect of February return on March return. If we use all monthly returns in the P-approach, then the effect of February return to March return in the last year has been reduced by the middle 12 months' returns, yielding virtually zero effect. However, if we only restrict the Feb-March effect in each calendar year, this effect can be revealed directly by the P-approach. It suggests that the stock market might have more memory about the same month in the last year than what happened in the last month (for instance, the January effect). Moreover, the month-to-month market autocorrelation is comparable to the Q-approach in magnitude.

Third, there exists a significant difference between the implied distribution and the risk-neutral distribution. Even though the derivatives data derive the implied distribution, the implied distribution still reflects the agent's perception of market return under the subjective probability measure.[14] In contrast, the risk-neutral distribution represents the distribution under the risk-neutral measure. On one hand, it can be shown that (see Martin (2021) and the proof of Proposition 3.1 in Appendix), the autocorrelation in the risk-neutral distribution is zero, under Assumption 3.1. On the other hand, we have shown significantly negative market autocorrelation from the

[14] Cochrane (2017) suggests that investors' survey report about the market return is consistent with the risk-neutral expectation.

implied distribution.[15] In a different context, Adam, Matveev, and Nagel (2021) strongly reject the risk-neutral expectation with stock market data. Here, we reject the risk-neutral expectation with derivatives market data by the Q-approach.

6 The Economic Value of the Q-approach

Thus far, we have explained the difference between the P-approach and the Q-approach about the serial dependence of the market returns. In this section, we discuss the economic value of the Q-approach.

Motivated by the empirical results in Section 4, we implement a market timing strategy. Because of the moderate short-term reversal revealed by the Q-approach, we first construct a *reversal signal* using both realized cumulative excess returns and the conditional derivative-based autocorrelation. Then, we trade the market with this reversal signal in the subsequent month. Specifically, we define the reversal signal at time t as,

$$\tilde{S}_{t,K}\left[r_{t-K\to t}, corr_{t-K}(r_{t-K\to t}, r_{t\to t+1})\right] = \begin{cases} 1, & \text{if} \quad r_{t-K\to t} > 0 \quad \& \quad corr_{t-K}(r_{t-K\to t}, r_{t\to t+1}) > 0, \\ 1, & \text{if} \quad r_{t-K\to t} < 0 \quad \& \quad corr_{t-K}(r_{t-K\to t}, r_{t\to t+1}) < 0, \\ 0, & \text{otherwise,} \end{cases} \tag{20}$$

where $r_{t-K\to t} = R_{t-K\to t} - R_{f,t-K\to t}$ is the realized cumulative excess return over the past K months, $corr_{t-K}(r_{t-K\to t}, r_{t\to t+1})$ is calculated by the Q-approach, and $K = 1, 2, 3, 4, 6,$ and 9 months. We name the strategy based on one reversal signal as the *single timing strategy*. In total, we have six market reversal signals at time t.

Following a market reversal signal, we implement a zero-cost strategy when $corr_{t-K}(r_{t-K\to t}, r_{t\to t+1})$ suggests a positive excess return $r_{t\to t+1}$. As an illustration, if we use the one-month reversal signal as a trading signal at time t and implement the corresponding market timing strategy, the realized return in the subsequent month is

$$\eta\left[\tilde{S}_{t,1}\right] = \begin{cases} r_{t\to t+1}, & \text{if} \quad \tilde{S}_{t,1} = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

It is also possible to implement the market timing strategy using all six reversal signals together, and we label it as the *combination timing strategy*. For instance, we long the market if $\sum_K \tilde{S}_{t,K}$ is greater than a threshold, ξ, an integer ranging from 2 to 5. Following the combination timing strategy, the realized return in the next month

[15] We highlight that correlation-type derivatives are not required to obtain the autocorrelation between two consecutive spot returns under the risk-neutral probability measure. In our study of the autocorrelation under the physical probability measure, we reduce it to publicly available derivative data such as stock market index option, VIX futures, and VIX options. Hence, we also do not need any correlation-type products in this approach.

is

$$\eta\left[\tilde{S}_{t,K},\forall K;\xi\right]=\begin{cases} r_{t\to t+1}, & \text{if} \quad \sum_K \tilde{S}_{t,K}\geqslant \xi, \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

It means that we long the market only if at least ξ reversal signals defined in Equation (20) indicate long signals.

We first study the realized return following the market timing strategy. Figure 7 plots the realized returns generated from the single market timing strategy. As a benchmark, we also plot the realized returns from a buy-and-hold strategy as in Gao, Han, Li, and Zhou (2018). The difference between our market timing strategy and the benchmark strategy is that we long the market only when the signal shows a positive market excess return in the following month, while the benchmark strategy is to long the market persistently. In other words, our market timing strategy is to stay away from the stock market if the signal from the Q-approach shows a future market downturn. Therefore, the relative performance of the market timing strategy depends on whether the reversal signal from the Q-approach indeed reveals valuable information about the market return in the following month.

As shown in Figure 7, the reversal signal from the Q-approach *predicts* the market downturn, particularly when the stock market declines significantly during 2008-2009, 2014, 2015, and 2018-2019. To sharpen our argument, we further plot the realized returns from the above two strategies during the NBER recessions from January, 2008 to June, 2009 in Figure 8. This recession period overlaps the 2008/09 global financial crisis. We see that the market timing strategy based on a one-month reversal signal avoids significant market crashes in January, June, September, October of 2008, and January of 2009.

We next analytically evaluate the timing strategy's performance. Based on the mean $\hat{\mu}_j$ and standard deviation $\hat{\sigma}_j$ of the out-of-sample realized returns by a strategy j, we compute three quantities (performance measures) as follows. (1) We measure the *out-of-sample Sharpe ratio* of each strategy, $\hat{s}_j=\frac{\hat{\mu}_j}{\hat{\sigma}_j}$; (2) we calculate the *certainty-equivalent (CEQ) return* by

$$C\hat{E}Q_j=\hat{\mu}_j-\frac{\gamma}{2}\hat{\sigma}_j^2, \tag{23}$$

where we choose $\gamma=1$ to be consistent with the Q-approach in Section 4; (3) we compute *return-loss* with respect to the buy-and-hold benchmark, as in DeMiguel, Garlappi, and Uppal (2009). Precisely, suppose $\{\hat{\mu}_b,\hat{\sigma}_b\}$ are the monthly out-of-sample mean and volatility of the excess returns from the buy-and-hold strategy, and $\{\hat{\mu}_j,\hat{\sigma}_j\}$ are the corresponding quantities for timing strategy j, the return-loss from strategy j is

$$\text{return-loss}_j=\frac{\hat{\mu}_b}{\hat{\sigma}_b}\times\hat{\sigma}_j-\hat{\mu}_j. \tag{24}$$

In other words, the return-loss is the additional return needed for strategy j to perform to be consistent with the benchmark. A negative value indicates that the strategy j outperforms the benchmark in terms of the Sharpe ratio.

Panel A of Table 7 reports performance evaluation statistics on (annualized) returns generated from the market timing strategies over the period December 2006−December 2019. First, the market timing strategy implies good realized returns, but does not necessarily lead to the highest realized return on average. The realized return on average is about 4.433% *per annum* by the single timing strategy, whereas the realized return from buy-and-hold is about 5.489% on average. This is reasonable because of the market's upward trend from 2006 to 2019, regardless of the financial crisis around 2008 or a market downturn in 2018. Since the market timing strategy only participates in the market in specific periods when the signal shows promising positive return, it might lose some upward opportunities. What is more important is to investigate (1) whether the market timing strategy is conservative and delivers a higher Sharpe ratio, and (2) whether it avoids bad market times.

For the first question, all timing strategies indeed produce smaller standard deviations than the benchmark, suggesting that the market timing strategy is always more conservative than the benchmark strategy. For instance, the standard deviation is 9.616% per annum for the single timing strategy, whereas the buy-and-hold strategy has an almost-twice-large standard deviation of 14.789%. As a result, the Sharpe ratio of the single timing strategy is 0.461, but the buy-and-hold strategy only offers a Sharpe ratio of 0.371. We also see that the last combination timing strategy delivers a higher Sharpe ratio of 0.467.

To evaluate the performance of the market timing strategy in bad market times for the second question, Panel B of Table 7 reports summary statistics during the NBER recessions. Not surprisingly, the buy-and-hold strategy suffers a dramatic loss, yielding a negative average return of -32.304% per annum, along with a standard deviation as high as 25.565%. Consequently, the Sharpe ratio of the buy-and-hold strategy is around -1.264. In contrast, the single timing strategy, $\eta\left[\tilde{S}_{t,1}\right]$, achieves an average return of -7.027% per annum, and a much smaller standard deviation of 14.420%. Although the Sharpe ratio from the single timing strategy is also negative, around -0.487, it exhibits a significant economic value relative to the benchmark. Economically, the -11.194% return-loss value of the single timing strategy suggests that investors are willing to pay as high as 11% per annum to switch from the buy-and-hold to the market timing strategy. Remarkably and consistently, during the market crisis, all single and combination timing strategies yield higher average returns, smaller standard deviations, higher Sharpe ratios, larger CEQs, and negative return-loss measures than the buy-and-hold strategy. In summary, Table 7 shows that the reversal signal identified by the Q-approach does reveal the market downturn from a forward-looking perspective, and the economic value can be substantial.

7 Other Applications of the Q-approach

In Section 3, we have shown that the serial dependence by the Q-approach can be derived by the conditional expectations of future returns. In this section, we discuss the shapes of the term structure of one-period future returns and expected risk premiums, as other applications of the Q-approach.

Let

$$f_{t,T} = \mathbb{E}_t \left[R_{t+T \to t+T+1} \right], \forall t, \forall T = 0, 1, \cdots. \tag{25}$$

Here, $f_t \equiv f_{t,0}$ is the conditional expected spot return $\mathbb{E}_t \left[R_{t \to t+1} \right]$. The expected future one-period return $f(t,T)$ forms a term structure of future returns analogous to the term structure of forward rates. To see it, conditional on the time-t perspective, $f_{t,T} = \mathbb{E}_t \left[\mathbb{E}_{t+T} \left[R_{t+T \to t+T+1} \right] \right] = \mathbb{E}_t [f_{t+T}]$, which is similar to the equation that the implied forward rate equals the expected spot rate.

Because of the small magnitude and relative stability of the risk-free return in each short-time period, the term structure of $f(t,T)$ is essentially similar to (in shape) the term structure of *expected risk premiums*, $f(t,T) - \mathbb{E}_t [R_{f,t+T \to t+T+1}]$, which is $f(t,T) - R_{f,t+T \to t+T+1}$ (by Assumption 3.1). Therefore, we obtain the term structure of one-period future returns and expected risk premiums by the Q-approach. Specifically, by Proposition 3.2, with a power-specified pricing kernel, we have, $\forall T = 1, 2, \cdots$,

$$f(t,T) = \frac{1}{R_{f,t+T \to t+T+1}} FPVIX^{(\gamma+1)}_{t,t+T \to t+T+1}. \tag{26}$$

Given Proposition 3.1, we can derive the term structure of one-period future returns and expected risk premiums with publicly available data.

Table 8 reports the summary statistics for the conditional expected one-month future returns by the Q-approach. We choose T to be 0, 1, 2, 3, 4, 6, and 9 months in $f(t,T)$. Panel A considers the full sample period from February 24, 2006 to December 31, 2019. For $T = 0$, the conditional expected spot return is on average 4.526% per annum. Since the risk-free rate, proxied by 1-month Treasury bill rate, is on average 1.117% during the same sample period, Table 8 suggests that the equity risk premium inferred from the Q-approach is around 3.409% per annum, a number close to conventional estimates of the equity premium. For example, Fama and French (2002) (Table IV in their paper) estimate the unconditional average equity premium over the sample period 1951 to 2000 to be 2.55% and 4.32%, based on dividend and earnings growth, respectively.

As reported in Table 8, the term structure of one-month future returns (and equity risk premiums) is mainly upward slopping on average in normal times. On average, the expected one-month future return and expected risk premium *increase* with respect to maturity T, except a slightly downward/flat feature when $T = 6$ months.

For example, the expected one-month future return one month later is, on average, 5.891% per annum; thus, the expected risk premium is about 4.774% per annum. Similarly, the expected risk premium for one-month four months later is around 5.736% per annum.

We then examine the shape of the term structure $f(t,T), T = 1,2,3,4,6,9$, when the market is in bad (good) times. Precisely, we use the NBER recessions period, January 1, 2008−June 30, 2009 to represent bad times, and the post-NBER recessions, July 1, 2009−December 31, 2019, to represent good times. As shown in Panel B of Table 8, the term structure of one-month future returns (and equity risk premiums) is *downward* sloping on average in bad times. In contrast, Panel C reveals an *upward* term structure of one-month future returns (and expected risk premiums), on average, in good times. Figure 9 illustrates the term structure of one-month future returns on average, in "good", "bad", or "overall" times, respectively.

Furthermore, Figure 10 displays the term structure of one-month future returns for all time t during the NBER recessions. More specifically, we divide the time period into four shorter periods: January 2008−October 2008; November 2008−January 2009; February 2009−April 2009; and May 2009−June 2009. We observe that the downward term structure of one-month future returns (expected risk premiums) between October 2008 and April 2009 (the most severe period of the financial crisis of 2007-2009) is significantly steep. On the contrary, Figure 11 shows the upward term structure of one-month future returns (expected risk premiums) at most of the time between 2009 and 2019.

It is interesting to compare our results with recent equity term structure studies in the literature. By viewing equity as a strip of zero-coupon equity (with dividend as payment) and examining the term structure of the zero-coupon equity to maturity, or the conditional expected annual return to long-maturity minus the annual return to short maturity equity, Binsbergen, Brandt, and Koije (2012) document that the equity term structure is downward sloping on average. Moreover, Gormsen (2021) shows that the term structure is downward sloping in good times, but upward sloping in bad times (counter-cyclical). Notice that the term structure of expected risk premium in our definition is different from those studies though. In our setting, $f(t,T)$ is the conditional expected value of a one-month future return of the stock index, $R_{t+T\to t+T+1}$, whereas Binsbergen, Brandt, and Koije (2012) and Gormsen (2021) focus on the one-period expected returns on dividend-claims. Binsbergen, Brandt, and Koije (2012) and Gormsen (2021) document these stylized facts with market derivative data such as index (call and put) options and dividend futures. Similarly, our approach uses VIX derivatives data and stock index options data. Importantly, we document another stylized fact that the term structure of one-period future returns and expected risk premiums are *pro-cyclical* in our setting.

A pro-cyclicality of the term structure of expected risk premiums can be understood as follows. By Proposition 3.2, the conditional expected one-month future returns in T months are essentially the futures prices of PVIX over the same period, as described in Equation (26). Because of the close relation between PVIX and

VIX, by the no-arbitrage principle and intuitively, the term structure of PVIX futures exhibits a similar shape as of VIX futures. Indeed, using publicly market data, Hu and Jacobs (2020) document that VIX futures prices tend to have an upward sloping term structure during normal times, and tend to become inverted or hump-shaped in times of market turbulence. The main difference between our results and Hu and Jacobs (2020) is that a PVIX futures contract is not tradable, and the pro-cyclicality property of the term structure of one-month future returns follows from no-arbitrage PVIX futures prices with publicly available VIX derivatives data.

8 Conclusion

This paper presents a Q-approach to study the serial dependence in the stock market using the derivatives market information. The distinct feature is its analytical expression of expected market future return (moments) in VIX derivatives (VIX futures and VIX options) and stock market index options. The Q-approach is real-time, provides a time-varying conditional view, and is robust to the stochastic discount factor process choice.

Using S&P 500 index options, CBOE VIX futures, and VIX options, we robustly demonstrate an upward market trend in the long run, but moderate short-term reversals across different holding periods. We further document that the market return dynamics inferred from the Q-approach are comparable to that from the historical stock market prices with fading memory methodology, but significantly different from the standard empirical method. Moreover, we show that the economic value of the Q-approach can be substantial since the reversal signal by the Q-approach can particularly reveal a market downturn in the future. Finally, by the Q-approach, we demonstrate the pro-cyclical term structure of expected risk premiums. Overall, we demonstrate important implications to the stock market of the derivatives market.

Appendix

Appendix A Monotonic properties of PVIX

Proposition Appendix A.1. *Monotonic properties of PVIX.*

1. *Assume* $\mathbb{E}_t[R_{t\to t+T}] > 1$ *(positive risk premium), then* $PVIX^{(\alpha)}_{t\to t+T}$ *is strictly increasing to the exponent* α *while the time to maturity* T *is fixed.*

2. *Under Assumption 3.1 and let* $X_{t,T} = \frac{1}{R_{f,t\to t+T}}\mathbb{E}^Q_t[R^\alpha_{t\to t+T}]$*. Then* $X_{t,T}$ *is monotonic increasing to* T *when both* t *and* α *are fixed.*

Proof. (1) Let $X = R_{t\to t+T}$. Then by the conditional-version of Holder's inequality, we have $\mathbb{E}_t[X^\alpha|^{\frac{1}{\alpha}} \leqslant \mathbb{E}_t[X^\beta|^{\frac{1}{\beta}}, \forall 1 < \alpha < \beta$. In terms of PVIX, it can be written as

$$\left(T \times PVIX^{(\alpha)}_{t\to t+T}\right)^{\frac{1}{\alpha}} \leqslant \left(T \times PVIX^{(\beta)}_{t\to t+T}\right)^{\frac{1}{\beta}}, \tag{A1}$$

yielding

$$PVIX^{(\alpha)}_{t\to t+T} \leqslant T^{\frac{\alpha}{\beta}-1}\left(PVIX^{(\beta)}_{t\to t+T}\right)^{\frac{\alpha}{\beta}}.$$

To prove $PVIX^{(\alpha)}_{t\to t+T} < PVIX^{(\beta)}_{t\to t+T}$, it suffices to show that $T \times PVIX^{(\beta)}_{t\to t+T} \geqslant 1$. To the end, by using Jensen's inequality, we have

$$T \times PVIX^{(\beta)}_{t\to t+T} = \mathbb{E}_t[X^\beta] \geqslant \mathbb{E}_t[X]^\beta > 1.$$

because of the positive excess risk premium.

Proof. (2) For any $t < S < T$, by Jensen's inequality, we have

$$\mathbb{E}^Q_S[R^\alpha_{S\to T}] \geqslant \left(\mathbb{E}^Q_S[R_{S\to T}]\right)^\alpha = R^\alpha_{f,S\to T},$$

where $R_{f,S\to T} = \mathbb{E}_S^Q[R_{S\to T}]$. Then, we have

$$\begin{aligned}
X_{t,T} &= \frac{1}{R_{f,t\to t+T}}\mathbb{E}_t^Q\left[R_{t\to t+S}^{\alpha}R_{t+S\to t+T}^{\alpha}\right] \\
&= \frac{1}{R_{f,t\to t+T}}\mathbb{E}_t^Q\left[R_{t\to t+S}^{\alpha}\mathbb{E}_{t+S}^Q[R_{t+S\to t+T}^{\alpha}]\right] \\
&\geqslant \frac{1}{R_{f,t\to t+T}}\mathbb{E}_t^Q\left[R_{t\to t+S}^{\alpha}R_{f,t+S\to t+T}^{\alpha}\right] \\
&= \frac{1}{R_{f,t\to t+S}}R_{f,t+S\to t+T}^{\alpha-1}\mathbb{E}_t^Q\left[R_{t\to t+S}^{\alpha}\right] \\
&\geqslant \frac{1}{R_{f,t\to t+S}}\mathbb{E}_t^Q\left[R_{t\to t+S}^{\alpha}\right] = X_{t,S}
\end{aligned}$$

since $R_{f,t+S\to t+T} \geqslant 1$.

Appendix B Proof of Proposition 3.1

The proof is divided into several steps.

Step 1. We first derive an approximation formula of VIX as follows

$$VIX_{t\to t+T}^2 \sim \frac{1}{T}\left(\mathbb{E}_t^Q\left[\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right)^2\right] - 1\right). \tag{B1}$$

By using the second-order expansion of $log(1+x) \sim x - \frac{1}{2}x^2$ when x closes to zero, and $\frac{R_{t\to t+T}}{R_{f,t\to t+T}}$ sufficiently closes to one, we obtain

$$log\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right) \sim \frac{R_{t\to t+T}}{R_{f,t\to t+T}} - 1 - \frac{1}{2}\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}} - 1\right)^2. \tag{B2}$$

By taking the conditional expectation under Q-measure, and $\mathbb{E}_t^Q\left[\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right] = 1$, we obtain

$$\mathbb{E}_t^Q log\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right) \sim \frac{1}{2} - \frac{1}{2}\mathbb{E}_t^Q\left[\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right)^2\right]. \tag{B3}$$

Recall the definition of VIX,

$$VIX_{t\to t+T}^2 = \frac{2}{T}L_t^Q\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right) \tag{B4}$$

where $L^Q_t(X) \equiv log\mathbb{E}^Q_t X - \mathbb{E}^Q_t logX$. Since $log\mathbb{E}^Q_t\left[\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right] = 0$, by (B3), we obtain

$$VIX^2_{t\to t+T} \sim \frac{1}{T}\left(\mathbb{E}^Q_t\left[\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right)^2\right] - 1\right). \tag{B5}$$

Step 2. We derive the futures price of PVIX for $\alpha = 2$. To simplify notations, we omit the exponent α in the definition of $FPVIX$.

We use the formula (B1) for the time period from $t+T_1$ to $t+T_1+T_2$,

$$VIX^2_{t+T_1\to t+T_1+T_2} \sim \frac{1}{T_2}\left(\mathbb{E}^Q_{t+T_1}\left[\left(\frac{R}{R_f}\right)^2\right] - 1\right).$$

Here, to simplify notation, we write $R = R_{t+T_1\to t+T_1+T_2}, R_f = R_{f,t+T_1\to t+T_1+T_2}$.

By applying the conditional expectation at t in the Q-measure of the last equation, we have

$$\begin{aligned}\mathbb{E}^Q_t[VIX^2_{t+T_1\to t+T_1+T_2}] &\sim \frac{1}{T_2}\left(\mathbb{E}^Q_t\left[\left(\frac{R}{R_f}\right)^2\right] - 1\right).\\ &\sim \left(\frac{R_{f,t\to t+T_1}}{R_{f,t\to t+T_1+T_2}}\right)^2\left(\underbrace{\frac{1}{T_2}\mathbb{E}^Q_t\left[R^2_{t+T_1\to t+T_1+T_2}\right]}_{FPVIX_{t,t+T_1\to t+T_1+T_2}}\right) - \frac{1}{T_2}\end{aligned}$$

To compute $FPVIX_{t,t+T_1\to t+T_1+T_2}$ on the right hand side, it suffices to compute the left fide, which is

$$\mathbb{E}^Q_t[VIX^2_{t+T_1\to t+T_1+T_2}] = \left(\underbrace{\mathbb{E}^Q_t[VIX_{t+T_1\to t+T_1+T_2}]}_{FVIX_{t,t+T_1\to t+T_1+T_2}}\right)^2 + Var^Q_t(VIX_{t+T_1\to t+T_1+T_2}).$$

Here, the first term on the right side of the last equation is the square of the VIX future by the risk-neutral pricing formula, and the second term is the conditional variance $Var^Q_t(VIX_{t+T_1\to t+T_1+T_2})$.

We now consider the VIX option with maturity $t+T_1$ and the underlying VIX is $VIX_{t+T_1\to t+T_1+T_2}$. Since the VIX is a tradable asset, by the fundamental pricing theorem in derivative theory, its future value process under $Q-$ measue is a martingale. Then, the conditional variance $Var^Q_t(VIX_{t+T_1\to t+T_1+T_2})$ equals to $\left(FVIX^2_{t,t+T_1\to t+T_1+T_2}\right)\sigma^2 T_1$, σ is the implied volatility of the at-the-money VIX option. Therefore,

$$\mathbb{E}^Q_t[VIX^2_{t+T_1\to t+T_1+T_2}] = FVIX^2_{t,t+T_1\to t+T_1+T_2}(1+\sigma^2 T_1). \tag{B6}$$

Finally, we obtain the futures price of PVIX for $\alpha = 2$:

$$FPVIX^{(2)}_{t,t+T_1 \to t+T_1+T_2} \sim \left(\frac{R_{f,t\to t+T_1+T_2}}{R_{f,t\to t+T_1}}\right)^2 \times \left[FVIX^2_{t,t+T_1\to t+T_1+T_2} + FVIX^2_{t,t+T_1\to t+T_1+T_2} T_1 \sigma^2 + \frac{1}{T_2}\right] \quad \text{(B7)}$$

Step 3. We compute the forward prices for higher power PVIX for $\alpha = 3$ by using the result in Step 2. A recursive procedure is explained in the next Step.

For this purpose, we use the third-order expansion of the function $log(1+x) \sim x - \frac{1}{2}x^2 + \frac{1}{3}x^3$ when x closes to zero. Therefore,

$$log\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right) \sim \frac{R_{t\to t+T}}{R_{f,t\to t+T}} - 1 - \frac{1}{2}(\frac{R_{t\to t+T}}{R_{f,t\to t+T}} - 1)^2 + \frac{1}{3}(\frac{R_{t\to t+T}}{R_{f,t\to t+T}} - 1)^3$$

we have (since $\mathbb{E}^Q_t[R_{t\to t+T}] = R_{f,t\to t+T}$)

$$\begin{aligned}
\mathbb{E}^Q_t\left[log\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right)\right] &\sim -\frac{1}{2}\mathbb{E}^Q_t\left[(\frac{R_{t\to t+T}}{R_{f,t\to t+T}} - 1)^2\right] + \frac{1}{3}\mathbb{E}^Q_t\left[(\frac{R_{t\to t+T}}{R_{f,t\to t+T}} - 1)^3\right] \\
&= -\frac{1}{2}\left(\mathbb{E}^Q_t[(\frac{R_{t\to t+T}}{R_{f,t\to t+T}})^2 - 2(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}) + 1)]\right) \\
&\quad + \frac{1}{3}\left(\mathbb{E}^Q_t[(\frac{R_{t\to t+T}}{R_{f,t\to t+T}})^3 - 3(\frac{R_{t\to t+T}}{R_{f,t\to t+T}})^2 + 3(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}) - 1]\right) \\
&= \frac{1}{3}\mathbb{E}^Q_t[(\frac{R_{t\to t+T}}{R_{f,t\to t+T}})^3] - \frac{3}{2}\mathbb{E}^Q_t[(\frac{R_{t\to t+T}}{R_{f,t\to t+T}})^2] + \frac{7}{6}
\end{aligned}$$

We use the above equation to replace $R_{t\to t+T}$ by $R = R_{t+T_1\to t+T_1+T_2}$, and $R_{f,t\to t+T}$ by $R_{f,t+T_1\to t+T_1+T_2}$ and conditional on time $t+T+1$, we obtain

$$VIX^2_{t+T_1\to t+T_1+T_2} \sim \frac{1}{T_2}\left(-\frac{2}{3}\mathbb{E}^Q_{t+T_1}[(\frac{R}{R_f})^3] + 3\mathbb{E}^Q_{t+T_1}[(\frac{R}{R_f})^2] - \frac{7}{3}\right) \quad \text{(B8)}$$

and

$$\mathbb{E}^Q_t[VIX^2_{t+T_1\to t+T_1+T_2}] \sim \frac{1}{T_2}\left(-\frac{2}{3}\mathbb{E}^Q_t[(\frac{R}{R_f})^3] + 3\mathbb{E}^Q_t[(\frac{R}{R_f})^2] - \frac{7}{3}\right) \quad \text{(B9)}$$

By Step 2, we have

$$FVIX^2_{t,t+T_1\to t+T_1+T_2}(1+\sigma^2 T_1) \sim \frac{1}{T_2}\left(-\frac{2}{3}\mathbb{E}^Q_t[(\frac{R}{R_f})^3] + 3\mathbb{E}^Q_t[(\frac{R}{R_f})^2] - \frac{7}{3}\right).$$

Therefore, we can derive $\mathbb{E}^Q_t[(\frac{R}{R_f})^3]$ and the futures price of PVIX for the exponent $\alpha = 3$.

Step 4. The forward price of PVIX for a general exponent N can be calculated recursively.

By the N-th order approximation of $log(1+x)$,

$$log(1+x) \sim \sum_{i=1}^{N} (-1)^{i-1} \frac{1}{i} x^i$$

for $x = \frac{R_{t+T_1 \to t+T_1+T_2}}{R_{f,t+T_1 \to t+T_1+T_2}} - 1$, we obtain

$$log\left(\frac{R}{R_f}\right) \sim \sum_{i=1}^{N} (-1)^{i-1} \frac{1}{i} \left(\frac{R}{R_f} - 1\right)^i$$

By using the same method in Step 2 and Step 3, we have

$$\mathbb{E}_{t+T_1}^{Q}\left[log(\frac{R}{R_f})\right] \sim \sum_{i=1}^{N} (-1)^{i-1} \frac{1}{i} \mathbb{E}_{t+T_1}^{Q}\left[\left(\frac{R}{R_f} - 1\right)^i\right].$$

Since

$$VIX_{t+T_1 \to t+T_1+T_2}^2 = -\frac{2}{T_2} \mathbb{E}_{t+T_1}^{Q}\left[log \frac{R}{R_f}\right] \sim \frac{2}{T_2} \sum_{i=1}^{N} (-1)^i \frac{1}{i} \mathbb{E}_{t+T_1}^{Q}\left[\left(\frac{R}{R_f} - 1\right)^i\right].$$

taking expectation conditional on t, the iterated law of expectation implies

$$\mathbb{E}_t[VIX_{t+T_1 \to t+T_1+T_2}^2] \sim \frac{2}{T_2} \sum_{i=1}^{N} (-1)^i \frac{1}{i} \mathbb{E}_t^{Q}\left[\left(\frac{R}{R_f} - 1\right)^i\right],$$

and then,

$$\frac{T_2}{2} FVIX_{t,t+T_1 \to t+T_1+T_2}^2 \left(1 + \sigma^2 T_1\right) \sim \sum_{i=1}^{N} (-1)^i \frac{1}{i} \mathbb{E}_t^{Q}\left[\left(\frac{R}{R_f} - 1\right)^i\right], \tag{B10}$$

Finally, $\mathbb{E}_t^{Q}\left[\left(\frac{R}{R_f}\right)^N\right]$ can be calculated by $\mathbb{E}_t^{Q}\left[\left(\frac{R}{R_f}\right)^i\right], i = 2, \cdots, N-1$, recursively. $\square$

Remark. Our approximation in Proposition 3.1 depends on Equation (B2) - (B3). We now explain why this approximation is sufficiently tight for empirical applications. For simplicity we use x to the random variable $\frac{R_{t \to T}}{R_{f,t \to T}} - 1$. Let $a \equiv \sup_x |log(1+x) - (x - \frac{x^2}{2})|$ for all possible scenarios of x. The number a is very small in magnitude because x is closes to zero. Moreover, for any $c > 0$,

$$\begin{aligned}
\mathbb{E}^Q\left[\left|log(1+x)-(x-\frac{x^2}{2})\right|\right] &= \mathbb{E}^Q\left[\left|log(1+x)-(x-\frac{x^2}{2})\right| : |x| \leqslant c\right] + \mathbb{E}^Q\left[\left|log(1+x)-(x-\frac{x^2}{2})\right| : |x| > c\right] \\
&\leqslant \frac{c^3}{3} + \mathbb{E}^Q\left[\left|log(1+x)-(x-\frac{x^2}{2})\right| : |x| > c\right] \\
&\leqslant \frac{c^3}{3} + aP(|x| > c).
\end{aligned}$$

Clearly, the smaller the parameter c the smaller the first term $\frac{c^3}{3}$. Even though the probability $P(|x| \geqslant c)$ becomes larger for a smaller value of c, but this probability itself is usually very small. In total, the upper bound of $\mathbb{E}^Q\left[|log(1+x)-(x-\frac{x^2}{2})|\right]$ is very small.

Numerically, if choose $|x| \leqslant 1\%$ for the monthly return (annual return bound is 12 percent), and the average VIX is 15%, then

$$\mathbb{E}_t^Q\left[\left|log(1+x)-(x-\frac{x^2}{2})\right|\right] \leqslant \frac{1}{3}(0.01)^3,$$

and

$$\left|\mathbb{E}_t^Q[log(1+x)]\right| = \frac{T}{2}VIX^2 = \frac{1}{2\times 12}(0.15)^2.$$

Therefore,

$$\left|\frac{\mathbb{E}_t^Q\left[\left|log(1+x)-(x-\frac{x^2}{2})\right|\right]}{\mathbb{E}_t^Q[log(1+x)]}\right| \leqslant \frac{1}{3}(0.01)^3 \times (2\times 12)\frac{1}{(0.15^2} = 0.04\%.$$

If we choose a large number for the month return, $|x| \leqslant 2\%$, which means annually stock return is bounded by 24 percent and -24 percent on both sides, and $VIX = 20\%$, then

$$\left|\frac{\mathbb{E}_t^Q\left[\left|log(1+x)-(x-\frac{x^2}{2})\right|\right]}{\mathbb{E}_t^Q[log(1+x)]}\right| \leqslant 0.16\%.$$

Therefore, the approximation pricing formula for the futures price of PVIX is sufficiently accurate for the market data, and Assumption 3.2 is not essentially required.

Appendix C Proof of Proposition 3.2

Proof of Proposition 3.2 (1). Before considering the futures price of PVIX for any exponent $\gamma > 0$, we need the following Lemma.

Lemma A. Under Assumption 3.1, then for any Lebesgue-measurable function $h(x)$,

$$Cov_t^Q(R_{t+T_1 \to t+T_1+T_2}, h(R_{t \to t+T_1})) = 0. \tag{C1}$$

Under additional Assumption 3.2, we have $Cov_t^Q(g(R_{t+T_1 \to t+T_1+T_2}), h(R_{t \to t+T_1})) = 0$.

Proof of Lemma A. The proof is similar to Martin (2021) and Chernov, Lochstoer, and Lundeby (2021). Let $h = h(R_{t \to t+T_1})$.

$$\begin{aligned}
Cov_t^Q(R_{t+T_1 \to t+T_1+T_2}, h) &= \mathbb{E}_t^Q\left[\left(R_{t+T_1 \to t+T_1+T_2} - \mathbb{E}_t^Q[R_{t+T_1 \to t+T_1+T_2}]\right) \times (h - \mathbb{E}_t^Q[h])\right] \\
&= \mathbb{E}_t^Q\left[\mathbb{E}_{t+T_1}^Q\left(R_{t+T_1 \to t+T_1+T_2} - \mathbb{E}_t^Q[R_{t+T_1 \to t+T_1+T_2}]\right) \times (h - \mathbb{E}_t^Q[h])\right] \\
&= \mathbb{E}_t^Q\left[\left(R_{f,t+T_1 \to t+T_1+T_2} - \mathbb{E}_t^Q[R_{f,t+T_1 \to t+T_1+T_2}]\right) \times (h - \mathbb{E}_t^Q[h])\right] \\
&= 0
\end{aligned}$$

where the last equation follows from the deterministic assumption (Assumption 3.1) of the risk-free interest rate. Under Assumption 3.2,

$$Cov_t^Q(g(R_{t+T_1 \to t+T_1+T_2}), h(R_{t \to t+T_1})) = \mathbb{E}_t^Q[g'(R_{t+T_1 \to t+T_1+T_2})] Cov_t^Q(R_{t+T_1 \to t+T_1+T_2}, h(R_{t \to t+T_1})) = 0.$$

□

We first derive $\mathbb{E}_t[R_{t+1 \to t+T}]$. We write

$$\frac{R_{t+1 \to t+T}}{m_{t,t+T}} = \left(\frac{S_{t+T}}{S_{t+1}} \frac{1}{m_{t+1,t+T}}\right) \frac{1}{m_{t,t+1}}$$

For the power utility function,

$$\frac{R_{t+1 \to t+T}}{m_{t,t+T}} = (R_{t+1 \to t+T})^{1+\gamma} \times (R_{t \to t+1})^{\gamma}.$$

By virtue of Lemma A, we have

$$Cov_t^Q\left((R_{t+1 \to t+T})^{1+\gamma}, R_{t \to t+1}^{\gamma}\right) = 0. \tag{C2}$$

Therefore,

$$
\begin{aligned}
\mathbb{E}_t[R_{t+1\to t+T}] &= \frac{1}{R_{f,t\to t+T}}\mathbb{E}_t^Q\left[\frac{R_{t+1\to t+T}}{m_{t,t+T}}\right] \\
&= \frac{1}{R_{f,t\to t+T}}\mathbb{E}_t^Q\left[(R_{t+1\to t+T})^{1+\gamma}\times(R_{t\to t+1})\right] \\
&= \frac{1}{R_{f,t\to t+T}}\mathbb{E}_t^Q\left[(R_{t+1\to t+T})^{1+\gamma}\right]\mathbb{E}_t^Q\left[\frac{1}{m_{t,t+1}}\right] \\
&= \frac{1}{R_{f,t+1\to t+T}}\mathbb{E}_t^Q\left[(R_{t+1\to t+T})^{1+\gamma}\right] \\
&= \frac{1}{R_{f,t+1\to t+T}}\mathbb{E}_t^Q\left[\mathbb{E}_{t+1}^Q\left[(R_{t+1\to t+T})^{1+\gamma}\right]\right] \\
&= \frac{T-1}{R_{f,t+1\to t+T}}FPVIX_{t,t+1\to t+T}^{(1+\gamma)}.
\end{aligned}
$$

Here we use the inverting Girsanov theorem that

$$
\mathbb{E}_t^Q\left[\frac{1}{m_{t,t+1}}\right] = R_{f,t\to t+1}\mathbb{E}_t[1] = R_{f,t\to t+1}.
$$

Similarly, in terms of two periods T_1 and T_2, we have

$$
\mathbb{E}_t[R_{t+T_1\to t+T_1+T_2}] = \frac{T_2}{R_{f,t+T_1\to t+T_1+T_2}}FPVIX_{t,t+T_1\to t+T_1+T_2}^{(1+\gamma)}, \tag{C3}
$$

By the same idea, the conditional moments of $R_{t+T_1,t+T_2}$ is

$$
\begin{aligned}
\mathbb{E}_t[R^n_{t+T_1\to t+T_1+T_2}] &= \frac{1}{R_{f,t\to t+T_1+T_2}}\mathbb{E}_t^Q\left[\left(\frac{S_{t+T_1+T_2}}{S_{t+T_1}}\right)^n\left(\frac{S_{t+T_1+T_2}}{S_t}\right)^\gamma\right] \\
&= \frac{1}{R_{f,t\to t+T_1+T_2}}\mathbb{E}_t^Q\left[\left(\frac{S_{t+T_1+T_2}}{S_{t+T_1}}\right)^{\gamma+n}\left(\frac{S_{t+T_1}}{S_t}\right)^\gamma\right] \\
&= \frac{1}{R_{f,t\to t+T_1+T_2}}\mathbb{E}_t^Q\left[\left(\frac{S_{t+T_1+T_2}}{S_{t+T_1}}\right)^{\gamma+n}\right]\mathbb{E}_t^Q\left[\left(\frac{S_{t+T_1}}{S_t}\right)^\gamma\right] \\
&= \frac{T_2}{R_{f,t+T_1\to t+T_1+T_2}}FPVIX_{t,t+T_1\to t+T_1+T_2}^{(\gamma+n)}.
\end{aligned}
$$

□

Proof of Proposition 3.2 (2). Notice that

$$
\frac{R_{t\to t+1}R_{t\to t+T}}{m_{t,t+T}} = \frac{S_{t+T}}{S_{t+1}}\frac{1}{m_{t+1,t+T}}\left(\frac{S_{t+1}}{S_t}\right)^2\frac{1}{m_{t,t+1}}.
$$

That is,

$$\frac{R_{t\to t+1}R_{t\to t+T}}{m_{t,t+T}} = \left(\frac{S_{t+T}}{S_{t+1}}\right)^{\gamma+1} \times \left(\frac{S_{t+1}}{S_t}\right)^{\gamma+2}$$

Applying Stein's lemma under the risk-neutral measure Q, we have

$$\begin{aligned}
\mathbb{E}_t[R_{t\to t+1}R_{t\to t+T}] &= \frac{1}{R_{f,t\to t+T}}\mathbb{E}_t^Q\left[\left(\frac{S_{t+T}}{S_{t+1}}\right)^{\gamma+1} \times \left(\frac{S_{t+1}}{S_t}\right)^{\gamma+2}\right] \\
&= \frac{1}{R_{f,t\to t+T}}\mathbb{E}_t^Q\left[\left(\frac{S_{t+T}}{S_{t+1}}\right)^{\gamma+1}\right] \times \mathbb{E}_t^Q\left[\left(\frac{S_{t+1}}{S_t}\right)^{\gamma+2}\right] \\
&= \frac{T-1}{R_{f,t\to t+T}}\mathbb{E}_t^Q[PVIX_{t+1\to t+T}^{(\gamma+1)}] \times PVIX_t^{(\gamma+2)} \\
&= \frac{T-1}{R_{f,t\to t+T}}FPVIX_{t,t+1\to t+T}^{(\gamma+1)} \times PVIX_t^{(\gamma+2)}.
\end{aligned}$$

For two periods T_1 and T_2, we have

$$\mathbb{E}_t[R_{t\to t+T_1}R_{t\to t+T_1+T_2}] = \frac{T_1T_2}{R_{f,t\to t+T_1+T_2}} \times PVIX_{t\to t+T_1}^{(\gamma+2)} \times FPVIX_{t,t+T_1\to t+T_1+T_2}^{(\gamma+1)}. \tag{C4}$$

$\square$

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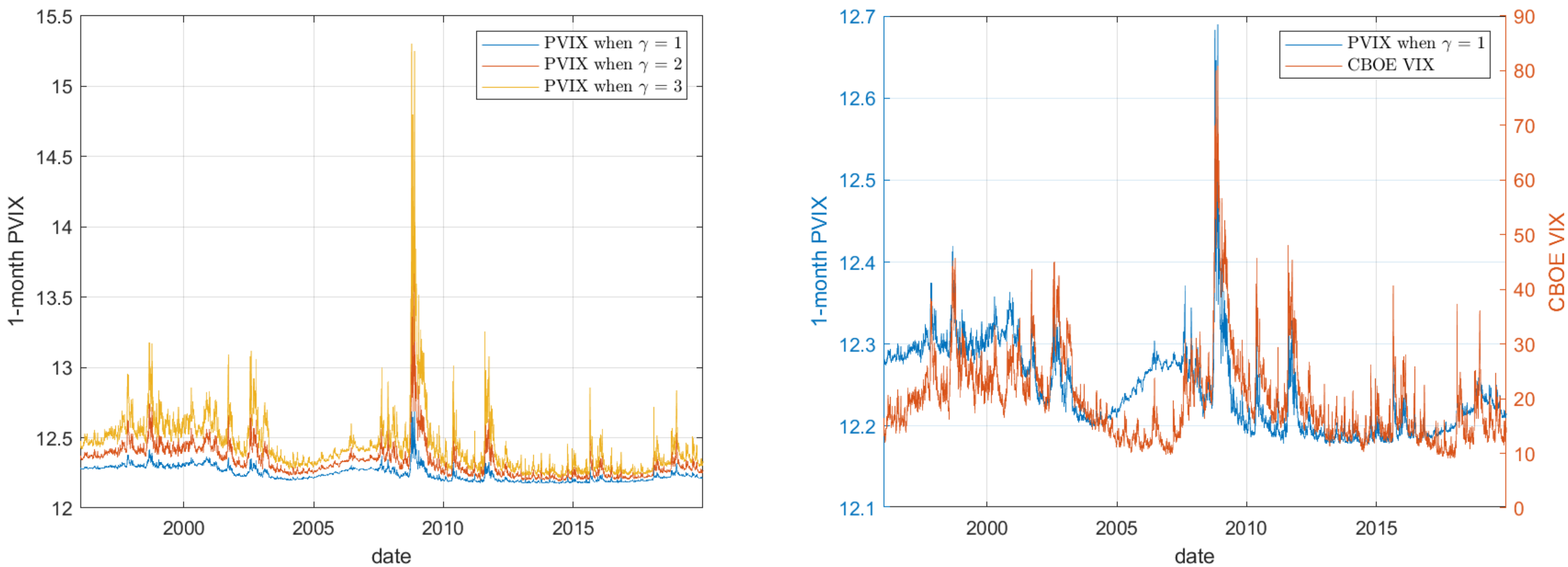
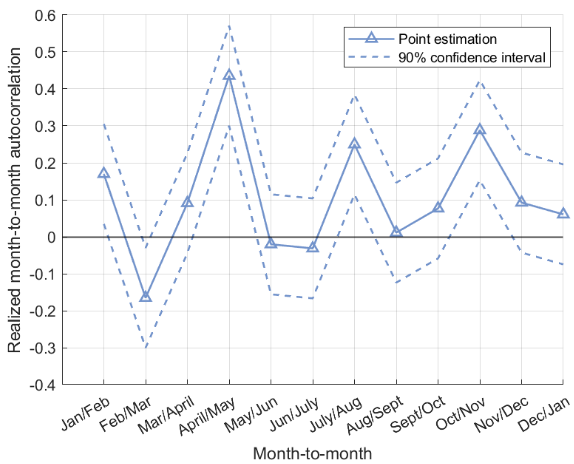


(a) 1-month PVIX indices

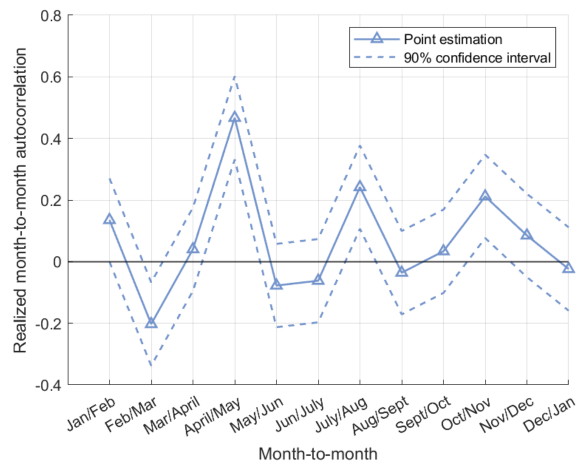
(b) 1-month PVIX & CBOE VIX

Figure 1: PVIX index & CBOE VIX

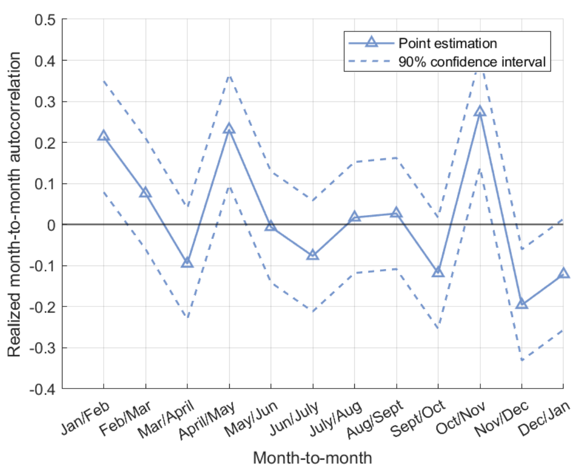
This figure plots 1-month PVIX indices computed from market prices of index options and CBOE VIX index. The sample period is from January 4, 1996 to December 31, 2019.



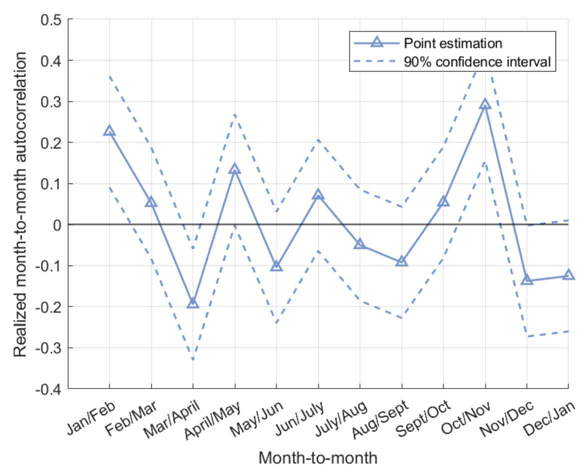
(a) Sample period 1871 - 2019



(b) Sample period 1920 - 2019



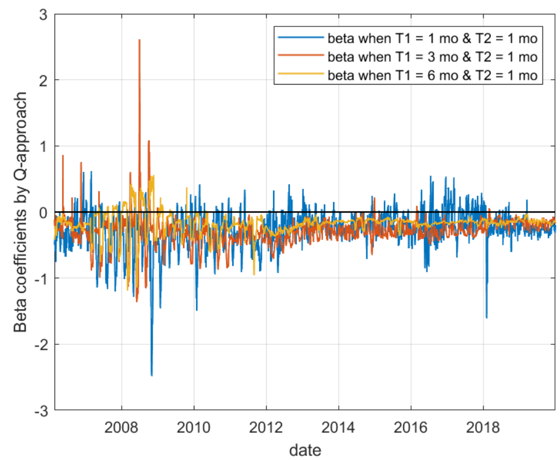
(c) Sample period 1970 - 2019



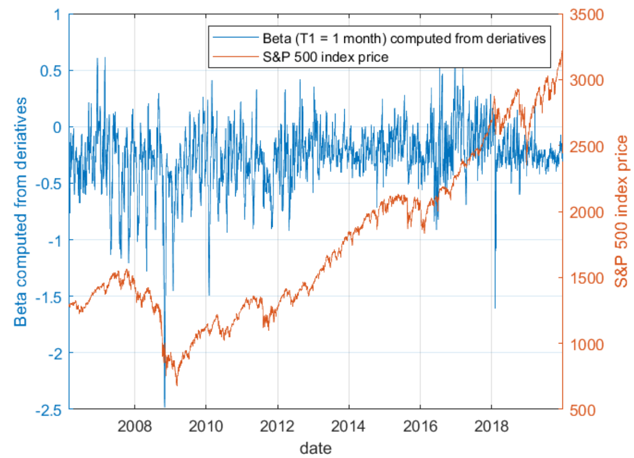
(d) Sample period 1990 - 2019

Figure 2: Realized market autocorrelation between adjacent calendar months

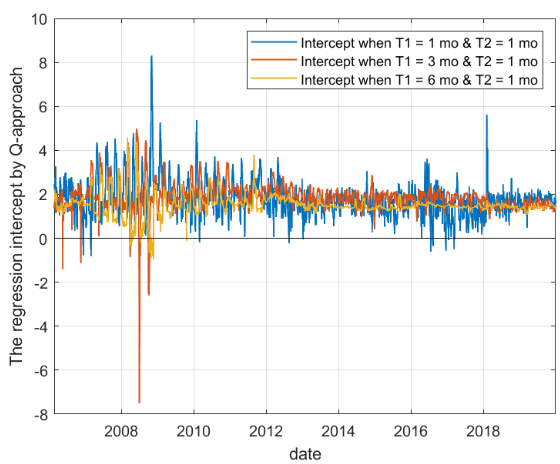
This figure plots the realized month-to-month autocorrelation of the S&P 500 monthly returns between two consecutive months. The area between the dotted line represents the 90% confidence interval for the sample autocorrelation by assuming the standard error equals one over the square root of the sample size. We consider four time periods: (a) 1871 – 2019, (b) 1920 – 2019, (c) 1970 – 2019, (d) 1990 – 2019. The data prior to January 1927 are obtained from Robert Shiller's website.



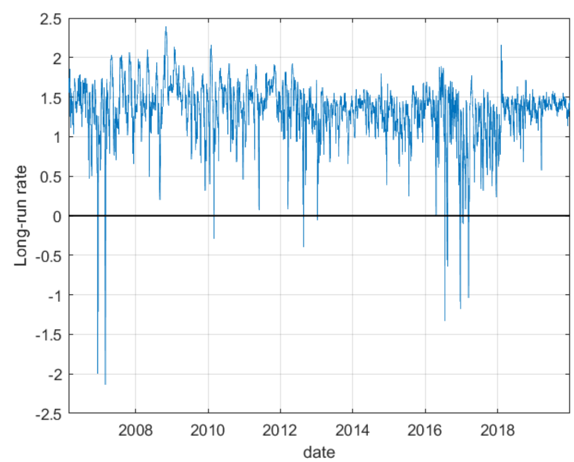
(a) Beta coefficient ($\beta_{t\to t+T_1}$)



(b) Beta & S&P 500 index



(c) The intercept ($\alpha_{t\to t+T_1}$)



(d) The long-run rate ($\frac{\alpha_t}{1-\beta_t}$)

Figure 3: Market return dynamics by Q-approach

This figure plots the real time regression coefficients, $\{\beta_t, \alpha_t\}$, and the long-run rate, $\frac{\alpha_t}{1-\beta_t}$, by Q-approach. The sample period is from February 24, 2006 to December 31, 2019.

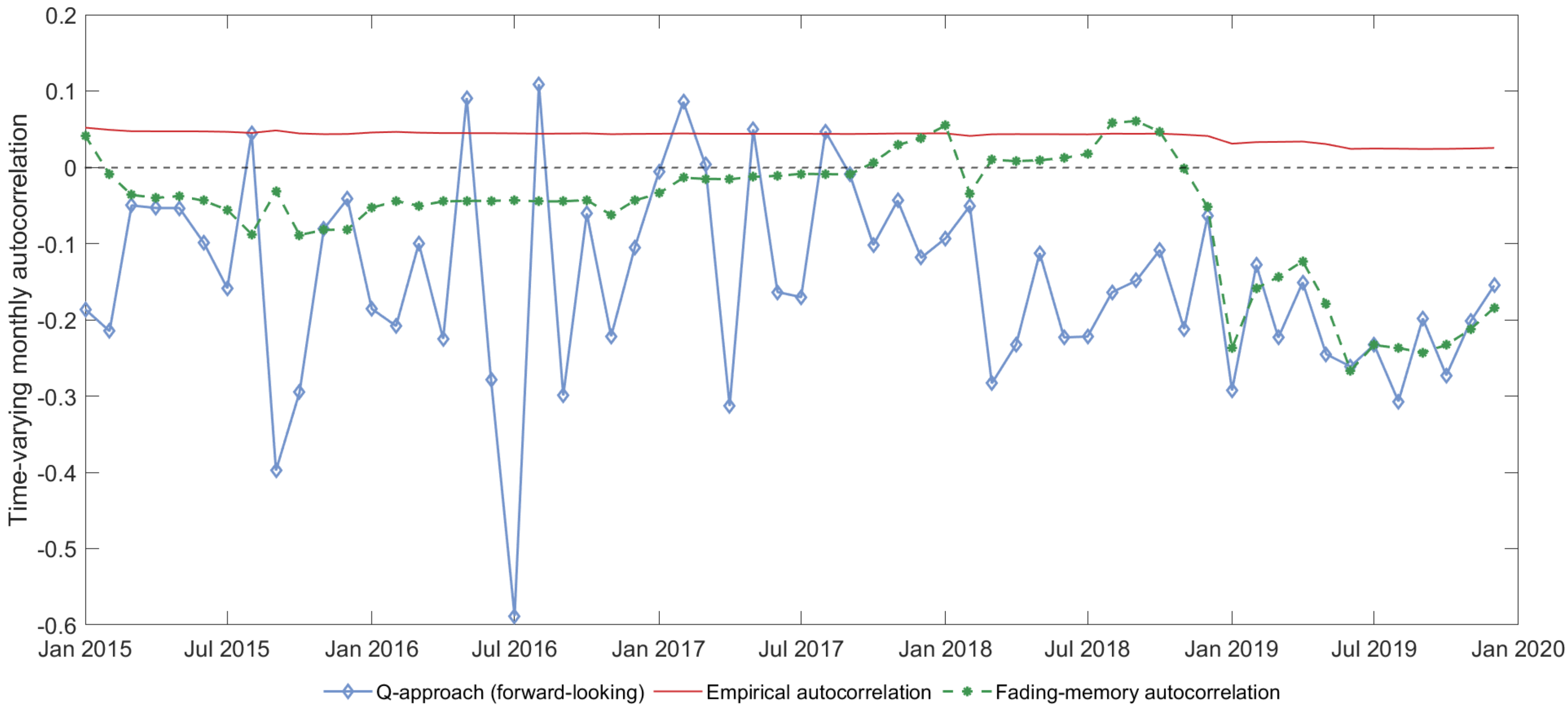


Figure 4: Time-varying market autocorrelation - recursive method

This figure compares the time-varying market autocorrelation computed from different empirical approaches. The sample (empirical) autocorrelation and fading-memory autocorrelation coefficients are computed recursively with an initial window of 10 years. The key variable of fading memory is set to be $\nu = 0.018$ as in Nagel and Xu (2021). The Q-approach-based autocorrelation, $corr_t(R_{t\rightarrow t+1mo}, R_{t+1mo\rightarrow t+2mo})$, is computed at the first day of each month using derivatives.

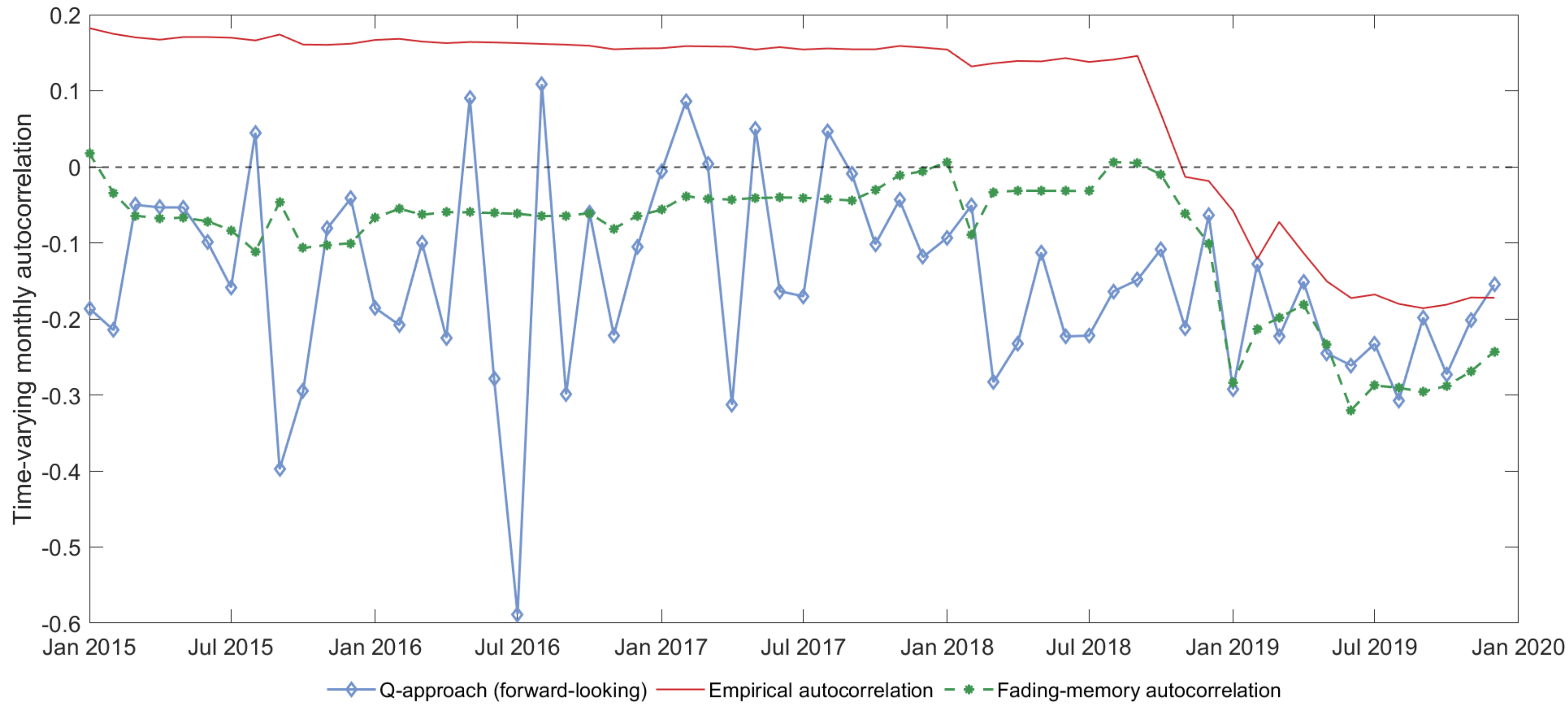


Figure 5: Time-varying market autocorrelation - rolling method

This figure compares the time-varying market autocorrelation computed from different empirical approaches. The sample (empirical) autocorrelation and fading-memory autocorrelation coefficients are computed using a 10-year rolling window. The key variable of fading memory is set to be $\nu = 0.018$ as in Nagel and Xu (2021). The Q-approach-based autocorrelation, $corr_t(R_{t\to t+1mo}, R_{t+1mo\to t+2mo})$, is computed at the first day of each month using derivatives.

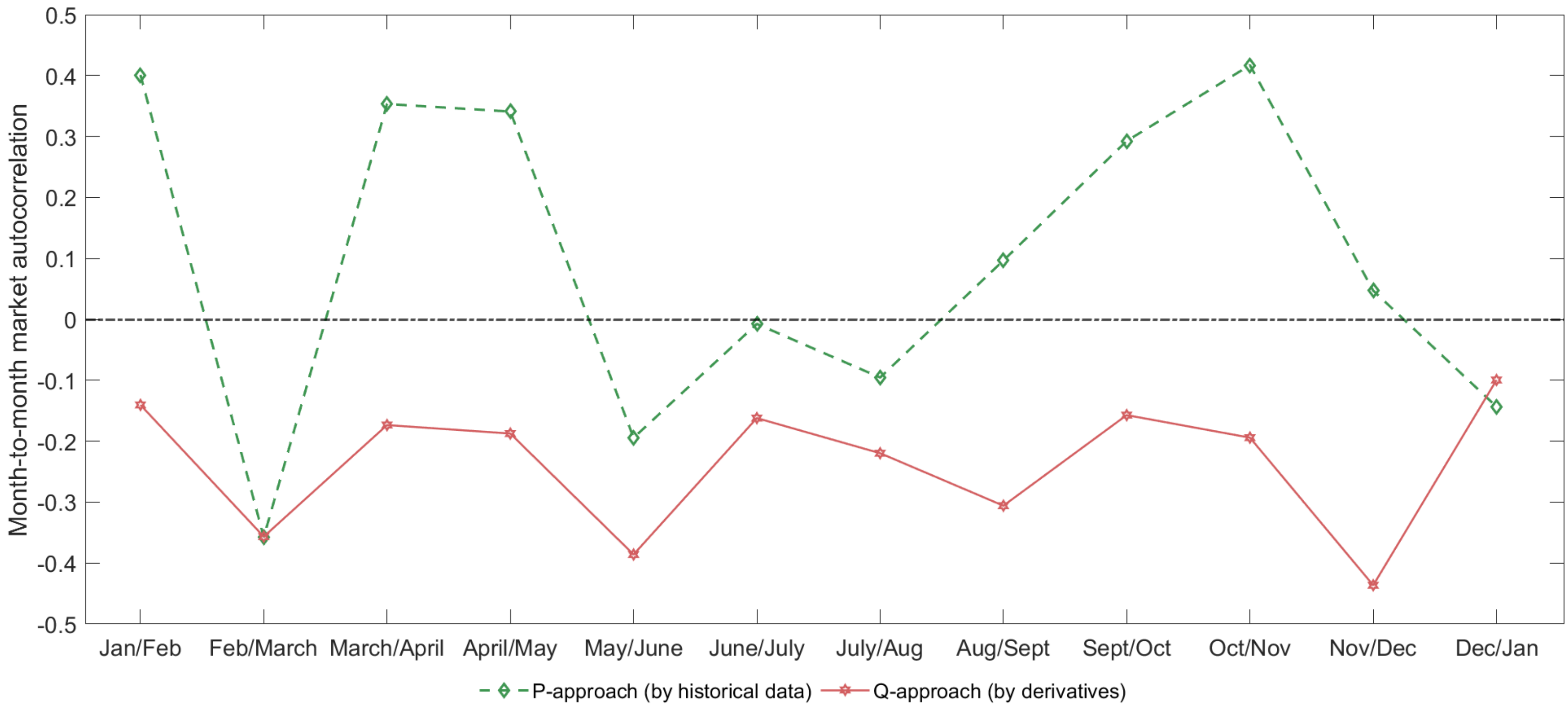


Figure 6: Month-to-month market autocorrelation by Q- and P-approaches

This figure plots the month-to-month autocorrelation on S&P 500 index between two consecutive months. By P-approach, we compute the sample autocorrelation using historical monthly return data; by Q-approach, we compute $corr_t(R_{t\to t+1mo}, R_{t+1mo\to t+2mo})$ by derivative data on the first day of each month, and then take the average within January, February, ..., and December. The sample period is from 2006 to 2019.

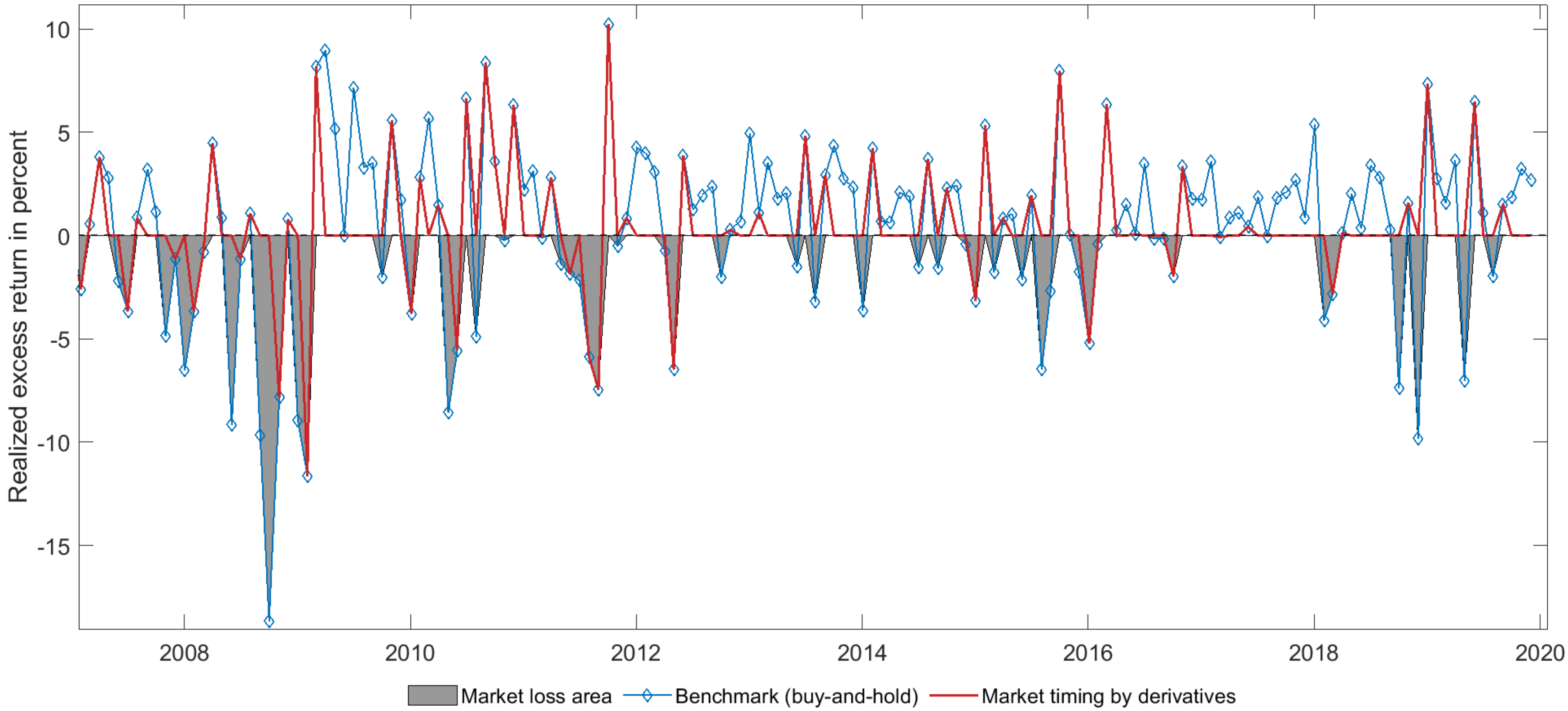


Figure 7: Market timing

This figure plots the realized out-of-sample excess returns generated from either buy-and-hold strategy (benchmark) or market timing strategy over the out-of-sample evaluation period from December 2006 to December 2019. The market timing strategy, $\eta\left[\tilde{S}_{t,1}\right]$ takes a long position in the market when the one-month reversal signal, $\tilde{S}_{t,1}\left[r_{t-1\to t}, corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})\right]$ equals one, and invests in the risk-free asset otherwise. Mathematically,

$$\tilde{S}_{t,1}\left[r_{t-1\to t}, corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})\right] = \begin{cases} 1, & \text{if} \quad r_{t-1\to t} > 0 \quad \& \quad corr_{t-1}(r_{t-1\to t}, r_{t\to t+1}) > 0, \\ 1, & \text{if} \quad r_{t-1\to t} < 0 \quad \& \quad corr_{t-1}(r_{t-1\to t}, r_{t\to t+1}) < 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $r_{t-1\to t}$ denote the cumulative excess returns over the past 1 month, and $corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})$ is the conditional autocorrelation by Q-approach computed at time $t-1$.

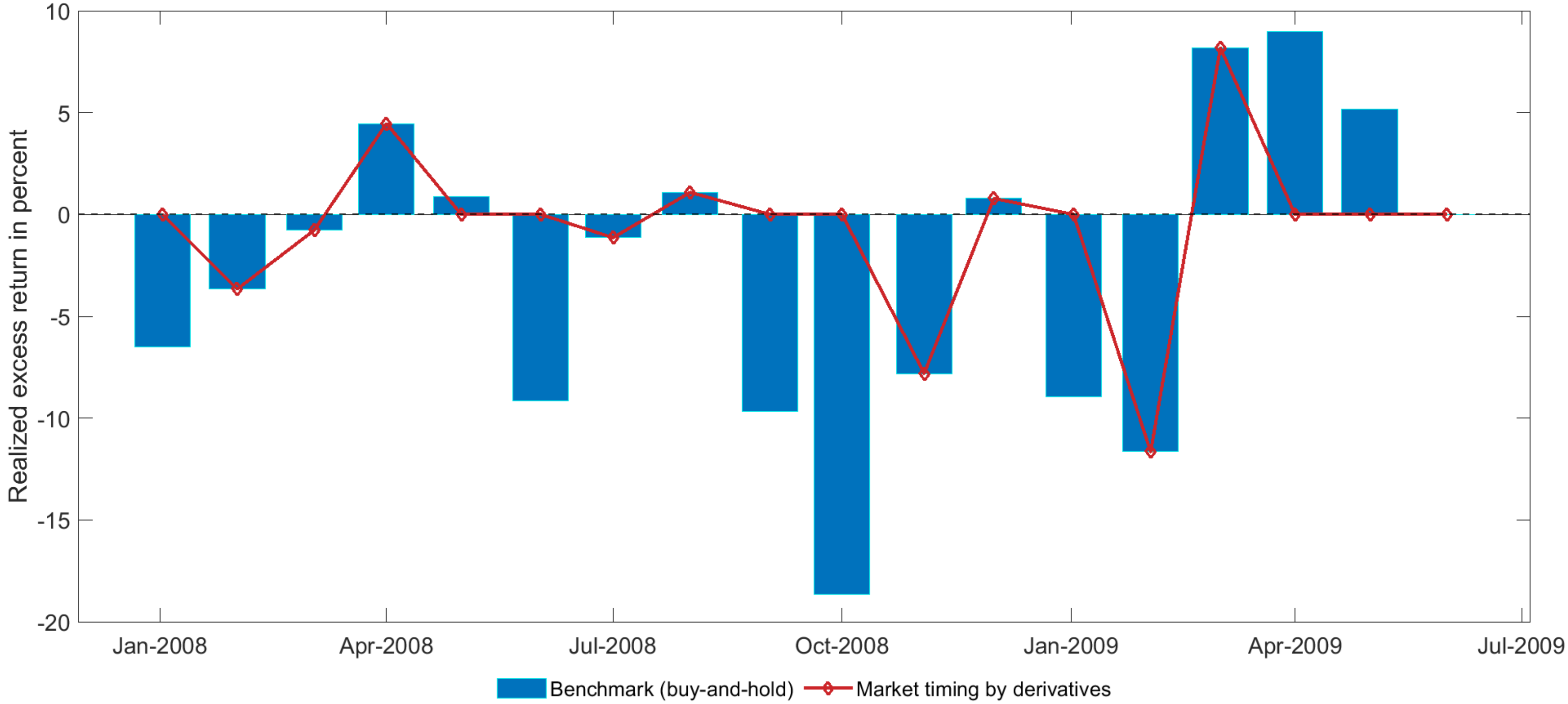


Figure 8: Market timing during NBER recessions

This figure plots the realized out-of-sample excess returns generated from either buy-and-hold strategy (benchmark) or the market timing strategy over the NBER recessions from January 2008 to June 2009. The market timing strategy, $\eta\left[\tilde{S}_{t,1}\right]$ takes a long position in the market when the one-month reversal signal, $\tilde{S}_{t,1}\left[r_{t-1\to t}, corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})\right]$ equals one, and invests in the risk-free asset otherwise. Mathematically,

$$\tilde{S}_{t,1}\left[r_{t-1\to t}, corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})\right] = \begin{cases} 1, & \text{if} \quad r_{t-1\to t} > 0 \quad \& \quad corr_{t-1}(r_{t-1\to t}, r_{t\to t+1}) > 0, \\ 1, & \text{if} \quad r_{t-1\to t} < 0 \quad \& \quad corr_{t-1}(r_{t-1\to t}, r_{t\to t+1}) < 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $r_{t-1\to t}$ denote the cumulative excess returns over the past 1 month, and $corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})$ is the conditional autocorrelation by Q-approach computed at time $t-1$.

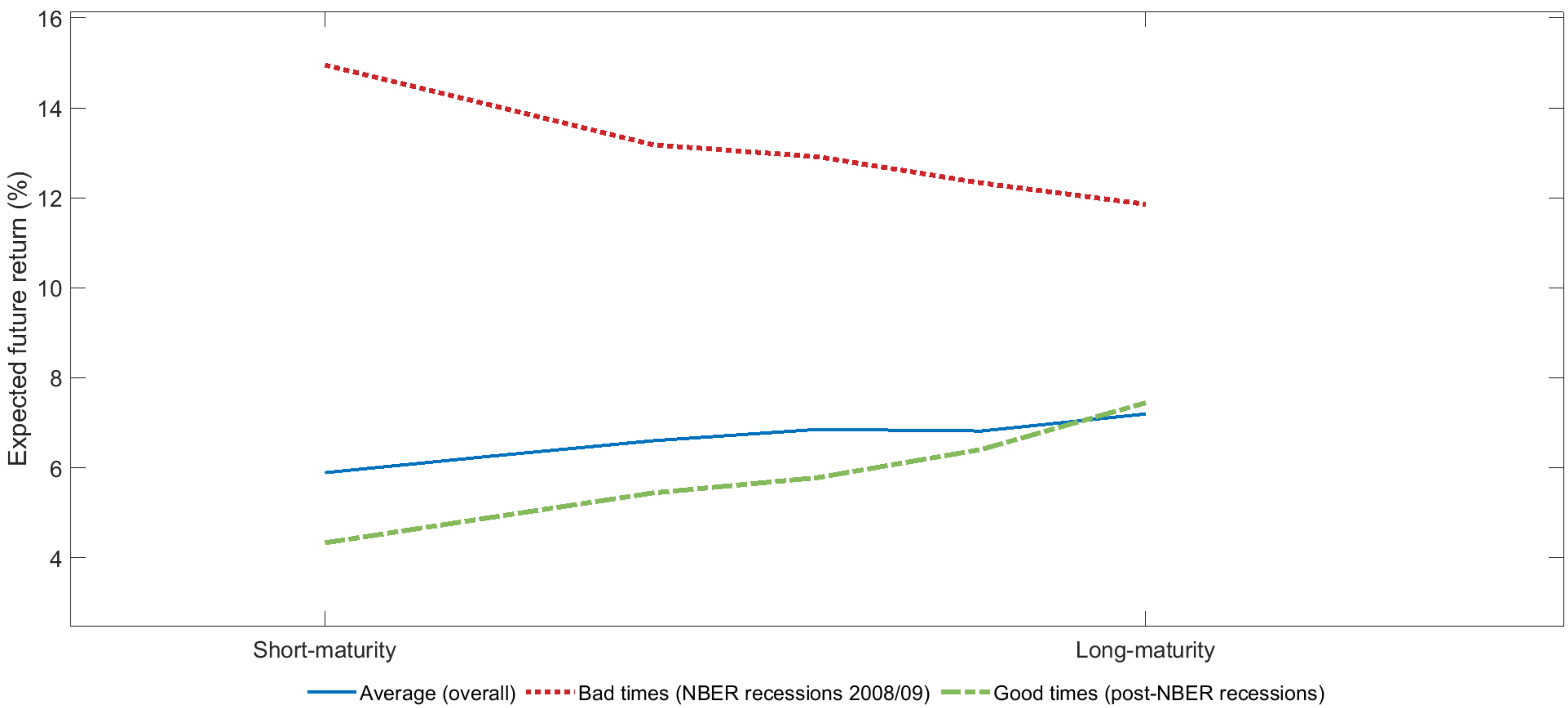


Figure 9: The term structure of expected future one-month return

This figure plots the term structure of the expected future one-month returns by Q-approach. The figure shows the unconditional average return (solid line), the average return in bad times from January 2008 to June 2009 during the NBER recessions (dashed line), and the average return in good times during the post NBER recessions (dash-dotted line).

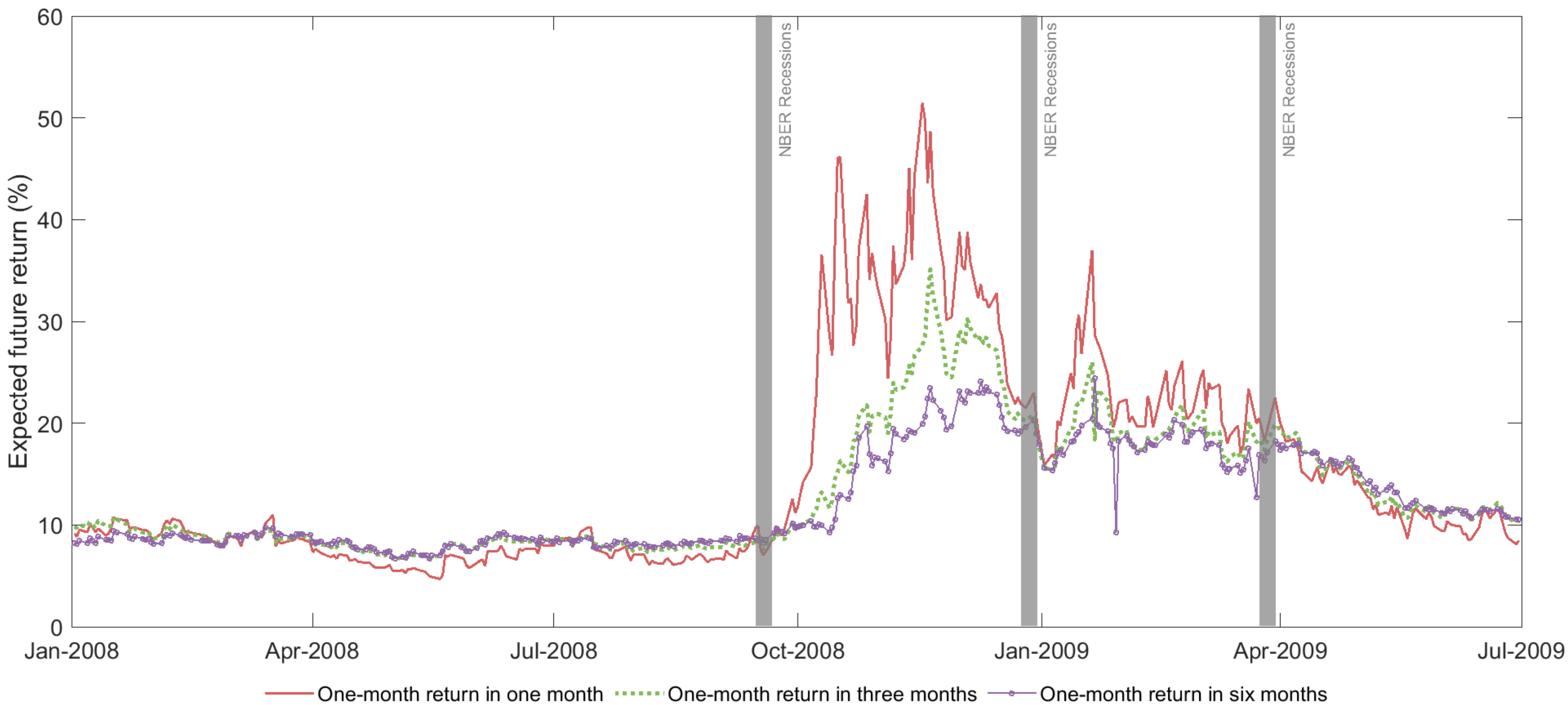


Figure 10: Expected future one-month returns by Q-approach during the NBER recessions

This figure plots the expected one-month returns in one, three, and six months by Q-approach during the NBER recessions from January 1, 2008 to June 30, 2009. All results are annualized and expressed in percentage.

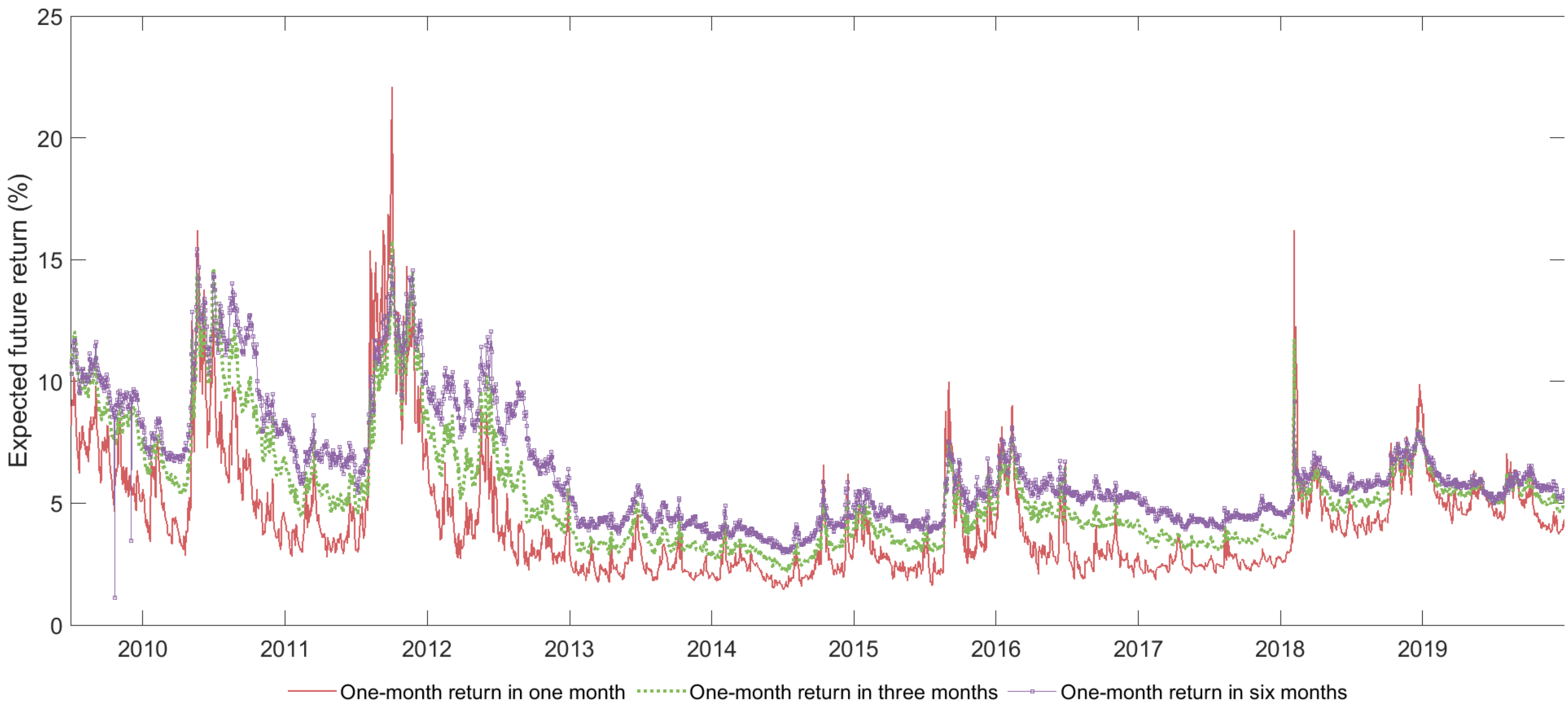


Figure 11: Expected one-month future returns by Q-approach post the NBER recessions

This figure plots the expected one-month returns in one, three, and six months by Q-approach during the *post* NBER recession period from July 1, 2009 to December 31, 2019. All results are annualized and expressed in percentage.

Table 1: PVIX index under a log-utility-based SDF

This table provide the summary statistics of power VIX index (PVIX) under a log-utility-based SDF (when $\gamma = 1$), including mean, median, standard deviation, skewness and kurtosis. Panel A uses the sample from January 4, 1996, the first available date in OptionMetrics; Panel B corresponds to the sample period utilized in the empirical analysis, from February 24, 2006 to December 31, 2019.

	Mean	Std dev	p25	p50	p75	Skew	Kurt
Panel A: Sample period: January 4, 1996 - December 31, 2019							
T (in months)							
1	12.244	0.052	12.198	12.232	12.284	1.289	7.999
2	6.161	0.048	6.118	6.151	6.201	0.720	3.714
3	4.134	0.046	4.092	4.125	4.172	0.560	2.624
4	3.120	0.046	3.078	3.110	3.158	0.488	2.235
6	2.107	0.044	2.067	2.097	2.143	0.471	2.035
9	1.429	0.042	1.395	1.419	1.454	0.708	2.414
Panel B: Sample period: February 24, 2006 - December 31, 2019							
T (in months)							
1	12.223	0.050	12.189	12.206	12.244	3.010	19.512
2	6.140	0.042	6.109	6.126	6.161	1.986	10.417
3	4.113	0.039	4.083	4.099	4.133	1.548	6.784
4	3.098	0.036	3.070	3.085	3.116	1.272	5.049
6	2.084	0.033	2.060	2.073	2.101	0.971	3.260
9	1.406	0.024	1.387	1.400	1.425	0.857	4.199

Table 2: No-arbitrage PVIX futures prices under a log-utility-based SDF

This table provide the summary statistics for prices of futures contract written on 1-month PVIX index. The futures contracts expire in either 1, 2, 3, 4, 6, or 9 months. We report mean, median, standard deviation, skewness and kurtosis. The sample period is from February 24, 2006 to December 31, 2019.

	Mean	Std dev	p25	p50	p75	Skew	Kurt
Panel A: 1-month PVIX							
	12.223	0.050	12.189	12.206	12.244	3.010	19.512
Panel B: PVIX futures price							
Maturity (in months)							
1	12.237	0.056	12.199	12.222	12.262	2.814	15.598
2	12.241	0.048	12.206	12.231	12.268	2.054	9.942
3	12.245	0.043	12.213	12.237	12.272	1.563	7.239
4	12.248	0.040	12.218	12.240	12.275	1.270	5.509
6	12.243	0.034	12.219	12.237	12.260	1.676	6.932
9	12.244	0.045	12.221	12.243	12.269	0.064	4.042

Table 3: Market autocorrelation by Q-approach

This table reports the statistics of the market autocorrelation of S&P 500 index,

$$corr_t(R_{t\to t+T_1}, R_{t+T_1\to t+T_1+T_2}) = \frac{Cov_t\left[R_{t\to t+T_1}, R_{t+T_1\to t+T_1+T_2}\right]}{\sqrt{Var_t(R_{t\to t+T_1})} \times \sqrt{Var_t(R_{t+T_1\to t+T_1+T_2})}}.$$

Panel A reports the key summary statistics for the real-time time-series of the market autocorrelation by Q-approach. We assess the statistical significance of the mean values with a t-test. Panel B reports the sample autocorrelation using historical data of monthly gross returns. The sample period is from February 24, 2006 to December 31, 2019. *** indicates significance at the 1% level.

T_1	1 month	2 months	3 months	4 months	6 months	9 months
T_2	1 month	1 month	1 month	1 month	1 month	1 month
Panel A: Autocorrelation computed from market prices of derivatives (Q-approach)						
Mean	-0.209***	-0.279***	-0.362***	-0.313***	-0.268***	-0.257***
	(-18.10)	(-24.40)	(-28.66)	(-24.42)	(-18.20)	(-14.73)
p25	-0.290	-0.364	-0.462	-0.383	-0.318	-0.346
p50	-0.196	-0.238	-0.346	-0.270	-0.251	-0.255
p75	-0.087	-0.151	-0.229	-0.194	-0.199	-0.204
Skew	-0.944	-0.645	0.252	-1.364	0.018	1.048
Kurt	5.467	7.696	8.711	5.814	8.116	8.160
Panel B: Autocorrelation computed from historical market returns (P-approach)						
$\hat{\rho}$	0.093	0.044	0.079	0.118	0.035	0.029
	(1.20)	(0.56)	(1.01)	(1.50)	(0.44)	(0.36)

Table 4: Regression beta coefficient by Q-approach

This table reports the time-series regression beta coefficient,

$$Y = \alpha_t + \beta_t X + \varepsilon,$$

where $X = R_{t \to t+T_1}, Y = R_{t+T_1 \to t+T_1+T_2}$. Panel A reports the key summary statistics for the real-time time-series of beta coefficients by Q-approach. We assess the statistical significance of the mean values with a t-test. Panel B reports the OLS estimator, $\hat{\beta}$, using historical data of monthly gross returns on S&P 500 index. The sample period is from February 24, 2006 to December 31, 2019. The t-statistics in the parentheses are based on the Newey-West standard errors. *** indicates significance at the 1% level.

T_1	1 month	2 months	3 months	4 months	6 months	9 months
T_2	1 month	1 month	1 month	1 month	1 month	1 month
Panel A: Beta computed from market prices of derivatives (Q-approach)						
Mean	-0.267***	-0.260***	-0.297***	-0.239***	-0.176***	-0.155***
	(-17.62)	(-21.14)	(-25.80)	(-21.66)	(-17.48)	(-14.68)
p25	-0.374	-0.349	-0.380	-0.292	-0.212	-0.205
p50	-0.248	-0.224	-0.285	-0.204	-0.159	-0.141
p75	-0.114	-0.139	-0.184	-0.140	-0.125	-0.106
Skew	-1.554	3.591	1.137	-1.994	-0.739	-0.416
Kurt	10.672	69.118	24.782	10.403	10.196	8.227
Panel B: Beta computed from historical market returns (P-approach)						
$\hat{\beta}$	0.093	0.030	0.044	0.056	0.013	0.009
	(0.77)	(0.40)	(0.64)	(0.84)	(0.26)	(0.18)

Table 5: Regression intercept term by Q-approach

This table reports the time-series regression intercept term,

$$Y = \alpha_t + \beta_t X + \varepsilon,$$

where $X = R_{t \to t+T_1}, Y = R_{t+T_1 \to t+T_1+T_2}$. Panel A reports the key summary statistics for the real-time time-series of the intercept term by Q-approach. We assess the statistical significance of the mean values with a t-test. Panel B reports the OLS estimator of $\hat{\alpha}$ using historical data of monthly gross returns on S&P 500 index. The sample period is from February 24, 2006 to December 31, 2019. The t-statistics in the parentheses are based on the Newey-West standard errors. *** indicates significance at the 1% level.

T_1	1 month	2 months	3 months	4 months	6 months	9 months
T_2	1 month	1 month	1 month	1 month	1 month	1 month
Panel A: The intercept computed from market prices of derivatives (Q-approach)						
Mean	1.787***	1.761***	1.865***	1.698***	1.511***	1.454***
	(39.90)	(48.02)	(54.91)	(51.62)	(49.93)	(45.65)
p25	1.336	1.407	1.538	1.408	1.362	1.313
p50	1.730	1.656	1.832	1.591	1.460	1.411
p75	2.098	2.023	2.101	1.846	1.619	1.606
Skew	1.578	-4.311	-1.533	2.175	0.593	0.287
Kurt	10.842	84.281	30.568	11.850	10.723	8.813
Panel B: The intercept computed from historical market returns (P-approach)						
$\hat{\alpha}$	0.913***	0.976***	0.961***	0.950***	0.993***	0.997***
	(7.37)	(12.57)	(13.41)	(13.56)	(18.24)	(18.94)

Table 6: Autocovariances of market returns by Q-approach

This table reports the two autocovariance indices on S&P 500 return by Q-approach,

$$Cov_t(R_{t\to t+T_1}, R_{t+T_1\to t+T_1+T_2}) = \mathbb{E}_t[R_{t\to t+T_1+T_2}] - \mathbb{E}_t[R_{t\to t+T_1}]\mathbb{E}_t[R_{t+T_1\to t+T_1+T_2}],$$
$$Cov_t(R_{t\to t+T_1}, R_{t\to t+T_1+T_2}) = \mathbb{E}_t[R_{t\to t+T_1}R_{t\to t+T_1+T_2}] - \mathbb{E}_t[R_{t\to t+T_1}]\mathbb{E}_t[R_{t\to t+T_1+T_2}].$$

We assess the statistical significance of the mean values with a t-test. The sample period is from February 24, 2006 to December 31, 2019. The t-statistics in the parentheses are based on the Newey-West standard errors. *** indicates significance at the 1% level.

T_1	1 month	2 months	3 months	4 months	6 months	9 months
T_2	1 month	1 month	1 month	1 month	1 month	1 month
Panel A: $Cov_t(R_{t\to t+T_1}, R_{t+T_1\to t+T_1+T_2})$ in percent						
Mean	-1.282***	-1.285***	-1.551***	-1.480***	-1.087***	-1.108***
	(-4.97)	(-8.84)	(-11.27)	(-7.14)	(-8.87)	(-8.95)
p25	-1.044	-1.418	-1.809	-1.444	-1.174	-1.490
p50	-0.434	-0.613	-0.912	-0.754	-0.714	-0.751
p75	-0.168	-0.353	-0.543	-0.473	-0.475	-0.546
Skew	-9.166	-4.073	-2.868	-9.757	-1.363	0.266
Kurt	109.710	49.240	25.255	126.188	25.580	26.747
Panel B: $Cov_t(R_{t\to t+T_1}, R_{t\to t+T_1+T_2})$ in percent						
Mean	3.093***	3.190***	3.252***	3.039***	2.732***	2.690***
	(8.12)	(12.41)	(14.24)	(12.87)	(17.57)	(19.45)
p25	1.053	1.431	1.612	1.568	1.604	1.646
p50	1.651	2.011	2.267	2.064	2.094	2.049
p75	3.033	3.473	3.725	3.268	2.986	3.122
Skew	6.300	4.808	3.735	5.777	3.921	3.378
Kurt	55.254	36.209	21.994	55.223	25.994	19.605

Table 7: Market timing

This table reports the economic value of timing the previous cumulative market excess return and conditional market autocorrelation by Q-approach. We consider six reversal signals, $\tilde{S}_{t,K}$, for $K = 1$, 2, 3, 4, 6, and 9 months such that,

$$\tilde{S}_{t,K}\left[r_{t-K\to t}, corr_{t-k}(r_{t-K\to t}, r_{t\to t+1})\right] = \begin{cases} 1, & \text{if} \quad r_{t-K\to t} > 0 \quad \& \quad corr_{t-k}(r_{t-K\to t}, r_{t\to t+1}) > 0, \\ 1, & \text{if} \quad r_{t-K\to t} < 0 \quad \& \quad corr_{t-k}(r_{t-K\to t}, r_{t\to t+1}) < 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $r_{t-K\to t}$ denote the cumulative excess returns over the past K months, and $corr_{t-k}(r_{t-K\to t}, r_{t\to t+1})$ is the conditional autocorrelation by Q-approach computed at time $t-K$.

The single timing strategy, $\eta\left[\tilde{S}_{t,1}\right]$ takes a long position in the market when the one-month reversal signal, $\tilde{S}_{t,1}\left[r_{t-1\to t}, corr_{t-1}(r_{t-1\to t}, r_{t\to t+1})\right]$ equals one, and invests in the risk-free asset otherwise. The combination timing strategy, $\eta\left[\tilde{S}_{t,K}, \forall K; \xi\right]$ utilizes all six reversal signals, and takes a long position in the market only if at least ξ out of six reversal signals take values of ones. We consider ξ to be 1, 2, 3, 4, and 5.

Panel A and B consider two different out-of-sample periods: 1) sample period: February 24, 2006 $-$ December 31, 2019; 2) NBER recession period: January 1, 2008 $-$ June 30, 2009, respectively. All results are annualized and are based on excess returns. The average value, standard deviation, and return-loss are expressed in percentage.

	Avg ex-Ret (%)	Std dev (%)	SRatio	CEQ	SRatio Diff	CEQ Diff	Ret-Loss (%)
Panel A: Sample period: December 2006$-$December 2019							
Buy and hold	5.489	14.789	0.371	0.044			
$\eta\left[\tilde{S}_{t,1}\right]$	4.433	9.616	0.461	0.040	0.090	-0.004	-0.863
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 2\right]$	2.389	11.175	0.214	0.018	-0.157	-0.026	1.759
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 3\right]$	1.997	9.986	0.200	0.015	-0.171	-0.029	1.709
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 4\right]$	1.733	9.343	0.185	0.013	-0.186	-0.031	1.735
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 5\right]$	3.347	7.166	0.467	0.031	0.096	-0.013	-0.687
Panel B: NBER recessions: January 2008$-$June 2009							
Buy and hold	-32.304	25.565	-1.264	-0.356			
$\eta\left[\tilde{S}_{t,1}\right]$	-7.027	14.420	-0.487	-0.081	0.776	0.275	-11.194
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 2\right]$	-23.918	20.746	-1.153	-0.261	0.111	0.095	-2.297
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 3\right]$	-11.842	18.780	-0.631	-0.136	0.633	0.220	-11.888
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 4\right]$	-11.842	18.780	-0.631	-0.136	0.633	0.220	-11.888
$\eta\left[\tilde{S}_{t,K}, \forall K; \xi = 5\right]$	-0.456	11.728	-0.039	-0.011	1.225	0.344	-14.364

Table 8: Conditional expected future one-month return by Q-approach

This table provide the summary statistics for the expected future one-month return by Q-approach. The maturities are 1, 2, 3, 4, 6, or 9 months. Spot return is when maturity equals zero. We report mean, median, standard deviation, skewness and kurtosis. Panel A, B, and C consider three different sample periods: (1) full sample: February 24, 2006−December 31, 2019; (2) Bad times (NBER recessions): January 1, 2008−June 30, 2009; and (3) Good times (post-NBER recessions): July 1, 2009−December 31, 2019. All results are annualized and expressed in percentage.

	Avg. Ret (%)	Std dev (%)	p25	p50	p75	Skew	Kurt
Panel A: Sample period: February 24, 2006−December 31, 2019							
Maturity (in months)							
0	4.526	4.451	1.958	3.310	5.707	4.479	33.029
1	5.891	5.202	2.813	4.557	6.887	3.810	22.653
2	6.254	4.419	3.386	5.160	7.392	3.077	16.245
3	6.602	3.802	3.993	5.680	7.845	2.530	12.536
4	6.853	3.553	4.350	5.951	8.102	2.155	9.718
6	6.818	3.371	4.386	5.780	8.204	1.895	7.380
9	7.197	3.891	5.093	6.412	9.116	0.837	4.739
Panel B: Bad times (NBER recessions) January 1, 2008−June 30, 2009							
Maturity (in months)							
0	11.771	9.047	5.595	8.137	14.670	1.979	7.184
1	14.952	10.142	7.527	9.864	20.475	1.359	4.103
2	14.071	7.860	7.959	10.163	19.375	1.131	3.365
3	13.178	6.205	8.286	10.262	18.021	1.075	3.290
4	12.915	5.440	8.528	10.385	17.486	0.868	2.660
6	12.329	4.801	8.435	9.718	17.007	0.712	2.086
9	11.859	4.436	8.549	9.702	16.067	0.997	4.787
Panel C: Good times (post-NBER recessions) July 1, 2009−December 31, 2019							
Maturity (in months)							
0	3.138	2.009	1.781	2.515	3.888	2.464	12.361
1	4.332	2.477	2.569	3.683	5.233	2.056	8.809
2	4.891	2.399	3.066	4.357	5.753	1.574	5.815
3	5.442	2.356	3.582	4.968	6.331	1.330	4.548
4	5.781	2.360	3.939	5.294	6.714	1.289	4.395
6	6.409	2.451	4.499	5.724	7.366	1.174	3.754
9	7.446	2.931	5.360	6.439	9.011	1.023	3.683