A Multi-Factor Model of Idiosyncratic Volatility*

Thijs van der Heijden[†], Qi Zeng[‡], Yichao Zhu[§]

June 13, 2018

Abstract

We propose a multi-factor model to explain the co-movement in the cross-section of idiosyncratic equity return volatilities (IVOL), motivated by a standard dynamic capital structure model in which the CAPM holds for asset returns. Empirically we show that three factors capture the IVOL co-movement: average IVOL (CIV) to account for time-series variation generated by time-varying average leverage, and leverage (LIV), and size (SIV) to explain the cross-section. The negative return-IVOL relation is mainly from the common component of IVOL, not the residual.

JEL Classification: G12, G32, C15

Keywords: Idiosyncratic Volatility, Volatility Co-movement, Risk Factors, Leverage, Capital Structure

*This paper originates from a chapter in Zhu's PhD thesis at The University of Melbourne. We thank Rob Brown, Bruce Grundy, Garry Twite, two anonymous PhD examiners and seminar participants at the Australian National University, Tel Aviv University, and the University of Melbourne for helpful comments.

[†]University of Melbourne; thijsv@unimelb.edu.au.

[‡]University of Melbourne; qzeng@unimelb.edu.au.

[§]Australia National University; Yichao.Zhu@anu.edu.au.

1 Introduction

Leverage plays an important role in explaining the cross-section of stock returns (Gomes and Schmid, 2010; Garlappi and Yan, 2011; Livdan, Sapriza, and Zhang, 2009).[1] It has also been shown to be related to the volatility of the stock returns (Merton, 1974; Choi and Richardson, 2016). We show that firm leverage is an important determinant of the co-movement in idiosyncratic stock return volatilities (IVOL) (Connor, Korajczyk, and Linton, 2006; Herskovic, Kelly, Lustig, and Van Nieuwerburgh, 2016; Duarte, Kamara, Siegel, and Sun, 2014).[2] Connor et al. (2006) find strong common dynamics in a panel of idiosyncratic volatilities, and suggest three potential explanations: "Something about the random technology generating firm-specific cash flows, or the dynamic flow of information about these firm-specific cash flows, or investor's changing reaction to firm-specific news, must underlie this common component in the volatility of asset-specific returns." We provide evidence for an alternative explanation for the presence of a common factor in idiosyncratic volatilities, namely common fluctuations in financial leverage.

Our simple explanation for the common movement in the cross-section of IVOL is motivated by capital structure dynamics. We demonstrate that leverage and firm size determine the firms' IVOL in a cross-section of firms whose capital structure adjusts imperfectly or infrequently to shocks to firms' asset values, which is a standard feature of many dynamic capital structure models with adjustment costs, e.g., Goldstein, Ju, and Leland (2001), and Strebulaev (2007). In these models, firms only update their capital structure once the leverage hits either an upper or a lower boundary. In between refinancing epochs, the leverage ratio varies directly with changes in the asset values as the nominal debt issued is fixed. To see how this set up leads to common movement in the IVOL cross-section, suppose first that unlevered firms' asset values follow a standard CAPM model with constant idiosyncratic volatility. Returns on the firms' equity in this model equal the return on assets multiplied by the leverage ratio, and so the volatility of equity returns will vary directly with leverage. The CAPM structure of asset returns implies that leverage ratios tend to co-move, which in turn leads to co-movement in IVOL.

[1] Whenever we refer to leverage, we mean financial leverage, measured as market leverage unless specifically indicated otherwise.

[2] In this paper, we follow the literature and refer to the volatility of residual stock returns in an asset pricing model as "idiosyncratic volatility". We stress that the notion "idiosyncratic" refers to the returns themselves not being priced, but that does not rule out the volatility of those idiosyncratic returns displaying co-movement that is potentially priced.

Firm size plays a role in our model because of the default option embedded in levered equity, the value of which is not homogeneous in firm value because it follows a power function.[3] Equity return volatility within a given refinancing cycle is determined by a time-varying scaling factor which depends on the default option value. Under the dynamic debt restructuring setting, firms with the same leverage can have different sizes because they have completed a different number of refinancing cycles. In other words, the weights, and the IVOLs, is a function of both leverage and size.

We operationalize our ideas within the dynamic capital structure framework of Goldstein et al. (2001). This model and Fischer, Heinkel, and Zechner (1989) are the base models for many of the more recent capital structure studies.[4] The model of Goldstein et al. (2001) has the advantage that it is homogeneous in firm value. In the model, a firm periodically levers up when its asset value hits a threshold if it does not default before that. Its analytical simplicity enables us to pinpoint the effects of the size and leverage both analytically and through simulation. We show that (1) both the leverage and size affect the firm's IVOL even under the assumption of constant asset volatility, (2) the previously documented CIV factor Herskovic et al. (2016) captures mainly the dynamics of the market average leverage level, and (3) the CAPM beta and IVOL exhibit an inverse relation because they are both affected by the same underlying force: leverage.

Motivated by the results of our theoretical model, we propose a three-factor model of equity IVOL, augmenting the CIV factor with a leverage (LIV) and size (SIV) factor. We construct the leverage factor (LIV) as the difference in IVOL between a portfolio of stocks with high leverage and a portfolio of stocks with low leverage, and the size factor (SIV) as the difference between the IVOL of the portfolio of small market cap stocks and the portfolio of large market cap stocks. We argue that LIV and SIV capture the dynamics of the cross-sectional dispersion in common IVOL, and show that the explanatory power of the CIV factor in the cross-section stems from its correlation with the LIV and SIV factors.

To test our prediction, we sort stocks into 25 leverage and size portfolios.[5] The three-factor model indeed captures a large proportion of the time series variation of IVOL. At portfolio level,

[3]Note that the model *is* homogenous in firm value from one financing cycle to the next which facilitates solving the dynamic model.

[4]For example, see Strebulaev (2007), Strebulaev and Yang (2013), among others.

[5]As a robustness check, we also consider IVOL portfolios and individual firms as test assets.

most of the adjusted R-squared values are above 75 percent. At the firm level, the average adjusted R-squared is 21 percent compared to just 15 percent using only the CIV factor.

More importantly, the loadings of CIV factor do not show any clear pattern across the different portfolios, while the loadings on the LIV and SIV factors exhibit strong monotonic patterns across portfolios. This result is not dissimilar to the way the three Fama-French factors explain equity returns: the HML and SMB factors do a much better job explaining the cross-section of returns than the market factor.

To further illustrate the relation between CIV and leverage, we first show that CIV itself is strongly correlated with LIV and SIV. Then we provide evidence that LIV and SIV have additional explanatory power above and beyond CIV, by regressing portfolio IVOL on CIV and then regressing the residuals of that regression on LIV and SIV. We find that the loadings of residuals on LIV and SIV are still significant. The converse is not true. Furthermore, we can substitute the CIV with the IVOL of pure leverage-sorted portfolio only and the results still holds: CIV has no explanation power for the residuals.

To see the application of our three-factor model, we return to the negative IVOL-return relation first studied by Ang, Hodrick, Xing, and Zhang (2006). We show that this negative relation is largely driven by the common component of the IVOL movement. Indeed, if we sort the stock according to IVOL predicted by the three-factor model, the negative relation still holds. But if we sort stocks according to the residuals of the regression of IVOL on the three factors, there is no obvious pattern. Duarte et al. (2014) finds similar results by taking out the component components in IVOL using Principal Component Analysis (PCA). What we show here is to pinpoint the components through our three-factor model. This suggest an important condition for any theoretical model that tries to explain the negative IVOL-return relation: it has to also explain why it is the co-movement component of the IVOL, not the residual component. And the common component is determined by firm's size and leverage. To our knowledge, none of the current theories explore this front.

Our theoretical model is not the first to embed the firm level dynamic capital structure model in a CAPM setting. Strebulaev and Yang (2013) already uses this setup in their simulation study of the firm leverage. However, their focus is a realistic quantitative study of firm capital structure choices and credit spread. Our focus is on the equity returns. The simpler and more basic setup of Goldstein et al. (2001) that we use here shows that our message is general. More surprisingly, using baseline parameters used in Goldstein et al. (2001), our simulations replicate many quantitative results reported in Herskovic et al. (2016).

We contribute to the literature by providing a theoretical argument linking leverage and size to the co-movement in firms' IVOL. Empirically we show a three-factor model that capture a large proportion of such IVOL co-movement. Our application of a three-factor model to the IVOL-return relation implies that studies related to IVOL need to account for this dynamic leverage-driven effect. Our work also provides theoretical and empirical support for recent work arguing that asset pricing anomalies should be tested using unlevered returns as in Choi and Richardson (2016) and Doshi, Jacobs, Kumar, and Rabinovitch (2016).

The remaining part of the paper is organized as follows. We first motivate, both empirically and theoretically, how leverage drives IVOL in Section 2. Then we present the theoretical model in Section 3. The results of the model motivate the usage of size and leverage as the factors, which we empirically construct and test in Section 4. Section 5 applies the three factor to the study of IVOL-return relation and Section 6 concludes. All the proofs are in the appendix.

2 IVOL, Leverage and Betas In the Cross-Section

We start by presenting some preliminary evidence, both empirical and theoretical, linking the co-movement of IVOL documented in Herskovic et al. (2016) to firms' leverage. In Table 1, we use all firms listed on NYSE/NASDAQ/AMEX during the period 1963–2015.[6] Every month we compute the return IVOL relative to the CAPM and sort stocks into IVOL deciles. We estimate CAPM beta using daily stock returns over the past 12 months. And we estimate the firm's IVOL loading on CIV by an unconditional regression of firm IVOL on CIV at the monthly level over the full sample period. The CIV factor is calculated as the equal-weighted cross-sectional average IVOL.

[Table 1]

[6] As usual the data are from CRSP and Compustat and we focus on common stock (security code 10 and 11) and exclude financial firms. We further get rid of penny stocks by requiring the stock prices to be at least $1.

As one can see from the table, the average loadings of the individual IVOLs on the CIV factor increases monotonically from 0.22 at the lowest IVOL decile to 3.23 at the highest IVOL decile. This partly supports the claim made in Herskovic et al. (2016) that the CIV factor captures the some cross-sectional variation in IVOL: higher IVOL is associated with higher loadings of the portfolio IVOL on the CIV factor. To motivate that leverage drives firm-level IVOL and the CIV factor loading, we also calculate the average market leverage of each decile. One can see that it also monotonically increases with IVOL from 0.23 at the lowest IVOL decile to 0.33 at the highest IVOL decile. Interestingly, the CAPM beta of portfolio returns also monotonically increases with IVOL: from 0.88 at the lowest IVOL decile to 1.43 at the highest IVOL decile. This monotonic increase in beta with IVOL is a standard result in a dynamic capital structure model; CAPM beta is driven by the same force that drives the individual firms' IVOL: (financial) leverage.

Theoretically, the intuition behind the effect of leverage on the firms' IVOL is the following. Suppose the unlevered equity (asset) return between t to $t+1$, $r^A_{i,t+1}$, of firm i follows a CAPM,

$$r^A_{i,t+1} = r_f + \beta^A_i R^A_{m,t+1} + \epsilon^A_{i,t+1}, \tag{1}$$

where r_f, $R^A_{m,t+1}$ are the risk-free rate and unlevered market return respectively, β^A is the unlevered equity (asset) beta, and $\epsilon_{i,t+1}$ is the residual risk of the firm's asset returns that are uncorrelated across firms and the market.

If the firm has debt level $D_{i,t}$ at time t, one can show that the levered equity returns follow a CAPM-like relation,

$$r^E_{i,t+1} = r_f + \frac{A_{i,t}}{A_{i,t} - D_{i,t}} \beta^A_i R^A_{m,t+1} + \frac{A_{i,t}}{A_{i,t} - D_{i,t}} \epsilon^A_{i,t+1}, \tag{2}$$

where $A_{i,t}$ is the firm's asset value. As long as the debt value is not just a constant proportion of the firm's asset value, the (idiosyncratic) levered equity returns are functions of the firm's leverage. Since asset returns and values are a function of the market factor, leverage will be as well and therefore the volatility of idiosyncratic equity returns (IVOL) will have a common factor structure. In the next section, we build a dynamic model to formalize this relation, and show that both size and leverage affect both IVOL and the CAPM return beta.

3 A Three-Factor IVOL Model of Equity Returns

The IVOL model builds on the single-firm framework of Goldstein et al. (2001) and embeds its setup in a single-factor economy. The Goldstein et al. (2001) model incorporates both the optimal leverage choices of a firm and dynamic updating of the capital structure. The driving force for taking on debt is the tax shield, and the optimal leverage follows by trading off the tax shield against the likelihood and cost of a default, as in many other dynamic capital structure models (see for example, Strebulaev, Whited, et al. (2012) and references therein.). When the firm asset value is high enough, the firm will lever up to capture more tax shield benefits. Since the firm level analysis is very similar to that in Goldstein et al. (2001), we relegate detailed derivations and proofs to Appendix A. Like Goldstein et al. (2001), we first consider a static variant of the model in which firms issue a consol bond only once. The setting allows us to focus on the intuition behind our results in a tractable setting, before moving on to the fully dynamic set up.

3.1 Model Setup: One-Factor APT for Asset Return Processes

Consider a one-factor model for firm assets/cash flows. We assume that there are N firms, indexed by $i = 1, \ldots, N$, whose cum-dividend value processes follow

$$\frac{dV_i(t) + \delta_i(t)dt}{V_i(t)} = (\mu_i + \delta_i)dt + \sigma_i(\rho_i dW(t) + \sqrt{1-\rho_i^2}dZ_i(t)), \qquad i = 1, \ldots, N, \tag{3}$$

where $\delta_i(t) = \delta_i V_i(t)$ is the payout-value ratio, $\mu_i,\ \delta_i,\ \sigma_i,\ \rho_i$ are all positive constants, and $W(t),\ Z_i(t)$ are standard Brownian motions that are uncorrelated with each other, for all $i = 1, \ldots, N$. There exists a risk-free bond with constant risk-free rate $r > 0$. The parameter ρ_i governs the relative importance of systematic and idiosyncratic shocks, while σ_i determines the magnitude of the total asset volatility.

We assume that the APT holds and that the cross-section is sufficiently large for the Law of Large Numbers to apply. In that case, the market return is only a function of the common shock $dW(t)$ and is the single pricing factor for asset returns. Define the value of the market portfolio of the assets, $V_m(t)$, the aggregate dividend, $\delta_m(t)$, and the portfolio weights, $w_i^A(t)$

as

$$V_m(t) \equiv \sum_i V_i(t), \quad \delta_m(t) \equiv \sum_i \delta_i(t), \quad \omega_i^A(t) \equiv \frac{V_i(t)}{V_m(t)},$$

then the cum-dividend market return of assets is given by

$$\frac{dV_m(t) + \delta_m(t)dt}{V_m(t)} = r_m^A(t)dt + \sigma_m^A(t)dW(t),$$

where

$$r_m^A(t) \equiv \sum_i \omega_i^A(t)(\mu_i + \delta_i), \quad \sigma_m^A(t) \equiv \sum_i \omega_i^A(t)\sigma_i\rho_i.$$

To make sure that the expected market return equals the risk-free rate under the risk-neutral measure defined below, we define the (time-varying) market price of risk as

$$\theta_m^A(t) \equiv \frac{r_m^A(t) - r}{\sigma_m^A(t)}. \tag{4}$$

Thus one can define a risk-neutral probability measure generated by the Brownian motion

$$d\tilde{W}(t) \equiv dW(t) + \theta_m^A(t)dt, \tag{5}$$

so that the market return is of the form

$$\frac{dV_m(t) + \delta_m(t)dt}{V_m(t)} = rdt + \sigma_m^A(t)d\tilde{W}(t). \tag{6}$$

For firm i, we define the price of the idiosyncratic risk Z_i as θ_i^A. The Brownian motion under the risk-neutral measure for individual firm i can then be written as

$$d\tilde{Z}_i(t) \equiv dZ_i(t) + \theta_i^A dt. \tag{7}$$

To prevent arbitrage, the expected return of each individual firm should equal r under this risk-neutral measure, or, equivalently,

$$\mu_i + \delta_i = r + \sigma_i \rho_i \theta_m^A + \sigma_i \sqrt{1-\rho_i^2}\theta_i^A. \tag{8}$$

As pointed out in Cochrane (2005), individual θ_i^As can be anything for any finite set of securities. In other words, idiosyncratic risk can be priced under a pure APT model. As a simplifying assumption and to tie our hands as much as possible, we will assume that the single-factor APT holds exactly at the firm level, i.e.,

$$\theta_i^A = 0 \quad \forall i. \tag{9}$$

Hence, idiosyncratic risk is not priced at the asset level, but we will show later that idiosyncratic equity *volatility* can still be priced, driven by common movements in financial leverage. This setup is not new in the literature (see for example, Strebulaev (2007)). However, to the best of our knowledge, we are the first to use this setup to study the IVOL in equity returns. Now, define the asset return beta as

$$\beta_i^A \equiv \frac{\mathrm{Cov}\left(\frac{dV_i(t)+\delta_i(t)dt}{V_i(t)}, \frac{dV_m(t)+\delta_m(t)dt}{V_m(t)}\right)}{(\sigma_m^A)^2} = \frac{\sigma_i\rho_i}{\sigma_m^A}, \tag{10}$$

then asset returns follow a CAPM where

$$\mu_i + \delta_i = r + \beta_i^A (r_m^A - r). \tag{11}$$

3.2 Optimal Static Capital Structure for A Single Firm

Now we allow each firm to be able to borrow through a consol bond with coupon C per period. We start by letting the firm decide the optimal capital structure only once in the beginning, and will study the dynamic capital structure later. Denote the tax rates for interest payment, dividend and corporate profits as τ_i, τ_d, and τ_c, respectively. So the effective rate for equity, τ_{eff} is given by

$$1 - \tau_{\text{eff}} = (1-\tau_c)(1-\tau_d). \tag{12}$$

When the firm defaults at asset value V, the deadweight default cost is proportional to firm value V; αV, where $\alpha > 0$. The restructuring cost is also proportional to firm value V; qV, where $q > 0$. As shown in Goldstein et al. (2001) and Appendix A.1, the optimal capital structure (coupon C^*) is given by (we drop the firm subscript i here for ease of exposition)

$$C^* = \frac{rV_0}{\lambda}\left[\left(\frac{1}{1+x}\right)\left(\frac{A}{A+B}\right)\right]^{\frac{1}{x}}, \tag{13}$$

where

$$\begin{aligned}
x &= \frac{1}{\sigma^2}\left[\left(\mu - \frac{\sigma^2}{2}\right) + \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}\right] > 0,\\
\lambda &= \frac{x}{1+x},\\
A &= (1-q)(1-\tau_i) - (1-\tau_{\text{eff}}),\\
B &= \lambda(1-\tau_{\text{eff}})\left(1-(1-q)(1-\alpha)\right).
\end{aligned}$$

For a given coupon C, the equity value at any point after the debt issuance and before any default is given by

$$E(C, V; V_B) = E_{solv} = (1-\tau_{\text{eff}})\left[V - V_B\left(\frac{V}{V_B}\right)^{-x} - \frac{C}{r}\left(1 - \left(\frac{V}{V_B}\right)^{-x}\right)\right], \tag{14}$$

where

$$V_B = \frac{x}{1+x}\frac{C}{r} \equiv \lambda\frac{C}{r}. \tag{15}$$

3.3 Equity Returns Under Static Capital Structure

Using the value for V_B from (14), we can rewrite the equity values in terms of the optimal coupon C^* from (13)

$$E(V, C^*) = (1-\tau_{\text{eff}})\left[V + G\left(\frac{V}{V_0}\right)^{-x} - \frac{C^*}{r}\right], \tag{16}$$

where

$$G = \frac{V_0}{x}\left[\left(\frac{1}{1+x}\right)\left(\frac{A}{A+B}\right)\right]^{1+\frac{1}{x}}.$$

The three terms between brackets in (16) have clear economic meaning. The first term is the total value of the assets. The last term reflects the value of a risk-free bond with coupon rate C^*. The second term contains the equity holder's default option value. In the following we use the above result to examine the properties of IVOL.

3.3.1 A Simplified Version

Consider a simplified version of (16), focusing on the first and the third term, and ignoring the default option value. This is a formal version of the motivational example we discussed in Section 2. In this case, we show analytically that leverage drives both the IVOL of the firms' equity returns as well as their CAPM beta. We will highlight the three testable implications from our framework: 1) the presence of a *CIV* factor, 2) a positive correlation between β and IVOL, and 3) a positive correlation between leverage and IVOL. Denote this equity value as E^s and define $D^s \equiv C/r$ as the face value of the consol bond (we use the superscript "s" to distinguish the variables in this section from their counterparts in the full model to be studied later),

$$E^s(V) \equiv (1 - \tau_{\text{eff}})(V - C/r) \equiv (1 - \tau_{\text{eff}})(V - D^s). \tag{17}$$

The after-tax instantaneous dividend payoff to the equity holders, denoted as $\Delta(t)$, equals $(1 - \tau_{\text{eff}})(\delta(t) - C)dt$. So the infinitesimal change in equity equals

$$(1 - \tau_{\text{eff}})(dV + (\delta(t) - C)dt) = (1 - \tau_{\text{eff}})\left([V(\mu + \delta) - D^s r]\, dt + V\sigma dz\right),$$

where (we recover the subscript i here)

$$dz_i(t) \equiv \rho_i dW(t) + \sqrt{1 - \rho_i^2} dZ_i(t). \tag{18}$$

The equity return process is given by

$$\frac{dE_i^s(t) + \Delta_i(t)dt}{E_i^s(t)} \equiv r_i^s dt + \sigma_i^s \left(\rho_i dW(t) + \sqrt{1 - \rho_i^2} dZ_i(t)\right)$$

with

$$r_i^s = \frac{V_i(t)}{V_i(t) - D_i^s}(\mu_i + \delta_i) - \frac{D_i^s}{V_i(t) - D_i^s} r,$$
$$\sigma_i^s = \frac{V_i(t)}{V_i(t) - D_i^s}\sigma_i.$$

There are two sources of shocks to the firm-level equity returns; a systematic one associated with $W(t)$ and an idiosyncratic one associated with $Z_i(t)$. The IVOL of the equity return is therefore given by

$$IVOL_i^s(t) = \sigma_i^s \sqrt{1-\rho_i^2} = \frac{V_i(t)}{V_i(t) - D_i^s}\sigma_i \sqrt{1-\rho_i^2}.$$

From here we can see the potential appearance of a common factor in IVOL because of the leverage term $\frac{V_i(t)}{V_i(t)-D_i^s}$. Recall that $V_i(t)$ contains the history of common shocks $W(t)$. D_i^s is fixed under the static capital structure setting we use in this section, and will be unchanged during one refinancing cycle in the dynamic version we study in Section 3.4. We can contrast this with the situation that D_i^s is proportional to $V_i(t)$ for all t. In that setting, the coefficient of the idiosyncratic risk becomes a constant, which would result in a constant idiosyncratic volatility. The key insight of our model is that as long as the leverage varies over time and is at least in part driven by shocks common to all firms in the cross-section, the idiosyncratic volatilities of the *equity* returns exhibit common variation even if the idiosyncratic *asset* volatilities do not. In this simple set up, IVOL is a positive monotonic transformation of leverage. We show below that this is no longer always the case when we also take the value of the default option into account.

It is important to note that the common variation does not stem from omitted risk factors. All asset returns are driven by a single common factor which is correctly accounted for when computing idiosyncratic return volatilities. The idiosyncratic shocks to firm asset values are completely independent across firms. The common movement in IVOL shows up as a multiplicative factor for the firm-level asset idiosyncratic volatility. Thus, additive econometric techniques such as principal component analysis (PCA) to adjust returns for risk will not be able to uncover completely "additional factors" that capture this common variation.

We can calculate directly the common factor defined in the empirical literature in this simple case. If all firms have the same parameters, then the common idiosyncratic volatility factor CIV as defined in Herskovic et al. (2016) equals a constant times the average leverage in the economy,

$$CIV = \frac{1}{N}\sum_i \sigma_i^s \sqrt{1-\rho_i^2} = \sigma\sqrt{1-\rho^2}\frac{1}{N}\sum_i \frac{V_i(t)}{V_i(t)-D_i^s}.$$

Since ρ can be thought of as a scaled market beta, it is natural to assume it is positive. In that case, the following holds.

Proposition 1. *If all firms have identical parameters and $\rho_i > 0\ \forall i$, then the covariance between firm-level IVOL and the* CIV *factor is positive for all firms and is larger for more highly levered firms.*

We can show that the CAPM still holds for equity returns in this simple case. Furthermore, IVOL is closely related to the equity beta. Define the market portfolio of equity as

$$E_m^s(t) \equiv \sum_i E_i^s(t) = (1-\tau_{\text{eff}})\sum_i (V_i(t) - D_i^s),$$

specify the relative weight of firm i as

$$w_i^s(t) \equiv \frac{E_i^s(t)}{E_m^s(t)} = \frac{V_i(t)-D_i^s}{\sum_i (V_i(t)-D_i^s)},$$

and define the market payout ratio as

$$\Delta_m(t) \equiv (1-\tau_{\text{eff}})\sum_i (\delta_i(t) - D_i^s r).$$

Then the market portfolio of equity evolves as

$$\frac{dE_m^s(t) + \Delta_m(t)dt}{E_m^s(t)} = r_m^s(t)dt + \sigma_m^s(t)dW(t),$$

where

$$
\begin{aligned}
r_m^s(t) &\equiv \sum_i w_i^s \left(\frac{V_i(t)}{V_i(t) - D_i^s}(\mu_i + \delta_i) - \frac{D_i^s}{V_i(t) - D_i^s} r \right), \\
&= r + \frac{V_m(t)}{E_m^s(t)}(r_m^A - r), \\
\sigma_m^s(t) &\equiv \sum_i w_i^s \frac{V_i(t)}{V_i(t) - D_i^s} \sigma_i \rho_i, \\
&= \frac{V_m(t)}{E_m^s(t)} \sigma_m^A.
\end{aligned}
$$

The covariance between individual equity returns and the market equity returns is given by

$$
\mathrm{Cov}\left(\frac{dE_i^s(t) + \Delta_i(t)dt}{E_i^s(t)}, \frac{dE_m^s(t) + \Delta_m(t)dt}{E_m^s(t)} \right) = \frac{V_i(t)}{V_i(t) - D_i^s} \sigma_i \rho_i \sigma_m^s(t),
$$

Then the individual firm's equity beta is given by

$$
\begin{aligned}
\beta_i^s(t) &= \frac{\mathrm{Cov}\left(\frac{dE_i^s(t)+\Delta_i(t)dt}{E_i^s(t)}, \frac{dE_m^s(t)+\Delta_m(t)dt}{E_m^s(t)} \right)}{(\sigma_m^s(t))^2} \\
&= \frac{V_i(t)}{V_i(t) - D_i^s} \frac{\sigma_i \rho_i}{\sigma_m^s(t)}
\end{aligned}
$$

As a result, the expected stock return for firm i is

$$
r_i^s(t) = r + \beta_i^s(t)(r_m^s(t) - r).
$$

Note the appearance of $\frac{V_i(t)}{V_i(t)-D_i^s}$ in both IVOL and the equity beta of the firm. The following cross-sectional result is immediate.

Proposition 2. *If all firms have identical parameters, then the cross-sectional correlation between IVOL and the equity beta is positive.*

The economic intuition for Proposition 2 is as follows. If the economy does poorly (well), then all firms will tend to become more (less) levered. Hence, if the economy does poorly (well), then IVOL will tend to increase (decrease) for all firms, and equity betas of relatively more-levered firms will tend to increase (decrease), while the equity betas of relatively less-levered firms will tend to decrease (increase). These effects are stronger the more the firm's leverage differs from the market-average leverage.

3.3.2 Full Version

We now study the full equity value in (16), including the default option value given by the nonlinear term V^{-x}. The same results obtain as in the previous section and the basic intuition about the common variation in the IVOL still holds, with leverage that linking IVOL and CAPM beta. In addition, the size of the firm also comes into play in this setting.

Note that the value process $V_i(t)$ and the constants G_i and C_i are all linear functions of the initial firm value $V_i(0)$. So to simplify the notation, in the following we scale them by the initial firm value $V_i(0)$. In other words, the equity value is given by

$$E_i(V_i, C_i^*, V_i(0)) = (1-\tau_{\text{eff}})V_i(0)(V_i(t) + G_iV_i(t)^{-x_i} - D_i^s),$$

where D_i^s is defined as above. The equity return process is then given by

$$\frac{dE_i(t) + \Delta_i(t)dt}{E_i(t)} \equiv r_i^E(t)dt + \sigma_i^E(t)(\rho_i dW(t) + \sqrt{1-\rho_i^2}dZ_i(t)),$$

where

$$\Delta_i(t) = \delta_i V_i(t) - C_i,$$

and the expected return and volatility of equity are given by

$$\begin{aligned} r_i^E(t) &= r + \frac{V_i(t)}{V_i(t) + G_iV_i(t)^{-x_i} - D_i^s}(\mu_i + \delta_i(t) - r) + \\ &\quad \frac{G_iV_i(t)^{-x_i}}{V_i(t) + G_iV_i(t)^{-x_i} - D_i^s}\left(-x_i\mu_i + \frac{1}{2}x_i(x_i+1)\sigma_i^2 - r\right), \\ \sigma_i^E &= \frac{V_i(t)(1 - x_iG_iV_i(t)^{-x_i-1})}{V_i(t) + G_iV_i(t)^{-x_i} - D_i^s}\sigma_i. \end{aligned}$$

In this setup, IVOL equals

$$IVOL_i^E(t) = \sigma_i^E\sqrt{1-\rho_i^2} = \frac{1 - x_iG_iV_i(t)^{-x_i-1}}{1 + G_iV_i(t)^{-x_i-1} - D_i^s/V_i(t)}\sigma_i\sqrt{1-\rho_i^2}. \tag{19}$$

In contrast to the simple version, IVOL is no longer a positive monotonic transformation of leverage. When leverage is sufficiently low, a lower leverage gives rise to a higher IVOL. This can be seen from (19) as follows. Holding D/V constant, a higher asset value V implies a lower V^{-x-1}, yielding a higher IVOL. If a firm has the same leverage ratio D/V but with different size V, then the relation between IVOL and leverage is not monotonic anymore. And this is the case under the dynamic capital structure.

Just as in the simple version, when all firms have identical parameters the common idiosyncratic volatility factor CIV equals a constant times the average leverage in the economy,

$$CIV(t) = \frac{1}{N}\sum_i \sigma_i^E(t)\sqrt{1-\rho_i^2} = \sigma\sqrt{1-\rho^2}\frac{1}{N}\sum_i \frac{1 - xGV_i(t)^{-x-1}}{1 + GV_i(t)^{-x-1} - D_i^s/V_i(t)}.$$

In this case, the constants x and G will be identical across firms as they only depend on the parameters. As before, a sufficient condition for the covariance between any firm's IVOL (leverage) and the CIV factor to be positive is $\rho_i > 0\ \forall i$. To see the relation between IVOL and equity beta, we define the market portfolio of equity, the relative market cap of firm i, and the market payout ratio as

$$\begin{aligned}
E_m(t) &\equiv \sum_i E_i(t) = (1-\tau_{\text{eff}})\sum_i V_i(0)(V_i(t) - D_i^s + G_i V_i(t)^{-x_i}), \\
w_i^E(t) &\equiv \frac{E_i(t)}{E_m(t)} = \frac{V_i(0)(V_i(t) - D_i^s + G_i V_i(t)^{-x_i})}{\sum_i V_i(0)(V_i(t) - D_i^s + G_i V_i(t)^{-x_i})}, \\
\Delta_m(t) &\equiv (1-\tau_{\text{eff}})\sum_i (\delta_i(t) - D_i^s r).
\end{aligned}$$

The market return evolves according to

$$\frac{dE_m + \Delta_m dt}{E_m} = r_m^E dt + \sigma_m^E dW(t),$$

where

$$\begin{aligned}
r_m^E &\equiv \sum_i \omega_i^E r_i^E = r + \frac{V_m}{E_m}(r_m^A - r) + \frac{V_m}{E_m}(-H_{m1} + H_{m2} - rH_{m3}), \\
\sigma_m^E &\equiv \sum_i \omega_i^E \sigma_i^E \rho_i = \frac{V_m}{E_m}(\sigma_m^A - \sigma_{m4}),
\end{aligned}$$

with

$$
\begin{aligned}
H_{m1} &\equiv \sum_i \omega_i^A G_i V_i^{-x_i-1} x_i \mu_i, \\
H_{m2} &\equiv \sum_i \omega_i^A G_i V_i^{-x_i-1} \left[\frac{1}{2} x_i (x_i+1) \sigma_i^2\right], \\
H_{m3} &\equiv \sum_i \omega_i^A G_i V_i^{-x_i-1}, \\
\sigma_{m4} &\equiv \sum_i \omega_i^A G_i V_i(t)^{-x_i-1} x_i \rho_i \sigma_i.
\end{aligned}
$$

As we can see, the default option value affects both the market expected returns as well as the market volatility. The covariance between firm-level equity returns and market equity returns is given by

$$
\text{Cov}\left(\frac{dE_i(t)+\Delta_i(t)dt}{E_i(t)}, \frac{dE_m(t)+\delta_m(t)dt}{E_m(t)}\right) = \frac{1-x_i G_i V_i^{-x_i-1}}{1+G_i V_i^{-x_i-1} - D_i^s/V_i} \sigma_i \rho_i \sigma_m^E.
$$

The equity beta of firm i is again a product of the asset beta, the average leverage in the market and the leverage of the firm,

$$
\begin{aligned}
\beta_i^E &= \frac{\text{Cov}\left(\frac{dE_i(t)+\Delta_i(t)dt}{E_i(t)}, \frac{dE_m(t)+\Delta_m(t)dt}{E_m(t)}\right)}{(\sigma_m^E)^2} \\
&= \frac{1-x_i G_i V_i^{-x_i-1}}{1+G_i V_i^{-x_i-1} - D_i^s/V_i} \frac{\sigma_m^A}{\sigma_m^E} \beta_i^A.
\end{aligned}
$$

The relation between IVOL and the equity beta derived in the simple model still holds.

Proposition 3. *If all the firms have identical parameters, then the cross-sectional correlation between IVOL and the equity beta is positive.*

3.4 Dynamic Capital Structure and Simulation Results

The results from the previous section allow us to study the common IVOL factor, the beta-IVOL relation, and the leverage-IVOL relation. However, the capital structure in the static model is not stationary which makes inference tricky. Specifically, without a second refinancing boundary where the firm issues additional debt to increase leverage after firm value has increased sufficiently, the average firm will become less and less leveraged over time as long as expected asset returns are positive. Introducing a second boundary V_U gets around this problem. The result is a stationary process for a firm: the return process for the firm will be identical during the intervals between the firm hits any two successive V_U before hitting V_B. Again we follow the analysis in Goldstein et al. (2001), and relegate details of the derivation to Appendix A.2.

Define $e(V_0)$ as the present value of all dividend claims, γ as the ratio of the firm value at two consecutive restructuring epochs, $p_U(V)$ as the present value of the contingent claim that pays \$1 when V hits V_U before hitting V_B, $D^0(V_0)$ as the current debt value, and $E(V_{0-})$ as the total equity value at the moment the initial debt is issued. After the initial issuance of the debt, the equity value at any time during the first period is given by

$$E(V) = e^0(V) + \gamma p_U(V)E(V_{0-}) - p_U(V)D^0(V_0). \tag{20}$$

The last term $(p_U(V)D^0(V_0))$ is the debt value that the firm would need to pay in expectation to the current debt holder to call back the debt.

The final step is to find the optimal capital structure, consisting of three values: the coupon C^*, the refinancing boundary V_U, and the bankruptcy boundary V_B, or, equivalently, C^*, $\gamma = V_U/V_0$, $\psi = V_B/V_0$. First we find the the optimal V_B or ψ as a function of C and V_U (or γ), and then solve for C and γ. We illustrate the idea of upward-refinancing and the typical path of a firm's asset value in Figure 1.

[Insert Figure 1 here]

3.4.1 Simulation Setup

We conduct extensive simulation studies to analyze the relation between leverage and IVOL in detail. The purpose of this exercise is twofold. First, we want to see quantitatively a standard model can replicate some of the salient properties of the IVOLs observed from the data. Secondly and more importantly, we want to find out whether there are any additional factors that can affect empirically observed IVOLs.

To exhibit the explanation power of the model, we use the parameters proposed in the original paper of Goldstein et al. (2001), which we reproduce in Table 2.[7] We do not attempt to calibrate cross-sectional variation in asset volatilities or leverage ratios here, and study a set of homogeneous firms. Firm heterogeneity is likely to give rise to more sophisticated relations between IVOL, betas and returns. For example, if asset volatilities vary across firms then it is much easier to let IVOL be a non-monotonic transformation of leverage. Our aim here is to show that a number of documented stylized facts can be generated even when imposing the additional restriction of homogeneous firms.

[Insert Table 2 here]

The initial payout ratio is $\delta/V_0 = 0.035 + 0.65C/V_0$, so the drift of the payout flow rate μ equals $r - \delta/V_0 = 0.01 - 0.65(C/V_0)$.[8] We simulate a cross-section of 5,000 firms over a period of 50 years. We consider one time step as one day, one month as 21 days, and one year as 252 days. For each firm, the initial asset value is \$100 and $\rho = 0.5$. This value of ρ implies that the proportion of the total asset variance contributed by the common shock Z equals 25%. Equity returns are computed as simple period-by-period returns. Table 2 provides an overview of the parameter values.

3.4.2 Summary Statistics

To reduce the influence of a single history on our overall conclusions, we run repeated simulations with the same initial setups 100 times and report the distributions of outcomes. Table 3 reports the summary statistics of the simulation results.

[Insert Table 3 here]

The initial values in the first column reproduce the numbers reported in Table 4 of Goldstein et al. (2001). For example, γ is 1.7, suggesting that the firm will wait until its value rises to

[7] We set the bankruptcy cost α at 5% as in Goldstein et al. (2001). We note that the bankruptcy cost used in Leland (1994) is 50%. Given the other parameter values, such a high bankruptcy cost yields $C^* = 0$ for all firms, suggesting that the bankruptcy cost is so high that the tax shield benefit is insufficiently large to make up for it.

[8] At any given point in time in our simulations, prices are computed under the risk-neutral measure in which any traded asset has an expected return equal to the risk-free rate. The dynamics of the firm asset value process are generated from the objective distribution, imposing a risk premium on the process of the common asset return factor W. This instantaneous drift of $\mathrm{d}V/V$ equals $r_f - (\delta/V_0) + \theta\rho\sigma$ where θ is the Sharpe ratio of the common shock $\mathrm{d}W$. We assume $\theta = 0.2$ in our simulations.

1.7 times its initial value and only then is it optimal to increase the firm's leverage. The initial leverage is 0.37.

The across-runs mean of the average asset value is \$347.13, and the 5^{th} and the 95^{th} percentile values are \$139.83 and \$871.46, respectively. The coupon payment grows from an initial value of \$1.85 to \$7.53 on average, given the refinancing scaling factor γ is 1.7. The average monthly return on equity is 1.06%. Whenever a firm goes bankrupt ($V_t \leq V_B$), we introduce a new firm with the same initial values into the sample in the next month, so the sample size is very stable.[9]

In Figure 2, we plot the time series of the main variables in the repeated simulations. The cross-sectional average debt value increases gradually because of the upward refinancing strategy. The equity value also increases over time. Leverage grows for approximately the first 10 years of a simulation run. After that, the distribution of leverage becomes stable at around 48%. In this paper, we report our main results using the samples with full simulation period. In unreported results, we repeat our tests using the sample which excludes the first 10 years and retains the remaining 40 years' data, and the main conclusions reached in our paper are unchanged.

[Insert Figure 2 here]

3.4.3 Estimating IVOL

We follow the approach in Herskovic et al. (2016) to estimate IVOL as the annualized variance of daily residual returns from an asset pricing model.[10] The asset pricing models we examine to determine residual returns are the CAPM and a five-factor principal component model (PCA), where we re-estimate the principal components each calendar year.[11] In both models, we exclude observations with equity value below \$1 or with a daily equity return greater than 300% to avoid extreme IVOL estimates. We then construct firm-year estimates of IVOL over 50 years. We exclude stocks with fewer than 100 trading days in a year. We also winsorize the top 0.5% of

[9] The average number of stocks reported in the tables is slightly smaller than the starting number of 5000. This gap is due to the sample filters imposed during the estimation of IVOL.

[10] Many existing studies (e.g., Ang et al., 2006) compute the IVOL as the standard deviation of the residual returns. We follow the approach in Herskovic et al. (2016) and use the variance. We find either measure of IVOL does not change the results in our paper.

[11] We include only the stocks with full observations (252 days) within a year to ensure a balanced panel in the principal component analysis.

IVOL estimates in each year.

Consistent with Herskovic et al. (2016), the estimates from both models are very close. In Table 4, we show that the full panel correlations between the annual IVOL estimated from the CAPM and PCA are highly correlated (99% on average across the simulation runs), and the main summary statistics such as means and standard deviations are also very close. Anticipating the results below where we show the strong co-movement in the cross-section of IVOL reaffirms our point that the co-movement is not the result of a missing risk factor, which would have been picked up by the PCA factors. In the following sections of the paper, we use the CAPM IVOL, estimated yearly from daily returns. In the Internet Appendix we reproduce the analyses using IVOL calculated from a model using principal components as factors, as well as using IVOL calculated monthly from daily returns.

[Insert Table 4 here]

3.4.4 Average IVOL In the Time-Series and Pairwise Correlations

In our first analysis, we visually show the strong co-movement in IVOL by plotting the time-series of average IVOL of stock portfolios. In each simulation run, we sort the stocks into quintiles by different firm characteristics, such as market capitalization and financial leverage; then we compute the equally-weighted average IVOL for each quintile. In Figure 2, we plot the time-series of average IVOL for the portfolios sorted by size (Figure 2(a)) and by leverage (Figure 2(b)) for a representative simulation run. The average IVOL of all size quintiles shows a clear co-movement pattern. The average IVOL of leverage-sorted portfolios shows a very similar co-movement pattern, except for the lowest leverage quintile whose average IVOL appears to be flat over the full sample period.

[Insert Figure 2 here]

Next, we investigate the average pairwise correlations between the average IVOL of the portfolios sorted by size or leverage. In Table 5, we report the results based on repeated simulation runs. For each simulation run, we sort firms by size (or by leverage) into quintiles and compute the average IVOL for each quintile in each year, then we compute the pairwise correlations of the average IVOL between the quintiles. Next, we compute the mean, 5^{th} and 95^{th} percentile values of the correlations for each quintile pair across the simulation runs. For example, the number in Column 4 and Row 1 in Panel A1 (size) of Table 5 shows that the mean correlation of the average IVOL between the largest and second-largest size quintile is 0.93; the numbers in brackets report the 5^{th} and 95^{th} percentile values of the correlation, 0.84 and 0.98, respectively.

[Insert Table 5 here]

In Panel A (size), the average pairwise correlations among average IVOL of the five size quintiles range from 0.81 to 0.96, suggesting that the co-movement in IVOL is very strong. In Panel B (leverage), the average pairwise correlations are also very high among the leverage quintiles 2 to 5. However, the pairwise correlation between the lowest leverage quintile with the other quintiles is close to zero on average and is insignificantly different from zero. This finding suggests that for the lowest leverage quintile, the average IVOL is essentially constant. This result supports our hypothesis that time-varying financial leverage drives the co-movement in IVOL.

In Equation (19), we show how IVOL is determined by the firm's financial leverage at the beginning of the period, assuming a constant asset return volatility. Similarly, the equally-weighted average IVOL of a portfolio formed in this section reflects the average leverage of stocks in a portfolio. Therefore the co-movement in average IVOL is a result of time-varying market-wide leverage, resulting from common shocks to firm value.

3.4.5 Explaining IVOL Using CIV

In our second test, we follow Herskovic et al. (2016), and run firm-by-firm time-series regressions of a firm's IVOL on the common factor in IVOL (CIV), measured as the equally-weighted IVOL in each cross-section. Hence CIV directly reflects the market equally-weighted average leverage in our setup.[12] In each simulation run, we run the time-series regression on each firm and compute the R^2. For each simulation run, we compute the average R^2 and report the mean, 5^{th} and 95^{th} percentiles across the simulation runs in Table 6.

[12]Note that in our illustration model, e.g., in (19), IVOL is the product of leverage and asset return volatility. If the asset return volatility is constant and identical across firms, then CIV is the equally-weighted average leverage across all firms multiplied by the constant asset return volatility.

[Insert Table 6 here]

In Table 6, Panel A row *Levered*, we show that when firms are levered, the common factor in IVOL explains 25% of the time variation in individual IVOL on average, and the 5^{th} and 95^{th} percentile values of average R^2 in repeated simulations are 18% and 31% respectively.[13] This result suggests that a substantial proportion of the time-series variation in individual IVOL can be explained by a single common factor.

To examine the relation between the common factor structure in IVOL and financial leverage, we redo this analysis for an unlevered sample. In the unlevered sample, we use the same initial parameter values and the same dynamics of V, but we force C and V_B equal to zero for all stocks throughout the simulation horizon, so the equity value E of a firm equals its asset value V.[14]

As shown in Table 6, when we use the unlevered sample the average R^2 reduces substantially to only 2%, suggesting that the time variation in individual IVOL is unable to be explained by a single factor, and also suggesting there is no common factor structure in IVOL when the firms are unlevered.[15]

3.5 The Relation Between IVOL and Leverage

In the simple model, IVOL is a monotonic transformation of leverage. As shown in Section 3.3.2, this is no longer true when the default option value is taken into account. In the simulations, we confirm this as follows. First, each month we sort stocks into percentiles by end-of-month leverage ($D/(D+E)$). For each of the 100 leverage-sorted portfolios, we compute the average IVOL of all stocks in the portfolio over the next month, and then compute the time-series average. We do this for each of the 100 simulation runs and report the average as well as the 5^{th} and 95^{th} percentiles across the simulation runs, for each portfolio, in Figure 3.

[13] The average R^2 from our simulation data is smaller than the figure of 35% reported in Herskovic et al. (2016) using empirical data. The magnitude of R^2 depends on the initial parameter values used in our simulation. In general, higher leverage would lead to a higher average R^2. Our simulation exercise does not attempt to match the empirical results in Herskovic et al. (2016).

[14] Firms do not achieve an optimal capital structure in this setup.

[15] The average R^2 is theoretically identical to zero for the unlevered sample as long as we consider spot IVOL. The non-zero average R^2 obtains because we estimate IVOL annually and also keep the β constant during that period. Using monthly betas and IVOL reduces the R^2 to zero (up to two decimal places).

In the case of dynamic capital structure, firms increase their debt level infrequently. Between any two debt restructuring epochs, a firm starts with high leverage ratio D/V after increasing the debt level. Subsequently, leverage decreases in expectation as firm value tends to increase until the next debt restructuring. Thus a given leverage (or D/V) includes firms with different nominal debt levels and firm values. The larger the asset value, the lower V^{-x-1} will be, and the higher IVOL will be. The combining effect of firm size V and leverage (or D/V) will result in a U-shaped relation between the cross-sectional average IVOL and leverage: at any moment in time, at the higher end of D/V ratio distribution, the effect of the leverage dominates, and the at the lower end of D/V ratio distribution, the effect of the size dominates. Both result in higher average IVOL across firms under different resetting time.

4 Empirical Results

Motivated by the dynamic capital structure model and simulation results, we propose a three-factor model to capture the cross-sectional IVOL variation with the following three factors: CIV, leverage and size.[16] We show that the CIV factor mainly captures the market wide average leverage movements while the size and leverage factors capture the cross-sectional differences in the common IVOL.

4.1 Data

The full sample includes all common stocks (SHRCD 10,11) listed on NYSE, AMEX and NASDAQ stock exchanges (EXCHCD 1,2,3), from December 1961 to December 2015. We obtain the stock return data from CRSP and obtain the financial data from COMPUSTAT. The Fama-French factors are from Kenneth French's website. To construct our sample, we apply a few filters: (1) we exclude stocks with price below \$5 before 2001 and below \$1 since 2001 to eliminate the potential minimum tick size effect when estimating IVOL; (2) following Strebulaev and Yang (2013), we exclude financial firms (SIC codes 6000-6999), and utility firms (SIC codes 4900-4999); we also exclude firms that with total book value of assets (COMPUSTAT item AT)

[16]There is a potential fourth factor: The debt restructuring frequency. As we have shown in the model, the nonlinear effect between leverage and IVOL is from the fact that different firms are at different leverage restructuring stages. However, the proxies for debt restructuring frequency are very noisy empirically. The few we tried did not show much explanatory power with R-squared values smaller than 1 percent.

of less than \$10 million in inflation-adjusted year 2000 dollars; (3) we require the observations to have valid IVOL, CAPM beta, market leverage and cash-adjusted leverage as defined below.

The annual IVOL is estimated as the standard deviation of the residual returns from Fama-French three-factor regressions using daily returns within a calendar year. The CAPM beta is estimated from CAPM model using daily returns over a calendar year. Following Strebulaev and Yang (2013), we define the market leverage ratio of firm as

$$ML = \frac{DLTT + DLC}{DLTT + DLC + CSHO \times PRCC\_F}, \tag{21}$$

and the cash-adjusted market leverage ratio as

$$MLC = \frac{DLTT + DLC - CHE}{DLTT + DLC + CSHO \times PRCC\_F}, \tag{22}$$

where $DLTT$ and DLC are the amount of long-term debt due in more than one year and the amount of debt in current liabilities due within one year, respectively; $CSHO$ is the fiscal year-end number of shares outstanding, $PRCC\_F$ is the fiscal year end common share price, and CHE are cash holdings and short term investments at the end of the fiscal year. We exclude all observations with missing data components.

We match the annual IVOL and CAPM beta with the leverage ratios from the last fiscal year-end. The final sample includes 2278 firms per year on average. In the sub-sample where we require a non-negative MLC, the sample size is 1460 firms per year on average.[17]

4.2 Factor Construction

The first factor is the CIV factor defined in Herskovic et al. (2016) as the equally-weighted average of monthly IVOL across all firms. Two new factors are constructed from leverage (LIV) and size (SIV) following the Fama and French (1993) methodology. More specifically, at the end of June of each year t, we form three leverage sorted portfolios with 30-40-30% of all stocks, respectively, based on the leverage of last fiscal year end; similarly, we measure the size at the end of June of each year t, and form two size portfolios based on the NYSE breakpoints. The

[17] In unreported results, we also compute ML and MLC using quarterly data and estimate IVOL and CAPM beta quarterly. These changes do not change the main findings.

3×2 leverage-size portfolios are formed as the intersections of the three leverage and the two size portfolios. Equally-weighted monthly IVOL on the portfolios are then computed from July of year t to June of year $t+1$. LIV is the IVOL on the mimicking portfolio for leverage factor in IVOL, is the difference each month between the simple average of the IVOL on the two high leverage portfolios (high leverage and small, high leverage and big) and the simple average of the IVOL on the two low leverage portfolios (low leverage and small, low leverage and big); similarly, SIV is the IVOL on the mimicking portfolio for size factor in IVOL, is the difference each month between the simple average of the IVOL on the three small portfolios (high leverage and small, medium leverage and small, low leverage and small) and the simple average of the IVOL on the three big portfolios (high leverage and big, medium leverage and big, low leverage and big).

Table 7 shows the summary statistics of the three factors. CIV factor has the largest value among the three factors, averaging 43 percent p.a., while SIV at 21 percent, indicating that at least on average small firms have substantially higher IVOL than large firms. Surprisingly the average value of LIV is only 2.6 percent, so on average the IVOL difference between low and high leverage firms is small. This is driven by a substantial fraction of months in which the high leverage portfolio IVOL is below the low leverage portfolio IVOL, with the 25th percentile of the LIV distribution being smaller than zero. We also include the descriptive statistics of IVOL for the low leverage and high leverage portfolios separately in Table 7. The IVOL of high leverage portfolios (LIVH) are mostly higher than those of low leverage portfolios (LIVL). However, there are some instances where the LIVL portfolio IVOL exceeds the LIVH portfolio IVOL (the 99 percentile of LIVL is 82.29% compared with 72.17% of LIVH). We will discuss the properties of the three factors in more details in the next section.

[Insert Table 7 here]

4.3 Playing Field

We use our three IVOL factors to analyse the IVOL for two sets of test assets. The first set is 5-by-5 portfolios formed according to leverage and size. The second is individual IVOL at firm level. The portfolio analysis aims to capture the common movement among the IVOL. It has the advantage to have less noise. We also study the power of the three IVOL factors at individual firm level. We focus on the patterns of the cross-sectional loadings here.

Similar to the method we used in the last section, at the end of June of each year t, we form 5 leverage sorted quintiles across all the firms based on the leverage of last fiscal year end; we also use the size at the end of June of each year t, and form 5 size quintiles based on the NYSE breakpoints. The 5-by-5 leverage-size portfolios are formed as the intersections of the 5 leverage and the 5 size portfolios.

Table 8 shows the summary statistics of the 25 portfolios. One can see that the IVOL is monotonically decreasing with size, with monthly annualized IVOL of 50% for small stocks and slightly over 20% for large stocks. However, the relation between IVOL and leverage is not monotonic overall. IVOL decreases with leverage for large size portfolios, while it increases with leverage for small size portfolios. One potential reason is that there are still some variation in size within each size quintile and they vary with the leverage. For example, within the largest size quintile, size varies by 50 percent (12.91 billion to 18.85 billion average size). The leverage, one the other hand is much more homogeneous within each leverage quintile: the largest variation is from the largest leverage quintile, which goes from 0.56 to 0.63. It would be ideal to increase the number of portfolios. However, that would render some portfolios, in particular for the largest size cells, too small to draw reliable inferences.[18] We revert to additional firm level analyses to show that our results are robust.

[Insert Table 8 here]

4.4 Time Series Regression: Portfolio Analysis

Our goal here is to explain the IVOL cross-section with three factors. Recall that Herskovic et al. (2016) argue that the CIV factor itself accounts for the majority of the cross-sectional variation in the IVOL common factor. Our model suggests that it is the leverage and size factors that account for this. Thus the foremost issue is the relative performances between CIV factor and the LIV/SIV factors. The second prediction from our model is that the loadings on the LIV factor monotonically increase with leverage, while the loadings on the SIV factor monotonically decrease with size. Finally, if our model is correct, the three factors can explain

[18]In the current sample, there are only 15 firms in the largest quintile with the highest leverage.

a large proportion of the time variations of IVOL.

Table 9 shows our main empirical results. In this table, we report the results of the following regression.

$$IVOL_{it} = \alpha_i + \beta_i^L LIV_t + \beta_i^S SIV_t + \beta_i^C CIV_t + \epsilon_{i,t}. \quad (23)$$

We report the regression coefficients and t-stats, as well as the adjusted R-squared.

[Insert Table 9 here]

The loadings on the LIV and SIV factors confirm to model predictions. Conditional on size, the loadings on the LIV factor indeed increases monotonically with leverage. For the smallest size portfolios, the loadings increase from -0.71 for lowest leverage portfolio to 0.63 for highest leverage portfolio. For the largest size portfolios, the loadings increase from -0.48 to 0.40 respectively. Similarly, conditional on leverage, the loadings on the SIV factor decrease monotonically. For the lowest leverage portfolios, the loadings decrease from 0.61 for the smallest size portfolio to -0.64 to the largest size portfolio. Within the highest leverage portfolios, the loadings decrease from 0.70 to -0.56 respectively. In terms of t-stats, most of the loadings are highly significant with the exception of β_{LIV} for the two middle leverage quintile portfolios. Interestingly, we can compare this obvious monotonic pattern of the loadings with the size and leverage themselves in the summary statistics. As observed above, the relation between IVOL and leverage is not quite monotonic.

Loadings on the CIV factor have no obvious patterns and show relatively little cross-sectional variation. In an unreported study when we run the regression of the IVOL of the 25 portfolios on the CIV factor only, resulting in a monotonic relation between loadings and size/leverage. Thus the CIV factor is dominated in the cross-section by the LIV and SIV factors.[19]

Finally, the time-series R-squared is around 70 to 80 percent, and increasing to about 95 percent for the smallest size portfolios. In an unreported study when we use CIV factor only, the R-squared is much lower at 40 to 60 percent on average while the largest R-square is about 90

[19]The patterns here are similar to the way the Fama-French three factors contribute to explaining the cross-section of average returns. In that case, the market return does have the power in explaining the time-variation of the returns if used as the single factor. However, it does not show up in the cross-section when size and book-to-market factors: the market betas are more or less similar across the Fama-French size and book-to-market sorted 25 portfolios.

percent for the smallest size portfolios. In summary, at least at the portfolio level, our three-factor model captures the average IVOL variation very well, and the LIV and SIV capture the dispersion in the IVOL cross-section.

4.5 Time Series Regression: Firm Level Analysis

For robustness check, we repeat our exercise in the previous section at the firm level. In this case, we run the regression of the firm level IVOL on CIV only and on three factors. Table 10 reports the results. Panel A reports the mean value and the distribution of the adjusted R-squared for the single factor and 3-factor model. The average adjusted R-squared of single CIV factor is 15%, while that of the 3-factor model is 21%.[20]. And the adjusted R-squared of 3-factor is broadly larger than that of the single CIV-factor.

[Insert Table 10 here]

Panel B presents another way to show the performances of the three-factor model compared with the single-factor model. Each month we sort all the stocks according to their IVOL into deciles. Then we calculate the average adjusted R-squared for each decile and average over the time. The three-factor model has a much larger adjusted R-squared between 22.5% to 24% across all the deciles compared with 14% to 17% for the CIV factor only.

4.6 What Is the CIV Factor?

In the previous section, we show that the CIV factor, after introduction of LIV and SIV factors, loses its explanatory power for the IVOL cross-section. This is consistent with our theoretical argument and simulation results in Section 3. However, the IVOL loadings on CIV factor, at both the portfolio and the firm level, are consistently large and significant in the data. In this section, we want to explore further the relation between the CIV factor and the leverage.

First, we regress CIV on LIV and SIV and Table 11 shows the result. The time-series regression coefficients are all highly significant and R-squared ranges from 58 percent for value-weighted factors to 77 percent equal-weighted foctors. The high correlation between CIV and LIV/SIV

[20] Herskovic et al. (2016) reports around 30% R-squared using IVOL measured at an annual frequency.

factors underlines the explanation power of CIV factor on the cross-sectional IVOLs, which we show disappears once we include LIV and SIV in the regression.

Recall in Table 7, the LIV factor has a much smaller value compared with the CIV and SIV factors, while the average IVOL of H or L leverage portfolios are much closer. To see this more closely, in Figure 4, we plot the time series of the three factors, as well as the average IVOL values of H and L leverage portfolios. All five time-series move together except for the five-year period between roughly July 1998 to July 2003. During this period, the average IVOL of low-leverage portfolio exceeds that of high-leverage portfolio, resulting in negative values for the LIV factor. This five-year period coincidences with the dot-com boom and subsequent burst, while the large variations of IVOL were mainly driven by the low-leveraged dot-com companies as well as sharp increases in IVOL at low-leverage industries such as high-tech and pharmaceuticals during this period.

[Insert Figure 4 here]

With the above caveat in mind, we still want to show the role of leverage for the IVOL cross-section. From Figure 4, we see that the average IVOL of either H or L leverage portfolios track the CIV factor quite closely throughout. We can thus use either the average IVOL of either H or L leverage portfolio instead of the CIV factor, and we hypothesize it should still be able to explain the co-movement of IVOL in the cross-section.

To explore the explanatory power of the leverage and size, we conduct two separate analyses. First, we show that LIV and SIV factors have additional explanation power over the CIV factor. To do so, we run the following two-stage regressions and compare their R-squared. We first run the regression of IVOL of the 25 leverage-size sorted portfolios on the CIV factor only. In the second step we regress the residuals of that first-stage regression on the LIV and SIV factors. Our model predicts that LIV and SIV should still have explanation power for the IVOL cross-section after accounting for the CIV factor.

Table 12, panel A presents the results. One can see that the R-squared of the residuals regression on LIV and SIV factors are still very high, ranging from 7% to 67%. All the regression coefficients (unreported here) are highly statistically significant.

[Insert Table 12 here]

Secondly, we argued before that the CIV factor mainly captures the effect of average leverage on IVOL. Its explanation power in the cross-section mainly comes from its correlation with the LIV and SIV. From the Figure 4, H or L co-moves very closely with the CIV factor. So we hypothesize that either H or L can capture the effects of both the average leverage and the cross-section together. To show this, we again conduct a two-step analysis. We first regress the IVOL of the 25 size-leverage portfolios on the average IVOL of the H leverage portfolio only. In the second step, we regress the residuals of the first step on the CIV factor.

Table 12, Panel B presents the results. We can see that CIV almost have no explanation power for the IVOL cross-section after accounting for the H leverage portfolio influence. R-squared ranges from zero to 2 percent for most of the regressions, and 4 or 5 percent for the smallest low leverage portfolios. Furthermore we show this is mainly the effect from leverage. Panel C shows this result. Here we use only SIV factor in the first stage regression, and regress the residual IVOL on CIV factor still have large explanation power in terms of residual regression, with R-squared ranging from 12% to 21%. In summary, our results indicate that the explanation power of CIV factor in the cross-section is mainly from the leverage effect.

5 The IVOL Puzzle

So far we have demonstrated that the IVOL co-movement is driven by the three factors: CIV, LIV and SIV. And LIV and SIV capture the cross-sectional IVOL variation. In this section, we apply our factor structure model to the negative relation between IVOL and future returns Ang et al. (2006). We find that the negative IVOL-return relation appears driven by the co-movement component of the IVOL rather than the residual component.

Table 13 reports the current month IVOL and the next month return for various IVOL sorts. Column 2 and 3 are the standard negative relation between IVOL and the next month returns. Stocks are sorted in to deciles each month by their IVOL, and returns are measured in excess of the risk free rate. There is not much difference in returns across the different deciles except the highest IVOL decile, for which the return is extremely low next month as reported in Ang et al. (2006).

[Insert Table 13 here]

For each firm, we run a time-series regression of the previous 36 monthly IVOL on the CIV, LIV and SIV factors. We then redo the portfolio formation by sorting stocks into deciles according to their predicted IVOL or residual IVOL from the time-series regression. Column 4 and 5 in Panel A of Table 13 show the results when stocks are sorted into deciles by regression residuals. The average next-month return for these decile portfolios is now higher for the highest decile than the lowest decile, in line with the Merton (1987) prediction.

In Column 6 and 7 of Table 13, stocks are sorted into deciles by their predicted IVOL. In this case, the negative relation between IVOL and future returns is recovered. In fact, the average return of highest predicted IVOL decile is extremely small, at -0.11% per month. In this table we report the equal-weighted portfolio, the results for value-weighted are qualitatively similar. The results in Table 13 suggest that the negative IVOl return relation is mainly driven by the common component in IVOL and therefore that leverage and size play an important role.

6 Conclusion

We present, both theoretically and empirically, a three-factor model for the cross-section of equity idiosyncratic volatility (IVOL). Theoretically we show that leverage and size affect a firm's equity IVOL when leverage is time-varying and affected by a common factor in asset returns. This is true in a simple set up where firm asset returns follow a CAPM with constant volatility. The three factors we propose are a leverage factor (LIV) and a size factor (SIV) in addition to the CIV factor as introduced in Herskovic et al. (2016). The CIV factor captures the market-wide average leverage effect. We show that when LIV and SIV are combined with CIV, the three factors together capture a large proportion of co-movement in the IVOL cross-section, and that LIV and SIV capture the cross-sectional dispersion.

Because of the highly nonlinear functional format of the IVOL, which is affected by a power function of the firm value V, our construction of LIV and SIV cannot capture all the effects from size and leverage. However, the potential improvements in including more factors seem to be limited. Further work is needed to incorporate the nonlinear relation and potentially other factors that may affect IVOL.

References

Ang, A., Hodrick, R. J., Xing, Y., Zhang, X., 2006. The Cross-Section of Volatility and Expected Returns. The Journal of Finance 61, 259–299.

Choi, J., Richardson, M., 2016. The Volatility of a Firm's Assets and the Leverage Effect. Journal of Financial Economics 121, 254–277.

Cochrane, J., 2005. Financial markets and the real economy. Tech. rep.

Connor, G., Korajczyk, R. A., Linton, O., 2006. The common and specific components of dynamic volatility. Journal of Econometrics 132, 231–255.

Doshi, H., Jacobs, K., Kumar, P., Rabinovitch, R., 2016. Leverage and the value premium. Available at SSRN 2493903 .

Duarte, J., Kamara, A., Siegel, S., Sun, C., 2014. The Systematic Risk of Idiosyncratic Volatility. Available at SSRN 1905731 .

Fama, E. F., French, K. R., 1993. Common risk factors in the returns on stocks and bonds. Journal of Financial Economics 33, 3–56.

Fischer, E., Heinkel, R., Zechner, J., 1989. Dynamic capital structure choice: Theory and tests. The Journal of Finance .

Garlappi, L., Yan, H., 2011. Financial Distress and the Cross-section of Equity Returns. Journal of Finance 66, 789–822.

Goldstein, R., Ju, N., Leland, H., 2001. An EBIT-Based Model of Dynamic Capital Structure*. The Journal of Business .

Gomes, J. F., Schmid, L., 2010. Levered returns. The Journal of Finance 65, 467–494.

Herskovic, B., Kelly, B., Lustig, H., Van Nieuwerburgh, S., 2016. The common factor in idiosyncratic volatility: Quantitative asset pricing implications. Journal of Financial Economics 119, 249–283.

Leland, H. E., 1994. Corporate Debt Value, Bond Covenants, and Optimal Capital Structure. The Journal of Finance 49, 1213–1252.

Livdan, D., Sapriza, H., Zhang, L., 2009. Financially constrained stock returns. The Journal of Finance 64, 1827–1862.

Merton, R. C., 1974. On the pricing of corporate debt: The risk structure of interest rates. The Journal of finance 29, 449–470.

Merton, R. C., 1987. A simple model of capital market equilibrium with incomplete information. The Journal of Finance 42, 483–510.

Strebulaev, I. A., 2007. Do Tests of Capital Structure Theory Mean What They Say? The Journal of Finance 62, 1747–1787.

Strebulaev, I. A., Whited, T. M., et al., 2012. Dynamic models and structural estimation in corporate finance. Foundations and Trends® in Finance 6, 1–163.

Strebulaev, I. A., Yang, B., 2013. The mystery of zero-leverage firms. Journal of Financial Economics 109, 1–23.

Figure 1: Example of refinancing

In this figure we show a typical sample path of the firm value in our simulations. The firm is chosen from the simulation of base sample. the initial value of firm is $V_0 = \$100$. Period 0 ends either by firm value reaching the initial bankruptcy boundary VB_0^*, at which point the firm declares bankruptcy, or by firm value reaching $VU_0 = \gamma * V_0$, at which point the debt is recalled and the firm again chooses an optimal capital structure. Note that the initial firm value at the beginning of period 1 is $V_0 = VU_0 = \gamma V_0$. Similarly, it's optimal to choose $VU_n = \gamma^n VU_0$ and $VB_n = \gamma^n VB_0$. The simulation runs for 50 years with 12600 time steps in total.

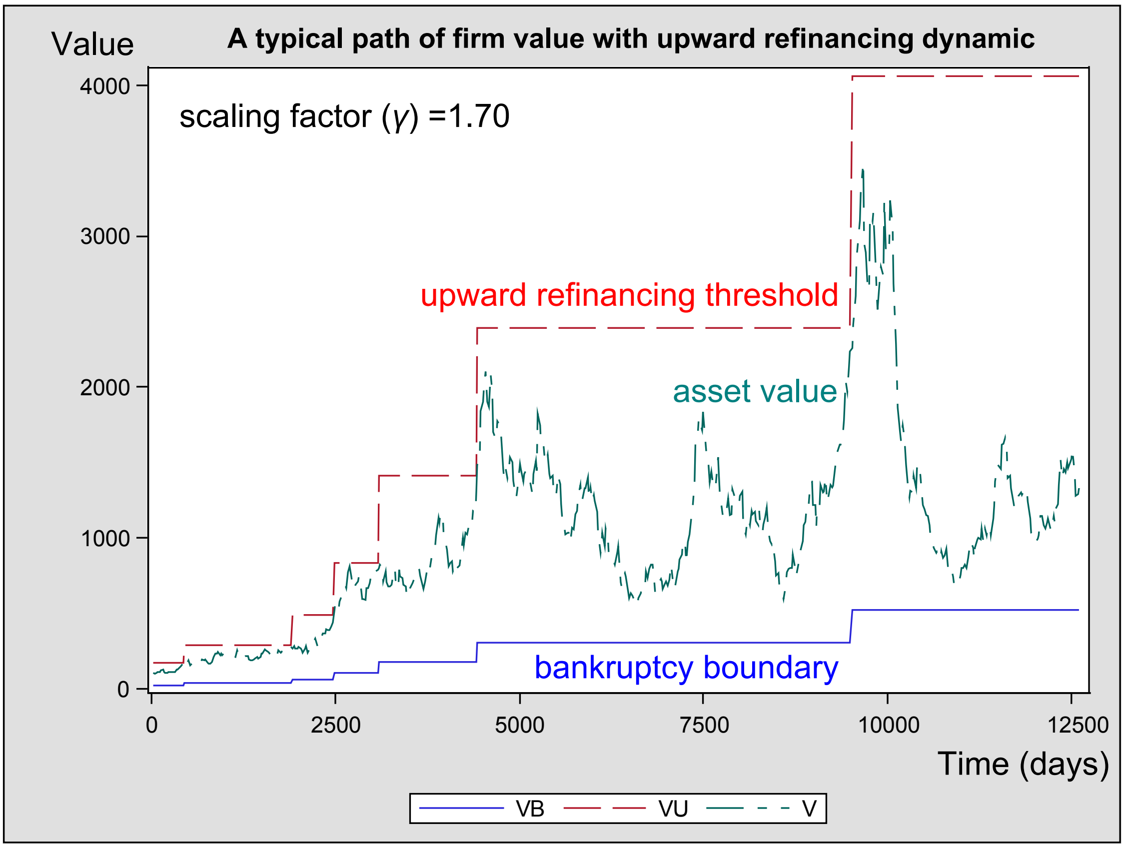
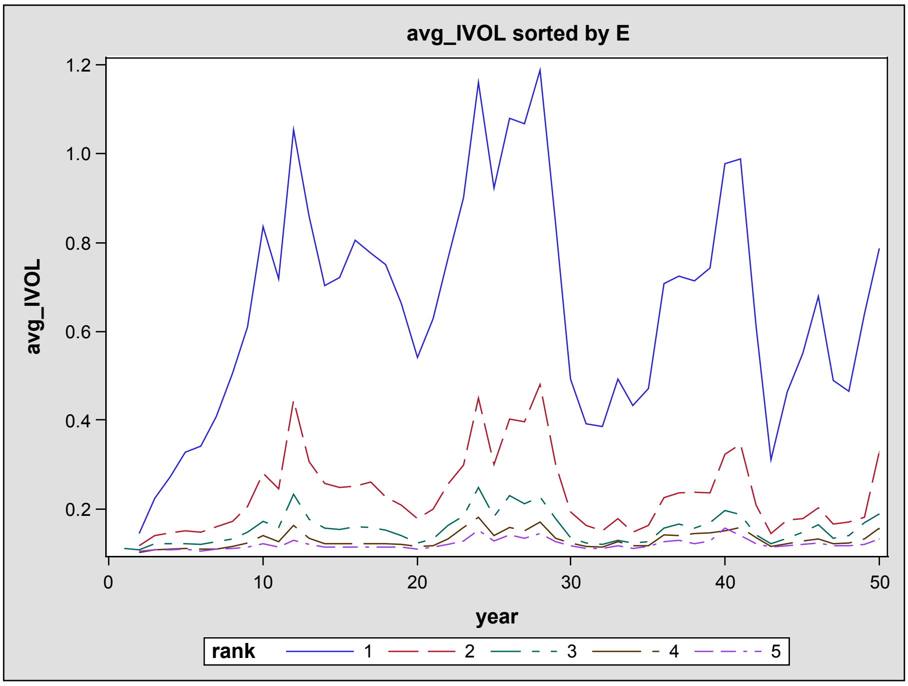


Figure 2: Comovement in portfolios' average IVOL from one simulation run

We compute the annual IVOL as the residual return volatility from the CAPM using daily equity returns. We construct the samples from repeated simulations on the optimal upward refinancing capital structure model in Goldstein et al. (2001). In each simulation run, we sort the stocks by their size (measured as equity value) or by financial leverage into quintiles, and then compute the average annual IVOL for each quintile. The top and bottom plots show the average annual IVOL of the quintiles sorted by size and leverage, respectively, for a typical simulation run. The parameter values used in the simulation are included in Table 2.

(a) Leverage



(b) Leverage

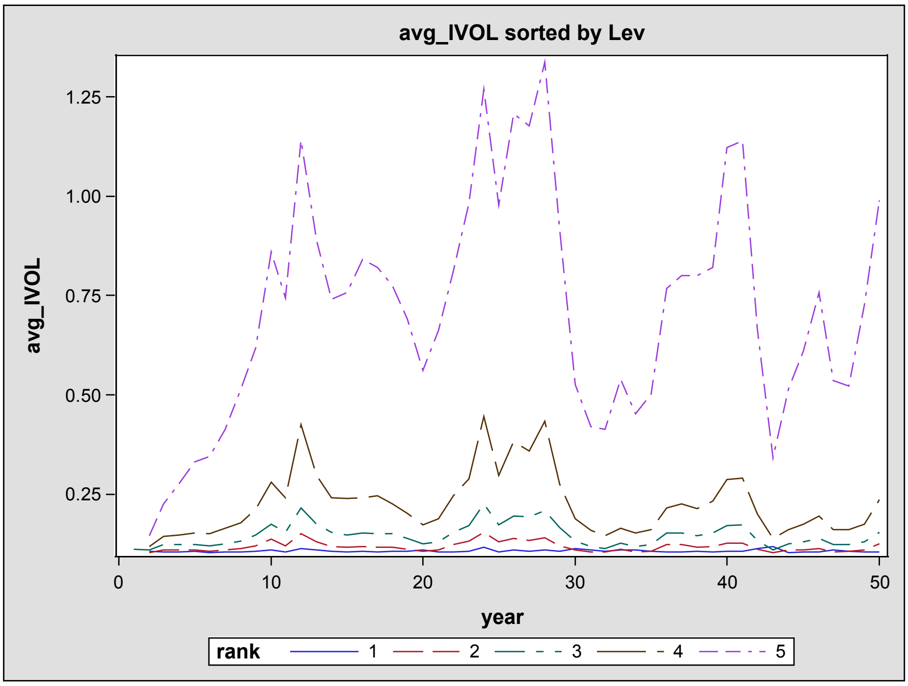


Figure 3: Simulated portfolios sorted by market leverage into percentiles

In this figure, in a given simulation run we sort the firms by market leverage ratios into percentiles each year, and compute the cross-sectional average IVOL and market leverage ratio (Lev). We then calculate the time-series average over the simulation period. We repeat this for 100 simulation runs and plot for each of the 100 percentile portfolios the average leverage against average IVOL across the simulation runs, as well as the 5th and 95th percentiles of IVOL.

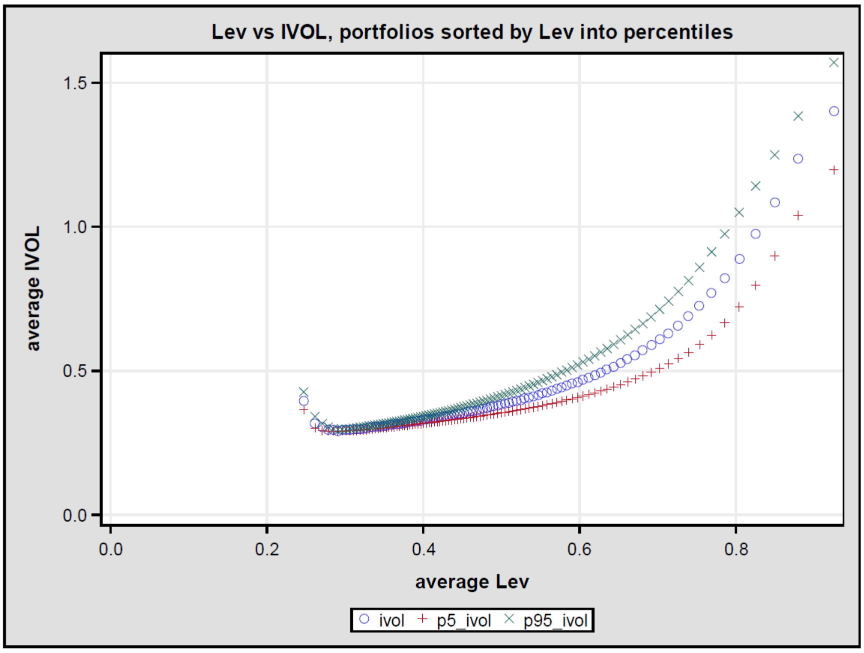


Figure 4: Time series of IVOL factors

In this figure we plot the time-series of the monthly values of CIV, SIV and LIV factors, as well as the average IVOL of the high leverage and low leverage portfolios.

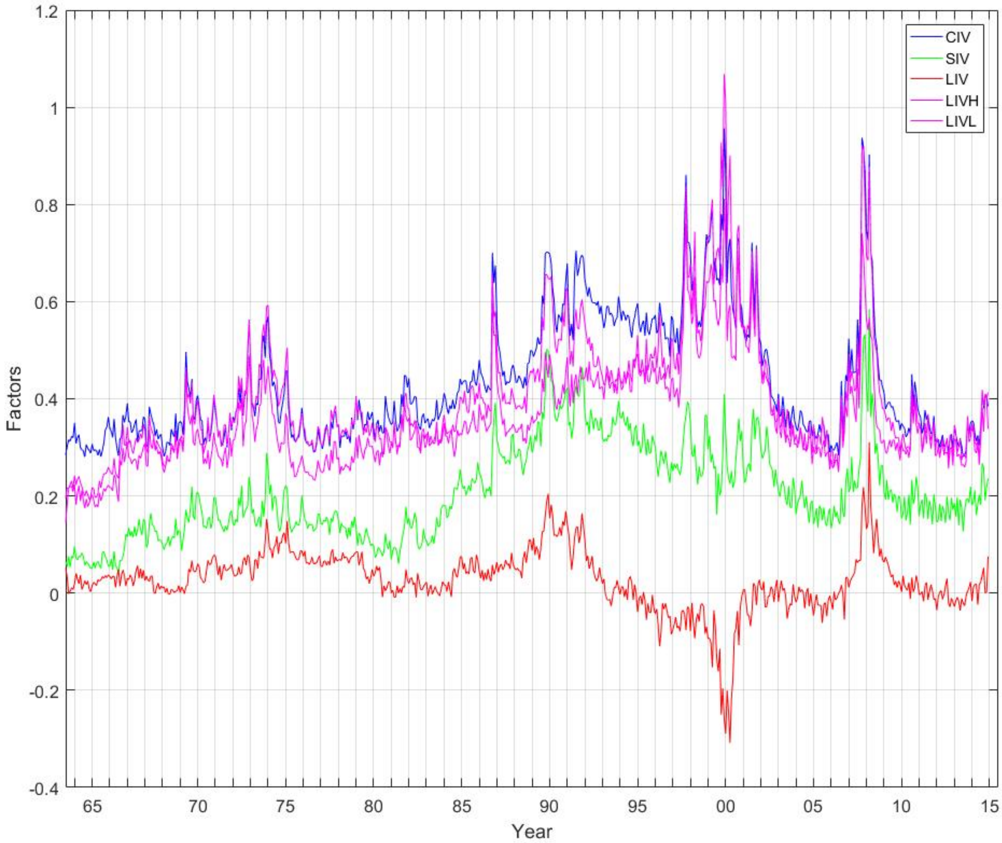


Table 1: Statistics of portfolios sorted by IVOL

This table shows the average of IVOL, CAPM beta leverage and β_{CIV} of the portfolios sorted by IVOL. The annual IVOL is estimated as the standard deviation of the residual returns from Fama-French three-factor regressions using daily returns within a calendar year. The CAPM beta is estimated from CAPM model using daily returns over a calendar year. Market leverage ratio (ML) of firm is computed as $ML = (DLTT + DLC)/(DLTT + DLC + CSHO \times PRCC\_F)$ from the last fiscal year-end. β_{CIV} is computed as the loading of portfolio IVOL on the CIV factor. Sample period is from July 1961 to December 2015. The following stocks are excluded from the sample: penny stocks (defined as the stocks with prices below $5 before April 2001 and the stocks with prices below $1 after April 2001), financial firms, utility firms and the firms with total book value of assets (COMPUSTAT item AT) of less than $10 million in inflation-adjusted year 2000 dollars.

IVOL Rank	$IVOL$	β_{CAPM}	ML	β_{CIV}
1	0.12	0.88	0.23	0.22
2	0.19	1.03	0.22	0.35
3	0.23	1.13	0.23	0.46
4	0.27	1.21	0.23	0.57
5	0.32	1.27	0.24	0.70
6	0.37	1.34	0.24	0.84
7	0.44	1.39	0.25	1.02
8	0.52	1.42	0.26	1.27
9	0.66	1.45	0.28	1.68
10	1.14	1.43	0.33	3.23

Table 2: Simulation parameter values

This table reports the parameter values used in the simulations of the Goldstein et al. (2001) model.

$V_0 = \$100$	Initial asset value
$\tau_c = 35\%$	Corporate tax rate
$\tau_i = 35\%$	Personal interest income tax rate
$\tau_d = 20\%$	Personal dividend income tax rate
$r_f = 4.5\%$	After tax risk free rate
$\sigma = 0.25$	Asset return volatility
$\alpha = 0.05$	Bankruptcy cost
$\epsilon = 0.5$	Tax shield effective rate when in default
$q = 0.01$	Restructuring (refinancing) cost rate
$\theta = 0.2$	Sharpe ratio of the common shock $\mathrm{d}Z$
$P/E = 20$	Price-to-earnings ratio
$\delta/V_0 = 0.035 + 0.65C/V_0$	Payout ratio
$\rho = 0.5$	Relative importance of common shocks in total volatility
$N = 5000$	Number of firms in cross-section
$T = 50$	Number of years in simulation (12600 days)
$N_{sim} = 100$	Number of simulation runs

Table 3: Summary statistics for the main variables in simulations

This table reports the initial value and the means of the main variables. We construct the sample from repeated simulations on the optimal upward refinancing capital structure model in Goldstein et al. (2001). Each simulation run includes 5000 initial firms over 50 years. In the column headed *Initial Value*, we report the optimal initial value of coupon payment (C), bankruptcy boundary (V_B) and refinancing scaling factor (γ). γ is endogenously determined and remains constant. V is the asset value of the firm. D and E are the market values of debt and equity, respectively. $Leverage = D/(D+E)$. $Creditspread = [C/D - r_f(1-\tau_i)] \times 10^4$. r^E is the monthly equity return. N_{stock} is the average number of stocks in the cross-section. In each simulation run, we compute the cross-sectional average for each variable and average over the full time-series. Column "Realization" reports the mean, as well as the 5^{th} and 95^{th} percentile values (in brackets) for each variable across the simulation runs. The parameter values used in the simulation are included in Table 2.

Variables	Initial Value	Realization
γ	1.70	1.70
V	100.00	347.13
		[139.83, 871.46]
C	1.85	7.53
		[3.36, 18.6]
D	20.92	80.43
		[34.93, 199.05]
E	35.61	115.23
		[43.69, 296.63]
$Leverage$	0.37	0.48
		[0.44, 0.51]
$Creditspread$	193.57	260.89
		[237.74, 282.97]
r^E		1.06
		[0.8, 1.29]
N_{stock}	5000	4928
		[4860.84, 4978.03]

Table 4: IVOL estimated from CAPM and PCA

In this table, we compare the annual IVOL estimated from CAPM and the IVOL estimated from Principal Component Analysis (PCA). IVOL is estimated as the idiosyncratic variance of the equity returns from CAPM or PCA using daily data. We report the mean of the average IVOL for both measures in repeated simulations. The numbers in brackets are the 5^{th} and 95^{th} percentile values. Row *CORR* shows the full panel correlations between the two measures of IVOL. We construct the sample from repeated simulations on the optimal upward refinancing capital structure model in Goldstein et al. (2001). The parameter values used in the simulation are included in Table 2.

	$IVOL_{CAPM}$	$IVOL_{PCA}$
MEAN	0.26	0.25
	[0.21, 0.31]	[0.20, 0.30]
STD	0.51	0.48
	[0.38, 0.70]	[0.36, 0.65]
CORR		0.99
		[0.98, 0.99]

Table 5: Pairwise correlations

This table reports the average pairwise correlations between the average annual IVOL of the portfolios sorted by size or leverage. In each simulation run, we sort the firms by size (or by leverage) into quintiles and compute the average IVOL for each quintile in each year, then we compute the pairwise correlations of the average IVOL between the quintiles. We repeat this procedure across repeated simulation runs. Next, we compute the mean, 5^{th} and 95^{th} percentile values (numbers in brackets) of the correlations for each quintile pair across the simulation runs. We construct the sample from repeated simulations on the optimal upward refinancing capital structure model in Goldstein et al. (2001). The parameter values used in the simulation are included in Table 2.

A Size	rank	1	2	3	4	5
	1	1.00				
	2	0.95	1.00			
		[0.92, 0.97]				
	3	0.91	0.96	1.00		
		[0.84, 0.96]	[0.92, 0.99]			
	4	0.87	0.91	0.94	1.00	
		[0.75, 0.95]	[0.78, 0.97]	[0.88, 0.98]		
	5	0.81	0.86	0.89	0.93	1.00
		[0.62, 0.94]	[0.65, 0.97]	[0.75, 0.97]	[0.84, 0.98]	
B Leverage	rank	1	2	3	4	5
	1	1.00				
	2	0.13	1.00			
		[-0.36, 0.73]				
	3	0.06	0.97	1.00		
		[-0.53, 0.73]	[0.95, 0.99]			
	4	0.10	0.94	0.98	1.00	
		[-0.51, 0.75]	[0.9, 0.97]	[0.97, 0.99]		
	5	0.01	0.85	0.91	0.94	1.00
		[-0.50, 0.59]	[0.71, 0.94]	[0.79, 0.97]	[0.86, 0.98]	

Table 6: Explaining firm-level IVOL using CIV

This table reports the results of using the common IVOL factor (CIV) to explain the time-series variations in firm-level IVOL. In each simulation run, we run time-series regressions $IVOL_{i,t} = a_i + b_i CIV_t + \epsilon_{i,t}$ for each individual stock over the full sample period and then compute the average R^2 in the cross-section. CIV is measured as the equally-weighted average IVOL. We report the mean of the average R^2 in the repeated simulation runs. The numbers in brackets are the 5^{th} and 95^{th} percentile values of R^2. To compare the relation between the common factor structure in IVOL and financial leverage, we refer to our simulation sample as the "*levered* sample". In the corresponding *unlevered* sample, we use the same initial parameter values and the same dynamics of asset value V, but we force the coupon payment C and the bankruptcy boundary V_B equal to zero for all stocks throughout the simulation horizon. We construct the sample from repeated simulations on the optimal upward refinancing capital structure model in Goldstein et al. (2001). The parameter values used in the simulation are included in Table 2.

	R^2	$\hat{a}$	$\hat{b}$
Levered	0.25	-0.14	1.87
	[0.18, 0.31]	[-0.25, -0.01]	[1.35, 2.39]
Unlevered	0.02	-0.05	1.22

Table 7: Summary statistics of the three IVOL factors

This table shows the summary statistics of the three factors CIV, SIV and LIV, as well as the two legs of LIV: LIVH and LIVL. Section 4.2 shows the detailed approach to construct these factors.

	Mean	Median	Std	P99	P90	P75	P25	P10	P1
CIV	0.4355	0.3857	0.1298	0.8609	0.6191	0.5267	0.3364	0.3101	0.2833
SIV	0.2089	0.1913	0.0965	0.4642	0.3511	0.2723	0.1381	0.0967	0.0518
LIV	0.0258	0.0254	0.0621	0.1785	0.0872	0.0599	-0.0015	-0.0345	-0.2003
LIVH	0.3938	0.3678	0.1147	0.7217	0.5528	0.4521	0.3138	0.2813	0.2032
LIVL	0.3680	0.3311	0.1267	0.8229	0.5272	0.4260	0.2900	0.2584	0.1824

Table 8: Summary statistics of the three IVOL factors in 5×5 leverage and size sorted portfolios

This table shows the average IVOL, leverage and size of $t \times 5$ portfolios sorted by size and leverage. At the end of June of each year t, we form 5 leverage sorted quintiles across all the firms based on the leverage of last fiscal year end; we also use the size at the end of June of each year t, and form 5 size quintiles based on the NYSE breakpoints. The 5 by 5 leverage-size portfolios are formed as the intersections of the 5 leverage and the 5 size portfolios.

	Size rank				
Lev rank	Small	2	3	4	Big
			IVOL		
Low	0.50	0.37	0.33	0.30	0.26
2	0.51	0.36	0.31	0.28	0.23
3	0.51	0.33	0.29	0.26	0.21
4	0.51	0.33	0.29	0.26	0.21
High	0.59	0.38	0.33	0.29	0.23
			Leverage		
Low	0.01	0.01	0.01	0.01	0.02
2	0.09	0.09	0.09	0.09	0.09
3	0.21	0.21	0.21	0.21	0.20
4	0.36	0.36	0.36	0.35	0.35
High	0.63	0.60	0.59	0.57	0.56
			Size		
Low	0.06	0.28	0.64	1.58	12.91
2	0.07	0.34	0.78	1.92	18.85
3	0.07	0.35	0.8	1.99	14.52
4	0.07	0.34	0.8	1.91	12.54
High	0.06	0.33	0.79	1.87	12.44
			N		
Low	321	103	61	43	35
2	327	105	68	54	55
3	280	88	70	59	66
4	309	89	62	51	43
High	376	65	37	26	15

Table 9: Regression results of three-factor model

This table shows the regression results of $IVOL_{it} = \alpha_i + \beta_i^L LIV_t + \beta_i^S SIV_t + \beta_i^C CIV_t + \epsilon_{i,t}$.

	Size rank					Size rank				
Lev rank	Small	2	3	4	Big	Small	2	3	4	Big
			β_{LIV}					$t(\beta_{LIV})$		
Low	-0.71	-0.71	-0.68	-0.66	-0.48	-20.89	-21.5	-17.65	-18.71	-16.08
2	-0.29	0.08	-0.13	0.04	-0.07	-8.10	2.40	-4.66	1.58	-3.07
3	-0.02	0.24	0.21	0.21	0.09	-0.58	7.17	6.94	6.98	3.59
4	0.30	0.33	0.37	0.20	0.10	8.75	9.68	11.36	5.95	3.22
High	0.63	0.68	0.65	0.29	0.40	17.22	14.1	14.25	7.22	8.70
			β_{SIV}					$t(\beta_{SIV})$		
Low	0.61	-0.21	-0.55	-0.59	-0.64	18.16	-6.39	-14.62	-16.97	-21.81
2	0.48	-0.53	-0.63	-0.71	-0.61	13.59	-16.37	-22.36	-26.72	-25.61
3	0.57	-0.58	-0.66	-0.71	-0.74	16.53	-17.58	-22.10	-23.51	-28.99
4	0.46	-0.46	-0.74	-0.70	-0.68	13.66	-13.90	-23.34	-21.45	-22.88
High	0.70	-0.62	-0.70	-0.75	-0.56	19.37	-13.03	-15.53	-19.07	-12.31
			β_{CIV}					$t(\beta_{CIV})$		
Low	0.86	0.97	1.13	1.04	0.96	34.66	40.41	40.63	40.59	44.52
2	0.89	0.91	0.94	0.88	0.75	34.23	37.85	44.98	44.78	42.66
3	0.85	0.88	0.89	0.81	0.82	33.42	36.23	40.57	36.11	43.35
4	0.89	0.89	0.92	0.85	0.84	36.03	35.90	39.28	35.60	38.11
High	0.98	1.08	0.95	1.03	0.77	36.54	30.95	28.36	35.47	23.06
			R^2					N		
Low	0.95	0.90	0.86	0.85	0.86	526	164	97	71	63
2	0.94	0.81	0.84	0.81	0.80	227	77	62	53	61
3	0.95	0.78	0.80	0.72	0.78	211	72	54	45	45
4	0.95	0.82	0.77	0.73	0.76	248	68	45	38	29
High	0.96	0.77	0.69	0.76	0.58	377	62	35	23	14

Table 10: firm level IVOL regression on CIV and three factors

We regress the firms' IVOL on CIV and the three factors at individual level, and report the R^2. In Panel A, we run firm level regression $IVOL_{i,t} = a_i + \beta_{CIV,i}CIV_t + \varepsilon_{i,t}$ and $IVOL_{i,t} = a_i + \beta_{CIV,i}CIV_t + \beta_{SIV,i}SIV_t + \beta_{LIV,i}LIV_t + \varepsilon_{i,t}$ and record the R^2 for each firm. This is unconditional full sample regression for each firm. Then we report the distribution of the R^2 in the cross-section, for regressions on CIV only and for regressions on the three IVOL factors. In Panel B, we form the stocks by their IVOL in each month into deciles (the composition of the decile portoflios may change over time). We report the average R^2 for each IVOL decile and finally we average over time.

Panel A Firm level										
	Mean	P1	P5	P10	P25	P50	P75	P90	P95	P99
CIV	0.15	-0.06	-0.03	-0.01	0.02	0.10	0.24	0.38	0.47	0.62
3-factor	0.21	-0.12	-0.03	0.00	0.07	0.19	0.33	0.46	0.54	0.68

Panel B Average R^2 of individual firms sorted into IVOL deciles										
	Low	2	3	4	5	6	7	8	9	High
CIV	0.14	0.15	0.16	0.16	0.17	0.17	0.17	0.16	0.16	0.15
3-factor	0.24	0.24	0.24	0.24	0.24	0.23	0.23	0.23	0.23	0.23

Table 11: CIV Regression on LIV and SIV Factors

This table reports the results from time-series regression of CIV on LIV and SIV. $CIV = a+bLIV+cSIV$ We report the results using LIV from different definitions of leverage and for both equal-weighted and value-weighted portfolios. Specifically, Lev1 is the one defined in the main text, where the equity value is calculated as the shares outstanding times the price at the end of last fiscal year from Compustat. Lev2 uses the market cap at end of last calendar year using data from CRSP. Lev 3 uses the market cap at the end of June this year using data from CRSP. t-stat are in the bracket.

	a	b	c	R-sq
Lev1-ew	0.1975	-0.4179	1.1904	0.7781
	(34.3434)	(-8.0099)	(46.3380)	.
Lev1-vw	0.0920	-0.40166	2.2363	0.5859
	(7.3676)	(-5.2586)	(29.0201)	.
Lev2-ew	0.1960	-0.3807	1.1934	0.7759
	(33.8368)	(-7.4366)	(45.9622)	.
Lev2-vw	0.0976	-0.4705	2.2011	0.5709
	(7.7422)	(-6.1731)	(28.2505)	.
Lev3-ew	0.1938	-0.2925	1.1940	0.7709
	(33.0925)	(-5.7026)	(43.9899)	.
Lev3-vw	0.0909	-0.5380	2.2250	0.5851
	(7.3951)	(-7.9540)	(29.3863)	.

Table 12: R^2 **from two-stage regressions**

This table reports the results from two-stage regression. In panel A, we first run the regression of IVOLs of 5×5 portfolios sorted by size and leverage on CIV factor only, then we regress the residuals of IVOLs from the first regression on LIV and SIV factors, and then we report the R^2 from the second regression. Similarly, in panel B, we run regression on LIVH in the first stage, and then regress the residuals on CIV in the second stage. In Panel C, we run regression on SIV in the first stage, and then regress the residuals on CIV in the second stage.

Panel A Regress residuals from CIV on SIV and LIV					
				Size rank	
Lev rank	Small	2	3	4	Big
Low	0.33	0.56	0.59	0.67	0.65
2	0.27	0.45	0.45	0.41	0.43
3	0.08	0.11	0.22	0.16	0.25
4	0.18	0.07	0.11	0.11	0.15
High	0.43	0.13	0.14	0.09	0.07
Panel B Regress residuals from LIVH on CIV					
				Size rank	
Lev rank	Small	2	3	4	Big
Low	0.05	0.02	0.01	0.01	0.01
2	0.04	0.01	0.00	0.00	0.00
3	0.04	0.00	0.00	0.00	0.00
4	0.02	0.00	0.01	0.00	0.00
High	0.01	0.01	0.02	0.00	0.00
Panel C Regress residuals from SIV on CIV					
				Size rank	
Lev rank	Small	2	3	4	Big
Low	0.18	0.19	0.19	0.19	0.19
2	0.19	0.19	0.21	0.20	0.21
3	0.20	0.20	0.21	0.20	0.21
4	0.18	0.16	0.17	0.18	0.20
High	0.16	0.14	0.12	0.17	0.15

Table 13: Revisit the result of IVOL puzzle

This table reports the next month return of the portfolios sorted by IVOL, IVOL predicted by three-factor model and the residual IVOL from the model. In Panel A, we sort the firms into deciles by at the end of month t by IVOL estimated over month t, then we compute the EW gross return in month t+1. In Panel B and C, we run firm level regression $IVOL_{i,t} = a_i + \beta_{CIV,i}CIV_t + \beta_{SIV,i}SIV_t + \beta_{LIV,i}LIV_t + \varepsilon_{i,t}$. To avoid looking ahead issue in computing IVOL-return relation, we do not run unconditional full sample regression here, instead we use a rolling-window where the loadings are estimated using the monthly IVOL and factors from month t-35 to t (36 months), and then use these loadings to compute the predicted IVOL (the part explained by the factors) and the residual IVOL (the residual) in month t. Then we sort the firms into deciles by predicted IVOL and resdiual IVOL seperately in to deciles, and then compute their EW gross return in month t+1

	Panel A IVOL		Panel B Residual		Panel C Predicted	
Rank	$IVOL$	r_{t+1}	$IVOL_R$	r_{t+1}	$IVOL_P$	r_{t+1}
1	0.12	1.08%	0.26	0.50%	0.18	1.18%
2	0.17	1.28%	0.25	1.00%	0.22	1.27%
3	0.21	1.25%	0.25	1.19%	0.25	1.37%
4	0.24	1.36%	0.25	1.28%	0.28	1.33%
5	0.28	1.31%	0.26	1.29%	0.31	1.38%
6	0.32	1.41%	0.28	1.38%	0.34	1.43%
7	0.37	1.22%	0.32	1.35%	0.38	1.39%
8	0.42	1.20%	0.37	1.37%	0.42	1.31%
9	0.51	0.97%	0.45	1.25%	0.46	1.01%
10	0.75	0.47%	0.69	0.92%	0.55	-0.11%

Appendix A Model Derivation

We now proceed to derive the optimal capital structure of a single firm in the context of Section 3.1. The model mainly follows Goldstein et al. (2001). First, we derive the solution to the static capital structure model where only a bankruptcy boundary is present. Then we solve the dynamic model which also features an upward refinancing boundary.

A.1 Static Capital Structure Model

Given the setup in Section 3.1, the cum-dividend expected return of any firm should be the risk-free rate r under the risk-neutral measure,

$$\frac{dV_i(t) + \delta_i(t)dt}{V_i(t)} = rdt + \sigma_i \left(\rho_i d\tilde{W}(t) + \sqrt{1-\rho_i^2} d\tilde{Z}_i(t) \right). \tag{24}$$

To simplify the notation, we define for the firm i from now on in this section we will ignore the subscript i):

$$\begin{aligned} d\tilde{z}(t) &\equiv \rho_i d\tilde{W}(t) + \sqrt{1-\rho_i^2} d\tilde{Z}_i(t), \\ \nu &\equiv r - \delta_i, \\ \sigma &\equiv \sigma_i. \end{aligned}$$

It follows that the asset process under the risk-neutral measure follows,

$$\frac{dV}{V} = \nu dt + \sigma d\tilde{z}(t), \tag{25}$$

with a total payout process that is proportional to the current asset value, $\delta(t) = \delta V(t)$.

For any claim on the asset value with intermediate payoff rate C, the value of the claim, $F(V,t)$, satisfies the following PDE

$$\frac{1}{2}\sigma^2 V^2 F_{VV} + \nu V F_V - rF + F_t + C = 0. \tag{26}$$

Consider that the firm only issues a consol debt, thus the value function is time-invariant. The resulting ODE becomes then

$$\frac{1}{2}\sigma^2 V^2 F_{VV} + \nu V F_V - rF + C = 0. \tag{27}$$

For a homogeneous ODE with the term C, the solutions are of the form $F(V) = A_1 V^{-y} + A_2 V^{-x}$,

where

$$x = \frac{1}{\sigma^2}\left[\left(\nu - \frac{\sigma^2}{2}\right) + \sqrt{\left(\nu - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}\right] > 0,$$

$$y = \frac{1}{\sigma^2}\left[\left(\nu - \frac{\sigma^2}{2}\right) - \sqrt{\left(\nu - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}\right] < 0.$$

The only boundary in this set up is the default boundary, V_B. For convenience, define the price of a claim that pays \$1 when the firm defaults as p_B. Since there is no intermediate payment for this claim, the general solution is given by

$$p_B(V) = A_1 V^{-y} + A_2 V^{-x}. \tag{28}$$

Consider the boundary conditions, $\lim_{V\to\infty} p_B(V) = 0, \lim_{V\to V_B} p_B(V) = 1$, the price is then given by

$$p_B(V) = \left(\frac{V}{V_B}\right)^{-x}. \tag{29}$$

Given this result, we now study the values of equity, debt and government claims, respectively. We start by considering the holder receiving all the payouts as long as the firm does not default, and zero in case of default. The value of the claim for this holder, denoted as V_{solv}, should be equal to the difference between the total value V and the value in case of default,

$$V_{solv} = V - V_B p_B(V). \tag{30}$$

Next we consider the holder receiving all the coupons, constant C, if the firm does not default, and zero if so. The value of a console bond with constant coupon C is given by C/r (recall we are in the risk-neutral world). So the value of the holder, denoted as V_{int}, is given by:

$$V_{int} = \frac{C}{r}[1 - p_B(V)]. \tag{31}$$

As such, we can easily write out the values accruing to the different claim holders who receive payments when there is no default, and zero otherwise

$$\begin{aligned} E_{solv}(V) &= (1-\tau_{eff})(V_{solv} - V_{int}), \\ G_{solv}(V) &= \tau_{eff}(V_{solv} - V_{int}) + \tau_i V_{int}, \\ D_{solv} &= (1-\tau_i)V_{int}, \end{aligned}$$

where

$$\tau_{eff} = (1-\tau_c)(1-\tau_d), \tag{32}$$

and τ_i, τ_d, τ_c are the tax rates for interest, dividends and corporate profits, respectively.

We next study the value of different claim holders when there is a default. Recall that the PV of contingent claim paying \$1 in case of default is given by $p_B(V)$. Given that the firm value equals V_B at default, the total PV of the claim to default is given by

$$V_{def}(V) = V_B p_B(V). \tag{33}$$

The default value V_B will be distributed to three parties: debt holders, government and bankruptcy cost. Denote the proportional bankruptcy cost by $\alpha > 0$, so that the PV of the bankruptcy cost, $BC_{def}(V)$, equals

$$BC_{def}(V) = \alpha V_{def}(V). \tag{34}$$

The remaining value, $(1-\alpha)V_B$, is distributed between the debt holder and the government through tax. Here we assume that the tax rate is charged as a dividend payment. Then the PVs of the default claims for the debt holder and government, respectively, equal

$$D_{def}(V) = (1-\alpha)(1-\tau_{eff})V_{def}(V),$$
$$G_{def}(V) = (1-\alpha)\tau_{eff}V_{def}(V).$$

The equity holder, of course, receives nothing in bankruptcy. So the total value of equity after debt issuance is given by

$$E(C, V; V_B) = E_{solv} = (1-\tau_{eff})\left[V - V_B\left(\frac{V}{V_B}\right)^{-x} - \frac{C}{r}\left(1 - \left(\frac{V}{V_B}\right)^{-x}\right)\right].$$

To obtain the optimal default value V_B, we impose the smooth-pasting condition as usual,

$$0 = \left.\frac{\partial E}{\partial V}\right|_{V=V_B}.$$

Then the optimal V_B^* is given by

$$V_B^* = \frac{x}{1+x}\frac{C}{r} \equiv \lambda\frac{C}{r}, \text{ with } \lambda = \frac{x}{1+x}. \tag{35}$$

Note that the optimal default boundary is a function of the coupon payment C.

Given this, we can rewrite the value of the equity as

$$E(V, C, V_B(C)) = (1-\tau_{eff})\left[V + \frac{1}{1+x}\lambda^x\left(\frac{C}{r}\right)^{x+1}V^{-x} - \frac{C}{r}\right].$$

The debt holder receives C as long as the firm does not default, and $(1-\alpha)(1-\tau_{eff})V_B$ upon default. So the value of the debt at any time is given by

$$D = D_{solv} + D_{def},$$

Let's consider the optimal coupon. Before the issuance of the debt, the equity holder, owning the whole firm, decides to issue the debt at the market value $D(V_0, C, V_B(C))$. There is also a restructuring cost q, so the net value to the equity holder at time $t = 0$ is given by

$$(1-q)D(V_0, C, V_B(C)) + E(V_0, C, V_B(C)). \tag{36}$$

The equity holder chooses the optimal capital structure (i.e., coupon C) by maximizing the above value. It follows that

$$C^* = \frac{rV_0}{\lambda}\left[\left(\frac{1}{1+x}\right)\left(\frac{A}{A+B}\right)\right]^{\frac{1}{x}}, \tag{37}$$

where

$$A = (1-q)(1-\tau_i) - (1-\tau_{eff}),$$
$$B = \lambda(1-\tau_{eff})(1-(1-q)(1-\alpha)).$$

A.2 Dynamic Capital Structure Model

Let us start at time $t = 0$. Again it is convenient to define a series of contingent claims. Since there are two boundaries now, we will have two contingent claims. Let $p_U(V)$ denote the present value of the contingent claim that pays \$1 when V hits V_U before hitting V_B, and $p_B(V)$ denote the present value of the contingent value that pays \$1 when V hits V_B before hitting V_U.

The value of the contingent claim $p_U(V)$ can be shown to equal

$$p_U(V) = -\frac{V_B^{-x}}{\Sigma}V^{-y} + \frac{V_B^{-y}}{\Sigma}V^{-x},$$

where

$$\Sigma \equiv V_B^{-y}V_U^{-x} - V_B^{-x}V_U^{-y}.$$

Similarly, the value of the contingent claim $p_B(V)$ is given by

$$p_B(V) = \frac{V_U^{-x}}{\Sigma}V^{-y} - \frac{V_U^{-y}}{\Sigma}V^{-x}.$$

Note that the values of these two claims are intensity variables. In other words, if all the V, V_B and V_U are scaled up by a constant factor γ, the values do not change. This is the property that we will use in the following.

Using the two contingent claim values, we can write the PV of other claims very easily. For example, for a claim that pays $\delta(t)$ as long as V does not hit V_U or V_B and zero when hit, the value is given by

$$V_{solv}^0 = V - p_B(V)V_B - p_U(V)V_U.$$

Here the superscript 0 refers to the period starting at $t = 0$ before hitting either V_U or V_B.

The total PV of the claims paying upon hitting one of the boundaries are given by

$$V_{def}^0 = p_B(V)V_B,$$
$$V_{res}^0 = p_U(V)V_U.$$

Note that the sum of the total claims is equal to the total value V of the firm

$$V_{solv}^0 + V_{def}^0 + V_{res}^0 = V.$$

The value of a claim that pays a constant interest C^0 before hitting either boundary and zero when any boundary is hit is given by

$$V_{int}^0 = \frac{C^0}{r}(1 - p_U(V) - p_B(V)).$$

Similar to the default boundary situation, different claim holders receive different claims when either boundary is hit. As before, let α and q denote the default and restructuring cost, respectively. Let us first consider the allocation of defaulting after the initial restructuring, which is similar to the previous model after the debt issuance. In other words, we first allocate the value $V - p_U(V)V_U$ to different claim holders

$$
\begin{aligned}
d^0(V) &= (1-\tau_i)V^0_{int}(V) + (1-\alpha)(1-\tau_{eff})V^0_{def}(V),\\
e^0(V) &= (1-\tau_{eff})(V^0_{solv}(V) - V^0_{int}(V)),\\
g^0(V) &= \tau_{eff}(V^0_{solv}(V) - V^0_{int}(V)) + \tau_i V^0_{int} + (1-\alpha)\tau_{eff}V^0_{def}(V),\\
bc^0(V) &= \alpha V^0_{def}(V).
\end{aligned}
$$

Now we consider the restructuring branch. The discussion above is about the process after the initial restructuring at $V(0)$, which we denote as V^0_U. When the firm hits the next restructuring boundary V^1_U, we define the constant:

$$
\gamma \equiv \frac{V^1_U}{V^0_U}.
$$

Goldstein et al. (2001) show that V^1_B also scales up by γ, and also $p^1_B(V^1_U) = p^0_B(V^0_U)$ and $p^1_U(V^1_U) = p^0_U(V^0_U)$. Since the optimal C^* also scales up by γ, the above split among different claims are identical in the next interval.

To summarize, initially the firm starts with $V(0) = V^0_U$. The firm then decides the capital structure choice C^0, and passes on the net proceeds of the debt issuance to to the initial equity holder. Then the firm value process follows (3) until either (1) it hits the default boundary V^0_B, or (2) it hits the restructuring boundary $V^1_U = \gamma V^0_U$, which starts a new period.

Denote by $e(V_0)$ the present value of all claims $e^0, e^1, e^2, \ldots$:

$$
\begin{aligned}
e(V_0) &\equiv e^0(V_0)(1 + \gamma p_U(V_0) + [\gamma^2 p_U(V_0)]^2 + \ldots\\
&= \frac{e^0(V_0)}{1-\gamma p_U(V_0)}.
\end{aligned}
$$

During a restructuring, the current debt is called back and a larger amount of new debt is issued. We assume that the debt is issued and called at par. Then the current value of the debt is equal to the PV of the cash flow before hitting V_U, $d^0(V_0)$, plus the PV of the call value, which is par

$$
D^0(V_0) = d^0(V_0) + p_U(V_0)D^0(V_0).
$$

It follows that

$$
D^0(V_0) = \frac{d^0(V_0)}{1-p_U(V_0)}.
$$

This debt issuance will be distributed to the equity holder, adjusting for the restructuring cost

q. So the present value of all future adjustment costs is given by

$$\begin{aligned} RC(V_0-) &= qD^0(V_0)(1 + \gamma p_U(V_0) + [\gamma p_U(V_0)]^2 + \dots \\ &= \frac{qD^0(V_0)}{1 - \gamma p_U(V_0)}. \end{aligned}$$

Putting everything together, the total value of the equity at the moment the initial debt is issued, is given by

$$E(V_{0-}) = \frac{e^0(V_0) + d^0(V_0) - qD^0(V_0)}{1 - \gamma p_U(V_0)}.$$

This is the sum of the present value of all future equity and debt claims net of the adjustment cost.

Therefore, the following no-arbitrage condition holds, stating that the after issuance equity value equals the before issuance equity value minus the (after restructuring cost) debt issuance

$$E(V_{0+}) = E(V_{0-}) - (1 - q)D^0(V_0).$$

Also around the time of the second restructuring, the following condition should hold

$$E(V_{U-}) = \gamma E(V_{0-}) - D^0(V_0).$$