Testing the Conditional CAPM using GARCH-type Models without any other Restrictions

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Abstract

We develop a new approach to testing conditional asset pricing models that avoids placing restrictions on the price of risk. The only assumption made in the model is on the form of the dynamics of the conditional covariance matrix of asset returns. Existing GARCH-based models require an assumption on the price of risk, which we avoid in our approach. We illustrate the methodology by testing a conditional version of the simple single-factor CAPM using monthly returns and show that accounting for time varying betas using our preferred model, which uses daily returns and accounts for the autocovariance in daily returns to model conditional monthly volatilities, reduces average absolute alphas by 32% compared with an unconditional model. Ultimately, we are unable to reject the null hypothesis that the conditional CAPM prices the size and industry sorted portfolios but reject the model for the Fama-French 25 portfolios.

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Conditional CAPM set

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1 Introduction

The Capital Asset Pricing Models (CAPM) of Sharpe (1964) and Lintner (1965) provides a single state, single factor, general equilibrium theory of the risk-return relation. Despite some initial empirical support, the weight of empirical evidence does not support the theory. For example, studies find that CAPM's single market factor cannot explain the returns of portfolios formed on the basis of size, (Banz (1981)), book-to-market ratio (Fama and French, (1993)) or return momentum (Jegadeesh and Titman (1993)). With the presence of these so called 'asset pricing anomalies', the literature has advanced multi-factor asset pricing models that include priced risk factor such as size, book-to-market and momentum; refer Carhart (1997) as an example.

But before searching for additional factors that may explain asset returns, the failure to account for time-variation in the conditional distribution of asset returns should be considered.[1] As highlighted by Lewellen and Nagel (2006) and others, the impact of failing to account for conditional information produces biased estimates of the average pricing error (commonly refer to as 'alpha'). For example, if an asset's conditional beta and market return positively (negatively) covary, the estimated alpha is upwardly (downwardly) biased. Therefore, these simple empirical and statistical facts, warrant the testing of the conditional CAPM.

Unfortunately for CAPM, the current state of empirical evidence does not support its conditional version. Harvey (1989), one of the original tests of conditional CAPM, rejects the conditional CAPM for the decile sorted size portfolios as the model's pricing errors maintain a significant amount of predictability, although it is noted that the all average pricing errors are insignificant when tested individually and jointly. Interestingly, the insignificance of these alphas highlights the alpha bias evident in other studies that have applied the unconditional CAPM to the size portfolios. Despite these findings, the approach employed by Harvey (1989) and others including Jagannathan and Wang (1996), and He, Kan, Ng and Zhang (1996) have been criticised for their assumption that agents form their return expectations in relation to a set of state variables. Therefore, as per Cochrane (2001), such tests of conditional CAPM are tests of the observed stated state variables rather than conditional CAPM.

To avoid specifying a set of state variables in testing the conditional CAPM, Lewellen and Nagel (2006) introduce a short window, non-overlapping, rolling regression approach to estimate contemporaneous conditional alphas and betas. The approach assumes a stable joint distribution within each short estimation window but allows for variation across sample windows. Therefore, as the window rolls through time, a time-series of conditional alphas and betas are generated.

[1] Given that the holding period used for calculating returns to test CAPM are relatively short, e.g. daily or monthly, the conditioning information is important relatively to longer term returns, refer Pagan (ref, p.30).

Significance tests of the estimated parameters can then be conducted in the Fama and Macbeth (1973) framework. Their results indicates that the size anomaly is, at best, weak within the conditional CAPM framework but the value and momentum anomalies remain significant.

A subsequent study by Boguth, Carlson, Fisher and Simutin (2011) that examined the Lewellen and Nagel's (2006) approach found that alphas were overstated because of an overconditioning bias. They showed that the bias was essentially due to the short-run regressions using the contemporaneous information set that is unknown to the investor under the conditional CAPM. On correcting this bias with an instrumental variable approach, where the instrument may be as simple as the lagged beta, Boguth et al. (2011) report alphas on the momentum portfolios that were up to 30% smaller. Despite this decrease in alphas, they remained significant.

Another approach that avoids assumptions on the state variables is to directly model the conditional moments of the joint distribution of asset returns and use these to infer the conditional beta. Models such as the multivariate version of the autoregressive conditional heteroskedasticity (ARCH) of Engle (1982) provide such capabilities. The initial applications of these models to the asset pricing literature was Bollerslev, Engle and Wooldridge (1988) and recent extensions include Bali and Engle (2010, 2014). These later of these studies offer similar findings to other tests of the average pricing errors of the conditional CAPM in relation to size and momentum, but Bali and Engle (2010, 2014) do report that the average pricing errors of the book-to-market portfolios are insignificant. However, it is noted in applying the ARCH models, several important restrictions are imposed to test the conditional CAPM. First, the joint distribution is often limited to a bivariate distribution of the asset and market returns. While not problematic for testing individual assets, it limits the ability of joint tests across assets. Second, in specifying the conditional CAPM, the reward-to-risk ratio does not vary with time.

Following on from the literature, we propose a simple new approach to testing conditional asset pricing models. The approach follows Lewellen and Nagel (2006), and Bali and Engle (2010, 2014) in that it avoids the use of state variables. However, it contributes to the literature by offering some specific improvements over these previous studies. First, as a test of the conditional CAPM, it avoids making assumptions about the price-of-risk or the reward-to-risk ratio by making all assumptions in relation to the modelling of the volatility dynamics. It is argued, the success of volatility modelling relative to return modelling in finance warrants this approach. Second, like Lewellan and Nagel (2006), we utilise high frequency information when forming estimates of conditional betas and pricing errors but only employ information within the agent's information set. This is achieved by adapting the Mixed Data Sampling (MIDAS) volatility modelling approach of Ghysels, Santa-Clara and Valkanov (2005a, 2005b) to the multivariate setting. Third, we model the second conditional moment of the full asset space in order to undertake a joint test of the significance across all portfolios. In doing so, we derive and justify a new test statistic that only relies on conditional volatility estimates. In addition, we specify the test statistic to account for parameter uncertainty in the volatility models. Finally, our results support the findings of early studies in relation to the importance of conditioning information to measuring pricing errors but we that this result is dependent on the volatility model employed. Specifically, by using high frequency data, pricing errors are reduced by up to 32%, which can be significant for the conclusions drawn for tests of the conditional CAPM.

2 Tests of the Conditional CAPM

Following from Harvey (1989), the N asset multivariate conditional CAPM can be specified as

$$\mathrm{E}\left[\mathbf{r}_t|\Omega_{t-1}\right] = \frac{\mathrm{E}\left[r_{m,t}|\Omega_{t-1}\right]}{\mathrm{VAR}\left[r_{m,t}|\Omega_{t-1}\right]}\mathrm{COV}\left[\mathbf{r}_t, r_{m,t}|\Omega_{t-1}\right], \tag{1}$$

where $\mathrm{E}\left[\mathbf{r}_t|\,\Omega_{t-1}\right]$ is $N\times 1$ vector of conditional expected excess asset returns, Ω_t is the information set from $t-1$, $\mathrm{E}\left[r_{m,t}|\,\Omega_{t-1}\right]$ and $\mathrm{VAR}\left[r_{m,t}|\Omega_{t-1}\right]$ are scalars and measure the conditional expected market return and variance, respectively, and $\mathrm{COV}\left[\mathbf{r}_t, r_{m,t}|\Omega_{t-1}\right]$ is the $N\times 1$ vector of conditional covariances between the return on each asset and the market. Defining the reward-to-risk ratio as $\lambda \equiv \frac{\mathrm{E}[r_{m,t}|\Omega_{t-1}]}{\mathrm{VAR}[r_{m,t}|\Omega_{t-1}]}$, which assumes λ is a constant, the model can be written as

$$\mathrm{E}\left[\mathbf{r}_t|\,\Omega_{t-1}\right] = \lambda\mathrm{COV}\left[\mathbf{r}_t, r_{m,t}|\Omega_{t-1}\right]. \tag{2}$$

To test this model, Harvey (1989) applied a generalised method of moment framework and assumed that the conditional expected return of the asset and the market were linear in a $l\times 1$ vector of lagged state variables, $\boldsymbol{Z}_{t-1}$. As an example, one set of moments conditions used was

$$\boldsymbol{\varepsilon}_t = \begin{bmatrix}\mathbf{u}_t \\ \mathbf{e}_t\end{bmatrix} = \begin{bmatrix}\left[\mathbf{r}_t - \mathbf{Z}_{t-1}\boldsymbol{\delta}\right] \\ \left[\mathbf{r}_t - \boldsymbol{\alpha} - \lambda\left(r_{mt} - \mathbf{Z}_{t-1}\boldsymbol{\delta}_m\right)\left(\mathbf{r}_t - \mathbf{Z}_{t-1}\boldsymbol{\delta}\right)\right]\end{bmatrix}, \tag{3}$$

where $\mathbf{u}_t$ is a $(N+1)\times 1$ vector of 'forecast' errors and $\mathbf{u}_t$ is a $N\times 1$ vector of 'pricing' errors while $\boldsymbol{\delta}$ and $\boldsymbol{\delta_m}$ coefficient matrices with $(l\times N)$ and $(l\times 1)$, respectively. When these moment conditions are interacted with the state variables, there are $(2N+1)\times l$ orthogonality conditions and $(N+1)\times(1+l)$ parameter to estimate. From this specification, the overall model can be tested in an overidentifying restriction test while the restriction that $\boldsymbol{\alpha} = \mathbf{0}$ can also be tested.

To overcome the criticisms associated with using state variables, Lewellen and Nagel (2006) directly estimate conditional alphas and betas from observed returns. In this instance, the conditional CAPM is specified as

$$\mathrm{E}\left[\mathbf{r}_t|\Omega_{t-1}\right] = \frac{\mathrm{COV}\left[\mathbf{r}_t, r_{m,t}|\Omega_{t-1}\right]}{\mathrm{VAR}\left[r_{m,t}|\Omega_{t-1}\right]}\mathrm{E}\left[r_{m,t}|\Omega_{t-1}\right]. \tag{4}$$

Defining $\boldsymbol{\beta}_t \equiv \frac{\mathrm{COV}[\mathbf{r}_t, r_{m,t}|\Omega_{t-1}]}{\mathrm{VAR}[r_{m,t}|\Omega_{t-1}]}$, the model can be written as,

$$\mathrm{E}\left[\mathbf{r}_t|\Omega_{t-1}\right] = \boldsymbol{\beta}_t \mathrm{E}\left[r_{m,t}|\Omega_{t-1}\right]. \tag{5}$$

Short window regressions are applied to a window of length τ where data is sampled at a frequency t where $t < \tau$. For example, using daily or weekly data, the regression for the i^{th} asset in a given quarter, τ is estimated as

$$r_{i,t} = \alpha_{i,\tau} + \beta_{i,\tau} r_{m,t} + \varepsilon_{i,t} \tag{6}$$

The significance of the alphas is then tested in a Fama and MacBeth (1973) framework,

$$\alpha_i = \frac{1}{\mathcal{T}}\sum_{\tau=1}^{\mathcal{T}} \alpha_{i,\tau}, \qquad \mathrm{VAR}\left(\alpha_i\right) = \frac{1}{\mathcal{T}^2}\sum_{\tau=1}^{\mathcal{T}} \left(\alpha_{i,\tau} - \alpha_i\right)^2. \tag{7}$$

Although not tested directly by Lewellen and Nagel (2006), the Fama and MacBeth (1973) framework is readily adjusted for the joint test that $\boldsymbol{\alpha} = \mathbf{0}$.

However, as identified by Boguth et al. (2011), this testing procedure suffers from an *overconditioning* bias as the conditional beta, β_τ, and the market return $r_{m,t}$ are estimated from information not contained within the investor's information set, Ω_{t-1}. As such, the mean of the *overconditioning* bias is defined as

$$\bar{\Delta}_{\alpha}^{OC} \equiv -\mathrm{COV}\left[\left(\beta_{i,\tau} - \beta_{i,\tau|t-1}\right), \left(r_{m,\tau} - \bar{r}_{m,\tau}\right)\right]. \tag{8}$$

Boguth et al. (2011) highlight that as the sample size grows $\beta_{i,\tau|t-1} \rightarrow \beta_{i,\tau}$ and the overconditioning bias should be small. Their empirical results show that correcting for the overconditioning bias by using only information in the investor's information set reduces estimated alphas by between 20% and 40% on momentum portfolios.

Finally, the recent work of Bali and Engle (2010, 2014) has also investigated the validity of the conditional CAPM using a bivariate GARCH in mean framework. Specifically, the system of equations are

$$
\begin{aligned}
r_{i,t} &= \alpha_i + \lambda h_{im,t} + \sqrt{h_{i,t}} u_{i,t} \\
r_{m,t} &= \alpha_m + \lambda h_{m,t} + \sqrt{h_{m,t}} u_{m,t} \\
h_{i,t} &= \omega_i + \alpha_i h_{i,t-1} u_{i,t-1}^2 + \beta_i h_{i,t-1} \\
h_{m,t} &= \omega_m + \alpha_m h_{m,t-1} u_{m,t-1}^2 + \beta_m h_{m,t-1} \\
h_{im,t} &= \rho_{im,t} \sqrt{h_{i,t}} \sqrt{h_{m,t}} \\
\rho_{im,t} &= \frac{q_{im,t}}{\sqrt{q_{ii,t}, q_{mm,t}}} \\
q_{im,t} &= \bar{\rho}_{im} + a_1 \left(u_{i,t-1} u_{m,t-1} - \bar{\rho}_{im} \right) + a_2 \left(q_{im,t-1} - \bar{\rho}_{im} \right)
\end{aligned}
$$

where $h_{i,t}$ and $h_{m,t}$ are the conditional variances, $h_{im,t}$ is the conditional covariance, $\rho_{im,t}$ is the conditional correlation, $u_{i,t}$ and $u_{m,t}$ are standardised residuals while all remaining terms that are not defined are parameters. As the Dynamic Conditional Correlation (DCC) model only uses lagged information in forming conditional expectation, this approach does not suffer from an *overconditioning* bias. In fact, Bali and Engle (2014) highlight this point by comparing their GARCH based appraoch to that of Lewellen and Nagel (2006). However, it is noted that in testing the pricing errors of the conditional CAPM that the approach assumes that the reward-to-risk ratio, λ, is constant through time. Moreover, while Bali and Engle (2010) report a Wald statistic for the joint test, no detail is provided on the calculation of this statistic. Despite these perceived limiations, they find in support for the conditional CAPM in all but the momentum portfolios.

3 A new test of the conditional CAPM

This paper proposes a new test of the conditional CAPM. The proposed test follows in the spirit of Bali and Engle (2010,2014) in that it utilises multivariate volatility models to estimate conditional betas given their success at modelling the conditional variance and covariance of returns relative to models of the conditional mean. However, the novelty of this new test is that it will a provide a joint test of the conditional CAPM that takes into consideration the covariances between pricing errors.

To formulate the test, the setting includes N assets that aggregate to the market portfolio. The $N \times 1$ vector of asset returns from the conditional CAPM at time t is expressed as

$$\mathbf{r}_t = \boldsymbol{\beta}_t \left(\boldsymbol{\theta} \right) r_{m,t} + \mathbf{e}_t. \tag{9}$$

where $\boldsymbol{\beta}_t \left(\boldsymbol{\theta} \right)$ is formed from the conditional volatility models that are governed by a $P \times 1$

parameter vector $\boldsymbol{\theta}$. If we follow a GMM specification and treat the volatility estimates as primitive, the moment conditions for estimating the average pricing error can be written as

$$\boldsymbol{g}_t(\boldsymbol{\alpha}) = \mathrm{E}\left[\boldsymbol{r}_t - \boldsymbol{\alpha} - \boldsymbol{\beta}_t(\boldsymbol{\theta})\, r_{m,t}\right]. \tag{10}$$

Given the specification of these moment conditions and that the system is exactly identfied, $\boldsymbol{\alpha} = T^{-1}\sum_{t=1}^{T}\mathbf{e}_t$. To get the GMM standard errors, $\boldsymbol{g}_t(\boldsymbol{\alpha})$ is differentiated with respect to the parameter vector,

$$\frac{\partial \boldsymbol{g}_t(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} = \boldsymbol{D} = -\boldsymbol{I}_N, \tag{11}$$

where $\boldsymbol{I}_N$ is the $N \times N$ identity matrix. The efficient estimator of the asymptotic variance of $\boldsymbol{g}_t(\boldsymbol{\alpha})$ is

$$\mathbf{S} = \mathrm{E}\left[\mathbf{e}_t\mathbf{e}_t'\right] - \boldsymbol{\alpha}\boldsymbol{\alpha}'. \tag{12}$$

Under the null, $\mathbf{S} = \mathrm{E}\left[\mathbf{e}_t\mathbf{e}_t'\right]$, such that

$$\mathrm{COV}(\boldsymbol{\alpha}) = T^{-1}\boldsymbol{D}^{-1}\boldsymbol{S}\boldsymbol{D}^{-1} = T^{-1}\mathrm{E}\left[\mathbf{e}_t\mathbf{e}_t'\right]. \tag{13}$$

In implementing this test, we note that as the N assets aggregate to the market portfolio, the return on the market portfolio at time $r_{m,t}$ can be written as $r_{m,t} = \mathbf{w}_{t-1}'\mathbf{r}_t$, where $\mathbf{w}_{t-1}$ previous periods $N \times 1$ vector of market value asset weights that sum to one. The $N \times N$ matrix of conditional covariances of asset returns is defined as $\boldsymbol{\Sigma}_t$. Given these inputs, the vector of conditional covariances between asset and market returns is $\boldsymbol{\Sigma}_t\mathbf{w}_{t-1}$ and the market variance is $\mathbf{w}_t'\boldsymbol{\Sigma}_t\mathbf{w}_t$. Then substituting terms into (9),

$$\mathbf{e}_t = \left(\mathbf{I}_N - \boldsymbol{\Sigma}_t\mathbf{w}_{t-1}\left(\mathbf{w}_{t-1}'\boldsymbol{\Sigma}_t\mathbf{w}_{t-1}\right)^{-1}\mathbf{w}_{t-1}'\right)\mathbf{r}_t.$$

Solving for $\mathrm{E}\left[\mathbf{e}_t\mathbf{e}_t'\right]$ and $\mathrm{COV}\left[\boldsymbol{\alpha}\right]$

$$\begin{aligned}
\mathrm{E}_t\left[\mathbf{e}_t\mathbf{e}_t'\right] &= \left(\mathbf{I}_N - \boldsymbol{\Sigma}_t\mathbf{w}_{t-1}\left(\mathbf{w}_{t-1}'\boldsymbol{\Sigma}_t\mathbf{w}_{t-1}\right)^{-1}\mathbf{w}_{t-1}'\right)\mathrm{E}_t\left[\mathbf{r}_t\mathbf{r}_t'\right]\left(\mathbf{I}_N - \boldsymbol{\Sigma}_t\mathbf{w}_{t-1}\left(\mathbf{w}_{t-1}'\boldsymbol{\Sigma}_t\mathbf{w}_{t-1}\right)^{-1}\mathbf{w}_{t-1}'\right)' \\
&= \boldsymbol{\Sigma}_t - \boldsymbol{\Sigma}_t\mathbf{w}_t\left(\mathbf{w}_t'\boldsymbol{\Sigma}_t\mathbf{w}_t\right)^{-1}\mathbf{w}_t'\boldsymbol{\Sigma}_t \\
&= \boldsymbol{\Omega}_t \\
\mathrm{E}\left[\mathbf{e}_t\mathbf{e}_t'\right] &= \bar{\boldsymbol{\Omega}} = T^{-1}\sum_{t=1}^{T}\boldsymbol{\Omega}_t \\
\mathrm{COV}\left[\boldsymbol{\alpha}\right] &= T^{-1}\bar{\boldsymbol{\Omega}}
\end{aligned}$$

Finally, given the estimates of $\boldsymbol{\alpha}$ and $\mathrm{COV}\left[\boldsymbol{\alpha}\right]$, the null hypothesis of no systematic pricing errors is simply tested by the Wald test,

$$\boldsymbol{\alpha}' \mathrm{COV}\left[\boldsymbol{\alpha}\right]^{-1} \boldsymbol{\alpha} \sim \chi^2_m,$$

where there are m restrictions under the null.

In the above approach, the test statistic ignored uncertainty in estimates of the volatility parameter vector. In order to account for this uncertainty when computing the standard errors for $\boldsymbol{\alpha}$, we introduce two derivations that produce the same result. Both note that we have already estimated the volatility parameters using maximum likelihood, but we only present the first approach here. The second approach is presented in Appendix A. Starting with,

$$\frac{\partial ll\left(\boldsymbol{\theta}; \boldsymbol{r}_t\right)}{\partial \boldsymbol{\theta}} = 0. \tag{14}$$

We then introduce an expression

$$\boldsymbol{g}_t\left(\boldsymbol{\theta}_A\right) = \left[\begin{array}{c} \frac{dll(\boldsymbol{\theta}; \boldsymbol{r}_t)}{d\boldsymbol{\theta}} \\ \boldsymbol{r}_t - \boldsymbol{\alpha} - \boldsymbol{\beta}_t\left(\boldsymbol{\theta}\right) r_{m,t} \end{array}\right] \tag{15}$$

where $\boldsymbol{\theta}_A$ is the $(P+N) \times 1$ parameter vector for all volatility model parameters and alphas. In undertaking this approach, it is observationally equivalent to setting $\boldsymbol{\alpha}$ to a vector of N zeros and using a weighting matrix that only weighted the first group of moments and using the non-efficient weighted specification test. GMM standard errors are computed as

$$\frac{\partial \boldsymbol{g}_t\left(\boldsymbol{\theta}_A\right)}{\partial \boldsymbol{\theta}_{\boldsymbol{A}}} = \boldsymbol{D} = \left[\begin{array}{cc} \frac{d^2 ll(\boldsymbol{\theta}; \boldsymbol{r}_t)}{d\boldsymbol{\theta} d\boldsymbol{\theta}'} & \mathbf{0}_{P\times N} \\ -\nabla \boldsymbol{\beta}_t \cdot f_t & -\boldsymbol{I}_N \end{array}\right] \tag{16}$$

where $\nabla \boldsymbol{\beta}_t = \frac{d\boldsymbol{\beta}_t(\boldsymbol{\theta})}{d\boldsymbol{\theta}'}$ is computed by numerically differentiating the function. A consistent estimator can be formed by using the sample average of the product of the derivative of beta with respect to the parameters times the factor return.[2]

To compute the standard errors of the estimates of $\boldsymbol{\alpha}$, we start with

$$COV\left(\boldsymbol{\theta}_A\right) = T^{-1} \boldsymbol{D}^{-1} \boldsymbol{S} \left(\boldsymbol{D}'\right)^{-1}. \tag{17}$$

The derivation of $COV\left(\boldsymbol{\theta}\right)$ is presented in Appendix B. However, the 2,2 sub-matrix element

[2] It is trivial to extend for time-varying pricing errors by modelling them as $\boldsymbol{\alpha}_t = \boldsymbol{\theta}_\alpha \cdot \boldsymbol{Z}_{t-1}$ and replacing the second part of the moment conditions with $(\boldsymbol{r}_t - \boldsymbol{\theta}_\alpha \cdot \boldsymbol{Z}_{t-1} - \boldsymbol{\beta}_t(\boldsymbol{\theta}_\beta) \cdot f_t) \otimes \boldsymbol{Z}_{t-1}$.

corresponds to $COV(\boldsymbol{\alpha})$, which is

$$\begin{aligned}\mathrm{COV}(\boldsymbol{\alpha}) = & T^{-1}\Bigg(\bar{\boldsymbol{\Omega}} + \overline{\nabla\boldsymbol{\beta}_t \cdot f_t}\boldsymbol{H}^{-1} I_{OP} \boldsymbol{H}^{-1}\left(\overline{\nabla\boldsymbol{\beta}_t \cdot f_t}\right)' + T^{-1}\frac{\partial ll(\boldsymbol{\theta}, \boldsymbol{r}_t)}{d\boldsymbol{\theta}}\boldsymbol{\varepsilon}_t'\boldsymbol{H}^{-1}\left(\overline{\nabla\boldsymbol{\beta}_t \cdot f_t}\right)' \\ & + \overline{\nabla\boldsymbol{\beta}_t \cdot f_t}\boldsymbol{H}^{-1} T^{-1}\frac{\partial ll(\boldsymbol{\theta}, \boldsymbol{r}_t)}{d\boldsymbol{\theta}'}\boldsymbol{\varepsilon}_t\Bigg).\end{aligned} \tag{18}$$

The test statistic is then simply,

$$\boldsymbol{\alpha}'\mathrm{COV}(\boldsymbol{\alpha})^{-1}\boldsymbol{\alpha} \sim \chi^2_N.$$

By accounting for parameter uncertainty in the volatility models, the covariance matrix for the pricing errors includes adjustments for the covariance matrix of the maximum likelihood estimates of the volatility parameters, $\boldsymbol{H}^{-1} I_{OP} \boldsymbol{H}^{-1}$, plus an adjustment for the covariance between the estimators.

4 Data

To apply and analyse our proposed approach to testing the conditional CAPM, portfolio returns data, measured at daily and monthly frequencies, for the period 2 January 1958 to 30 December 2016 is obtained from Ken French's data library `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/`.[3] The portfolios for which data are collected include: the decile sorted size and industry portfolios and the Fama-French 25 (quintile, double sorted size and book-to-market portfolios). Additional data is collected for the market value of each portfolio, average firm size in each portfolio, the risk-free rate of return and the market return. In total, the daily data series consists of 14,726 return observations for each portfolio while the equivalent monthly data series has 702 return observations.

As an indication of the properties of the data collected, basic descriptive statistics on daily and monthly returns on the size portfolios, market and risk-free asset are presented Table 1. The size portfolio returns, refer Panel A, are consistent with the finance literature in that the smallest firms generally display the greatest mean and standard deviation of returns, although this decline is not always monotonic. The Jarque-Bera test rejects normality of returns and there are some large return autocorrelations at the first lag for the smaller portfolios.

[INSERT TABLE 1 HERE]

[3]Ken French's data library provides return data from 1 July 1926. However, the periods associated with the stock market crash of 1929, great depression, second world war and recovery period after the second world war are not include due to either their clearly excessive levels of volatility and the merger of CRSP and Compustat databases.

5 Size Results

5.1 Existing tests of unconditional and conditional CAPM

Table 2 presents parameter estimates and test results of the unconditional and conditional CAPMs. The unconditional results of the individual portfolios presented in Panel A indicate that alpha and beta tend to decrease in size, although not monotonically. Six of the estimated alphas display some degree of statistical significance, which provides evidence against the unconditional CAPM. The joint tests in Panel B confirm this result as the unconditional CAPM is rejected at the 1% level.

The remaining columns of Table 2 report test results of the conditional CAPM based on the methodology of Lewellen and Nagel (2006). Clearly, the average α_i and β_i based on this methodology stand in stark contrast to their unconditional counterparts. All alphas are larger in magnitude, generally by a factor of two or more, and almost all are highly significant. It is then of no surprise that this significant under pricing is then evident through relatively small average betas. Moreover, the fact that the smallest betas are reported for the smaller portfolios highlights a major disconnection between the unconditional and conditional models as reported here. A comparison to results of Lewellen and Nagel (2006), albeit limited due to different datasets and the fact they did not report results for monthly rolling regressions, show that the alphas do somewhat reconcile. However, there is no reconciling the reported betas. Finally, and unsurprisingly, the joint Wald test clearly rejects the conditional CAPM.

[INSERT TABLE 2 HERE]

5.2 New Test with Symmetric GARCH-type Models

Subsequent to the rejection of both the unconditional and conditional CAPMs, we then apply our new estimation and testing approach to the conditional CAPM. The derivation of the alpha estimate and joint test statistic in Section 3 did not rely on any specific model of multivariate volatility. As such, a variety of volatility models could be used in testing the conditional CAPM. An extensive literature on conditional volatility models exists and we leave reference of these models to the detailed survey papers of Andersen, Bollerslev, Christoffersen & Diebold (2006), Bauwens, Laurent & Rombouts (2006), and Silvennoinen & Teräsvirta (2009). In this study, we simply select a set of multivariate volatility models capable of modelling moderately sized dimensions $N \leq 25$ but of known varying quality. Our initial set of models, which we describe in brief detail, are all symmetric in that volatility responds equally to positive and negative returns of the same magnitude.

The first model used to capture the time-varying nature of volatility is a simple equally weighting moving average (EQMA) model,

$$\mathbf{H}_t = \frac{1}{M}\sum_{j=1}^{M} \boldsymbol{\varepsilon}_{t-j}\boldsymbol{\varepsilon}'_{t-j}, \tag{19}$$

where M is the number of observations within the window and $\boldsymbol{\varepsilon}_{t-j}$ is a lagged $N \times 1$ demeaned return vector. As the window moves through time, conditional volatility estimates change with the addition of new information and the removal of the oldest information. For this study, we set $M = 60$ as it corresponds to five years of monthly data.

The second model is the exponentially weighted average (EWMA) of RiskMetrics (1996). As suggested by its name, this model employs an exponential weighting scheme that places greater weight of more recent observations. Specifically,

$$\begin{aligned}\mathbf{H}_t &= \sum_{j=1}^{\infty} \lambda^{j-1}(1-\lambda)\,\boldsymbol{\varepsilon}'_{t-j}\boldsymbol{\varepsilon}_{t-j}\\ &= \lambda\mathbf{H}_{t-1} + (1-\lambda)\,\boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}'_{t-1},\end{aligned}$$

where the parameter λ controls the decay weights. Following from RiskMetrics (1996), λ is set at 0.97 for monthly data.

The third model is the Dynamic Conditional Correlation (DCC) of Engle (2002). The DCC provides a parametric alternative to the simple filters described above and, unlike other parametric approaches, it can model the volatility dynamic in moderate dimensions, $N \leq 25$, due to its decomposition of the conditional covariance matrix into conditional volatility and conditional correlation components,

$$\mathbf{H}_t = \mathbf{D}_t\mathbf{R}_t\mathbf{D}_t \tag{20}$$

where $\mathbf{D}_t$ is a $N \times N$ diagonal matrix of conditional standard deviations and $\mathbf{R}_t$ is a $N \times N$, well-defined, positive definite conditional correlation matrix. Through this decomposition, estimation is performed with a two-step quasi-maximum likelihood procedure that first estimates each of the conditional volatility equations and then subsequently estimates the conditional correlation equation. Reference can be made to Engle and Sheppard (2001) for detail on this procedure. A further benefit of this estimation approach is that a variety of conditional volatility and correlation specifications can be used within DCC. To start, we model conditional volatility with the classic GARCH(1,1) of Bollerslev (1986),

$$h_{i,t} = \omega_i + \alpha_i\varepsilon^2_{i,t-1} + \beta_i h_{i,t-1}, \tag{21}$$

where ω_i, α_i and β_i are parameters for each asset i. The conditional correlation is then modelled with Engle's (2002) mean reverting specification,

$$\mathbf{Q}_t = (1 - \alpha - \beta)\,\bar{\mathbf{Q}} + \alpha \mathbf{z}_{t-1}\mathbf{z}'_{t-1} + \beta \mathbf{Q}_{t-1} \tag{22}$$

where α and β are parameters, $\mathbf{Q}_t$ is a $N \times N$ matrix of conditional covariances of standardised returns, which are $z_{i,t-1} = \frac{\epsilon_{t-1}}{\sqrt{h_{i,t-1}}}$, and $\bar{\mathbf{Q}}$ is the unconditional covariance matrix of the standardised returns. Finally, to ensure that correlations are well-defined on the main diagonal,

$$\mathbf{R}_t = \mathbf{Q}_t^{-1/2}\mathbf{Q}_t\mathbf{Q}_t^{-1/2}. \tag{23}$$

The final model considered is an extension of the Mixed Data Sampling (MIDAS) specification of Ghysels, Santa-Clara and Valkanov (2005a, 2005b) to the multivariate space. There exists several potential advantages of the MIDAS specification. First, it can use higher frequency data when modelling a variable of interest that is observed or measured at a lower frequency. For example, this study uses high frequency daily return data to estimate monthly volatility. The perceived advantage of using this approach is twofold in that there is more data to estimate the model and that more recent data is available. Finally, MIDAS provides a parsimonious approach to estimating decay weights through its use of the beta or exponential functions.

To employ the MIDAS in the multivariate setting, we simply follow the DCC decomposition, $\mathbf{H}_t = \mathbf{D}_t\mathbf{R}_t\mathbf{D}_t$. Conditional variances are modelled in the standard MIDAS specification of Ghysels, Santa-Clara and Valkanov (2005b), although we use the beta function rather than the exponential function when estimating weights and allow for the scale parameter, ϕ_i, to be estimated,

$$h_{i,t} = \phi_i \Sigma_{k=0}^{k^{max}} b\left(k, \theta_{i,1}; \theta_{i,2}\right) \varepsilon_{i,t-k}^2. \tag{24}$$

where $b\left(k, \theta_{i,1}; \theta_{i,2}\right)$ is the beta function,

$$b\left(k, \theta_{i,1}; \theta_{i,2}\right) = \frac{f\left(\frac{k}{k^{max}}, \theta_{i,1}; \theta_{i,2}\right)}{\Sigma_{j=1}^{k^{max}} f\left(\frac{j}{k^{max}}, \theta_{i,1}; \theta_{i,2}\right)}, \tag{25}$$

where $f\left(z, a, b\right) = \frac{z^{a-1}(1-z)^{b-1}}{\beta(a,b)}$ where $\beta\left(a, b\right) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, Γ is the gamma function and all weights sum to one given the normalisation that occurs. When estimating the model, we restrict $\phi_i > 0$, $\theta_{i,1} = 1$ and $\theta_{i,2} > 1$. The later of these restrictions ensures that the beta function produces a positive declining weighting scheme for lagged squared returns. Finally, we also estimate the model with Ghysels, Santa-Clara and Valkanov (2005b) restriction that $\phi_i = 22$, which is consistent with assumption that returns are i.i.d.

The modelling of conditional correlation follows a similar process to that of the basic volatilities,

$$\mathbf{Q}_t = \Sigma_{k=0}^{k^{max}} b\left(k, \theta_1; \theta_2\right) \mathbf{z}_{t-k}\mathbf{z}'_{t-k}. \tag{26}$$

No scale parameter is required as it is redundant due the transformation $\mathbf{R}_t = \mathbf{Q}_t^{-1/2}\mathbf{Q}_t\mathbf{Q}_t^{-1/2}$ and the θ parameters are restricted as above. In applying th maximum likelihood, only returns at the lowest sampling frequency (e.g. monthly) are directly used to estimate the conditional correlation equation. This ensures the internal consistency of the model as we cannot standardised daily returns because we do not have estimates of daily conditional volatilities.

Parameter estimates and related statistics of the DCC and MIDAS models are presented in Table 3. To distinguish between the two estimated MIDAS models, we refer to MIDAS model where only the weight parameter is estimated as MIDAS$_w$ and the one where both weight and scale are estimated as MIDAS$_{ws}$. Panel A of Table 3 shows that most estimated parameters are significant. Interestingly, when the scale parameter is estimated for the MIDAS model, it is always larger than 22 and in some instances it is more than three times larger. Formal tests of the relative statistical quality of the volatility equations are conducted using the nested and non-nested likelihood ratio tests described in Vuong (1989). These tests indicate that both MIDAS specifications are rejected when tested against the simple GARCH model, although the MIDAS$_{ws}$ is only rejected for the smaller portfolios. Relative to each other, MIDAS$_{ws}$ is clearly the superior MIDAS specification. Interestingly, despite rejecting the restricted MIDAS model, Panel B indicates that the choice of MIDAS specification has limited impact on the dynamics of the correlation equation as the estimated parameters are almost identical.

Further evaluation of the parameters indicates a high level of persistence in volatility as the sum of the GARCH(1,1) parameters α_i and β_i is close to one. Persistence within the MIDAS specification is somewhat harder to identify given the scale parameter. However, the weighting parameter provides some insight as larger parameters indicate less persistence with more weight placed on more recent observations. From Panel A, it is then evident that the larger weighting parameters for MIDAS$_{ws}$ places greater emphasis on the most recent observations.

[INSERT TABLE 3 HERE]

To see the full impact of the estimated parameters on the weighting functions, the scale term must also be considered. Figure 1 plots the scale adjusted weights applied to lagged squared returns for both MIDAS models for the Small, Medium and Big portfolios. For the Big portfolio the weights are largely the same for the MIDAS models given both models have similar parameter values. However, for the Small and Medium portfolios the weights differ substantially. First, the impact of the larger scale parameter, which is up to three times larger for MIDAS$_{ws}$, is clearly evident as the MIDAS$_{ws}$ places substantially more weight on lagged observations than its restricted counterpart. Second, for the Medium portfolio, the larger weighting parameter estimates have resulted in much more weight being placed on the most recent observations and virtually all the weight is placed on the first 100 observations, which corresponds approximately to the previous four months of observations. Given these plots and the likelihood ratio tests reported above, it is concluded that the restriction that $\phi_i = 22$ is too restrictive for all but the largest of size portfolios.

[INSERT TABLE 1 HERE]

Table 4 reports average pricing errors and associated statistics for the conditional CAPM when based on the EWMA, EQMA, DCC and MIDAS models. Panel A reports and tests the average pricing errors of the individual size portfolios while Panel B reports the average absolute pricing error across all portfolios and our new joint test statistics. The first result of note is that the conditional CAPM based on the EQMA or EWMA models performs poorly across half of the portfolios as the average pricing error of the conditional model is larger than equivalent one for the unconditional model. Although the larger pricing error does not necessarily translate into additional rejections of the model, the conditional CAPM is rejected for half of the portfolios. The joint tests reported in Panel B confirm this result as the conditional CAPM is rejected at the 10% level of significance for both EQMA and EWMA.

Results improve for the conditional CAPM when it is based DCC model. In Panel A, eight of the ten portfolios now have smaller average pricing errors than the unconditional model and only three are significant. Moreover, despite the overall decline in average absolute alpha being a modest 7%, the joint tests do not reject the conditional CAPM based on DCC. Interestingly, the similarity of the unadjusted and adjusted joint test statistics indicates that the effect of parameter uncertainty in the volatility model is limited in this case.

The final results of Table 4 to report are those based on the MIDAS models. These results are quite disparate and warrant some discussion. To start, the conditional CAPM based on the MIDAS$_w$ performs terribly on all but the largest of portfolios. When compared to the results from the DCC based model, the pricing error can be up to twice the size and the overall average absolute pricing error is approximately 50% larger. Individual and joint tests further highlight this terrible performance as the conditional CAPM is rejected for most portfolios and the joint tests are rejected at all conventional levels of significance. These results stand in stark contrast to MIDAS$_{ws}$. By simply providing flexibility in the scale parameter estimated, this MIDAS specification produces the smallest average pricing errors for all portfolios, which results in average absolute pricing error that is approximately 60% smaller than those of the unconditional model and the conditional model based on DCC. Unsurprisingly, none of the individual tests or the joint test statistics reject the conditional CAPM. In addition, the adjusted joint test indicates that the test statistic is not overly sensitive to the volatility parameter estimates. Clearly, these results indicate that the MIDAS model can reduce the pricing errors of the conditional CAPM, although the MIDAS specification employed is of significance importance.

[INSERT TABLE 4 HERE]

While the above results support the MIDAS_{ws} version of the conditional CAPM, sub-sample analysis indicates a significant flaw in the MIDAS_{ws} model. Figure (2) plots the average pricing errors of the unconditional CAPM and the conditional CAPM based on DCC, MIDAS_{w} and MIDAS_{ws} for the full sample, first half and second half of the samples in panels A, B and C, respectively. What is evident from these plots is that MIDAS_{ws} performs poorly in both halves of the sample and that the small average pricing error of MIDAS_{ws} over the full sample is not driven by 'better' pricing *per se*. Instead, the full sample results are a direct result of excessive over-pricing in the second half of the sample that has produced large negative pricing errors for the smaller portfolios. The result of which is artificially low full sample results. Hence, while MIDAS_{ws} performed marginally better than other models in the first half of the sample, we cannot conclude that it produces better results for the conditional CAPM given its relative poor performance in the later half of the sample.

[INSERT FIGURE 2 HERE]

To explain the over-pricing by the conditional CAPM based on MIDAS_{ws} in the later half of the sample, we plot the volatility (standard deviation) of the Small portfolio for the DCC and MIDAS_{ws} models against the difference in the standard deviation of monthly and the square-root of monthly realised variance; Figure 3. What this plot reveals is that the fitted volatility of MIDAS_{ws} deviates dramatically from that of DCC in the later part of the sample. This deviation occurs at exactly the same time that the difference in volatility measures inverts from being generally positive, where squared returns are greater than realised volatility, to being generally negative. Unfortunately for the MIDAS_{ws} model, its specification lacks the flexibility needed to overcome this structural change in the volatility relation. Hence, the scale parameter that bridges the gap between the volatility measures for the first part of the sample produces excessive estimates of volatility in the later part of the sample because of a structural change in the underlying volatility relation. The end result of which is systematic over pricing by the conditional CAPM.

[INSERT FIGURE 3 HERE]

5.3 Testing with autocovariance corrected MIDAS

To overcome the problems of MIDAS$_{ws}$ in situations where the autocovariance of daily return changes, we propose a simple adjustment. Specifically, following from French, Schwert and Stamburg (1987), we include an autocovariance term for lagged returns in the MIDAS model to help bridge the gap between squared monthly returns and the realised measure. Referring to the model as MIDAS$_{wsac}$, where the additional ac in the subscript indicates an estimate for autocovariance, we specify the model as

$$h_{i,t} = \phi_{i,1}\Sigma_{k=0}^{k^{max}} b\left(k, \theta_{i,1}; \theta_{i,2}\right) \varepsilon_{i,t-k}^2 + \phi_{i,2}\mathrm{I}_0\Sigma_{k=0}^{k^{max}-1} b\left(k, \theta_{i,3}; \theta_{i,4}\right) \varepsilon_{i,t-k}\varepsilon_{i,t-k-1}, \qquad (27)$$

where $\phi_{i,2}$ is a strictly positive scale parameter and I_0 is an indicator function that equals one only when $\Sigma_{k=0}^{k^{max}-1} b\left(k, \theta_{i,3}; \theta_{i,4}\right) \varepsilon_{i,t-k}\varepsilon_{i,t-k-1} > 0$ to ensure that variances remain positive. It is noted that the indicator function still allows for negative values within the truncation period as long as the weighted sum of all autocovariances is non-negative.

Table 5 reports the parameter estimates, average pricing errors and related statistics for the MIDAS$_{wsac}$ model. For this version of MIDAS, the autocovariance parameters are all significant and their inclusion has a dramatic effect on the scale parameter. In particular, the scale parameter on squared returns reduces for all portfolios and is never significantly larger than 22, although it is significantly smaller than 22 on three occasions. These results show that the autocovariance term is an important as it captures part of the volatility dynamic that previously was captured by the scale term. Figure 4 further highlights the value of the autocovariance term by plotting the weights on lagged squared returns for the MIDAS$_{wsac}$ and MIDAS$_{ws}$ models. Consistent with reported parameter values, the Small portfolio sees a dramatic reduction in weights with the inclusion of the autocovariance term while more modest reductions are noted for the Medium and Big portfolios. Although the weights on the larger portfolios show modest changes, the likelihood ratio tests reported in Table 5 show that the MIDAS$_{wsac}$ specification is superior to its restricted counterpart. Moreover, unlike earlier tests that the MIDAS$_{ws}$ was inferior to the GARCH model, the non-nested likelihood ratio does not find that the MIDAS$_{wsac}$ is inferior to GARCH model. In fact, MIDAS$_{wsac}$ is found to be superior for the Big portfolio.

In terms of average pricing errors, the average absolute alpha reported in Table 5 is larger than spurious ones reported for MIDAS$_{ws}$ but, at 0.078, it is still 32% smaller than that reported for DCC. Hence, it is of no surprise that joint tests cannot reject the conditional CAPM.

[INSERT TABLE 5 HERE]

[INSERT FIGURE 4 HERE]

Figure 5 plots the average pricing errors of MIDAS$_{wsac}$ against the unconditional CAPM and DCC. The primary results of these plots is that MIDAS$_{wsac}$ produces smaller average pricing errors than its two counterparts without the over-pricing errors that afflicted MIDAS$_{ws}$ in the second half of the sample. Finally, Figure 6 demonstrates that this improved performance of MIDAS$_{wsac}$ is driven, in part, by its ability not to deviate from other volatility measures when the autocovariance relation inverts.

[INSERT FIGURE 5 HERE]

[INSERT FIGURE 6 HERE]

5.4 Testing with a Asymmetric GARCH-type Models

Our final addition into the modelling of conditional volatility is to include asymmetric effects into the conditional volatility equations of the DCC and MIDAS models. It is well established in the literature that the reaction of volatility to negative returns is larger than positive returns of the same magnitude. Often referred to as the leverage effect, this effect is easily incorporated into the GARCH and MIDAS specifications. For GARCH, we employ the asymmetric model of Glosten, Jagannathan and Runkle (1993),

$$h_{i,t} = \omega_i + (\alpha_i + \mathrm{I}_0\delta_i)\,\varepsilon_{i,t-1}^2 + \beta h_{i,t-1}, \tag{28}$$

where I_0 is one whenever $\varepsilon_{i,t-1} < 0$ and δ is a parameter. The asymmetric MIDAS specification follows Ghysels, Santa-Clara and Valkanov (2005b),

$$h_{i,t} = 22\left[\phi_{i,2}\Sigma_{k=0}^{k^{max}} b\,(k, \theta_{i,1}; \theta_{i,2})\,\mathrm{I}_0^- \varepsilon_{i,t-k}^2 + (2 - \phi_{i,2})\,\Sigma_{k=0}^{k^{max}} b\,(k, \theta_{i,3}; \theta_{i,4})\,\mathrm{I}_0^+ \varepsilon_{i,t-k}^2\right] \tag{29}$$

where the $\phi_{i,2}$ parameter now controls the weight placed on squared negative and positive returns, and I_0^- and I_0^+ are indicator functions that identify positive and negative returns. This specification allows for different weighting functions to be applied to squared negative and positive returns. However, as is our experience reported below, estimating this model can prove challenging as the asymmetric scale term often sits on a boundary condition which causes identification issues that result in difficulties in estimating the weighting parameter and associated standard errors. Therefore, when extending the model to allow for an estimated scale, we propose an alternative approach to overcome estimation issues. In our extension, we simply use separate scale parameters for negative and positive squared returns but apply the weighting function to each. In this case, if either of the scale terms is zero, the model is still identified. Specifically,

$$h_{i,t} = \phi_{i,1}\Sigma_{k=0}^{k^{max}} b\left(k, \theta_{i,1}; \theta_{i,2}\right) \mathrm{I}_0^{-} \varepsilon_{i,t-k}^2 + \phi_{i,2}\Sigma_{k=0}^{k^{max}} b\left(k, \theta_{i,1}; \theta_{i,2}\right) \mathrm{I}_0^{+} \varepsilon_{i,t-k}^2. \tag{30}$$

We did consider using separate weighting functions and scale parameters that allow the scale parameters to be negative. However, even after ensuring all volatility estimates are positive, this approach does not overcome identification issues as the set of possible scale parameter values includes zero. Finally, we also estimate the model with autocovariance term,

$$\begin{aligned} h_{i,t} = {} & \phi_{i,1}\Sigma_{k=0}^{k^{max}} b\left(k, \theta_{i,1}; \theta_{i,2}\right) \mathrm{I}_0^{-} \varepsilon_{i,t-k}^2 + \phi_{i,2}\Sigma_{k=0}^{k^{max}} b\left(k, \theta_{i,1}; \theta_{i,2}\right) \mathrm{I}_0^{+} \varepsilon_{i,t-k}^2 \\ & + \phi_{i,3}\mathrm{I}_0\Sigma_{k=0}^{k^{max}-1} b\left(k, \theta_{i,3}; \theta_{i,4}\right) \varepsilon_{i,t-k}\varepsilon_{i,t-k-1}. \end{aligned} \tag{31}$$

It is noted that we employ the same parameter restrictions on the scale and weighting parameters as described early. Therefore, even in the most general specification estimated, it is still somewhat parsimonious as it includes only five estimated parameters (3 scale and 2 weighting).

Parameter estimates for the DCC and MIDAS models are presented in Table 6. Estimates of the GJR-GARCH model show that significant asymmetric effects are present in seven of the ten size portfolios. Interestingly, the Small, '5' and Big portfolios have no significant ARCH terms. The leverage effect is also present across the MIDAS models, however, its impact shows some degree a variability. For example, MIDAS$_{w-asym}$, which has the scale term set to 22, the parameter that determines the weight applied to squared negative and positive returns, places significantly more weight on squared negative returns across all ten portfolios. That being said, all estimated volatility equations apart from the Big portfolio have at least one estimated parameter sitting on a boundary condition. MIDAS$_{wsac-asym}$ produces similar results with seven of the volatility equations placing no weight on squared positive returns. The only result that is a little less clear in terms of the asymmetric effect is for MIDAS$_{ws-asym}$. In this instant, the three smallest portfolios place greater weight on positive squared returns. It is important to note that these results do differ from those reported Ghysels et al. (2005b) who found that the asymmetry favoured positive returns within MIDAS. However, time periods and portfolios do differ, although it is noted that the result for the Big, which is closes to Ghysels et al. (2005b) market portfolio, still differs.

Three additional results from Table 6 warrant further comment. First, the autocovariance term in MIDAS$_{wsac-asym}$ remains significant with the leverage effect. Second, likelihood ratio tests find that only the MIDAS$_{wsac-asym}$ is not inferior to the GJR-GARCH. They also confirm the statistical importance of the asymmetric term. Finally, as noted with earlier estimates, the different approaches to modelling conditional volatility appear to have minimal impact on the conditional correlation equation.

[INSERT TABLE 6 HERE]

Table 7 and Figure 7 present average pricing errors for the models that include asymmetric terms. The most striking results here are that the asymmetric term does not necessarily improve pricing. Obviously, as beta is a ratio of conditional covariance to variance, the ratio of 'better' estimates does not necessarily produce a 'better' beta. That being said, the differences between the symmetric and asymmetric models appear trivial. However, it is noted that the change from the unadjusted to adjusted test statistic is relatively large for the MIDAS$_{w-asym}$ and this most likely reflects the model parameters sitting on boundary conditions and their large standard errors. Moreover, based on the unadjusted test statistic, the conditional CAPM based on DCC is now rejected at the 10% level when GJR-GARCH is used as the univariate volatility model. Finally, the asymmetric MIDAS still performs poorly on the small portfolios when the autocovariance term is excluded. This result provides further support for using the autocovariance term in MIDAS models when used to estimate conditional monthly volatility based on daily returns.

[INSERT TABLE 7 HERE]

[INSERT FIGURE 7 HERE]

6 FF25 and Industry Portfolio Results

As an examination of the robustness of the results reported, we extend our analysis to the Fama-French 25 (FF25) and Industry portfolios. For the sake of brevity and given earlier results, we simply present summary results for the unconditional CAPM and conditional CAPM based on the DCC$_{GJR}$, MIDAS$_{wsac}$ and MIDAS$_{wsac-asym}$ models. Table 8 presents the average absolute alpha for the four versions of CAPM and the test statistics of the conditional models. Panel A shows that the reduction in the average absolute pricing error for conditional CAPM based on the DCC and MIDAS models is 12%, 29% and 24%, respectively. Although somewhat smaller in magnitude, these results are similar to the Size portfolio results where the reduction of average pricing error of the conditional CAPM was greatest for the MIDAS models. However, despite the reduction in pricing errors, the conditional CAPM is still rejected for the FF25 for all models at the 1% level in all cases bar the adjusted test statistic for the MIDAS$_{wsac}$ where it is rejected at the 5% level.

The equivalent Industry portfolio results are reported in Panel B. Once again, the reductions in average absolute pricing error is greatest for the MIDAS specification, although the reduction is a more modest 14%. That being said, the joint test statistics reveal that conditional CAPM is only rejected when its estimates are from the DCC model. In part, these results reflect the additional variation in pricing errors that comes from the MIDAS model. However, as most examples have seen the MIDAS models produce the smallest average pricing errors, it is evident that our conclusions come from an improvement in pricing accuracy, at least on average.

[INSERT TABLE 8 HERE]

7 Conclusion

This paper extends tests of the conditional CAPM based on GARCH type models such that the test only makes assumptions on the volatility dynamics rather than the price of risk. In doing so, the testing procedure avoids the over-conditioning bias of earlier papers by only using information within the investor's information set and removes limiting assumptions such as a constant reward-to-risk ratio. A derivation of the Wald type test statistic is provided where the test statistic does and does not account for uncertainty in the volatility parameter estimates. Finally, the effect of the volatility model on the testing is considered.

Applied to the size sorted portfolios, several new and relevant results are evident. First, relative to the unconditional alpha, the conditional alphas reported are up to 32% smaller. However, this result is dependent on the choice in volatility model as this reduction is only reported for the MIDAS model that employs daily return data, estimates a scale parameter and accounts for the autocovariance in daily returns. The models that employ monthly returns show smaller improvements, at best, over the unconditional model in terms of average pricing errors. However, the improvement was sufficient enough for the conditional CAPM not to be rejected for the size portfolios. Second, while parameter uncertainty of the volatility impacts the tests, its effects are generally limited in most cases bar those where parameters are close to boundary conditions. As such, the basic test statistic that treats volatility as known works reasonably well if one considers that the parameters of the volatility model are reasonably well estimated. Finally, consistent with other findings in the literature, the conditional CAPM cannot be rejected based on tests of the average pricing errors of the size and industry portfolios but it is rejected for the Fama-French 25 portfolio.

8 Appendix A

If we specify the moment conditions for the estimation of the volatility model parameters and the conditional CAPM as

$$\boldsymbol{g}_t(\boldsymbol{\theta}_A) = \begin{bmatrix} \frac{dll(\boldsymbol{\theta};\boldsymbol{r}_t)}{d\boldsymbol{\theta}} \\ \boldsymbol{r}_t - \boldsymbol{\beta}_t(\boldsymbol{\theta})\, r_{m,t} \end{bmatrix} \tag{32}$$

Following from Cochrane (2001, p.205), we use a prespecified weighting matrix, $\boldsymbol{W}$, that uses the moments from the volatility models to estimate the volatility parameters but then uses the remainin moment from the conditional CAPM to test the conditional CAPM. Specifically,

$$\boldsymbol{g}_t(\boldsymbol{\theta}_A)'\,\boldsymbol{W}\boldsymbol{g}_t(\boldsymbol{\theta}_A) = \boldsymbol{g}_t(\boldsymbol{\theta}_A)' \begin{bmatrix} \boldsymbol{I}_{P\times P} & \boldsymbol{0}_{P\times N} \\ \boldsymbol{0}_{N\times P} & \boldsymbol{0}_{N\times N} \end{bmatrix} \boldsymbol{g}_t(\boldsymbol{\theta}_A). \tag{33}$$

Given $\boldsymbol{W}$ is a presepecified weighting matrix, the asymptotic covariance matrix of $\boldsymbol{g}_t(\boldsymbol{\theta}_A)$ is

$$\mathrm{COV}\left[\boldsymbol{g}_t(\boldsymbol{\theta}_A)\right] = T^{-1}\left(\boldsymbol{I}_{(P+N)\times(P+N)} - \boldsymbol{D}\left(\boldsymbol{D}'\boldsymbol{W}\boldsymbol{D}\right)^{-1}\boldsymbol{D}'\boldsymbol{W}\right)\boldsymbol{S}\left(\boldsymbol{I}_{(P+N)\times(P+N)} - \boldsymbol{W}\boldsymbol{D}\left(\boldsymbol{D}'\boldsymbol{W}\boldsymbol{D}\right)^{-1}\boldsymbol{D}'\right) \tag{34}$$

Defining,

$$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{H} \\ -\overline{\nabla\boldsymbol{\beta}_t f_t} \end{bmatrix} \tag{35}$$

and

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{I}_{OP} & \boldsymbol{S}_{(1,2)} \\ \boldsymbol{S}'_{(1,2)} & \bar{\boldsymbol{\Omega}} \end{bmatrix}, \tag{36}$$

where $\boldsymbol{H} = \frac{d^2 ll(\boldsymbol{\theta};\boldsymbol{y})}{d\boldsymbol{\theta}d\boldsymbol{\theta}'}$, $\boldsymbol{I}_{OP}$ is the informaiton matrix and $\boldsymbol{S}_{(2,1)} = \mathrm{E}(\frac{\partial ll(\boldsymbol{\theta},\boldsymbol{y}}{d\boldsymbol{\theta}}\boldsymbol{\varepsilon}'_t)$. Noting the following results,

$$\boldsymbol{D}'\boldsymbol{W} = \begin{bmatrix} \boldsymbol{H} & \boldsymbol{0}_{P\times N} \end{bmatrix}$$

$$\boldsymbol{D}'\boldsymbol{W}\boldsymbol{D} = \boldsymbol{H}^2$$

$$\boldsymbol{D}\left(\boldsymbol{D}'\boldsymbol{W}\boldsymbol{D}\right)^{-1}\boldsymbol{D}'\boldsymbol{W} = \begin{bmatrix} \boldsymbol{I}_{P\times P} & \boldsymbol{0}_{P\times N} \\ -\overline{\nabla\boldsymbol{\beta}_t f_t}\boldsymbol{H}^{-1} & \boldsymbol{0}_{N\times N} \end{bmatrix}$$

$$\boldsymbol{W}\boldsymbol{D}\left(\boldsymbol{D}'\boldsymbol{W}\boldsymbol{D}\right)^{-1}\boldsymbol{D}' = \begin{bmatrix} \boldsymbol{I}_{P\times P} & -\overline{\nabla\boldsymbol{\beta}'_t f_t}\boldsymbol{H}^{-1} \\ \boldsymbol{0}_{N\times P} & \boldsymbol{0}_{N\times N} \end{bmatrix},$$

such that

$$
\begin{aligned}
\text{COV}\left[\boldsymbol{g}_t\left(\boldsymbol{\theta}_A\right)\right] &= T^{-1}\left(\boldsymbol{I}_{(P+N)\times(P+N)} - \begin{bmatrix} \boldsymbol{I}_{P\times P} & \mathbf{0}_{P\times N} \\ -\overline{\nabla\boldsymbol{\beta}_t f_t}\boldsymbol{H}^{-1} & \mathbf{0}_{N\times N} \end{bmatrix}\right) \boldsymbol{S} \left(\boldsymbol{I}_{(P+N)\times(P+N)} - \begin{bmatrix} \boldsymbol{I}_{P\times P} & -\overline{\nabla\boldsymbol{\beta}_t' f_t}\boldsymbol{H}^{-1} \\ \mathbf{0}_{N\times P} & \mathbf{0}_{N\times N} \end{bmatrix}\right) \\
&= T^{-1}\begin{bmatrix} \mathbf{0}_{P\times P} & \mathbf{0}_{P\times N} \\ \overline{\nabla\boldsymbol{\beta}_t f_t}\boldsymbol{H}^{-1} & \boldsymbol{I}_{N\times N} \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_{OP} & \boldsymbol{S}_{(1,2)} \\ \boldsymbol{S}'_{(1,2)} & \bar{\boldsymbol{\Omega}} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{P\times P} & \overline{\nabla\boldsymbol{\beta}_t' f_t}\boldsymbol{H}^{-1} \\ \mathbf{0}_{N\times P} & \boldsymbol{I}_{N\times N} \end{bmatrix} \\
&= T^{-1}\begin{bmatrix} \mathbf{0}_{P\times P} & \mathbf{0}_{P\times N} \\ \overline{\nabla\boldsymbol{\beta}_t f_t}\boldsymbol{H}^{-1}\boldsymbol{I}_{OP} + \boldsymbol{S}'_{(1,2)} & \overline{\nabla\boldsymbol{\beta}_t f_t}\boldsymbol{H}^{-1}\boldsymbol{S}_{(1,2)} + \bar{\boldsymbol{\Omega}} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{P\times P} & \overline{\nabla\boldsymbol{\beta}_t' f_t}\boldsymbol{H}^{-1} \\ \boldsymbol{I}_{N\times P} & \mathbf{0}_{N\times N} \end{bmatrix} \\
&= T^{-1}\begin{bmatrix} \mathbf{0}_{P\times P} & \mathbf{0}_{P\times N} \\ \mathbf{0}_{N\times P} & \overline{\nabla\boldsymbol{\beta}_t f_t}\boldsymbol{H}^{-1}\boldsymbol{I}_{OP}\boldsymbol{H}^{-1}\overline{\nabla\boldsymbol{\beta}_t' f_t} + \boldsymbol{S}'_{(1,2)}\boldsymbol{H}^{-1}\overline{\nabla\boldsymbol{\beta}_t' f_t} + \overline{\nabla\boldsymbol{\beta}_t f_t}\boldsymbol{H}^{-1}\boldsymbol{S}_{(1,2)} + \bar{\boldsymbol{\Omega}} \end{bmatrix}
\end{aligned}
$$

9 Appendix B

The inverse of the matrix $\boldsymbol{D}$ can be computed using the partitioned matrix inverse formula, which simplifies greatly as the 1,2 term is zero. In fact we only need the lower row:

$$\boldsymbol{D}^{-1}_{(1,1)} = \boldsymbol{H}^{-1}$$

where we are simplifying notation with

$$\boldsymbol{H} = \frac{d^2 ll(\boldsymbol{\theta}; \boldsymbol{y})}{d\boldsymbol{\theta} d\boldsymbol{\theta}'}.$$

$$\boldsymbol{D}^{-1}_{(2,2)} = \boldsymbol{I}_{N\times N}$$

$$\boldsymbol{D}^{-1}_{(1,2)} = \boldsymbol{0}_{P\times N}$$

$$\boldsymbol{D}^{-1}_{(2,1)} = -\boldsymbol{I}_{N\times N} \cdot \overline{\nabla \boldsymbol{\beta}_t \cdot f_t} \cdot \boldsymbol{H}^{-1}$$

We have the 2,2 sub-matrix giving

$$cov(\boldsymbol{\alpha}) = T^{-1}[-\overline{\nabla \boldsymbol{\beta}_t \cdot f_t} \cdot \boldsymbol{H}^{-1}, \boldsymbol{I}_N] \cdot \begin{bmatrix} \boldsymbol{I}_{OP} & \boldsymbol{S}_{(1,2)} \\ \boldsymbol{S}'_{(1,2)} & \boldsymbol{\Sigma} \end{bmatrix} \cdot \begin{bmatrix} -(\overline{\nabla \boldsymbol{\beta}_t \cdot f_t} \cdot \boldsymbol{H}^{-1})' \\ \boldsymbol{I}_N \end{bmatrix}$$

, Where the moment variance-covariance matrix is given by

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{I}_{OP} & \boldsymbol{S}_{(1,2)} \\ \boldsymbol{S}'_{(1,2)} & \boldsymbol{\Sigma} \end{bmatrix}$$

and the off-diagonal term is

$$\boldsymbol{S}_{(2,1)} = E(\frac{\partial ll(\boldsymbol{\theta}, \boldsymbol{y}}{d\boldsymbol{\theta}} \boldsymbol{\varepsilon}'_t)$$

.

$$cov(\boldsymbol{\alpha}) = T^{-1}[-\overline{\nabla \boldsymbol{\beta}_t \cdot f_t} \cdot \boldsymbol{H}^{-1} \cdot \boldsymbol{I}_{OP} + \boldsymbol{S}_{(2,1)}, -\overline{\nabla \boldsymbol{\beta}_t \cdot f_t} \cdot \boldsymbol{S}_{(1,2)} + \boldsymbol{\Sigma}] \cdot \begin{bmatrix} -(\overline{\nabla \boldsymbol{\beta}_t \cdot f_t} \cdot \boldsymbol{H}^{-1})' \\ \boldsymbol{I}_N \end{bmatrix}$$

so we have

$$cov(\boldsymbol{\alpha}) = T^{-1}\left(\boldsymbol{\Sigma} + \overline{\nabla \boldsymbol{\beta}_t \cdot f_t} \cdot \boldsymbol{H}^{-1} \cdot \boldsymbol{I}_{OP} \cdot \boldsymbol{H}^{-1} \cdot (\overline{\nabla \boldsymbol{\beta}_t \cdot f_t})' + \boldsymbol{S}_{(2,1)} \cdot \boldsymbol{H}^{-1} \cdot (\overline{\nabla \boldsymbol{\beta}_t \cdot f_t})' - \overline{\nabla \boldsymbol{\beta}_t \cdot f_t} \cdot \boldsymbol{S}_{(1,2)}\right)$$

The terms following the $\boldsymbol{\Sigma}$ account for parameter uncertainty, as if we knew the parameters driving the dynamics of the conditional $\boldsymbol{\beta}_t$s then the covariance matrix of the sample abnormal returns will simply be $T^{-1}\boldsymbol{\Sigma}$.

$$cov(\boldsymbol{\alpha}) = T^{-1}\left(\boldsymbol{\Sigma} + \overline{\nabla \boldsymbol{\beta}_t \cdot f_t} \cdot cov(\hat{\boldsymbol{\theta}}_{MLE}) \cdot (\overline{\nabla \boldsymbol{\beta}_t \cdot f_t})' + \boldsymbol{S}_{(2,1)} \cdot \boldsymbol{H}^{-1} \cdot (\overline{\nabla \boldsymbol{\beta}_t \cdot f_t})' - \overline{\nabla \boldsymbol{\beta}_t \cdot f_t} \cdot \boldsymbol{H}^{-1} \cdot \boldsymbol{S}_{(1,2)}\right)$$

The presence of the covariance matrix of the MLE parameter estimates driving the dynamics of $\boldsymbol{\theta}$ helps make this point more clearly. So the variance of the abnormal returns equals the variance of the pricing errors if we new the betas exactly plus a delta method adjustment for the estimation error in the parameter estimates, plus an adjustment for the covariance between the estimators. This will produce larger standard errors because we need to estimate the parameters driving the beta dynamics.

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	Unconditional		L.N.	
	α	β	α	$\bar{\beta}_t$
Panel A: Portfolio results				
Small	0.198	1.111***	0.462***	0.719***
	(0.157)	(0.035)	(0.166)	(0.014)
2	0.110	1.213***	0.304**	0.931***
	(0.134)	(0.030)	(0.141)	(0.015)
3	0.188*	1.203***	0.367***	0.961***
	(0.112)	(0.025)	(0.120)	(0.013)
4	0.125	1.171***	0.292***	0.965***
	(0.103)	(0.023)	(0.110)	(0.012)
5	0.175**	1.161***	0.334***	0.967***
	(0.088)	(0.020)	(0.095)	(0.010)
6	0.125*	1.110***	0.283***	0.938***
	(0.073)	(0.017)	(0.079)	(0.007)
7	0.125**	1.110***	0.277***	0.943***
	(0.062)	(0.014)	(0.069)	(0.007)
8	0.096*	1.091***	0.191***	0.969***
	(0.054)	(0.012)	(0.059)	(0.005)
9	0.075*	1.010***	0.096**	0.971***
	(0.041)	(0.009)	(0.044)	(0.004)
Big	– 0.025	0.928***	– 0.135	1.040***
	(0.037)	(0.008)	(0.043)	(0.004)
Panel B: Joint tests				
Wald	26.690***		26.691***	
G.R.S.	2.624***			

Table 2: Tests of the unconditional CAPM and the conditional CAPM for the Size portfolios. Panel A reports estimated alphas, betas and their associated standard errors (in parentheses) for the unconditional and conditional CAPM. The unconditional model parameters are estimated using ordinary least squares and standard errors are corrected for heteroskedasticity. The conditional parameters are estimated in the rolling regression approach of Lewellen and Nagel (2006) where daily data is used to estimate coefficients within a given monthly window. The reported standard errors are based on the methodology of Fama-Macbeth (1973). Panel B reports the joint test statistics for the unconditional (Wald and G.R.S.) and conditional model (Wald). Significance at the 10%, 5% and 1% levels are indicated by *, ** and ***, respectively.

	Daily									Monthly								
Portfolio	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\rho}_1$	$\hat{\rho}_2$	$\hat{\rho}_3$	$JBTest$	ω	α	β	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\rho}_1$	$\hat{\rho}_2$	$\hat{\rho}_3$	$JBTest$	ω	α	β
Panel A: Size Portfolios																		
Small	12.681	14.100	0.200	0.082	0.087	0.001	0.016	0.172	0.813	14.211	21.381	0.233	0.009	−0.020	0.001	2.915	0.098	0.831
2	12.951	17.232	0.087	0.031	0.048	0.001	0.014	0.134	0.857	13.928	21.434	0.143	−0.022	−0.047	0.001	2.672	0.073	0.861
3	13.933	17.035	0.105	0.020	0.036	0.001	0.015	0.128	0.861	14.727	20.438	0.131	−0.049	−0.043	0.001	2.668	0.060	0.867
4	13.212	16.781	0.107	0.014	0.035	0.001	0.014	0.119	0.870	13.826	19.661	0.131	−0.045	−0.040	0.001	2.537	0.070	0.856
5	13.541	16.705	0.113	0.003	0.027	0.001	0.014	0.116	0.872	14.083	19.025	0.120	−0.030	−0.026	0.001	2.566	0.079	0.841
6	12.974	15.705	0.131	−0.003	0.026	0.001	0.014	0.117	0.870	13.424	17.824	0.126	−0.059	−0.016	0.001	2.006	0.065	0.863
7	13.004	15.788	0.129	−0.009	0.019	0.001	0.012	0.114	0.875	13.356	17.488	0.124	−0.031	0.001	0.001	1.649	0.087	0.853
8	12.655	15.909	0.108	−0.018	0.014	0.001	0.010	0.103	0.888	12.902	17.046	0.095	−0.045	−0.006	0.001	1.374	0.093	0.855
9	12.087	15.531	0.078	−0.031	0.012	0.001	0.009	0.094	0.898	12.160	15.597	0.099	−0.039	0.004	0.001	1.077	0.102	0.850
Big	10.604	15.845	0.016	−0.041	−0.003	0.001	0.007	0.078	0.916	10.413	14.371	0.022	−0.031	0.048	0.001	0.859	0.104	0.849
Panel B: Market and Risk-free Asset																		
Market	11.157	15.297	0.058	−0.029	0.006	0.001	0.009	0.089	0.903	11.166	15.060	0.074	−0.040	0.025	0.001	0.988	0.098	0.854
Risk-Free	4.485	0.194	0.998	0.997	0.995	0.001	0.000	0.261	0.739	4.495	0.895	0.971	0.952	0.940	0.001	0.000	0.258	0.742

Table 1: Descriptive statistics for daily and monthly returns on the Size portfolios are reported in Panel A while the market portfolio and risk-free asset monthly returns are reported in Panel B. All data is collected from Ken French's data library and covers the period of January 1958 to June 2016. For each return frequency, the first two columns of statistics report the annualised mean and standard deviation of returns. The next four columns report the autocorrelation of returns for the first three lags and the p-value from the Jarque-Bera test ($JBTest$) of normality. Finally, the estimated parameters of the GARCH(1,1) are reported in the final three columns for insight into the persistence of squared returns. The GARCH model is estimate using maximum likelihood.

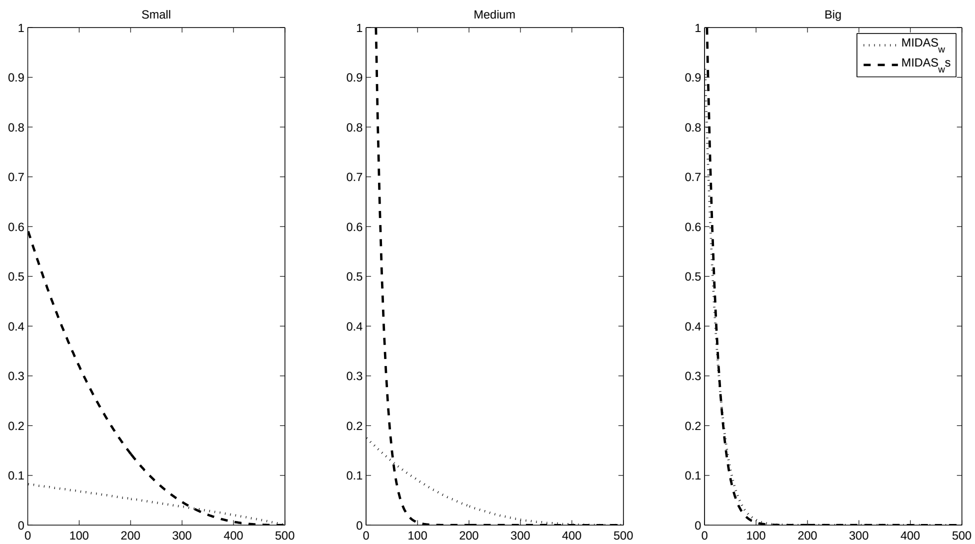

Figure 1: MIDAS weight plots for the MIDAS$_w$ and MIDAS$_{ws}$ for the Small, Medium (portfolio '5') and Big portfolios. Weights are adjusted for the estimated scale parameter.

	(A) DCC			(B) MIDAS$_w$		(C) MIDAS$_{ws}$		L.R. Tests		
	ω	α	β	ϕ_1	θ_2	ϕ_1	θ_2	$H0: A = B$	$H0: A = C$	$H0: B = C$
Panel A: Conditional Variances										
Small	3.022**	0.099***	0.825***	22.000	1.872***	78.321***	3.777	4.586***	3.751***	784.952***
	(1.180)	(0.035)	(0.038)	(–)	(0.398)	(9.394)	(3.521)			
2	2.859***	0.077***	0.850***	22.000	2.201***	68.702***	43.171***	3.773***	2.542**	430.075***
	(0.981)	(0.027)	(0.023)	(–)	(0.638)	(8.449)	(13.367)			
3	2.898***	0.073**	0.845***	22.000	2.545***	61.025***	43.751***	3.996***	2.667***	316.047***
	(0.990)	(0.029)	(0.030)	(–)	(0.761)	(7.228)	(12.581)			
4	2.788***	0.083***	0.834***	22.000	3.003***	55.722***	39.707**	4.081***	2.823***	271.983***
	(0.953)	(0.030)	(0.032)	(–)	(1.100)	(6.396)	(15.931)			
5	2.978***	0.101***	0.805***	22.000	3.990**	50.056***	29.419***	3.722***	2.462**	228.236***
	(1.076)	(0.037)	(0.045)	(–)	(1.817)	(5.755)	(10.482)			
6	2.237***	0.093***	0.826***	22.000	4.089**	46.798***	29.895**	3.688***	2.097**	179.647***
	(0.778)	(0.033)	(0.039)	(–)	(1.838)	(5.199)	(12.755)			
7	1.765***	0.105***	0.830***	22.000	4.366*	44.193***	28.425**	3.581***	1.616	149.245***
	(0.675)	(0.035)	(0.044)	(–)	(2.535)	(5.059)	(12.897)			
8	1.441***	0.119***	0.826***	22.000	5.838**	40.822***	34.584**	3.498***	1.216	112.594***
	(0.549)	(0.031)	(0.040)	(–)	(2.752)	(4.107)	(15.976)			
9	1.085**	0.124***	0.826***	22.000	6.986**	34.353***	29.372*	2.630***	1.144	52.938***
	(0.513)	(0.031)	(0.044)	(–)	(3.301)	(3.520)	(17.486)			
Big	0.817**	0.120***	0.836***	22.000	21.238**	24.024***	25.472**	– 1.332	– 1.100	2.125
	(0.356)	(0.024)	(0.031)	(–)	(9.165)	(1.973)	(10.728)			
Panel B: Conditional Correlation										
Q_t		0.019***	0.960***		1.216***		1.274***			
		(0.005)	(0.014)		(0.085)		(0.088)			

Table 3: Volatility parameter estimates and likelihood ratio tests for the DCC and MIDAS models for the Size portfolios. Estimates of DCC parameters are based on monthly data while the MIDAS models use daily data for conditional volatility parameters but monthly data for the conditional correlation parameters. Models are estimated using a two-step maximum likelihood approach where the first step estimates the conditional volatility equations while the second step estimates the conditional correlation equation. Reported standard errors (in parentheses) for the volatility parameters are calculated numerically. Vuong (1989) likelihood ratio test statistics reported for non-nested models and standard likelihood ratio test reported for nested models. Significance at the 10%, 5% and 1% levels are indicated by *, ** and ***, respectively.

	Unconditional	EQMA	EWMA	DCC	$MIDAS_w$	$MIDAS_{ws}$
Panel A: Individual Portfolio Results						
Small	0.198	0.238	0.220	0.185	0.351^{***}	0.034
		(0.151)	(0.154)	(0.152)	(0.103)	(0.192)
2	0.110	0.144	0.128	0.098	0.180	− 0.004
		(0.130)	(0.131)	(0.130)	(0.114)	(0.193)
3	0.188	0.200^{*}	0.186^{*}	0.167	0.231^{**}	0.075
		(0.109)	(0.110)	(0.111)	(0.105)	(0.164)
4	0.125	0.138	0.125	0.106	0.162	0.034
		(0.100)	(0.101)	(0.101)	(0.101)	(0.150)
5	0.175	0.179^{**}	0.169^{*}	0.150^{*}	0.207^{**}	0.098
		(0.087)	(0.088)	(0.088)	(0.092)	(0.127)
6	0.125	0.136^{*}	0.131^{*}	0.119	0.150^{*}	0.057
		(0.071)	(0.072)	(0.073)	(0.079)	(0.104)
7	0.125	0.128^{**}	0.126^{**}	0.118^{*}	0.126^{*}	0.056
		(0.060)	(0.060)	(0.061)	(0.071)	(0.088)
8	0.096	0.098^{*}	0.100^{*}	0.101^{*}	0.097	0.064
		(0.053)	(0.054)	(0.055)	(0.064)	(0.077)
9	0.075	0.052	0.055	0.066	0.034	0.033
		(0.040)	(0.040)	(0.043)	(0.054)	(0.062)
Big	−0.025	− 0.047	− 0.045	− 0.041	− 0.059	− 0.015
		(0.034)	(0.035)	(0.035)	(0.036)	(0.045)
Panel B: Joint Results						
$\overline{\lvert\alpha\rvert}$	0.124	0.136	0.128	0.115	0.160	0.047
ψ		16.726^{*}	16.261^{*}	14.975	27.519^{***}	3.301
ψ^{adj}		*N/A*	*N/A*	14.720	27.163^{***}	3.076

Table 4: Average pricing errors of the Size portfolios for the unconditional CAPM and conditional CAPM where the later version is based on the EWMA, EQMA, DCC, $MIDAS_w$ and $MIDAS_{ws}$ models. Panel A reports the average pricing errors for each portfolio and the corresponding standard error (in parentheses). Panel B reports the average absolute pricing error and the joint test statistic based on the test proposed in this paper. Test statistics are reported unadjusted and adjusted for parameter uncertainty in the conditional volatility models. Significance at the 10%, 5% and 1% levels are indicated by *, ** and ***, respectively.

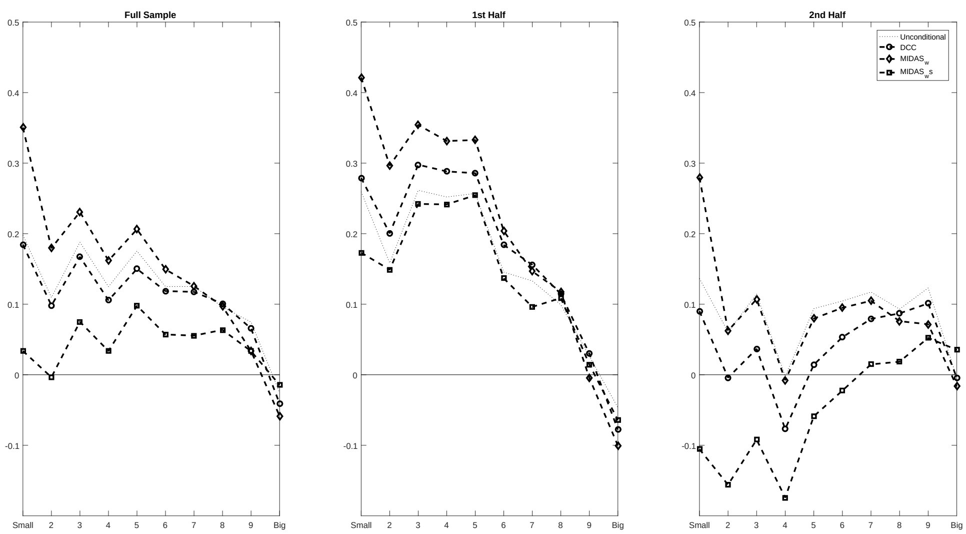

Figure 2: Average pricing error plots of the unconditional and conditional CAPM for the Size Portfolios for the full, first half and second half of the sample. Conditional CAPM plots are based on the DCC, MIDAS_w and MIDAS_{ws} conditional volatility models.

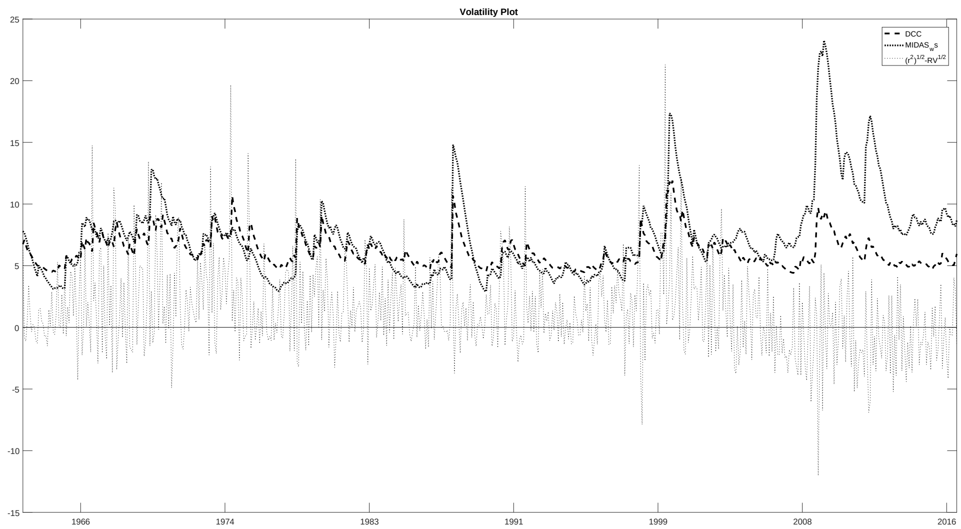

Figure 3: Volatility plot for the Small portfolio for the DCC (heavy dotted line) and MIDAS$_{ws}$ (heavy dashed line), and plot of difference between monthly return standard deviation and square-root of monthly realised volatility (light dotted line).

	(D) MIDAS$_{wsac}$				L.R. Tests		
	ϕ_1	θ_2	ϕ_2	θ_4	H0: A = D	H0: C = D	α
Panel A: Conditional Variances and Ave. Pricing Errors							
Small	29.775***	64.690**	159.664***	1.951**	1.041	92.389***	0.161
	(8.141)	(31.182)	(43.534)	(0.881)			(0.389)
2	24.843***	53.625**	141.699***	2.668**	0.804	68.061***	0.064
	(4.703)	(27.007)	(31.096)	(1.303)			(0.363)
3	22.187***	51.282**	126.743***	2.197***	0.699	73.838***	0.115
	(3.874)	(25.128)	(27.691)	(0.768)			(0.333)
4	21.402***	47.907	124.017***	3.134***	0.504	74.292***	0.051
	(3.855)	(36.739)	(25.092)	(0.740)			(0.324)
5	19.228***	28.222**	118.671***	4.389**	0.092	68.524***	0.100
	(2.726)	(12.011)	(21.414)	(1.823)			(0.303)
6	18.121***	37.091*	112.387***	4.839***	−0.390	68.101***	0.065
	(2.747)	(22.025)	(20.757)	(1.556)			(0.284)
7	16.762***	24.815**	116.290***	6.223***	−0.433	62.403***	0.076
	(2.570)	(10.860)	(20.325)	(1.783)			(0.297)
8	18.941***	47.230	99.207***	3.749	−0.814	57.243***	0.090
	(3.494)	(50.552)	(19.037)	(2.676)			(0.279)
9	16.428***	40.330**	97.342***	4.644**	−1.237	68.283***	0.055
	(2.140)	(17.233)	(14.981)	(2.323)			(0.234)
Big	17.733***	30.804***	75.324***	3.962***	−2.089**	31.767***	−0.023
	(1.798)	(11.119)	(19.294)	(0.799)			(0.203)
Panel B: Conditional Correlation and Ave. Abs. Pricing Error							
$\overline{\lvert\alpha\rvert}$							0.080
Q_t		1.264***					
		(0.111)					
ψ							8.334
ψ_{adj}							7.458

Table 5: Volatility parameter estimates, likelihood ratio tests and average pricing errors for the MIDAS$_{wsac}$ models for the Size portfolios. Estimates of the MIDAS model use daily data for conditional volatility parameters but monthly data for the conditional correlation parameters. Model is estimated using a two-step maximum likelihood approach where the first step estimates the conditional volatility equations while the second step estimates the conditional correlation equation. Reported standard errors (in parentheses) for the volatility parameters are calculated numerically. Vuong (1989) likelihood ratio test statistics reported for non-nested models and standard likelihood ratio test reported for nested models. Average pricing errors based on the conditional CAPM and the joint test statistics based on the test proposed in this paper. Test statistics are reported unadjusted and adjusted for parameter uncertainty in the conditional volatility models. Significance at the 10%, 5% and 1% levels are indicated by *, ** and ***, respectively.

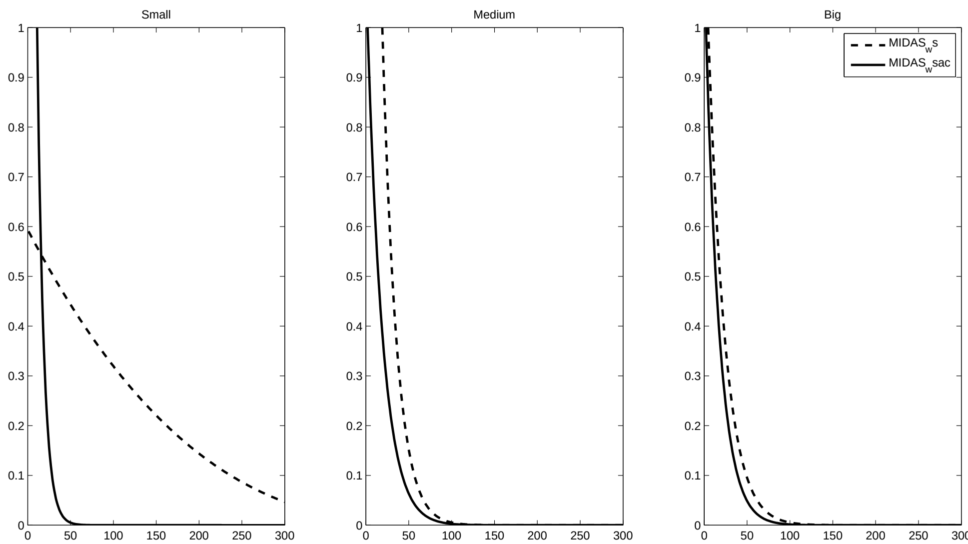

Figure 4: MIDAS weight plots for the MIDAS_{ws} and MIDAS_{wsac} for the Small, Medium (portfolio '5') and Big portfolios. Weights are adjusted for the estimated scale parameter.

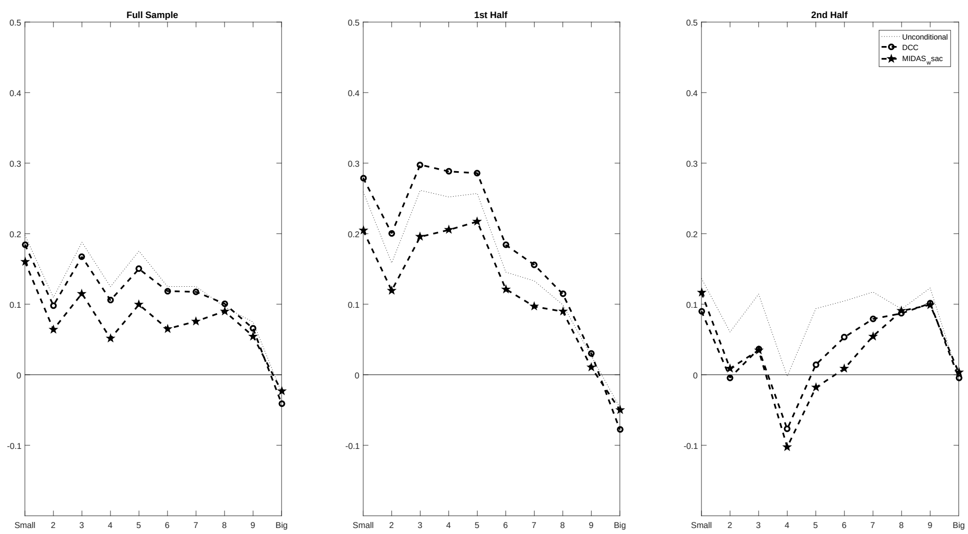

Figure 5: Average pricing error plots of the unconditional and conditional CAPM for the Size Portfolios for the full, first half and second half of the sample. Conditional CAPM plots are based on the DCC and MIDAS$_{wsac}$ conditional volatility models.

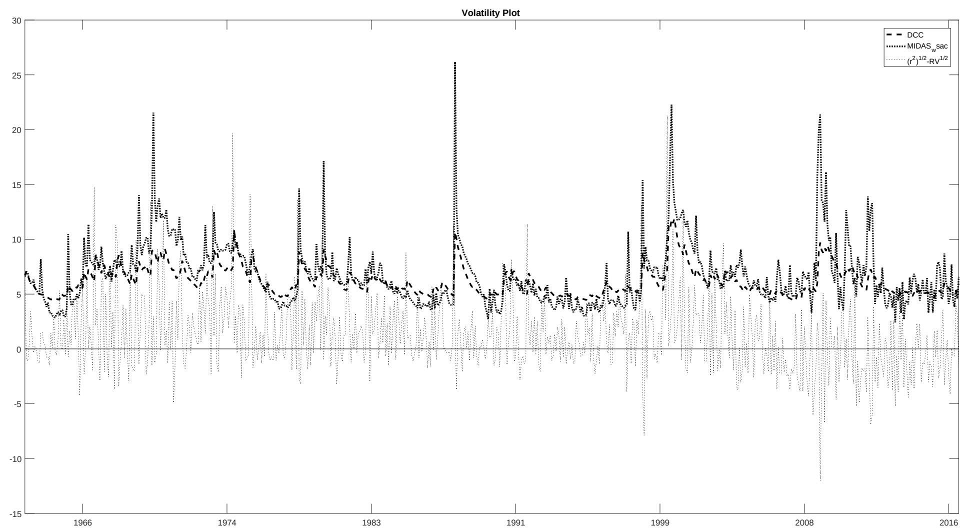

Figure 6: Volatility plot for the Small portfolio for the DCC (heavy dotted line) and MIDAS$_{wsac}$ (heavy dashed line), and plot of difference between monthly return standard deviation and square-root of monthly realised volatility (light dotted line).

	DCC_{GJR}	$MIDAS_{w-asym}$	$MIDAS_{ws-asym}$	$MIDAS_{wsac-asym}$
	α	α	α	α
Panel A: Individual Portfolio Results				
Small	0.168	0.334***	0.036	0.175
	(0.154)	(0.109)	(0.192)	(0.179)
2	0.077	0.164	−0.007	0.076
	(0.133)	(0.120)	(0.194)	(0.160)
3	0.152	0.222**	0.063	0.087
	(0.115)	(0.110)	(0.163)	(0.132)
4	0.100	0.152	0.021	0.064
	(0.105)	(0.105)	(0.148)	(0.124)
5	0.184*	0.204**	0.109	0.131
	(0.095)	(0.096)	(0.129)	(0.109)
6	0.152*	0.155*	0.076	0.085
	(0.079)	(0.081)	(0.104)	(0.089)
7	0.124*	0.136*	0.066	0.092
	(0.064)	(0.073)	(0.090)	(0.078)
8	0.094	0.107	0.072	0.105
	(0.058)	(0.066)	(0.078)	(0.071)
9	0.081*	0.061	0.050	0.039
	(0.045)	(0.055)	(0.062)	(0.058)
Big	−0.046	−0.065	−0.021	−0.026
	(0.037)	(0.037)	(0.045)	(0.041)
Panel B: Joint Results				
$\overline{\lvert\alpha\rvert}$	0.118	0.160	0.052	0.088
ψ	16.121*	23.263***	3.761	9.050
ψ^{adj}	14.504	15.275	3.194	8.204

Table 7: Average pricing errors of the Size portfolios for the unconditional CAPM and conditional CAPM where the later version is based on the DCC_{GJR}, $MIDAS_{w-asym}$, $MIDAS_{ws-asym}$ and $MIDAS_{wsac-asym}$ models. Panel A reports the average pricing errors for each portfolio and the corresponding standard error. Panel B reports the average absolute pricing error and the joint test statistic based on the test proposed in this paper. Test statistics are reported unadjusted and adjusted for parameter uncertainty in the conditional volatility models. Significance at the 10%, 5% and 1% levels are indicated by *, ** and ***, respectively.

	(E) DCC_{GJR}				(F) $MIDAS_{w-asym}$				(G) $MIDAS_{ws-asym}$			(H) $MIDAS_{wsac-asym}$					L.R. Tests			
	ω	α	β	δ	ϕ_1	θ_2	ϕ_2	θ_4	ϕ_1	θ_2	θ_4	ϕ_1	ϕ_2	θ_2	ϕ_3	θ_4	H0: E = F	H0: E = G	H0: E = H	H0: D = H
Panel A: Conditional Variances																				
Small	2.312***	0.055	0.860***	0.053	22.000	1.715***	1.722***	169.000	3.745	81.320**	75.958***	30.378**	29.149*	64.408**	159.561***	1.947**	3.920***	3.991***	1.310	0.007
	(0.808)	(0.037)	(0.027)	(0.041)	–	(0.324)	(0.309)	(499.281)	(3.171)	(35.146)	(26.509)	(14.910)	(15.369)	(31.769)	(42.862)	(0.884)				
2	2.319***	0.014	0.878***	0.092**	22.000	1.998***	2.003***	169.000**	35.623***	43.831***	88.883***	39.745***	10.741	46.052**	133.625***	2.479*	3.472***	2.558**	0.855	5.411**
	(0.771)	(0.022)	(0.018)	(0.041)	–	(0.001)	(0.461)	(70.948)	(13.801)	(16.908)	(20.080)	(12.405)	(10.913)	(20.610)	(26.301)	(1.320)				
3	3.212***	0.000	0.831***	0.149***	22.000	1.997***	2.487***	169.000*	25.722	20.029	92.103***	33.045***	0.000	4.686	140.153***	2.422**	3.575***	2.493**	1.386	– 5.145
	(1.111)	(0.000)	(0.042)	(0.057)	–	(0.001)	(0.733)	(90.203)	(27.555)	(33.152)	(19.886)	(5.665)	(0.017)	(5.321)	(29.859)	(0.987)				
4	3.769***	0.000	0.791***	0.179***	22.000	1.972***	2.725***	169.000*	19.475	9.214	90.640***	41.452***	0.004	31.495***	117.566***	3.134***	3.696***	2.510**	0.194	12.846***
	(1.401)	(0.000)	(0.054)	(0.062)	–	(0.018)	(0.920)	(97.769)	(18.120)	(28.003)	(19.362)	(6.185)	(0.007)	(11.943)	(21.153)	(1.047)				
5	6.341	0.000	0.625***	0.345*	22.000	2.000***	3.624**	169.000	20.460*	12.318	81.246***	40.139***	0.000	29.361***	105.740***	3.783***	3.334***	2.033**	– 0.301	14.153***
	(5.202)	(0.000)	(0.235)	(0.205)	–	(0.002)	(1.468)	(118.480)	(10.505)	(14.425)	(14.949)	(5.878)	(0.003)	(9.187)	(18.387)	(1.114)				
6	4.918	0.000	0.643***	0.367**	22.000	2.000***	5.064**	169.000	17.678***	0.000	84.240***	35.131***	3.841	33.731***	97.154***	4.440**	3.132***	1.172	– 0.877	16.476***
	(3.045)	(0.000)	(0.162)	(0.173)	–	(0.001)	(2.516)	(128.755)	(6.535)	(0.002)	(10.166)	(10.872)	(10.733)	(10.740)	(18.103)	(1.810)				
7	2.572**	0.000	0.792***	0.214**	22.000	1.970***	5.480*	1.000	19.397***	0.000	82.092***	36.947***	0.000	28.236***	100.086***	5.827***	2.975***	0.958	– 0.752	14.115***
	(1.227)	(0.000)	(0.065)	(0.084)	–	(0.024)	(3.289)	(0.759)	(6.229)	(0.003)	(9.882)	(6.201)	(0.008)	(9.598)	(19.648)	(1.715)				
8	1.887***	0.000	0.803***	0.240***	22.000	2.000***	5.878**	1.000	24.825**	6.939	70.534***	39.826***	0.000	40.271***	85.372***	3.187***	2.855***	0.669	– 1.384	18.872***
	(0.664)	(0.000)	(0.045)	(0.068)	–	(0.006)	(2.463)	(0.779)	(11.361)	(10.638)	(9.653)	(5.384)	(0.013)	(12.610)	(16.380)	(1.181)				
9	1.804	0.000	0.771***	0.287***	22.000	1.998***	7.603**	1.000	18.763***	0.000	64.731***	24.352***	0.000	2.941**	111.857***	4.717***	1.952*	0.559	– 0.417	– 12.956
	(1.231)	(0.000)	(0.100)	(0.109)	–	(0.008)	(3.384)	(2.862)	(7.213)	(0.004)	(6.492)	(3.196)	(0.007)	(1.194)	(17.673)	(1.812)				
Big	1.057*	0.044	0.826***	0.139*	22.000	1.520***	23.691**	4.666**	19.909	8.592	38.734***	30.766***	4.114	24.097***	75.759***	3.936***	– 1.238	– 1.091	– 2.120**	6.667***
	(0.609)	(0.044)	(0.040)	(0.077)	–	(0.271)	(9.698)	(2.286)	(13.903)	(12.349)	(10.471)	(7.641)	(7.718)	(8.814)	(19.841)	(1.127)				
Panel B: Conditional Correlation																				
Q_t		0.017***	0.960***			1.259***				1.181***			1.270***							
		(0.004)	(0.015)			(0.117)				(0.081)			(0.101)							

Table 6: Volatility parameter estimates and likelihood ratio tests for the DCC and MIDAS models for the Size portfolios. Estimates of DCC_{GJR} parameters are based on monthly data while the MIDAS models use daily data for conditional volatility parameters but monthly data for the conditional correlation parameters. Models are estimated using a two-step maximum likelihood approach where the first step estimates the conditional volatility equations while the second step estimates the conditional correlation equation. Reported standard errors (in parentheses) for the volatility parameters are calculated numerically. Vuong (1989) likelihood ratio test statistics reported for non-nested models and standard likelihood ratio test reported for nested models. Significance at the 10%, 5% and 1% levels are indicated by *, ** and ***, respectively.

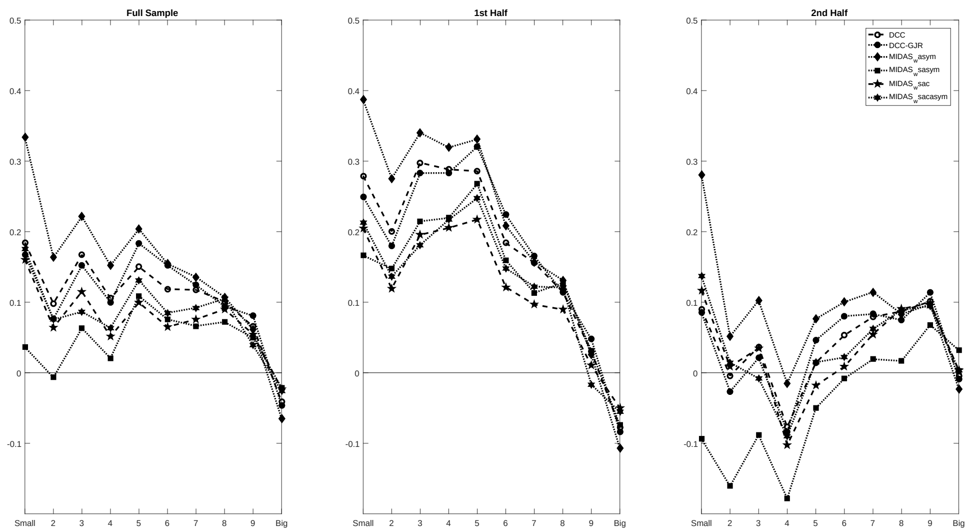

Figure 7: Average pricing error plots of the unconditional and conditional CAPM for the Size Portfolios for the full, first half and second half of the sample. Conditional CAPM plots are based on the DCC, DCC_{GJR}, $MIDAS_{w-asym}$, $MIDAS_{ws-asym}$, $MIDAS_{wsac}$ and $MIDAS_{wsac-asym}$ conditional volatility models.

	Unconditional	DCC_{GJR}	$MIDAS_{wsac}$	$MIDAS_{wsac-asym}$
Panel A: FF25 Joint Results				
$\overline{\vert\alpha\vert}$	0.259	0.226	0.185	0.197
ψ		114.8***	55.6***	91.8***
ψ_{adj}		111.1***	41.0**	73.2***
Panel B: Industry Joint Results				
$\overline{\vert\alpha\vert}$	0.133	0.126	0.114	0.126
ψ		20.5**	14.5	15.5
ψ_{adj}		19.9**	14.2	14.6

Table 8: Average absolute pricing errors of the FF25 and Industry portfolios for the unconditional CAPM and conditional CAPM where the later version is based on the DCC_{GJR}, $MIDAS_{wsac}$ and $MIDAS_{wsac-asym}$ models. Panels A and B report the average absolute pricing errors and the joint test statistics for the FF25 and Industry portfolios, respectively. Test statistics are reported unadjusted and adjusted for parameter uncertainty in the conditional volatility models. Significance at the 10%, 5% and 1% levels are indicated by *, ** and ***, respectively.