Reference Health and Investment Decisions[1]

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December 8, 2022

Abstract

This paper considers how reference health influences medical spending and decreasing health and their interaction with consumption and investment choices. We focus on the reference point's adaptiveness in a dynamic model of an agent facing a series of health shocks. The agent trades off current consumption against out-of-pocket medical costs and the expected lifetime utility stream while optimizing portfolio composition. While a static reference point implies heavy medical spending, an adaptive reference health reduces medical spending while boosting consumption, assets and, lifetime utility.

Keywords: Investments, medical spending, optimal control, behavioral economics, reference adaption.

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1 Introduction

Benchmark values influence the economic decisions of individuals. People judge a stock trade as a gain or a loss relative to the original purchase price of the stock or judge their wages against the salaries of their peers. Such benchmark values, or reference points, form a fundamental building block of behavioral economics and descriptive theories of choice under risk, such as prospect theory and cumulative prospect theory by Kahneman and Tversky (1979) and Tversky and Kahneman (1992) and the models of reference-dependence of Köszegi and Rabin (2006, 2007).

This paper considers how reference points influence medical spending and decreasing health over the lifetime, and their interaction with intertwined economic choices such as consumption and investment. Our particular focus lies on the adaptiveness of the reference point: as health and wealth evolve with a sizable random component, we consider the dynamics of an adaptive reference point that evolves with the individual health decay. Hugonnier, Pelgrin, and St Amour (2013) model and empirically confirm the interdependence of health and asset holdings. Yogo (2016) documents that stochastically decaying health and out-of-pocket health expenditures significantly affect household investment decisions. Mortality and decaying health also influence insurance and annuitization choices in behavioral settings (Gottlieb, 2012; Reichling and Smetters, 2015).

The first contribution of this paper is a dynamic model of medical spending, health, consumption, and investment choices that features both static and adaptive health reference levels. An agent faces a series of health shocks, for which he or she incurs out-of-pocket medical costs. At every shock, the agent's choice of how much to spend on medication on the one side allows to recover the health damage partly, but on the other side limits the assets available for future medical expenditure and consumption. Additionally, the agent continuously selects consumption and invests the remaining assets in a risky and risk-free asset in proportions of his or her choice.

The second contribution of our paper is the mechanism of health reference updating. As the health decays with every shock, depending on the adaptiveness of the reference health, the agent either still judges her health against the original health before the shock or adapts the reference health downward to include the new, after-shock health level. In an extreme case, that agent perfectly adapts to the after-shock health level, that is, the new health level does not pose a loss to the agent. For not perfectly adaptive reference health, any health loss considers a permanent disutility, as the perceived quality of life diminishes.

The agent's instantaneous utility poses our third contribution. The agent, at every point in time, considers the following trade-off: The continuous adjustment of both consumption and the investment portfolio composition determines the agent's instantaneous utility, which depends on both the current consumption as well as the health and reference health level. High health allows the agent to enjoy consumption more, whereas high reference health decreases the consumption utility, as the agent considers his current health status as a loss. The agent balances this utility against his postulated after-shock utility stream: based on his reference adaption, the agent judges his future lifetime utility stream and a bequest utility, and how the current choices affect it. We design a particular utility function to capture the empirically derived properties of health references on consumption, health, and medical spending by Harris and Kohn (2018).

We characterize the agent's expected lifetime utility in a continuous-time optimal control framework. Our utility function allows us to derive the special cases of critical, that is, terminally ill health, and the boundary case of no health shocks, in semi-closed form. We further characterize the optimal controls and the expected lifetime utility as an ordinary differential equation expanding Merton (1969) and provide an efficient numerical scheme to determine the general solution.

In our numerical analysis, we show that a static reference point implies a strong desire to limit health reductions already at high health levels. Consequently, the agent spends much on medication in the early periods. Compared to the case of a static reference point, the adaptive reference agent obtains a much more rectangularized utility. Because the reference health reflects that previous health levels become unattainable, focusing on achieving more realistic health levels and accepting them as sufficient enhances the utility of consumption and the current health status, that is, the psychological flexibility allows the agent to feel better. This agent's ability to adapt to decreasing health allows him or her to draw high utility from consumption and limit medical spending. This medical spending profile not only allows the agent to spend less on medication over the lifetime, but it also enhances the enjoyability of life itself as the agent feels better after a heath shock.

2 A model of stochastic health decay and life-time portfolio choices

This section outlines our model. We set off by describing the agent's health decay and how it is influenced by medical expenditure. Then, we turn to the agent's choice problem consisting of medical and financial investments, and consumption choice. Following that, we introduce the health-reference-dependent utility specification and discuss the interaction of risk aversion with health preferences.

2.1 Health decay and medical spending

Consider an agent who faces stochastically decaying health H, which starts at an initial level of $H_0 > 0$ at the starting time $t = 0$. In random intervals, his health drops by a random shock θ that occurs with a probability of $\lambda(H)$ in an instantaneous time step $[t, t+dt]$. Better health reduces the likelihood of a health shock, that is, the intensity $\lambda(H)$ decreases in H. The random shock has a support $\Theta = [\underline{\theta}, \overline{\theta}]$ with $0 < \underline{\theta} < \overline{\theta}$ and a distribution F_θ. A point process N_t announces the health shocks, which are independent and identically distributed. The jump times T_n of the point process N_t capture the times of individual health shocks, while θ_n denotes their magnitude. Death occurs if the health falls below a critical value H_D, with $0 < H_D < H_0$. The time of death is the stopping time

$$T_D = \inf\{t \geq 0 : H_t \leq H_D\} \; = \; \inf\{T_n : H_{T_n} \leq H_D\}\,. \tag{1}$$

Once a health shock occurs, the agent may seek treatment. Treatment improves health by investing in medication $k \geq 0$ that increases it by $V(k,\theta) \geq 0$. The resulting total health change at the shock equals $V(k,\theta) - \theta$. No medication has no effect, that is, $V(0,\theta) = 0$. Medication has a positive and marginally decreasing treatment effect. Specifically, marginal benefits are strictly positive at zero medication, and zero for infinite medication, yielding $\lim_{k\to 0} V_k(k,\theta) > 0$ and $\lim_{k\to\infty} V_k(k,\theta) = 0$. Heavier health shocks imply higher benefits of medication, that is, $V_{k,\theta}(k,\theta) > 0$. Despite medication, health shocks leave some lasting damage $\varepsilon_H > 0$, which bounds the medical impact from below as

$$\theta - V(k,\theta) \geq \varepsilon_H\,. \tag{2}$$

To be specific, for a given health shock θ and amount k spent on the related medical treatment, we consider the following concrete functional form[1] of the treatment effect $V(k,\theta)$:

$$V(k,\theta) = (\theta - \varepsilon_H)\left(1 - e^{-\frac{k}{\theta}}\right)\,. \tag{3}$$

This medication effect does not distinguish health differences of the agent. Appendix I.2 presents an extension to a health-dependent minimal lasting damage $\varepsilon_H(h)$.

[1] This specification can be generalized. We discuss this in Appendix I.1.

The net health shock equals

$$\theta - V(k, \theta, h) = \theta\, e^{-\frac{k}{\theta}} + \varepsilon_H(h) \left(1 - e^{-\frac{k}{\theta}}\right) . \tag{4}$$

Consequently, the original health level $H_0 > 0$ decays at some point to the death threshold H_D as a consequence of the health shocks. Overall, the health process evolves as follows:

$$H_t = H_0 - \sum_{n=1}^{N_t} \left(\theta_n - V(k_n, \theta_n, H_{T_n-})\right) . \tag{5}$$

2.2 Financial investment and consumption

Besides managing his health and medical expenditure, the agent invests his assets and spends money on consumption. Between health shocks, his wealth is invested in an asset portfolio X that contains a risk-free and a risky asset as in Merton (1969). He consumes at a rate c_t. The risky asset follows a geometric Brownian motion, which implies the portfolio dynamics

$$dX_t = (a_t(\mu - r) + r)X_t dt - c_t dt + a_t \sigma X_t dW, \qquad X_0 = x > 0. \tag{6}$$

Here, a_t is the (potentially time-dependent) investment strategy that splits the assets between the risk-free return r and the risky return $\mu > r$. The risky part of the portfolio depends on the risk represented by the Brownian motion W scaled by the volatility $\sigma > 0$. Accounting for his medical spending at the health shocks, we obtain the overall wealth as

$$X_t = x + \int_0^t \left(X_s \left[a_s (\mu - r) + r\right] - c_s\right) \mathrm{d}s + \int_0^t X_s\, a_s\, \sigma\, \mathrm{d}W_s - \sum_{n=1}^{N_t} k_n \,. \tag{7}$$

The agent has limited liability, that is, his wealth cannot fall below zero, giving $X_t > 0$.

2.3 Agent decisions, health reference, and utility

The above model presents a stochastic control problem to the agent. He controls his financial investment, consumption and medical spending to maximize his lifetime utility. Death is certain, as every health shock leaves some permanent damage. After a finite number of health shocks, the agent dies. Medical spending defers death,[2] but its costs limit the agent's consumption and financial investment.

The agent draws utility from consumption and health over his lifespan. He values health H_t both in absolute terms and relative to a reference level B_t, that is, the agent's utility is $u_t(c_t, H_t, B_t)$ at time t. The agent forms an expectation of his future consumption and health based over his expected lifetime. Formally, (H, B, X) is a controlled process, and with control (a, c, k), the agent's lifetime utility equals

$$J(h,b,x;a,c,k) = \mathbb{E}_{(h,b,x)}\left[\int_0^{T_D} e^{-\rho t} u_t(c_t, H_t, B_t)\,\mathrm{d}t\right] + \mathbb{E}_{(h,b,x)}\left[e^{-\rho T_D} U_D(X_{T_D})\right] \tag{8}$$

with $\rho > 0$ as the discount rate. It consists of the actual expected lifetime utility in the first term and the expected utility of the random bequest wealth X_{T_D} which appears at the random death time T_D in the second term. The agent evaluates the bequest utility based on his or her current health and reference point. This implies that the assessment of the death utility may change. At the time of dealth T_D, the death utility $U_D(x)$ captures the agent's bequest motive as the total expected utility beyond his or her death assumed for the heirs as

$$U_D(x) = \mathbb{E}_{(h,b,x)}\left[\int_{T_D}^{\infty} e^{-\rho(t-T_D)} u(c_D(x), h, b)\,\mathrm{d}t\right], \tag{9}$$

[2]A potential extension of our model is to allow for spending on a healthy life style that at a cost decreases the intensity of the health shock process. Then the agent faces the trade-off between reducing the frequency of health shocks and treatment of them.

where the consumption $c_D(x)$ depends exclusively on the bequest wealth x which at the time of death is known. We assume that after T_D a risk-free value preserving strategy is implemented

$$0 \overset{!}{=} \mathrm{d}X_t = (rX_t - c_D(X_t))\,\mathrm{d}t \quad \Longleftrightarrow \quad c_D(x) = rx\,,$$

that is, the death utility is the utility of the certain consumption stream left to the heirs by investing all assets in the risk-free asset and immediately consuming the interest. Note that the utility of this stream falls below the utility while being alive.

In line with Harris and Kohn (2018) and Schünemann, Strulik, and Trimborn (2017) who consider that individuals' health references are formed with their past health realizations, we propose the following health reference updating rule

$$B_{t\in[T_n,T_{n+1})} = (1-\omega)B_{T_{n-1}} + \omega H_{T_n}, \text{ with } T_0 = 0 \text{ and } B_{t\in[0,T_1)} = H_0. \tag{10}$$

Here, $\omega \in [0,1]$ measures the speed of the health reference updating. The agent starts with his initial health level as his health reference and updates it along his health changes at health shocks. Every health shock updates his health reference by averaging the previous reference level and the updated after-shock health level with weights ω and $1-\omega$, respectively. There are two extreme reference updating cases: First, with $\omega = 0$, the agent benchmarks his health to the initial level over his lifetime reference point $B_t = H_0$. Second, with $\omega = 1$, the agent instantly updates his reference following health changes, that is, his health reference coincides with his health level $B_t = H_t$. By iterating, the agent's reference health depends on the health history as follows

$$B_t = (1-\omega)^{N_t} H_0 + \sum_{n=1}^{N_t} (1-\omega)^{N_t - n}\omega H_{T_n}. \tag{11}$$

Now, we specify the agent's utility drawing from his consumption and health, both in absolute terms and benchmarked to a reference level. Taking the empirically identified signs of the marginal cross-utilities of consumption, health, and reference health by Harris and Kohn (2018), which are $u_{CH} < 0$[3], $u_{CB} > 0$ and $u_{HB} > 0$, we define the utility function

$$u(c,h,b) = \frac{(ce^{h-b})^{1-\gamma}}{1-\gamma}, \tag{12}$$

with $\gamma > 1$ is the risk-aversion parameter. The agent enjoys more consumption and better health, however, their marginal effects decrease, that is $u_C > 0$, $u_H > 0$, $u_{CC} < 0$, and $u_{HH} < 0$. With $u_{CH} < 0$, the agent sees consumption and health as substitutes: When the agent is at a worse health stage, he or she feels more compensated by additional consumption, and at the same time, his or her additional health improvement matters more when he or she already has less to consume. The health reference, which reveals the agent's health history, also affects the marginal benefits from additional consumption and health: With $u_{CB} > 0$ and $u_{HB} > 0$, a higher health trajectory lets the agent enjoy additional consumption and health more. Nevertheless, the health reference is a benchmark based on which the agent evaluates his current health. A higher reference level has a negative effect on the agent's utility, that is $u_B < 0$.

Moreover, the above utility (12) implies constant relative risk-version of $\gamma > 1$ in consumption and constant absolute risk-aversion of $\gamma - 1 > 0$ in health. The consumption loss the agent can tolerate grows proportionally to his total consumption, however, the tolerated health loss stays constant across different health stages. The lower the risk aversion parameter, the lower the minimum negative utility is, making living less attractive. Intuitively, when γ is closer to 1, the agent is more risk-neutral to any health loss, then giving up life becomes less unacceptable.

[3] In Harris and Kohn (2018), $u_{CH} < 0$ was identified for high compsumption, while for low consumption, they found $u_{CH} > 0$. To be comparable to Hugonnier, Pelgrin, and St Amour (2013) which assumes health and consumption are substitutes, we adopted $u_{CH} < 0$ to define the utility function in this paper.

We rewrite the utility function (12) as

$$u(c,h,b) = \frac{c^{1-\gamma}}{1-\gamma} K(h,b,\gamma), \tag{13}$$

where $K(h,b,\gamma) = e^{(h-b)(1-\gamma)}$. Plugging this into the death utility equation (9) gives

$$U_D(x) = \mathbb{E}_{(h,b,x)} \left[\int_{T_D}^{\infty} e^{-\rho\,(t-T_D)} \frac{(rx)^{1-\gamma}}{1-\gamma} e^{(h-b)(1-\gamma)} \,\mathrm{d}t \right] = -e^{(h-b)(1-\gamma)} \frac{x^{1-\gamma}}{\frac{\rho}{r}\, r^{\gamma} (\gamma-1)}\,.$$

Hence, the lifetime utility in equation (8) can be rewritten to

$$J(h,b,x;a,c,k) = \mathbb{E}_{(h,b,x)} \left[\int_0^{T_D} e^{-\rho\, t} u_t(c_t, H_t, B_t)\,\mathrm{d}t \right] - \frac{e^{(h-b)(1-\gamma)}}{\frac{\rho}{r}\, r^{\gamma} (\gamma-1)} \mathbb{E}_{(h,b,x)} \left[e^{-\rho\, T_D} X_{T_D}^{1-\gamma} \right] \tag{14}$$

The development of the asset portfolio, the stochastic health decay, and the reference level are Markovian. The health process is a point process; consequently, the health decay is memoryless. The expected time until the next health shock only depends on the intensity $\lambda(H)$. The actual time elapsed since the last health shock carries no information. This structure implies that the agent's decisions are also Markovian: they depend only on the information at the current time in the form of the agent's current health, reference level, and financial wealth. The historical development of how he reached this state is irrelevant. In particular, the only driver of the agent's choices is the current status of the state space (H,B,X). This includes the health status H_t carries the interpretation of proximity to death, whereas physical age plays no role, in line with the findings of Zweifel, Felder, and Meiers (1999) and Zweifel, Felder, and Werblow (2004).

3 Optimal medical spending, investment, and consumption

The specified model implies two choices the agent faces. First, he considers how much to spend on medication once a health shock arrives. Second, he selects a consumption and investment strategy between health shocks. The two choices are interconnected: Medical spending restores his health at the expense of future consumption. However, higher health makes future consumption more enjoyable. Higher consumption brings immediate utility now but limits both potential future medical spending and indirectly the consumption utility via lower health. The investment choice trades off the chances that the higher and riskier return of the risky asset offers in obtaining more money for consumption and medical spending on average, accompanied by the risk of lower returns in some cases.

Consider the medical spending choice first. Health is constant between the health shocks at T_n and decreasing. Thus, we focus first at the shock times T_n, and consider the choice at this point in time. At T_n the agent knows the health level h and wealth x just before the health shock θ_n with observable magnitude θ_n arrives. The choice of amount k_n spent on medication results in a remaining life time utility of

$$J(h-\theta_n+V(k_n,\theta_n,h), f(h,b,\omega,\theta,k_n), x-k_n; a,c,k)\,. \tag{15}$$

Selecting a medical spending of k_n to threat health shock θ_n restores the health by $V(k_n,\theta_n,h)$, but reduces the wealth, and, subsequently, the future spending ability by k_n. The function $f(h,b,\omega,\theta,k_n)$ determines the formulation of health reference. Specifically, it is the weighted average of the reference before the shock and the updated new health level at the shock: $f(h,b,\omega,\theta,k_n) = (1-\omega)b+\omega(h-\theta_n+V(k_n,\theta_n,h))$ with $\omega \in [0,1]$. An $\omega = 1$ implies that the agent's reference health perfectly adapts following health shocks and his medication investment decisions at the shocks, consequently, he optimizes his choices based on this adapted health reference.

We now turn to the optimization. Denote by $k^{\star}(h,b,x,\theta_n,\omega;a,c,k)$ the argument maximizing (15) at (h,b,x) for shock size θ_n in the case of ω, given the subsequent strategy (a,c,k). Suppose that an optimal strategy $(a^{\star},c^{\star},k^{\star})$ exists with value function

$$U(h,b,x) = \sup_{(a,c,k)} J(h-\theta+V(k,\theta,h), f(h,b,\omega,\theta,k), x-k; a,c,k)\,, \tag{16}$$

then $k^{\star}$ depends on (h,b,x), θ, and ω and satisfies

$$k^{\star}(h,b,x,\theta,\omega) = \arg\max_{0\leq k\leq x} U(h-\theta+V(k,\theta,h), f(h,b,\omega,k), x-k)\,.$$

Since $X \geq 0$, we have that $k \leq x$, and thus $k^{\star}(h,b,x,\theta,\omega)$ exists given U and V are sufficiently smooth but may not be unique.[4] We summarize our result in the following proposition.

Proposition 1. *For our given assumptions, the (possibly not unique) optimal medical expenditure choice exists and is characterized as*

$$k^{\star}(h,b,x,\theta,\omega) = \arg\max_{0\leq k\leq x} U(h-\theta+V(k,\theta,h), f(h,b,\omega,k), x-k)\,. \tag{17}$$

We formalize the agent's value function for any given health and wealth level, that is, we write his utility as a combination of his present utility from his current consumption and investment choice, and an expectation of his future utility. At any time t where no health shock takes place and

[4] In case $k^{\star}(h,b,x,\theta,A)$ is not unique, a criterion has to be postulated in order to select a specific maximizing argument. Motivated by prolonging life, the maximal expenditure k for can be taken yielding the highest post-shock health level.

for starting value (h,b,x), the value function U can be written as

$$U(h,b,x) = \mathbb{E}_{(h,b,x)}\left[\int_0^{T_1} e^{-\rho t} u(c_t,h,b)\,\mathrm{d}t\right] + \mathbb{E}_{(h,b,x)}\Big[e^{-\rho T_1}$$
$$U(h-\theta_1+V(k^\star(h,b,X_{T_1},\theta_1,\omega),\theta_1,h), f(h,b,\omega,k^\star(h,b,X_{T_1},\theta_1,\omega)), X_{T_1}-k^\star(h,b,X_{T_1},\theta_1,\omega))\Big]. \tag{18}$$

The first part of the utility is the agent's expected stream of utility from his current consumption choice, discounted by his time preference parameter, for his current health up to the next health shock. The second part is the expected value of his future utility at the next health shock based on his optimal medical spending choice. For any potential health shock magnitude, the agent selects an optimal spending that trades off this future consumption ability against his health loss for this particular shock.

The above equation uses time homogeneity. Since T_1 is an exponential random time with intensity λ and the distribution of the next health shock θ_1 is given by F_θ, we integrate both quantities out as these are independent of the Wiener process W driving the agent's wealth process, that is,

$$U(h,b,x) = \mathbb{E}_{(h,b,x)}\left[\int_0^{\infty} e^{-[\rho+\lambda(h)]t} u(c_t,h,b)\,\mathrm{d}t\right] + \mathbb{E}_{(h,b,x)}\Big[\int_0^{\infty} e^{-[\rho+\lambda(h)]t}\lambda(h)$$
$$\int U(h-\theta+V(k^\star(h,b,X_t,\theta,\omega),\theta,h), f(h,b,\omega,k^\star(h,b,X_t,\theta,\omega)), X_t-k^\star(h,b,X_t,\theta,\omega))\,\mathrm{d}F_\theta(\theta)\,\mathrm{d}t\Big]. \tag{19}$$

To simplify the following discussion, we define the expected future utility, given the optimal medical spending choice, at the next health shock by integrating over the health shock distribution F_θ as follows:

$$\mathbb{U}(h,b,x) = \int U(h-\theta+V(k^\star(h,b,x,\theta,\omega),\theta,h), f(h,b,\omega,k^\star(h,b,x,\theta,\omega)), x-k^\star(h,b,x,\theta,\omega))\,\mathrm{d}F_\theta(\theta)\,. \tag{20}$$

The quantity $\mathbb{U}(h,b,x)$ is calculated based on $(U(h',b',\cdot))_{\{h-\varepsilon(h)\geq h'\geq H_D,\, b'=(1-\omega)b+\omega h'\}}$. Also, h and b act as parameters and we can characterize $U(h,b,x)$ for fixed h and b, and variable x using optimal control arguments, that is, the Hamilton Jacobi Bellman formalism.

Fix h and b and let $x \geq 0$ be variable, then assume that $U(h',b',x)$ is given for $h' \leq h-\varepsilon$ and $b' = (1-\omega)b+\omega h'$, where ε is the minimal effect of a health shock given the best possible medication, see (2). From there and using (17), the optimal medication expense $k^\star(h,b,x,\theta,\omega)$ can be derived for $h' \leq h-\varepsilon$ and $b' = (1-\omega)b+\omega h'$, and all $x \geq 0$ as well as θ in the support of health shock distribution F_θ. Hence $\mathbb{U}(h,b,x)$ can be computed for all $x \geq 0$ based on (20).

We can now write the value function as

$$U(h,b,x) = \mathbb{E}_{(h,b,x)}\left[\int_0^\infty e^{-[\rho+\lambda(h)]t}\,(u(c_t,h,b)+\lambda(h)\,\mathbb{U}(h,b,X_t))\,\mathrm{d}t\right]. \tag{21}$$

Standard arguments allow us to integrate until infinity, but adjust the discount factor to account for the probability of a health shock. The first term now is the stream of the discounted utility of consumption for a given health level and a given reference point, and the second term is the expected utility at the next health shock, weighted with the instantaneous probability for it. For this to hold, the optimal holding in the risky asset $a^\star$ and the optimal consumption $c^\star$ are characterized using the Hamilton-Jacobi-Bellman equation

$$0 = \max_{(a,c)} \mathscr{A}U(h,b,x) + u(c_t,h,b) + \lambda(h)\,\mathbb{U}(h,b,x) - [\rho+\lambda(h)]\,U(h,b,x)\,. \tag{22}$$

The generator of the wealth process in between health shocks $\mathscr{A}$ is

$$\mathscr{A}\,g(x) = (x\,[a\,(\mu - r) + r] - c+)\,g_x(x) + \frac{1}{2}\,a^2\,\sigma^2\,x^2\,g_{xx}(x)\,. \tag{23}$$

The first order conditions determine the optimal strategies

$$a^\star(h,b,x) = \frac{\mu - r}{\sigma^2}\,\frac{U_x(h,b,x)}{-x\,U_{xx}(h,b,x)}, \qquad c^\star(h,b,x) = \left(\frac{U_x(h,b,x)}{K(h,b,\gamma)}\right)^{-\frac{1}{\gamma}}.$$

The value function U, for fixed h, satisfies

$$\begin{aligned} 0 = \frac{1}{2}\,\frac{(\mu - r)^2}{\sigma^2}\,\frac{U_x(h,b,x)^2}{-U_{xx}(h,b,x)} + r\,x\,U_x(h,b,x) + \frac{\gamma}{1-\gamma}\,U_x(h,b,x)^{\frac{\gamma-1}{\gamma}}\,K(h,b,\gamma)^{\frac{1}{\gamma}} \\ + \lambda(h)\,\mathbb{U}(h,b,x) - [\rho + \lambda(h)]\,U(h,b,x)\,. \end{aligned}$$

We collect our results in the following proposition.

Proposition 2. *For our given assumptions, the optimal investment and consumption choices are*

$$a^\star(h,b,x) = \frac{\mu - r}{\sigma^2}\,\frac{U_x(h,b,x)}{-x\,U_{xx}(h,b,x)}, \qquad c^\star(h,b,x) = \left(\frac{U_x(h,b,x)}{K(h,b,\gamma)}\right)^{-\frac{1}{\gamma}}. \tag{24}$$

The value function is characterized through the ordinary differential equation

$$\begin{aligned} 0 = \frac{1}{2}\,\frac{(\mu - r)^2}{\sigma^2}\,\frac{U_x(h,b,x)^2}{-U_{xx}(h,b,x)} + r\,x\,U_x(h,b,x) + \frac{\gamma}{1-\gamma}\,U_x(h,b,x)^{\frac{\gamma-1}{\gamma}}\,K(h,b,\gamma)^{\frac{1}{\gamma}} \\ + \lambda(h)\,\mathbb{U}(h,b,x) - [\rho + \lambda(h)]\,U(h,b,x)\,. \end{aligned} \tag{25}$$

4 Critical health and no health shocks

In this section, we consider two interesting special cases. First, we analyze the choices of a critically ill agent, and second turn to the case if health shocks do not exist. These special cases provide useful boundaries and benchmark values for the previously discussed general case.

4.1 Special case: Critical health

An interesting special case is the one where the agent is critically ill, which we define as a health level at which any further health shock immediately implies death despite any medical treatment. We now turn to analyze the utility and optimal choice in this state.

To define this critical health area, first fix h and note that for $h \leq H_D$ the health level is below the death threshold and hence $T_D = 0$ giving $U(h,b,x) = U_D(x)$ for all $x \geq 0$, as we assume $\gamma > 1$. The utility of death is fixed. Moreover, the expected future utility after any health shock for a critically ill agent is also fixed. Hence, for a critical health level h with

$$H_D < h \leq H_D + \varepsilon_H \,, \tag{26}$$

we have that $\mathbb{U}(h,b,x) = U_D(x,h,b)$, for all $x \geq 0$, as the next health shock leads to death regardless of the spending on health. In this case, the ordinary differential equation has a semi-closed-form solution.

Proposition 3. *For a critically ill agent, given that*

$$\frac{\rho + \lambda(h)}{\gamma} - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{1-\gamma}{\gamma^2} - r\frac{1-\gamma}{\gamma} - > 0$$

holds, the value function $U(h,b,x)$ is of the form

$$U(h,b,x) = G(h,b)\,x^{1-\gamma}, \tag{27}$$

with $G(h,b)$ determined, setting $L(h,b) = ((1-\gamma)\,G(h,b))^{-\frac{1}{\gamma}}$ (noting that $G<0$), with $L>0$, by

$$0 = \frac{1}{2}\,\frac{(\mu-r)^2}{\sigma^2}\,\frac{(1-\gamma)}{\gamma} + r\,(1-\gamma) - [\rho+\lambda(h)] + \gamma L(h,b)\,e^{-\frac{1-\gamma}{\gamma}(h-b)} + \frac{\lambda(h)\,r}{\rho\,r^{\gamma}}\,L(h,b)^{\gamma},$$

which has a unique solution. The optimal consumption and investment choice become

$$a^{\star} = \frac{\mu-r}{\sigma^2\gamma}, \qquad c^{\star} = \left(\frac{(1-\gamma)G(h,b)}{K(h,b,\gamma)}\right)^{-\frac{1}{\gamma}} x.$$

Medical spending equals $k^{\star}=0$, because medication cannot provide further help.

The proof of this proposition is presented by Appendix II.1.

4.2 Special case: No health shock

A further benchmark case is the one where health shocks are not possible. This case is a generalization of the famous Merton (1969) investment problem, however with a health-dependent utility function. It represents an upper bound of the agent's utility because medical spending is not necessary, leaving the agent free to maximize his consumption utility over his infinite lifetime. It also allows us to measure to which degree medical spending can help the agent to achieve this utopic case.

Consider the agent's utility $U(h,b,x)$. In a world without any health shocks, that is, for $\lambda(h)=0$, for all h, we obtain an upper bound $\overline{U}(h,b,x)$. However, then we are in the standard setting and the upper bound is of the form $\overline{U}(h,b,x) = \overline{G}(h,b)x^{1-\gamma}$, for $x \geq 0$, and some $\overline{G}(h,b)$ depending on the health and reference level h and b. We obtain the following result.[5]

Proposition 4. *In case that there is no health shock, for our given assumptions, the optimal investment and investment choices are*

$$a^{\star}(h,b,x) = \frac{\mu - r}{\sigma^2}\frac{1}{\gamma}, \qquad c^{\star}(h,b,x) = \left(\frac{\rho}{\gamma} - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{1-\gamma}{\gamma^2} - r\frac{1-\gamma}{\gamma}\right)x. \tag{28}$$

The agents value function equals $\overline{U}(h,b,x) = \overline{G}(h,b)x^{1-\gamma}$, for $x \geq 0$, and

$$\overline{G}(h,b) = \frac{K(h,b,\gamma)}{(1-\gamma)\left(\frac{\rho}{\gamma} - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{1-\gamma}{\gamma^2} - r\frac{1-\gamma}{\gamma}\right)^{\gamma}}. \tag{29}$$

The proof of this proposition is collected by Appendix II.1.

5 Numerical analysis

This section discusses how reference-dependence health affects economic choices. First, we present the base-case parametrization of our model. The focus then lies on the agent's optimal choice for the benchmark case. The next section considers how the reference adaption influences choices.

Numerically, we use a standard Euler-scheme to solve the second-order ordinary differential equation in Proposition 2 in Equation (25). Appendix IV collects the technical details of the discretization and derivation of necessary boundary conditions. Table 1 collects the economic parametrization of the base case. The numerical parametrization is presented in Table IV.1 in Appendix IV.

[5] In the current situation $\mathbb{U}$ is not important as its weight $\lambda = 0$.

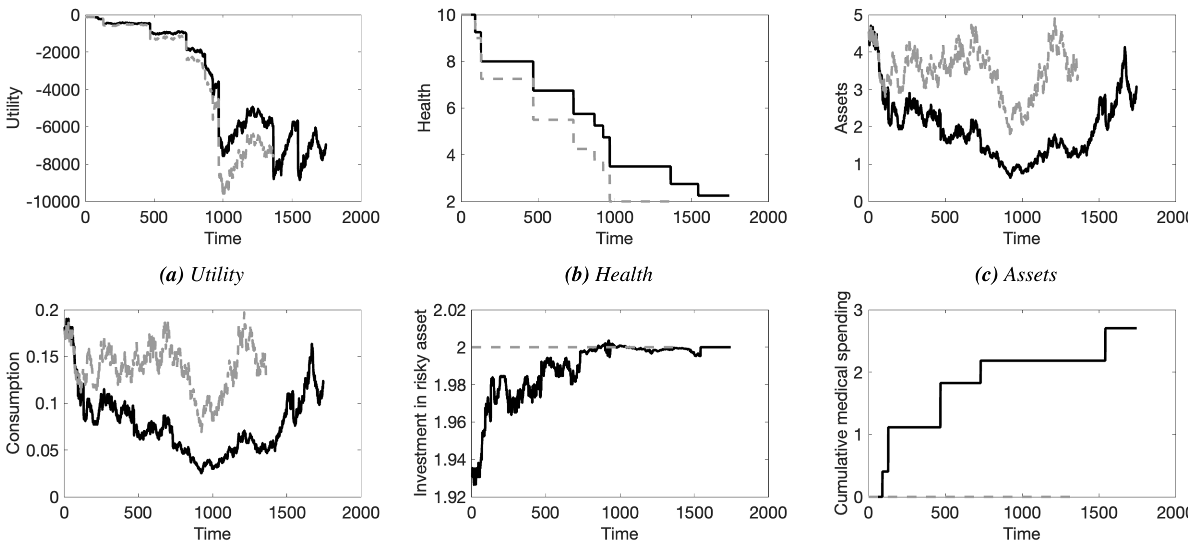
Table 1: *Model parameters for the base case.*

Parameter	Value	Parameter	Value
Risk-free rate	$r = 3\%$	Death threshold	$H_D = 2$
Risky asset return	$\mu = 6\%$	Initial health	$H_0 = 10$
Volatility	$\sigma = 10\%$	Health shock intensity	$\lambda = 0.32\%$
Risk aversion	$\gamma = 1.5$	Minimal health shock	$\underline{\theta} = 0.5$
Initial reference point	$B_0 = 10$	Maximal health shock	$\overline{\theta} = 2.0$
Discount parameter	$\rho = 3\%$	Minimal damage	$\varepsilon_H = 0.25$
Reference weight	$\omega \in [0, 1]$		

5.1 Static reference dependence

Consider first the case of a purely static reference dependence of the agent's health assessment, similar to the analysis by Harris and Kohn (2018). The agent evaluates his health development against the initial health, making a deteriorating health a sequence of losses. Figure 1 displays a sample path of overall utility (Panel 1a), health (Panel 1b), and assets (Panel 1c) developments, accompanied by the agent's optimal consumption (Panel 1d), investment (Panel 1e), and accumulated medical spending (Panel 1f). The solid black line represents the case of optimal medication, whereas the dashed grey line considers the same path without any medication.

Unsurprisingly, medical spending enhances both the instantaneous utility and health at every time step in this example. It mitigates the health shocks' damages and keeps health high; see Panel 1b. Although this spending reduces both the agent's assets and consumption (Panels 1c and Panel 1d), the higher health overcompensates the utility loss from consumption. In this benchmark case, the agent has a static reference health level that equals the initial health. This implies a strong desire to limit health reductions already at high health levels. Consequently, the agent spends much on medication for large shocks as these allow a substantial recovery of the health; see Panel 1f and Panel 1b. For small shocks, the agent refrains from medication as these shocks cause relatively less damage that can be recovered. Finally, as health deteriorates, the agent increases his fraction



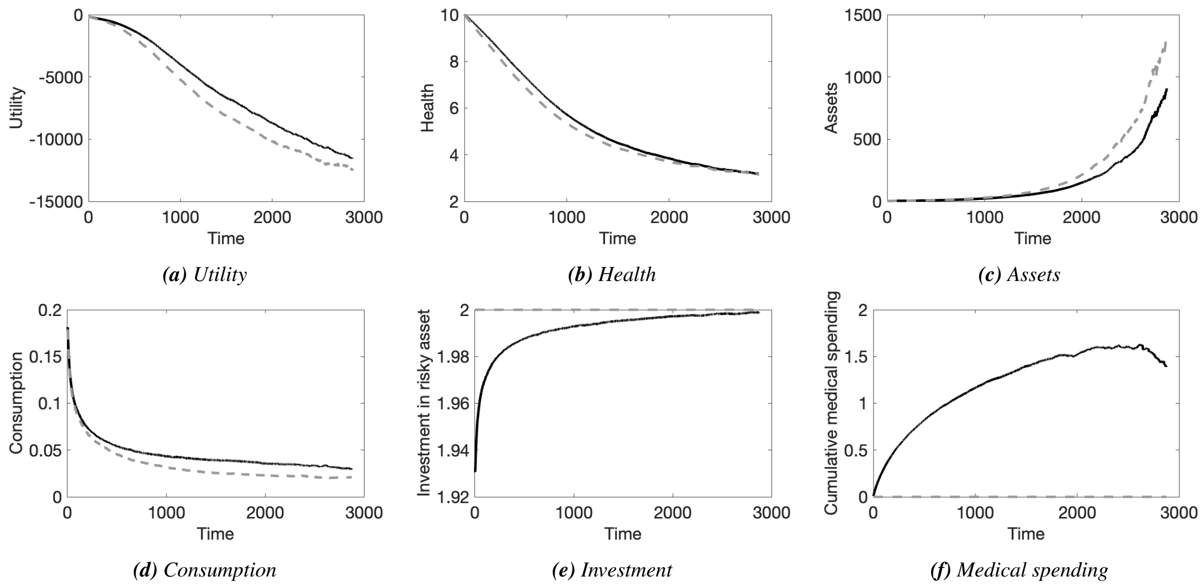
(a) Utility *(b) Health* *(c) Assets*

(d) Consumption *(e) Investment* *(f) Medical spending*

Figure 1: Sample path, static reference health. *This figure shows a sample development of utility (Panel 1a), health, (Panel 1b), assets (Panel 1c), consumption (Panel 1d), investment (Panel 1e), and accumulated medical spending (Panel 1f). Each graph shows the case of optimal medication (solid black line) and without medication (dashed grey line). The reference point does not update, that is $\omega = 0$. All other parameters are as in Table 1.*

of risky investments but generally remains below the one without medication (Panel 1e). The investment in risky assets is driven by two channels: First, the fraction invested risky increases in wealth. Second, for a given wealth level, the appetite for investment risk is non-linear in health: As the health deteriorates, the agent reduces the risky investment. As health becomes critically low, risky investments raise again.

While Figure 1 illustrates the agents behavior for a specific example, Figure 2 presents a more general perspective from a Monte Carlo sample, that averages 10,000 paths at each time step. Each Panel in Figure 2 presents the equivalent information to Figure 1 for the averaged paths. We see that the averaging confirms the intuition from our previous analysis. Medical spending significantly boosts both the utility at any given time and consequently the life-time utility (Panel 2a). This effect is intimately connected to the agent's health: as the health remains high longer (Panel 2b), it allows



(a) Utility *(b) Health* *(c) Assets*

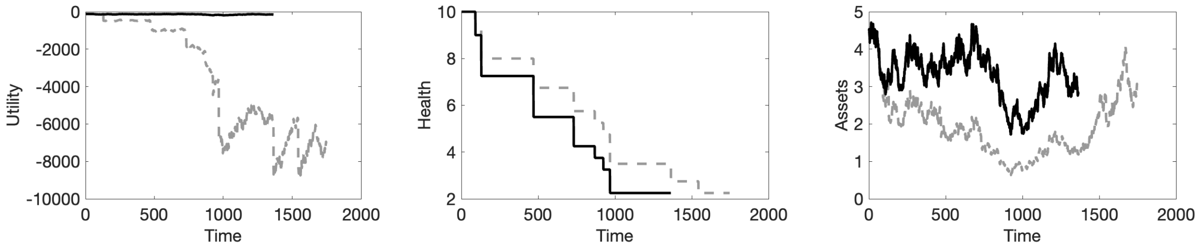
(d) Consumption *(e) Investment* *(f) Medical spending*

Figure 2: Overall development, static reference health. *This figure shows a average development of utility (Panel 2a), health, (Panel 2b), assets (Panel 2c), consumption (Panel 2d), investment (Panel 2e), and accumulated medical spending (Panel 2f). Each graph shows the case of optimal medication (solid black line) and without medication (dashed grey line) for a Monte Carlo time-step-by- time-step averaging. The reference point does not update, that is $\omega = 0$. All other parameters are as in Table 1.*

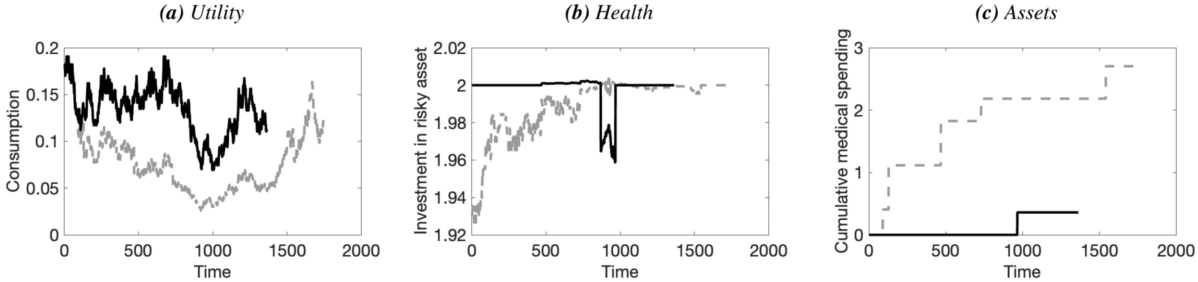
the agent to draw a sufficiently high utility from health that offsets the reduction in consumption spending to cover health care (Panel 2d). The agent also raises the fraction invested in risky assets as health decreases but remains below the no-medication case (Panel 2e).

5.2 Adaptive reference health

We now consider the case of an adaptive reference health level, that is, the agent's reference health perfectly adapts to the new health level. Consequently, the agent's optimal choices reflect this adaption. To visualize the influence of an adaptive reference health on the agent's utility and choices, Figure 3 presents the same sample path as in the previous section; however, the solid line now



(a) Utility *(b) Health* *(c) Assets*

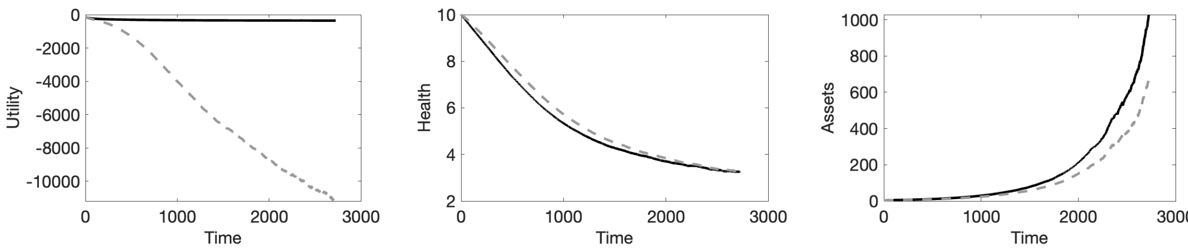


(d) Consumption *(e) Investment* *(f) Medical spending*

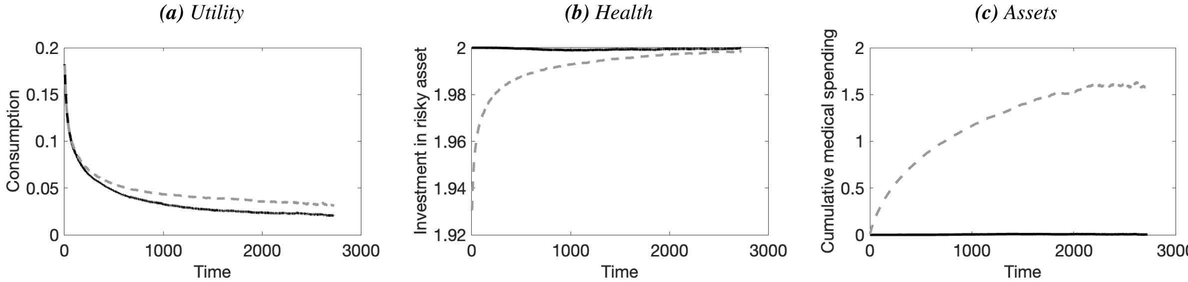
Figure 3: Sample path, adaptive and static reference health. *This figure shows a sample development of utility (Panel 3a), health, (Panel 3b), assets (Panel 3c), consumption (Panel 3d), investment (Panel 3e), and accumulated medical spending (Panel 3f). Each graph shows the case of an adaptive reference health (solid black line) and a stationary reference health (dashed grey line). The reference point updates with* $\omega = 1$. *All other parameters are as in Table 1.*

displays the case of a fully adaptive reference health, whereas the dashed line stands for the case of a stationary one. Similarly, Figure 4 presents the matching averaged Monte Carlo sample.

For the path presented in Figure 3, the agent obtains a steady high utility until late in life. Compared to the case of a static reference point, the adaptive reference agent obtains a much more rectangularized utility (Panel 3a): While the static reference point causes a steadily decreasing utility profile, the adaptive reference agent enjoys high utility until the last few health shocks. This agent's ability to adapt to decreasing health allows him or her to draw high utility from consumption (Panel 3d), as lower medical spending (Panel 3f) boost his assets. This medical spending profile not only allows the agent to spend less on medication over the lifetime, it also focuses on the treatment of health shocks late in life. It implies that an agent with an adaptive reference health lives a slightly shorter but substantially happier life because of his psychological flexibility. The agent further



(a) Utility *(b) Health* *(c) Assets*



(d) Consumption *(e) Investment* *(f) Medical spending*

Figure 4: Overall development, adapting and static reference health. *This figure shows a average development of utility (Panel 4a), health, (Panel 4b), assets (Panel 4c), consumption (Panel 4d), investment (Panel 4e), and accumulated medical spending (Panel 4f). Each graph shows the case of an adaptive reference health (solid line) and a stationary reference health (dashed line) for a Monte Carlo time-step-by-time-step averaging. The reference point updates with* $\omega = 1$*. All other parameters are as in Table 1.*

invests slightly riskier compared to the static reference health agent (Panel 3e). The investment risk tends to go down as the health decreases.

The agent's adaptive reference point systematically boosts the utility, particularly when health is failing (Panel 4a). Because the reference health reflects that previous health levels become unattainable, focusing on achieving more realistic health levels and accepting them as sufficient enhances the utility of consumption and the current health status, that is, the psychological flexibility allows the agent to feel better. The adaptive reference point shifts medical spending over time.

6 Conclusion

In this paper, we build a dynamic model of reference health and study how it affects medical spending, lifetime health, consumption, assets, and investments. We incorporate adaptiveness of this reference health, that is, the reference point changes with the agent's decaying health over time. Adaptive reference health meaningfully adjusts lifetime medical spending by reducing medical spending. It slightly reduces the agent's life span by shifting resources from medical spending to consumption which the agent enjoys more because a failing health does not cause a strong disutility. In contrast, the agent with static health reference maintains near optimal health as long as possible at the price of a lower consumption and assets.

Our analysis leaves some model extensions to future research. One possibility is the inclusion of investment in sport or illness prevention instead of medication: This type of health investment is working on slowing down its natural decay. We can consider including this type of investment and allow it to influence either the likelihood of health shocks or the health damage boundaries of health shocks at a cost. Furthermore, the model can feature the fear of death. To include a fear of death analysis, the death utility can vary in a non-positive value range, reflecting different levels of fear to study their impacts on the agent's optimal decisions.

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Reference Health and Investment Decisions
Appendix

Chunli Cheng Christian Hilpert Alexander Szimayer Peter Zweifel

December 8, 2022

List of Appendices

Appendix I Medical Treatment

This appendix collects two extensions that generalize the impact of medication by allowing for more complicated relations between money and health improvements and health-dependent health improvements.

I.1 Generalized treatment impact

For a given health shock θ and amount k spent on the related medical treatment, the effect of the treatment $V(k,\theta)$ can be captured by the following general parametrization:

$$V(k,\theta) = (\theta - \varepsilon_H)\,(1 - e^{-\kappa\,\theta^{\delta}\,k})\,, \tag{I.1}$$

where $\delta \in \mathbb{R}$ and $\kappa > 0$. The net health shock is then

$$\theta - V(k,\theta) = \theta\, e^{-\kappa\,\theta^{\delta}\,k} + \varepsilon_H\,(1 - e^{-\kappa\,\theta^{\delta}\,k})\,. \tag{I.2}$$

The amount $k_{0.5}(\theta)$ that is required to halve a given health shock θ, with $\theta > 2\,\varepsilon_H$, is

$$k_{0.5}(\theta) = \frac{\theta^{-\delta}}{\kappa}\,\ln\frac{2\,\theta - 2\,\varepsilon_H}{\theta - 2\,\varepsilon_H}\,. \tag{I.3}$$

From there we see that δ shapes the relationship between wealth and health. For $\delta = -1$, the monetary costs for halving a given health shock is proportional to the extent of the health shock itself. For $\delta < -1$, these costs are over-proportionally increasing. Whereas for $\delta > -1$, the opposite holds. Setting $\kappa = 1$ without loss of generality, as we can affect the scale by the choice of F_θ, and $\delta = -1$, to keep things simple, we obtain the specification in the main paper.

I.2 Health-dependent treatment impact

So far, for a given health shock θ, the health improvement from a given amount of medical spending κ is the same for the agent no matter in which health stage he is after experiencing the health shock. One way of including that the medical benefits differ in the health level is to reform the lasting damage ε_H to be health-dependent $\varepsilon(h)$, where h is the health level just before the health shock. To be specific, we consider that $\varepsilon(h)$ decreases in health: The lower the health, the more lasting damage the agent incurs. We update the medical treatment effect (3) accordingly as follows:

$$V(k,\theta,h) = (\theta - \varepsilon(h))\,(1 - e^{-\frac{k}{\theta}}). \tag{I.4}$$

Consequently, we can update the derivatives to be

$$V_k(k,\theta,h) = \frac{\theta - \varepsilon(h)}{\theta}\, e^{-\frac{k}{\theta}} \quad \text{and} \quad V_{kk}(k,\theta,h) = -\frac{\theta - \varepsilon(h)}{\theta^2}\, e^{-\frac{k}{\theta}}, \tag{I.5}$$

which increase and decrease in health, respectively. The medication effect specification (I.4) presents that a given amount of medical spending bring more benefits to a healthier agent. It tells that compared to seeking medical treatment immediately, postponing it till a later less healthy stage requires more medical spending for retaining a fixed amount of health recovery.

Here we can consider a simple functional form of $\varepsilon(h) = \frac{\varepsilon}{1+h}$, $\varepsilon > 0$, with which we write

$$V(k,\theta,h) = \left(\theta - \frac{\varepsilon}{1+h}\right)\left(1 - e^{-\frac{k}{\theta}}\right). \tag{I.6}$$

To ensure positive health improvement from positive medication at any health h and health shock θ level, ε shall be set to guarantee $\frac{\varepsilon}{1+H_D} < \underline{\theta}$.

Appendix II Special Cases

This appendix collects the derivations of various special cases.

II.1 Derivation for critically ill agent

Given the agent is critically ill, the value function $U(h,b,x)$ is of the form

$$U(h,b,x) = G(h,b)\,x^{1-\gamma}, \tag{II.1}$$

with $G(h,b) > 0$ for $\gamma > 1$. To see this, follow the ansatz (II.1) with partials

$$U_x(h,b,x) = (1-\gamma)\,G(h,b)\,x^{-\gamma} \text{ and } U_{xx}(h,b,x) = -\gamma(1-\gamma)\,G(h,b)\,x^{-\gamma-1}\,. \tag{II.2}$$

Plugging the latter in (25) yields

$$\begin{aligned}
0 = &\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{(1-\gamma)^2\,G(h,b)^2\,x^{-2\gamma}}{\gamma(1-\gamma)\,G(h,b)\,x^{-\gamma-1}} + (rx+\beta h)\,(1-\gamma)\,G(h,b)\,x^{-\gamma} \\
&+ \frac{\gamma}{1-\gamma}K(h,b)^{\frac{1}{\gamma}}\left((1-\gamma)\,G(h,b)\,x^{-\gamma}\right)^{\frac{\gamma-1}{\gamma}} \\
&- [\rho+\lambda(h)]\left[G(h,b)\,x^{1-\gamma}\right] - \lambda(h)\frac{r}{\rho r^{\gamma}(\gamma-1)}G(h,b)^{-1}e^{(h-b)(1-\gamma)},
\end{aligned}$$

To simplify, we divide by $K(h,b)\,x^{1-\gamma}$

$$0 = \frac{1}{2}\,\frac{(\mu-r)^2}{\sigma^2}\,\frac{1-\gamma}{\gamma} + r\,(1-\gamma) + \gamma\left(\frac{K(h,b)}{(1-\gamma)\,G(h,b)}\right)^{\frac{1}{\gamma}} - [\rho+\lambda] - \frac{\lambda\, r e^{(h-b)(1-\gamma)}}{\rho r^{\gamma}(\gamma-1)} G(h,b)^{-1}\,.$$

Rearranging gives

$$\frac{\rho+\lambda}{\gamma} - \frac{1}{2}\,\frac{(\mu-r)^2}{\sigma^2}\,\frac{1-\gamma}{\gamma^2} - r\,\frac{1-\gamma}{\gamma} + \left(\frac{K(h,b)}{(1-\gamma)\,G(h,b)}\right)^{\frac{1}{\gamma}} - \frac{\lambda\, r e^{(h-b)(1-\gamma)}}{\rho r^{\gamma}(\gamma-1)} G(h,b)^{-1}$$

Set $L(h,b) = ((1-\gamma)G(h,b))^{-\frac{1}{\gamma}}$ (noting that $G(h,b) < 0$), then $L(h,b) > 0$ and

$$\frac{\rho+\lambda}{\gamma} - \frac{1}{2}\,\frac{(\mu-r)^2}{\sigma^2}\,\frac{1-\gamma}{\gamma^2} - r\,\frac{1-\gamma}{\gamma} + (K(h,b))^{\frac{1}{\gamma}} L(h,b) - \frac{\lambda\, r e^{(h-b)(1-\gamma)}}{\rho r^{\gamma}} L(h,b)^{\gamma}\,, \tag{II.3}$$

which has exactly one positive solution.

Appendix III Theoretical boundaries

From the derivation of the no-medication case in the previous subsection, the boundary conditions follow immediately

$$0 = \lim_{x\searrow 0} \underline{U}(h,b,x) \leq \lim_{x\searrow 0} U(h,b,x) \leq \lim_{x\searrow 0} \overline{U}(h,b,x) = 0\,,$$

and since $\underline{U}(h,b,x) \leq U(h,b,x)$ but also $\underline{U}(h,0) = 0 = U(h,0)$, we must have

$$\infty = \lim_{x\searrow 0} \underline{U}_x(h,x) \leq \lim_{x\searrow 0} U(h,b,x) \leq \infty\,.$$

Table IV.1: *Computational parameters for the base case.*

Parameter	Value	Parameter	Value
Health steps	41	Asset steps	101
Health step size	0.25	Minimal asset value	0.5
Reference grid steps	41	Reference step size	0.25
Health shock grid steps	7	Health shock step size	0.25
Minimal asset value	0.5	Maximal asset value	5.0

We summarize the boundary conditions as follows

$$\lim_{x \searrow 0} U(h,b,x) \leq \lim_{x \searrow 0} \overline{U}(h,b,x) = 0\,, \tag{III.1}$$

$$\lim_{x \searrow 0} U_x(h,b,x) \geq \lim_{x \searrow 0} \underline{U}_x(h,x) = +\infty\,. \tag{III.2}$$

Appendix IV Numerical Method

Recall that $\gamma > 1$. How to obtain the function $U(h,b,x)$ and based on this the optimal strategy $(a^\star, c^\star, k^\star)$?

For $h \leq H_D$ the value function is set to the death utility, that is, $U_D(h,b,x)$, for all $x \geq 0$. For the poor health states, $H_D < h \leq H_D + \varepsilon_H$, U is given by (II.1) with $G(h,b)$ determined in equation (II.3). For $h > H_D + \varepsilon_H$, $\mathbb{U}$ is computed using the already known $U(h',b,x,)$ for $h' \leq h - \varepsilon_H$ where simultaneously the optimal health expenditure $k^\star(h,b,x,\theta)$ is solved for using (17). The value function $U(h,b,x)$ is computed using the PDE (25). Then this step is repeated when increasing h. The structure of the value function in (II.1) is in general not preserved when $\mathbb{U}(h,b,x)$ is computed, thus no closed-form solution exists.

Let us denote the domain the distribution F_θ describing the health shocks by $[\underline{\theta}, \overline{\theta}]$, with $\varepsilon_H \leq \underline{\theta} < \overline{\theta}$. Thus, if a health shock occurs, its smallest possible impact $\underline{\theta}$ exceeds the value ε_H. Define the grid of the state space (h, b, x) by the respective mesh Δh and Δx. For N_x let

$$x_i = i\,\Delta x\,, \quad \text{for } i = 0, \ldots, N_x\,. \tag{IV.1}$$

For $\underline{N}_h$ and $\overline{N}_h$ define

$$h_i = H_D + i\,\Delta h\,, \quad \text{for } i = -\underline{N}_h, \ldots, 0, \ldots., \overline{N}_h\,. \tag{IV.2}$$

Similarly,

$$b_m = H_D + m\,\Delta h\,, \quad \text{for } m = 0, \ldots., \overline{N}_h\,. \tag{IV.3}$$

Health states below the death threshold H_D are required for a formal reason. In case a health shock for $h \approx H_D$ occurs, the utility is recursively calculated by jump conditions from utility at states with a lower health level. Thus $\underline{N}_h = \overline{\theta}/\Delta h$ such that $H_D - \underline{N}_h\,\Delta h = H_D - \overline{\theta}$ covering the biggest possible health shock $\overline{\theta}$ from the lowest alive state H_D. Next, the support $[\underline{\theta}, \overline{\theta}]$ of the health shock distribution F_θ is discretized by

$$\theta_i = i\,\Delta h\,, \quad \text{for } i = \underline{N}_\theta, \ldots, \overline{N}_\theta\,, \tag{IV.4}$$

with $(\underline{N}_\theta - 1)\,\Delta h < \underline{\theta} \leq \underline{N}_\theta\,\Delta h$ and $\overline{N}_\theta\,\Delta h < \overline{\theta} \leq (\overline{N}_\theta + 1)\,\Delta h$. The corresponding probabilities are determined by

$$p_i = \mathbb{P}(\theta_i - 0.5\,\Delta h < \theta \leq \theta_i + 0.5\,\Delta h)\,,\ \text{for } i = \underline{N}_\theta + 1, \ldots, \overline{N}_\theta - 1\,, \tag{IV.5}$$

and $p_{\underline{N}_\theta} = \mathbb{P}(\theta \leq \theta_{\underline{N}_\theta} - 0.5\,\Delta h)$ as well as $p_{\overline{N}_\theta} = \mathbb{P}(\theta_{\overline{N}_\theta} - 0.5\,\Delta h < \theta)$.

We initialize the numerical approach by observing that in death states $h_i \leq H_D$, or, $i \leq 0$

$$\hat{U}(h_i, b_m, x_j) = U_D(h_i, b_m, x_j)\,, \quad \text{for } i = -\underline{N}_h, \ldots, 0,\, m = 0, \ldots \overline{N}_h, \text{ and } j = 0, \ldots, N_x\,. \tag{IV.6}$$

Also, for the zero wealth state $x_0 = 0$, the agent has zero utility as no further consumption can be financed, that is,

$$\hat{U}(h_i, b_m, x_0) = 0\,, \quad \text{for } i = 1, \ldots, \overline{N}_h \text{ and } m = i, \ldots, \overline{N}_h. \tag{IV.7}$$

The second step of the initialization is to look at poor health levels h_i, with $H_D < h_i \leq H_D + \varepsilon_H$ according to (26). The approximation of the value function $\hat{U}$ follows from (II.1) and is given by

$$\hat{U}(h_i, b_m, x_j) = G(h_i)\, x_j^{1-\gamma}\,, \quad \text{for } i = 1, \ldots, N_h^\star, m \geq i, \ldots, N_h^\star, \text{ and } j = 1, \ldots, N_x\,, \tag{IV.8}$$

where G has the discrete version

$$G(h_i, b_m) = \frac{K(h_i, b_m, \gamma))}{(1-\gamma)\left(\frac{\rho+\lambda}{\gamma} - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{1-\gamma}{\gamma^2} - r\,\frac{1-\gamma}{\gamma}\right)^\gamma}\,, \quad \text{for } i = 1, \ldots, N_h^\star\, m \geq i, \ldots, \overline{N}_h, \tag{IV.9}$$

and $N_h^\star$ is characterized by $\Delta h\,(N_h^\star - 1) < \overline{\theta} \leq \Delta h\, N_h^\star$.

The next major step is to solve the differential equation (25) in x for each given h and b numerically. This requires the computation of the health jump effect given by $\mathbb{U}$ along the lines of (20) beforehand, which in turn is based on the optimal investment in medical treatment $k^\star$ given a health shock as outline in (17).

To address the latter define for a given health shock $\theta = i\,\Delta h$, with $i = \underline{N}_\theta, \ldots, \overline{N}_\theta$ the investment $k_j(\theta_i)$ to generate a beneficial effect on the health level of magnitude $j\,\Delta h$ by

$$k_l(\theta_i) = \theta_i \ln \frac{\theta_i - \varepsilon_H}{\theta_i - \varepsilon_H - l\,\Delta h}\,, \qquad \text{for } l = 0, \ldots, \overline{N}_k(\theta_i)\,, \tag{IV.10}$$

with $\theta_i - \varepsilon_H - \Delta h < \overline{N}_k(\theta_i)\,\Delta h \leq \theta_i - \varepsilon_H$, such that $V(k_l(\theta_i), \theta_i) = l\,\Delta h$, or, equivalently, $\theta_i - V(k_j(\theta_i), \theta_i) = (i - j)\,\Delta h$. Preparing the numerical equivalent to (17) define

$$l^\star(h_i, b_m, x, \theta_j, \omega) = \underset{l=0,\ldots,\overline{N}_k(\theta_j)}{\arg\max}\; \hat{U}(h_i - \theta_j + V(k_l(\theta_j), \theta_j), f(h_i, b_m, \omega, k_l(\theta, x - k_l(\theta_j))\,, \tag{IV.11}$$

where $\hat{U}$ is interpolated linearly in x. In case the maximum in (IV.11) is not unique, the largest expense is selected resulting in the longest life with identical utility. The numerical equivalent to (17) is thus

$$k^\star(h_i, b_m, x, \omega, \theta_j) = k_{l^\star(h_i, b_m, x, \omega, \theta_j)}(\theta_i)\,. \tag{IV.12}$$

Then, for $(h_i, b_m, x_j) = (i\,\Delta h, m\,\Delta h, j\,\Delta x)$, the numerical version of (20) becomes

$$\hat{\mathbb{U}}(h_i, b_m, x_j) = \sum_{l=\underline{N}_\theta}^{\overline{N}_\theta} \hat{U}((i - l + l^\star(h_i, x_j, \theta_l))\Delta h, (1-\omega)m\Delta h + \omega(i - l + l^\star(h_i, x_j, \theta_l))\Delta h, x - k^\star(h_i, x_j, \theta_l))\, p_l\,, \tag{IV.13}$$

where $\hat{U}$ is interpolated linearly in x. Now, we are equipped to solve the second order ordinary differential equation (25). For doing so, we need two boundary conditions.

IV.1 Boundary Conditions

The boundary conditions $U(h, 0+) = 0$ and $U_x(h, 0+) = +\infty$ in (III.1) and (III.2), respectively, cannot be used directly for the numerical approach. Instead, we focus on the case of vanishing wealth.

Then, based on the discussion in Section III, it is reasonable to assume $U(h,b,x) \approx G(h,b)\,x^{1-\gamma}$ with partial derivative in wealth dimension $U_x(h,b,x) \approx (1-\gamma)\,G(h,b)\,x^{-\gamma}$. Vanishing wealth in the discretized setup is given by $x = \Delta x = x_1$. For $i = N_h^\star + 1, \ldots, \overline{N}_h$, we approximate

$$\hat{U}(h_i, x_1) = G(h_i, b_m)\,x_1^{1-\gamma} \text{ and } \hat{U}_x(h_i, x_1) \;=\; (1-\gamma)\,G(h_i, b_m)\,x_1^{-\gamma}. \tag{IV.14}$$

Thus, (IV.14) are the needed boundary conditions, that are specified in terms of $G(h_i, b_m)$. The general ordinary differential equation in Proposition 2 becomes

$$0 = \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{1-\gamma}{\gamma}x_j^{1-\gamma}G(h_i,b_m) + rG(h_i,b_m)x_j^{1-\gamma} + \frac{\gamma}{1-\gamma}K(h_i,b_m)^{\frac{1}{\gamma}}((1-\gamma)G(h_i,b_m))^{\frac{\gamma-1}{\gamma}}$$
$$+\lambda(h_i)\mathbb{U}(h_i,b_m,x_j) - (\rho + \lambda(h_i))G(h_i,b_m)x_j^{1-\gamma}$$

Here, the expected after shock utility $\mathbb{U}(h_i, b_m, x_j)$ is iteratively determined as before. Then, the value of $G(h_i, b_m)$ is solved numerically.