Real Options and Financial Flexibility

Gustavo Grullon, Ali Kakhbod, A. Max Reppen, Tarik Umar, and Hao Xing

Abstract

We solve a model of firm dynamics that shows a firm's financial flexibility affects the timing of investment decisions. Firms may delay investment to accumulate cash, which increases financial flexibility, or raise costly external capital to exercise immediately their real options. Small firms, valuing financial flexibility more, require more cash before exercising. Our models predicts that the value of waiting is hump-shaped in cash. Low-cash firms must delay investment longer to accumulate cash and prefer financing investments externally. High-cash firms already have high financial flexibility and thus benefit less from waiting. We empirically examine these predictions.

JEL classification: G30, G31

Keywords: Real options, option to delay, costly financing, financial constraints, investment

☆First Draft March 2022. Emails: Gustavo Grullon (grullon@rice.edu), Ali Kakhbod (akakhbod@berkeley.edu), A. Max Reppen (amreppen@bu.edu), Tarik Umar (tarik.umar@rice.edu), and Hao Xing (haoxing@bu.edu).

September 12, 2022

1. Introduction

We know little about how a firm's financial flexibility affects the timing of firm investments. Companies often have the ability to wait to time an investment—a real option (Dixit and Pindyck, 1994). The literature on real options since McDonald and Siegel (1986) generally assumes that firms operate in frictionless capital markets. A convenience of this assumption is that valuing a real option is like valuing an American option (Black and Scholes, 1973; Merton, 1973). However, with costly financing, the total cost of exercising the option includes the effect of investing on a firm's financial flexibility. Firms manage their cash reserve, or financial flexibility, to economize over issuance costs. Waiting delays investment's benefits but allows the firm to accumulate cash and allows earnings fundamentals to improve. Surveys of chief financial officers (CFOs) by Graham (2000) and Graham and Harvey (2001) confirm that CFOs attach great importance both to real options and to maintaining financial flexibility.

Recently, the seminal Julien Hugonnier, Semyon Malamud and Erwan Morellec (2015) paper solves a dynamic model of a firm with a real option to delay investment allowing for cash accumulation and external financing. In that model, capital supply is uncertain so that accumulating cash allows a firm to potentially exercise a real option prior to *costless* external financing becoming available.

We solve a dynamic model of a firm facing *costly* financing and possessing an option to delay a lumpy investment. Like Hugonnier, Malamud and Morellec (2015), exercising the option permanently increases the mean of the firm's stochastic cash flow process. Also, the exercise cost is fixed and irreversible. Unlike that paper, our model firm has a dynamic physical capital stock as one state variable and a cash reserve as the second state variable. The dynamic physical capital stock follows the standard capital accumulation model with infinitely divisible investments, depreciation, and convex adjustment costs, which encourage the firm to smooth investment over time. Thus, our model firm makes regular investments in its dynamic capital stock punctuated by a lumpy and transformative investment—a pattern observed empirically (Doms and Dunne, 1998; Caballero, 1999; Cooper, Haltiwanger and Power, 1999). Hugonnier, Malamud and Morellec (2015) does not model investment beyond the exercise decision.

By modeling a dynamic physical capital stock, we can uniquely examine how a firm's real option dynamics vary over a firm's life cycle. Our firm changes as it grows because our model allows us to incorporate several realistic assumptions, such as diminishing returns to scale (e.g., Caballero, 1991) and financing frictions that decline in intensity with size (e.g., Altınkılıç and Hansen, 2000). Together, these modeling differences significantly enrich the determinants of a firm's demand for flexibility.

Although the classical tools of option pricing theory can no longer be directly applied, and although the analysis of the exercise decision and its financing requires solving a considerably more involved two-dimensional partial differential equation, we are able to solve the problem numerically and provide a proof of convergence in the appendix.

We find support in the data for several novel predictions.

First, our model uniquely allows us to characterize how a firm's threshold level of financial flexibility for exercising the real option changes as a firm grows. The model predicts the threshold level of cash scaled by the size of the physical capital stock (the cash-to-capital ratio) for exercise is declining and convex in firm size (capital). The threshold is higher when a firm is small because financial flexibility is more valuable when a firm is small, making the firm more reluctant to pay the fixed exercise cost. As the capital stock increases, the value of financial flexibility declines convexly because of diminishing returns to scale and issuance costs that decline relative to size. The fact that the decision to exercise a real option depends on cash is in contrast to benchmark models of real-option investment in corporate finance, where the decision to exercise depends only on future cash flows (e.g., McDonald and Siegel, 1986; Dixit and Pindyck, 1994). More recently, in Hugonnier, Malamud and Morellec (2015), there is a single cash threshold determining exercise because capital and the investment opportunity are static.

We find support for this first prediction in the data. Cash holdings at exercise are declining and convex in capital. We use growth investments as a proxy for real option exercises. Specifically, we calculate a firm's net investment as a firm's expenditures on capital less depreciation scaled by lagged capital. Our results are robust to classifying net investments above various percentiles as option exercises. As discussed more below, given that only recently models of real options consider a firm's financial flexibility, it is unsurprising that no prior empirical work examines how a firm's financial flexibility determines a firm's real option's behavior.

Second, our model predicts that the value of waiting is non-monotonic in a firm's financial flexibility. Specifically, the value of the option to delay investment is hump-shaped in a firm's cash holdings. We compute the value of the real option as the difference in the value of a firm with the ability to delay investment and the value of the firm with a now-or-never investment opportunity. When cash is low, the real option has little value because the firm expects a lengthy wait to accumulate the minimum desired cash holdings for internally financing the exercise cost. Additionally, a low-cash firm is likely to require external financing anyways to avoid inefficient liquidation and thus is more likely to finance the exercise cost externally. When cash is high, the option also has little value because the firm has high financial flexibility so that waiting to accumulate additional cash is less beneficial relative to the forgone increase in cash flows from not exercising immediately. Instead, waiting is more valuable when a firm has intermediate cash levels. More precisely, the option is most valuable when the cash reserve is right around the cost of exercising the real option. At this cash level, immediately paying the exercise cost increases the marginal value of cash dramatically and may even require costly financing that could be avoided by waiting a reasonable amount of time. The option value is more sensitive to financial flexibility when a firm is small because its marginal value of cash is higher. Hugonnier, Malamud and Morellec (2015) does not calculate the value of the real option.

We find support for this second prediction in the data. It is well known that real option values increase with expected cash flow volatility, and Gustavo Grullon, Evgeny Lyandres and Alexei Zhdanov (2012) confirms that firms with more real options have more positive relations between stock returns and innovations in firm volatility. Our model likewise predicts an increase in option values with volatility, and the sensitivity of option values to changes in volatility is hump-shaped in a firm's financial flexibility. Consistent with our prediction, we empirically find that the positive sensitivity of stock returns to changes in firm volatility is hump-shaped in a firm's cash holdings and more so when a firm is smaller. We also find that the hump-shape is stronger for firms that the literature generally characterizes as having more real options, such as young firms, high-R&D firms, high-growth firms, and firms in industries with plenty of real options. Additionally, consistent with firms with intermediate financial flexibility valuing waiting more, we empirically find that firms with intermediate cash holdings cut investment more in response to increases in a firm's stock volatility. In other words, growth investments are negatively related to firm volatility, consistent with several prior empirical studies (Leahy and Whited, 1996; Guiso and Parigi, 1999; Bulan, 2005; Bloom, Bond and Van Reenen, 2007), and this negative relation is U-shaped in a firm's liquidity.

Overall, our paper connects two classical strands of literature, one on real options started by McDonald and Siegel (1986) and one on corporate cash balances started by Miller and Orr (1966). Early static models find that financing frictions, which create a demand for cash, decrease investment but ignore real flexibility and thus cannot speak to the effect of costly financing on the timing of investment (e.g., Fazzari, Hubbard and Petersen, 1988; Froot, Scharfstein and Stein, 1993; Kaplan and Zingales, 1997). Décamps and Villeneuve (1994) considers a financially constrained firm with an asset in place generating cash-flows that are subject to i.i.d shocks and a growth option, which similarly raises the drift of the cash-flow process. However, their firm has no access to external funding and thus must finance the exercise cost with internal funds. Boyle and Guthrie (2003) studies real options in the presence of financial constraints (a collateral constraint), but their firm may continue operations even when it accumulates arbitrarily large negative earnings, although it is not able to raise funds. An earlier study by Mauer and Triantis (1994) considers a real options problem for a levered firm that faces recapitalization costs to change the mix of debt and equity to maximize the interest tax shield. However, like Leland (1994), the firm does not otherwise incur external financing costs. Boot and Vladimirov (2019) focuses on agency issues when examining a financially constrained firm with an asset in place that generates random cash flows and a new investment opportunity. Gomes and Schmid (2021) develop a model to understand the joint exercise and leverage policies of firms, but do not study liquidity management. In the strategic dynamic contexts, several papers solve a real options problem under incomplete information (Miao and Wang, 2007; Grenadier and Malenko, 2010, 2011; Grenadier, Malenko and Malenko, 2016). Recently, Patrick Bolton, Neng Wang and Jinqiang Yang (2019) models a firm with a growth option allowing for cash accumulation but does not model a dynamic physical capital stock.[1]

Our empirical results add to the relatively small empirical literature on real options. Prior work examines whether managerial decisions and market prices for assets are consistent with real option theory. These papers focus on a few industries such as real estate (Quigg, 1993; Cunningham, 2006; Bulan, Mayer and Somerville, 2009), oil and gas (Paddock, Siegel and Smith, 1988; Kellogg, 2014; Décaire, Gilje and Taillard, 2020), and mining (Moel and Tufano, 2002). In general, these studies show that investors value the ability to wait because they pay more than measures of current intrinsic value. While these papers consider a project's future cash flows when determining when to exercise a real option, our model and empirical work highlight that it is important to consider the

[1] Additionally, our paper also extends the broader dynamic liquidity management literature (e.g., Girgis, 1968; Gomes, 2001; Hennessy and Whited, 2005, 2007; Riddick and Whited, 2009; Bolton, Chen and Wang, 2011; Décamps et al., 2011; Anderson and Carverhill, 2012; He and Milbradt, 2014; Kakhbod et al., 2021; Dai et al., 2021). By contrast, we focus on real options and particularly how costly financing creates a demand for liquidity that affects when a firm decides to exercise a real option.

financial flexibility of the firm owning the real option. Overall, much additional work is warranted, and accordingly, Lambrecht (2017) states, "We need more empirical studies that test whether firms behave according to what real options theory predicts."

2. Model

The firm's operating revenue depends on its capital stock and productivity. We assume that the firm's productivity, before exercising the growth option, evolves according to:

$$dZ_t = \mu dt + \sigma dW_t, \tag{1}$$

where W is a one-dimensional Brownian motion under the risk neutral measure and μ and σ are positive constants. Thus, productivity shocks dZ_t, before the option exercise, are i.i.d. with mean μdt and variance $\sigma^2 dt$.

The firm has a growth option that improves the average productivity and can be exercised at a time of its choosing. Exercising the option costs the firm Φ (in monetary units). The option exercise is irreversible and permanently increases the firm's productivity. After exercising the growth option, the firm's productivity process becomes

$$dZ_t = \tilde{\mu} dt + \tilde{\sigma} dW_t, \tag{2}$$

where $\tilde{\mu} > \mu$. Unless otherwise stated, we assume $\tilde{\sigma} = \sigma$. Thus, the growth option involves a form of "real scaling" of the firm's cash flow process, where the scaling does not come from an expansion in capital stock, but instead from a higher average productivity of capital. This improvement in productivity upon exercising the option resembles that in Hugonnier, Malamud and Morellec (2015) and several other papers (Décamps and Villeneuve, 2007, 2013; Vath, Pham and Villeneuve, 2008; Bolton, Wang and Yang, 2019).

Given the firm's productivity Z, the firm's cumulative cash flows Y follow the dynamics

$$dY_t = k_t^{\alpha} dZ_t, \tag{3}$$

where k is the size of capital stock of the firm. The production exhibits decreasing returns to scale with the scale parameter $\alpha \in (0, 1)$ following Bertola and Caballero (1994).[2]

[2]Our model can accommodate $\alpha = 1$ or $\alpha > 1$; however, we find strong support in the data for the model's predictions with regards to investment assuming $\alpha < 1$. Also, diminishing returns to scale is quite common in the literature. See Caballero (1991), Basu and Fernald (1997), Gomes (2001), and Grullon and Ikenberry (2021).

The firm may regularly make investments that add to the physical capital stock k. As is standard in capital accumulation models, for an investment process i, the dynamics of the capital stock follows

$$dk_t = \Big(i_t - \delta k_t\Big) dt, \tag{4}$$

where $\delta \geq 0$ is the depreciation rate. We assume that investment is irreversible, i.e., $i \geq 0$. Investment is subject to a convex adjustment cost

$$g(k,i) = \frac{\theta}{2}\left(\frac{i}{k}\right)^2 k, \tag{5}$$

for a positive constant θ that measures the degree of the adjustment cost. Modeling a dynamic physical capital stock is one difference with Hugonnier, Malamud and Morellec (2015).

The firm is financed by equity and long-term debt. The firm pays a coupon at a constant rate b for the long-term debt. Having $b \geq 0$ allows us to study the impact of debt on a firm's payout, investment, growth, and risk management decisions. Examining a levered firm is another modeling difference with Hugonnier, Malamud and Morellec (2015).

The firm faces costly external equity financing. When the firm issues equity of a lump size I, the cost is

$$\lambda(I) = \lambda_f + \lambda_p I, \tag{6}$$

where λ_f and λ_p are constants, representing the constant component and the proportional components of issuance costs, respectively. In our model, the relation between the fixed issuance cost and capital (λ_f/k) is decreasing in capital because the fixed cost λ_f does not scale with capital.[3] By contrast, Hugonnier, Malamud and Morellec (2015) assumes no issuance costs but rather an uncertain supply of external funds. Thus, in their model, because the investment opportunity is static, when external financing becomes available, the firm exercises the option.

The firm is also subject to default risk. When the firm exhausts the cash reserve, the equityholders compare the benefit of equity issuance and continuing (continuation value) with the residual value for equityholders after liquidation and applying proceeds

[3]This modeling feature contrasts with the more standard approach of multiplying the fixed issuance cost by capital in the dynamic liquidity management literature because we do not assume homogeneity in size. Bolton, Chen and Wang (2011) acknowledge that a limitation of their model is that, "In practice, external costs of financing scaled by firm size are likely to decrease with firm size." Also, prior work finds declining costs relative to proceeds and size (Lee et al., 1996; Altınkılıç and Hansen, 2000).

first to paying off the debt (liquidation value). If the latter outweighs the former, the firm defaults. When the firm defaults at time τ, its capital stock k_τ is fire sold. The recovery rate ℓ is assumed to be constant. The liquidation value ℓk_τ is used to pay off the long-term debt with the face value b/r_{debt}, where r_{debt} is the cost of financing for long-term debt. If there is any value after paying the long-term debtholder, the remaining value, $\left(\ell k_\tau - b/r_{\text{debt}}\right)_+$ is distributed to the equityholders.[4]

Due to the external financing and liquidation costs, the firm manages a cash reserve to economize over issuance costs and to avoid default. The firm determines its investment and cash management strategies, which include when to pay a dividend and when to raise equity. The value of the cash reserve follows the dynamics

$$dc_t = (r - \lambda_c)c_t dt + dY_t - bdt - i_t dt - g(k_t, i_t)dt - dD_t + dI_t. \tag{7}$$

Here r is the interest rate, λ_c is the cash holding cost (liquidity premium), D is the cumulative dividend payout, and I is the cumulative equity issuance. Both D and I are non-decreasing processes. Cash earns a return equal to the risk free rate (r) net of a carry cost of holding cash (λ_c).[5] Even though cash earns a lower rate of return, the firm holds cash for precautionary reasons to lower the expected issuance or default costs. The firm manages an optimal cash policy to trade off the risk management benefits of maintaining a cash reserve against the delay in dividend payouts. The firm defaults when it runs out the cash reserve but decides not to issue equity. Therefore, the default time of the firm is

$$\tau = \inf\{t \geq 0 : c_t < 0\}.$$

When the firm exercises the growth option, it pays the fixed cost Φ from its cash reserve. Therefore, the firm needs its cash reserve to be at least Φ when the growth option is exercised. To reach Φ, the firm may accumulate cash flows over time or issue costly

---

[4] $a_+ = \max\{a, 0\}$.

[5] This assumption is standard in models with cash. For example, see Bolton, Chen and Wang (2011); Bolton, Wang and Yang (2019), Kim, Mauer and Sherman (1998) and Riddick and Whited (2009). If $\lambda_c = 0$, then the firm finds it optimal to hold as much cash as it can (indefinitely postponing the dividend) to prevent costly equity issuance. The equity is still valuable because equityholders could always choose to extract the cash via a dividend. The more realistic case is when $\lambda_c > 0$. Cash may earn low returns because interest earned on a firm's cash holdings is taxed at the corporate tax rate, which generally exceeds the personal tax rate (Graham, 2000; Faulkender and Wang, 2006). Agency problems may lower cash returns (Jensen, 1986; Harford, 1999; Dittmar and Shivdasani, 2003; Pinkowitz, Stulz and Williamson, 2006; Dittmar and Mahrt-Smith, 2007; Harford, Mansi and Maxwell, 2008; Caprio, Faccio and McConnell, 2011; Gao, Harford and Li, 2013; Back, Kakhbod and Xing, 2021).

equity. When choosing to pay Φ, the firm needs to evaluate the benefit of preserving the cash holding with the benefit of using it to pay the cost of option exercise to upgrade the productivity. The benefit of preserving cash is the flexibility to fund investment and meet liabilities without costly issuance and to avoid default.

2.1. Firm's problem

Suppose that the firm chooses to exercise the real option to upgrade its production technology at a stopping time ν. After the option exercise, the firm continues to choose investment, dividend payout, and equity issuance to maximize the present value of dividend payouts net of equity issuance costs. The firm's value right after option exercise is then

$$G(k_\nu, c_\nu) = \sup_{i\geq 0, D, \{\sigma_j, I_j\}} \mathbb{E}_\nu \left[\int_\nu^\tau e^{-r(s-\nu)} dD_s - \sum_j e^{-r(\sigma_j-\nu)} \Big(I_j + \lambda(I_j) \Big) + 1_{\{\tau<\infty\}} e^{-r(\tau-\nu)} \Big(\ell k_\tau - b/r_{\text{debt}} \Big)_+ \right], \tag{8}$$

where $\{\sigma_j\}$ is a sequence of stopping times when the lump sum of equity of size I_j is issued at time σ_j. We call G the option exercise value and treat (k, c) as its state variables.

Back to time zero, the firm needs to plan not only future investment, dividend payout, and equity issuance, but also when, if ever, to exercise the real option. Hence, the firm's value at time zero is:

$$\begin{aligned} \sup_{\substack{i\geq 0, D, \nu \\ \{\sigma_j, I_j\}}} \mathbb{E} \Bigg[& \int_0^{\tau\wedge\nu} e^{-rs} dD_s - \sum_j 1_{\{\sigma_j<\nu\}} e^{-r\sigma_j} \Big(I_j + \lambda(I_j) \Big) \\ & + 1_{\{\tau<\nu\}} e^{-r\tau} \Big(\ell k_\tau - b/r_{\text{debt}} \Big)_+ + 1_{\{\tau\geq\nu\}} e^{-r\nu} \Big(G(k_\nu, c_\nu - \Phi) \Big) \Bigg]. \end{aligned} \tag{9}$$

We restrict the option exercise time ν to be in the class of stopping times such that $c_\nu > \Phi$. In the problem (9), the first line corresponds to the discounted dividend value net of the equity issuance costs before the option exercise time. If the firm defaults before option exercise, the firm receives the liquidation value $(\ell k_\tau - b/r_{\text{debt}})_+$. If the firm decides to exercise the real option before its default, the firm receives the option exercise value, but its cash reserve after exercising is reduced by Φ. These correspond to the two terms in the second line of (9), respectively. We denote V as the firm's value function before option exercise and consider (k, c) as its state variables.

It follows from the dynamic programming that the value function V satisfies the HJB equation:

$$0 = \min\Bigg\{ rV - \sup_{i\geq 0}\Bigg\{ \Big[i-\delta k\Big]\partial_k V + \Big[(r-\lambda_c)c + k^\alpha\mu - b - i - g(k,i)\Big]\partial_c V + \frac{1}{2}k^{2\alpha}\sigma^2\partial^2_{cc}V\Bigg\},$$
$$\partial_c V - 1, V(k,c) - \sup_{I\geq 0}\Big[V(k,c+I) - I - \lambda(I)\Big], V(k,c) - G(k,c-\Phi)\Bigg\}, \quad c \geq \Phi, \tag{10}$$

$$0 = \min\Bigg\{ rV - \sup_{i\geq 0}\Bigg\{ \Big[i-\delta k\Big]\partial_k V + \Big[(r-\lambda_c)c + k^\alpha\mu - b - i - g(k,i)\Big]\partial_c V + \frac{1}{2}k^{2\alpha}\sigma^2\partial^2_{cc}V\Bigg\},$$
$$\partial_c V - 1, V(k,c) - \sup_{I\geq 0}\Big[V(k,c+I) - I - \lambda(I)\Big]\Bigg\}, \qquad c \in (0,\Phi). \tag{11}$$

In the equations above, if $c \geq \Phi$, the firm chooses among four alternatives: continuation (the group of terms on the first line of the right-hand side in (10)), dividend payout (the first group of terms in the second line), equity issuance (the second group in the second line), and option exercise (the last group in the second line). When $c \in (0,\Phi)$, the firm only chooses among the first three options in (11).

In the continuation group, rV represents the required rate of return on equity, which equals the risk free rate demanded by risk neutral investors. The term $[i-\delta k]\partial_k V$ is firm's marginal benefit of net investment on equity value. The term $[(r-\lambda_c)c + k^\alpha\mu - b - i - g(k,i)]\partial_c V$ is the marginal benefit of cash on equity value. The term $\frac{1}{2}k^{2\alpha}\sigma^2\partial^2_{cc}V$ captures the effect of the volatility of cash holdings due to production volatility on equity value.

In the dividend payout group, the firm postpones dividend payout until the marginal cost of reducing the cash reserve matches the marginal benefit of dividend payout, i.e., $\partial_c V = 1$.

In the equity issuance group, at each point (k,c) in the state space, the equity holders compare the value of the firm without issuance $V(k,c)$ to the best value for issuance $\sup_{I\geq 0}\Big[V(k,c+I) - I - \lambda(I)\Big]$, where $V(k,c+I) - I - \lambda(I)$ is the firm value post issuance net of issuance costs. The firm only issues equity when the best value for issuance is strictly larger than the current value.

Finally, in the option exercise group, the equity holders compare the current value of the firm $V(k,c)$ with the value after option exercise $G(k,c-\Phi)$, where the cash position is reduced by the cost Φ. The equity holders only exercise the real option when the exercise value is larger than the current firm value.

The dynamic programming principal implies that all four groups are nonnegative, that only one group equals to zero at each point (k, c) in the state space, and that the corresponding action is optimal for the firm.

By the same argument, the option exercise value satisfies

$$0 = \min\left\{ rG - \sup_{i\geq 0}\left\{ \left[i - \delta k\right]\partial_k G + \left[(r-\lambda_c)c + k^{\alpha}\tilde{\mu} - b - i - g(k,i)\right]\partial_c G + \frac{1}{2}k^{2\alpha}\tilde{\sigma}^2\partial^2_{cc}G \right\},\right.$$
$$\left. \partial_c G - 1, G(k,c) - \sup_{I\geq 0}\left[G(k,c+I) - I - \lambda(I)\right] \right\}. \quad (12)$$

Comparing to (10), option exercise is no longer available and (μ, σ) is replaced by $(\tilde{\mu}, \tilde{\sigma})$. Boundary conditions for (10), (11), and (12) will be specified in the appendix.

3. The Model Solution

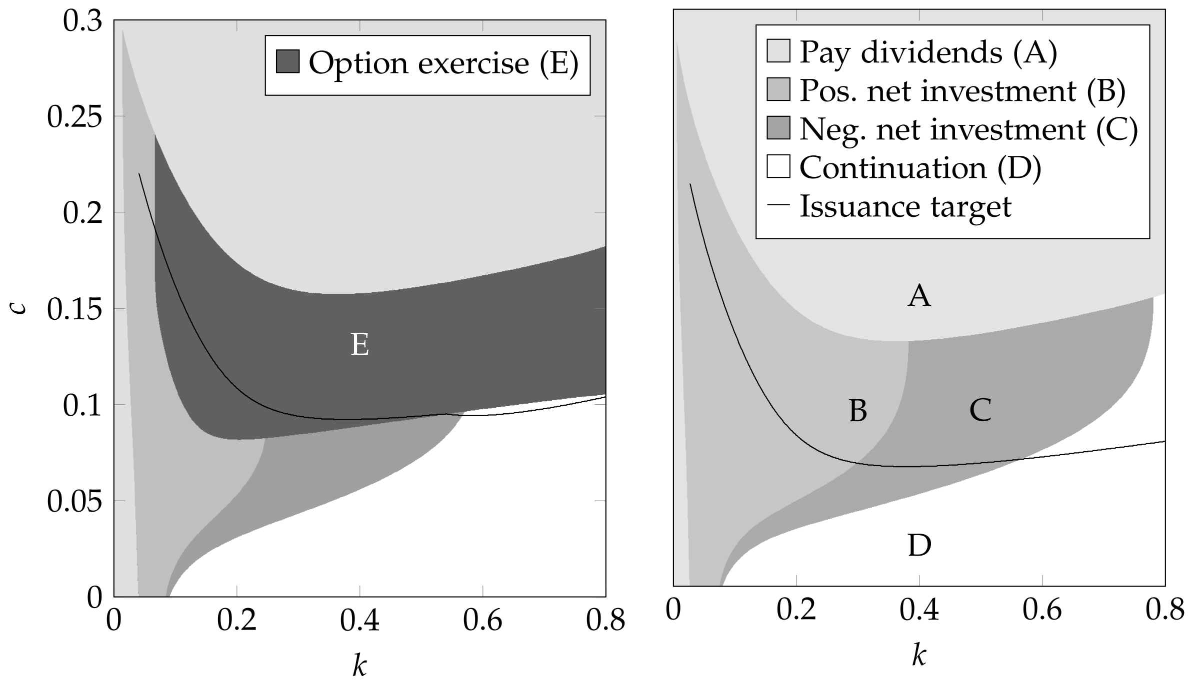
In this section, we present and discuss the model solution for the baseline set of parameters presented and discussed in Table 1. A series of figures reveal how costly financing creates a demand for precautionary cash holdings and illustrates how that demand affects a firm's decision to exercise a real option. Also, the figures reveal how the effects of financing costs matter differently when a firm is small versus when a firm is large. To help illustrate the factors affecting the firm's decision, we vary the different parameters in the model to examine their impact on a firm's predicted exercise behavior.

[Insert Table 1 Here]

3.1. Numeric results

Figure 1 depicts a firm's optimal choices for the baseline set of parameters. Figure 1a (1b) presents a firm's optimal choices prior to (after) exercising the real option. The y-axis captures the size of a firm's cash reserve c, and the x-axis captures the size of a firm's physical capital stock k. The legend notates which regions of capital and cash correspond to which firm behaviors.

When cash is sufficiently high, the firm is in the payout region (labeled "Pay dividends" in legend and denoted by the letter A). When the state process (k, c) reaches the boundary of the payout region (also known as "the dividend boundary"), minimal dividend is paid out to reflect the state process below the region. If (k, c) starts inside the payout region, then a lump sum dividend is paid out so that the state process (k, c) lands on the



(a) Before Exercise

(b) After Exercise

Figure 1: **Optimal Exercise Region**

Parameters used are summarized in Table 1.

dividend payout boundary.[6,7]

There are three investment regions depicted. In the region labeled "Pos. net investment" (denoted by the letter B), the firm finds it optimal to grow the physical capital k by choosing an investment amount i that exceeds the amount of depreciation δk. Recall that the firm's physical capital exhibits diminishing returns to scale so that the marginal returns to investing are higher when the firm's capital stock is smaller. In the region

[6]Interestingly, in Hugonnier, Malamud and Morellec (2015), when a firm has low cash and the exercise cost is high, the firm permanently abandons financing a real option's exercise internally, immediately pays a lump sum dividend, and waits for costless external financing to become available. The authors highlight that this dynamics is evidence that barrier strategies may not always be optimal because the firm may pay dividends when cash is low in addition to when cash is high. In our model, regardless of the exercise cost, when cash is low, the firm would not payout a dividend but rather invest in the firm's dynamic physical capital stock, which is omitted in their model. The real option always has strictly positive value and is never abandoned.

[7]The payout region forms a column on the far left when capital is very low. This column is the strategic default region. When the state process (k, c) reaches this boundary, rather than continuing the firm, equityholders pay out the remaining cash reserve as a lump sum dividend and default afterward.

labeled "Neg. net investment" (denoted by the letter C), the firm finds it optimal to shrink the physical capital stock k by choosing an investment amount i that is lower than the amount of depreciation δk. Consistent with diminishing returns to scale, as capital k increases, the firm invests at a lower rate. The firm's net investment rate also tends to decline as cash declines. Because of costly financing and default, when a firm's cash is low, the value of financial flexibility is high because the likelihood of paying the cost of issuance or default is increasing. A high demand for financial flexibility competes with the marginal returns to investing in the physical capital. In the region labeled "Continuation" (denoted by the letter D), the marginal value of cash is high enough relative to the marginal returns to investing that the firm finds it optimal to not invest at all, such that $i = 0$. Thus, in the continuation region, the firm's capital shrinks at the rate δ and the cash reserve continues to change with the cash flow process. For these investment regions, Internet Appendix Figure B.1 shows a heat map of net investment rates.

The firm may also choose to issue equity. The firm continuously compares the best value of the firm with issuance to the value of the firm without issuance to determine whether to issue. In Figure 1a, the optimal time to issue is when cash is close to zero. For this reason, there is no visible issuance boundary at which the firm finds issuance optimal. However, Internet Appendix Figure D.5 shows that when issuance costs are lower, the firm may choose to issue at higher cash levels because the benefits of waiting to economize over issuance costs are lower relative to the costs of delaying the benefits of investing. When issuing is optimal, the firm issues a lump sum amount of equity and lands on the black line labeled "Issuance target" in the legend. The lumpiness of issuance is to economize over fixed issuance costs.

In our model, the firm also possess a real option to upgrade the productivity of the physical capital. Figure 1a shows that when the state process (k, c) reaches the "Option exercise" region (denoted by the letter E) that the firm finds exercising the option optimal given its cash and capital stocks. At exercise, the firm pays the exercise cost Φ and the production technology immediately switches to the new productivity process with $\tilde{\mu}$ and $\tilde{\sigma}$. Figure 1a shows that in the presence of costly financing, the decision to exercise depends jointly on a firm's cash and capital positions. The firm may also enter the exercise region via an equity issuance as the issuance target may intersect the exercise region. After exercising the option, Figure 1b shows a firm's optimal dynamics.

Unlike traditional real option models, exercising the real option does not necessarily become optimal as k increases. In other words, there is no threshold level of capital above which exercising is always optimal. Even for high levels of capital k, when cash c is low, the firm prefers to not exercise the real option because paying the exercise cost Φ increases the likelihood of paying the issuance costs.

Figure 1 also shows that a firm's marginal value of cash is higher when a firm is smaller. For instance, when capital is low, the firm requires more cash before exercising the real option and also chooses to raise more cash via equity issuance conditional on issuance. There are few reasons for this behavior: First, issuance costs are high when k is low because of the fixed component of issuance costs. Thus, the precautionary savings motive is high. Second, the firm needs enough cash to fund the coupon b to avoid defaulting. Because k is small, the firm's cash flows are low and the ability to pay the coupon depends more on the cash reserve than the cash flows. Third, the incentives to invest in the existing capital stock are relatively higher when k is low because of diminishing returns to scale. Again, because k is low, investment is funded more from the cash reserve than the cash flows. For these reasons, the marginal value of cash increases when k is low, which discourages paying Φ. Additionally, when k is low the benefits of exercising are smaller because the resulting increase in expected cash flows from exercising depends on k and k is smaller relative to the exercise cost Φ. Hence, the exercise boundary in Figure 1 rises in cash as capital declines.

4. Empirical Support

The analyses thus far has been mostly theoretical. In the preceding sections, we build and solve a dynamic model of a firm with a real option and two states — cash and capital. This section provides empirical support for several novel predictions about the relation between option values and liquidity and option exercising and liquidity.

4.1. Data

A primary data source is the annual Compustat data file, providing detailed financial statement information on public firms. After filtering the data, there are 8,006 firms and 93,628 firm-years. (See Internet Appendix Table C.1 for details on the filtering.) The sample period is 1971 through 2020. For certain tests, the data series is more limited in time frame because of availability of certain key variables.

[Insert Table 2 Here]

The two primary state variables in the model are a firm's cash and capital positions. To proxy for the cash state variable, we use a firm's cash and cash equivalents (*che*) from the quarter-end balance sheet. The capital position is a firm's property, plant, and equipment net of depreciation (*ppent*). Table 2 shows that the median cash-to-capital ratio is about 22.9%.

To identify real option exercises, we rely on a firm's net investment rate in physical capital. We calculate a firm's net investment (capital expenditures less depreciation in a quarter) scaled by lagged capital for each period and then standardize these values within the firm. Table 2 shows that the median net investment rate is 4.5% per year.

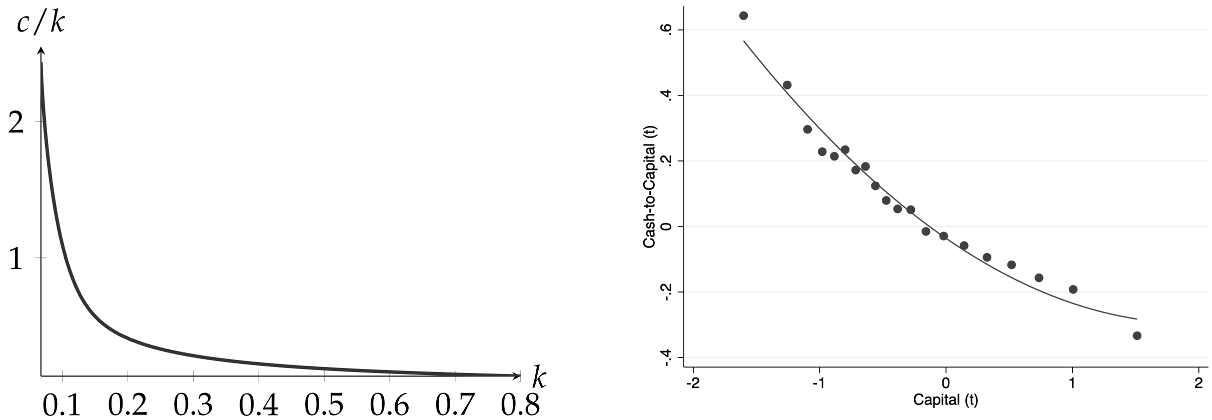
We also rely on stock market data from CRSP. We obtain daily stock returns from the CRSP daily return files. We use daily returns when calculating monthly and annual firm volatility. We obtain daily factor returns and risk-free rates from Ken French's website (Link). See Internet Appendix Table C.2 for our sample selection criteria. Our sample covers 15,221 firms; spans April, 1964 to December, 2020; and contains about 1.86 million stock-month observations. Table 2 shows that the median daily volatility is 2.7%. The median beta is 0.81, the median book-to-market ratio is 0.55, and the median turnover is about 0.6% of shares outstanding.

4.2. Hypothesis 1: The minimum cash-to-capital ratio for exercise is declining and convex in capital

Our model predicts that the required minimum cash for exercising the real option relative to a firm's size is declining and convex in capital. Figure 2a illustrates this prediction. The rationale is that the value of financial flexibility (the marginal value of cash) is high when capital is low for several reasons: (1) because existing cash flows are low, the firm relies mostly on the cash reserve to fund investments and meet liabilities; (2) solvency is low because of the fixed debt; (3) issuance costs are high when a firm is small; and (4) the marginal returns to growing the dynamic physical capital stock are high because of diminishing returns to scale. Consequently, when capital is low, the firm is more reluctant to pay the fixed exercise cost Φ to improve the productivity of the physical capital k. As capital increases, the minimum cash relative to capital declines convexly because of diminishing returns to scale and the decline fixed issuance costs relative to size.

This reasoning leads to the following prediction:

H1: The minimum cash-to-capital ratio for exercise is declining and convex in capital.



(a) The Y-axis is the cash-to-capital ratio along the exercise boundary in Figure 1; the X-axis is the corresponding level of capital. The parameters used are summarized in Table 1.

(b) The Y-axis is a firm's cash-to-capital ratio at the end of year t (standardized within firm). The X-axis is a firm's net property, plant, and equipment at the end of year t (standardized within firm). We limit the sample to firm years preceding firm years with net investment rates (capital expenditures less depreciation all scaled by lagged capital) in the top 50% of net investment rates in the full sample (above 2.7% growth). We then sort firm-year observations into twenty bins according to a firm's size standardized within firm (the black dots). For each bin, we report the average size of the firm-years in that bin and their average cash-to-capital ratios. Controls for year fixed effects.

Figure 2: Predicted Exercise Boundary from Figure 1 vs. Actual

Figures 2b provides empirical support for the model's prediction. We limit the sample to firm years preceding firm years with net investment rates (capital expenditures less depreciation all scaled by lagged capital) in the top 50% of net investment rates in the full sample (above 2.7% growth). For this sample, we show cash-to-capital ratios on the y-axis and capital levels on the x-axis. To construct the figure, we first standardize a firm's cash-to-capital ratios and capital levels within firm using the full sample. Thus, a value of zero for either variables reflects a firm's average, and deviations from the firm's average are compared to the volatility of that variable at a specific firm. Then, for the subsample of firm-years preceding a high net investment period, we sort firm-year observations into twenty bins according to a firm's size standardized within firm (the black dots). For each bin, we report the average size of the firm-years in that bin and their average cash-to-capital ratios. Consistent with our model's prediction, the threshold amount of cash for exercising appears to be declining and convex in capital.[8]

[8]In Internet Appendix Figure C.1, we show similar results using acquisition spending to proxy for

To evaluate this hypothesis more rigorously, we estimate the following empirical specification:

$$\frac{\text{Cash}_{i,t}}{\text{Capital}_{i,t}} = \beta_1 \text{Capital}_{i,t} + \beta_2 \text{Capital}^2_{i,t} + \delta_t + \epsilon_{i,t}. \quad (13)$$

The outcome is the cash-to-capital ratio at the end of year t for firm i (standardized within firm). The sample includes each firm-year t immediately preceding a firm-year $t+1$ in which a firm exercises a real option. To proxy for real option activity, we require the net investment rates in year $t+1$ to exceed the sample median of 2.6% in Panel A and the 75th percentile of 13.1% in Panel B. *Capital* is a firm's property, plant, and equipment net of depreciation at the end of year t standardized within firm. We standardize capital within firm so that the quadratic form of capital captures the distance from a firm's average size rather than an average across firms in the sample. We control for year fixed effects (δ_t). $\epsilon_{i,t}$ is the unexplained variation. Standard errors are clustered by year.

Table 3 shows the results. Column (1) provides significant evidence that the cash-to-capital ratio of a firm is declining in a firm's capital in a convex manner. The coefficient on *Capital* is negative, and the coefficient on *Capital*^{2} is positive and statistically significant. Columns (2) and (3) show largely similar results before and after 1998. Columns (4) to (10) show largely similar results cross SIC-1 industries.

[Table 3 Here]

4.3. Hypothesis 2: Real option values are hump-shaped in a firm's cash reserve

Our model predicts that the value of a firm's real option to delay investment is hump-shaped in the firm's cash reserve and more so when a firm is small. Figure 4 shows how the value of the real option scaled by capital changes with cash when a firm is small ($k = 0.1$) and when a firm is large ($k = 0.8$). The value of the option to delay starts close to zero when cash is zero because (1) the firm must wait a longer time to fund the exercise cost internally, which delays any benefits of exercising the option and because (2) the firm imminently needs to raise costly financing anyways to avoid default. As the cash reserve increases, the value of the option increases because the need to raise costly financing

real option exercises. We do not use acquisitions as our main proxy of real options exercises for several reasons: (1) many acquisitions are completed using stock consideration rather than cash and (2) acquisition prices include a control premium. In Internet Appendix Figure C.2, we also show similar results using alternative net-investment-rate thresholds for option exercises. In Internet Appendix Figure C.3 we show similar results accounting for intangible capital.

anyways to avoid default decreases and because exercising immediately may increase the marginal of cash significantly and even require costly external financing. Around the cost of exercising the option, the value of the option to wait reaches its maximum value. As the cash reserve increases further, the value of the option to delay declines again because the firm's cash position approaches the optimal cash reserve for exercising the option. When cash is high, the difference in the value of the firm exercising the option immediately and the value of the firm exercising the option imminently (after a short delay) is small because the firm has sufficient cash to fund the exercise cost Φ and a reasonable remaining amount of cash after exercise for precautionary savings. In other words, the benefit of waiting in terms of reducing the impact of paying the exercise cost on a firm's financial flexibility (marginal value of cash) is lower when cash is high.

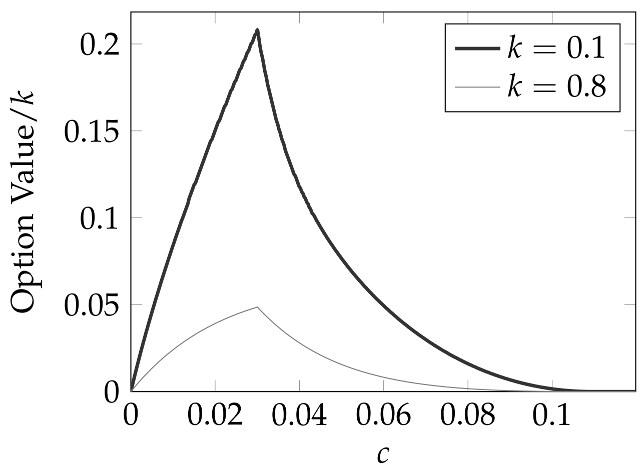


Figure 3: Option Values and Financial Flexibility

This figure shows how the predicted value of the real option to delay investment scaled by capital k changes with a firm's financial flexibility, or the size of its cash reserve. We show how option values change with cash when capital is low ($k = 0.1$) and high ($k = 0.8$).

Figure 4: **Option values vary with a firm's financial flexibility**

Figure 4 also shows that the hump-shape in option values scaled by capital is generally larger when a firm is smaller. This change in the sensitivity of option values to cash over a firm's life cycle occurs because the marginal value of cash is higher when a firm is smaller.

This reasoning leads to the following hypothesis:

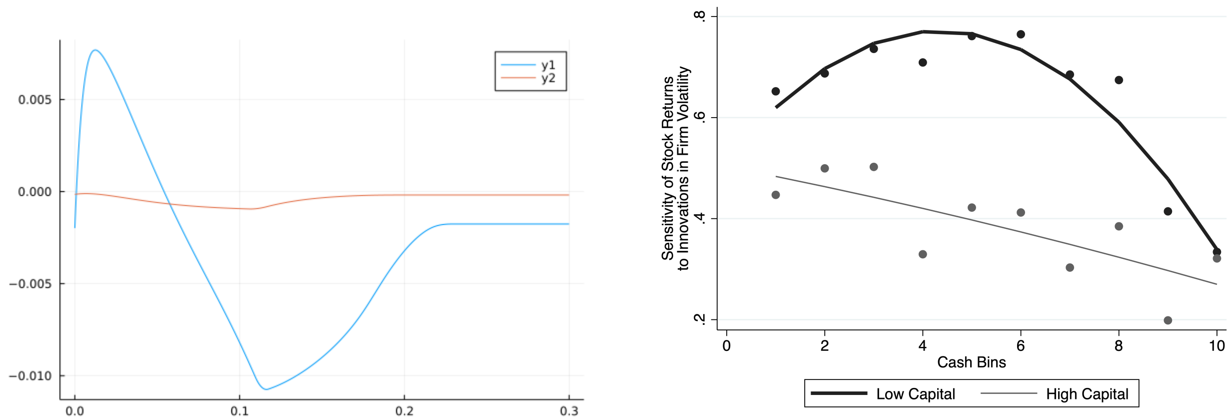
H2: The value of a firm's real option is hump-shaped in cash, and more so when the firm is small.

To examine **H2** empirically, we need a methodology that reveals the value of a firm's real options. However, it is challenging to directly observe the value of a firm's real options. That is, stock prices incorporate the value of assets-in-place and real options. Nevertheless, it is possible to tease out the value of a firm's real options from stock returns because the value of a firm's real options is increasing in the volatility of expected cash flows, which is σ in our model. The volatility of stock prices is expected to be related to the volatility of expected cash flows, which justifies the use of measures of stock return volatility as proxies for underlying volatility, as in Leahy and Whited (1996) and Bulan (2005). The main rationale for this positive relation between real option values and volatility is that, since firms can change their operating and investment decisions to mitigate the effects of bad news (e.g., defer investments) and amplify the effects of good news (e.g., expedite investments), an increase in the volatility of an underlying process increases the value of the real option. Since flexibility induces the convexity of firm value with respect to the value of its underlying assets, firm value is an increasing function of volatility, due to Jensen's inequality.

In our model, the firm may choose to delay exercising the option if its financial flexibility is too low. A firm may also choose to delay the investment if its physical capital stock is too low as this determines the expected amount of increase in cash flows from exercising the growth option, which it compares to the direct exercise cost and the indirect impact on a firm's financial flexibility. Higher volatility increases the likelihood of having to raise costly financing, which makes paying Φ immediately less appealing. Instead, when volatility increases, the firm has a stronger preference to wait either to grow the cash reserve or to become a larger firm, which decreases issuance costs relative to firm size.

Consistent with this reasoning, Duffee (1995) finds a positive stock volatility-return relation. Grullon, Lyandres and Zhdanov (2012) suggests that this relation is driven by the positive effect of volatility on the value of real options. Grullon, Lyandres and Zhdanov (2012) shows that for firms likely to derive more of their value from real options that firm value is more positively related to a firm's idiosyncratic volatility shocks. By contrast, for firms with few real options, that paper finds no positive relation between idiosyncratic volatility and firm value.

For these reasons, to evaluate **H2**, we examine how the sensitivity of a firm's stock returns to its volatility changes with a firm's cash position. Because Figure 4 shows how an option value varies with a firm's cash and not how its sensitivity to volatility varies with a firm's cash, we construct Figure 5(a) to show more directly what our model predicts is the relation between the sensitivity of option values to volatility and a firm's financial flexibility.



(a) Predicted

This figure shows the model's prediction of the change in the value of the real option when the volatility of a firm's cash flows increases and how this sensitivity varies with a firm's cash holdings. The value of the option is the difference in the value of the firm with the ability to delay a one-time investment opportunity and the value of the firm with a now-or-never investment opportunity. We scale the option values by capital. To examine the sensitivity with volatility, we compare option values when volatility is $\sigma = 0.1$ and $\sigma = 0.09$ (the default).

(b) Actual

We bin our full firm-month sample into two groups based on a firm's net property, plant, and equipment (standardized within firm) in quarter t. The "Low Capital" group includes all firm-month observations when a firm is below its average size. Then, for each size group, we form ten cash bins based on a firm's cash reserve at the end of quarter t standardized within firm. The Y-axis is the sensitivity of firm stock returns to innovations in firm-specific stock volatility calculated for each bin using Equation 16. The X-axis denotes the ordinal values of the cash bins.

Figure 5: **Sensitivity of option values to innovations in firm volatility by a firm's size and cash reserve**

To measure firm i's volatility in month t, we follow Ang et al. (2006, 2009), Duffee (1995), and Grullon, Lyandres and Zhdanov (2012) among others, and calculate the standard deviation of the firm's daily returns during month t:

$$VOL_{i,t} = \sqrt{\frac{\sum_{\tau \in t}(r_{i,\tau} - \overline{r_{i,t}})^2}{n_t - 1}}. \tag{14}$$

$r_{i,\tau}$ is the natural logarithm of day $\tau \in t$ gross excess return on firm i's stock, $\overline{r_{i,t}}$ is the mean of the logarithms of gross daily returns on firm i's stock during month t, and n_t is the number of nonmissing return observables during month t. We use logarithmic returns to mitigate the potential mechanical effect of return skewness (see Duffee, 1995) on the relation between returns and contemporaneous return volatilities. The change in volatility in month t, $\Delta VOL_{i,t}$, is computed as the difference between the estimated volatility in month t and the estimated volatility in month $t-1$:

$$\Delta VOL_{i,t} = VOL_{i,t} - VOL_{i,t-1} \tag{15}$$

We estimate the following regression of individual firm returns, $r_{i,t}$, net of the risk-free rate, $r_{f,t}$, on contemporaneous changes in firm-level volatility, $\Delta VOL_{i,t}$, and a vector of firm characteristics, $x_{i,t}$, most of which are known at the beginning of month t. As in Grullon, Lyandres and Zhdanov (2012), the characteristics include a firm's market beta, book-to-market ratio, market equity, turnover (volume/shares outstanding), and past return over the prior six months (t-2, t-8). We include time fixed effects, η_t, for each month t to isolate the cross-section and cluster the standard errors by month t.

$$r_{i,t} - r_{f,t} = \alpha t + \beta_t \Delta VOL_{i,t} + \delta_t x_{i,t} + \eta_t + \epsilon_{i,t} \tag{16}$$

Figure 5b provides evidence consistent with **H2**. Specifically, we group all firm-months in our sample into two groups based on a firm's net property, plant and equipment (standardized within firm). Thus, the first group contains firm-month observations when a firm's size is below its average, capturing behavior when the firm is small. Then for each size group, we create ten cash bins based on a firm's cash reserve (standardized within firm). Each cash bin contains about 100-thousand firm-month observations. We then estimate specification (16) for each bin and store the sensitivity of stock returns to innovations in firm volatility. The figure shows a clear hump-shape relation in this sensitivity, which is consistent with real options being less valuable when cash is low and high. Also consistent with **H2**, the figure shows that the hump-shape of real option values in cash is stronger when a firm is smaller.[9]

The hump-shape of option values with a firm's cash reserve should be stronger for firms with more real options. We follow the literature and examine if the hump-shape is stronger for small firms in the cross section (not within firm differences in size but differences in levels), when firms spend more on research and development spending relative to their sales, when firm's are younger than their average age in the sample, when firms see relatively higher future sales growth than their average, and when firms operate in industries with plenty of real options, following the approach in Grullon, Lyandres and Zhdanov (2012). Figures 6a-e show that firms predicted to have more real

[9]We find similar results using a Fama-MacBeth specification (See Internet Appendix C.4), using the cash-to-capital ratio instead of cash (See Internet Appendix C.5), and when accounting for intangible capital (See Internet Appendix C.6).

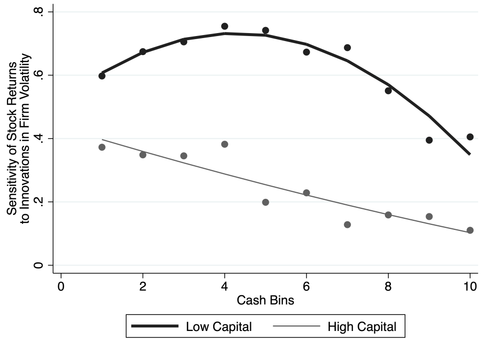
options according to these proxies generally have higher sensitivities of stock returns to innovations in firm specific stock volatility and exhibit a hump-shape in these sensitivities with cash.

As additional robustness, if the option to delay is more valuable for intermediate levels of cash, we might also predict that firms with intermediate cash levels are more likely to delay investing in response to increases in firm volatility. Figure 7 provides visual evidence consistent with this reasoning. Figure 7 shows the sensitivity of growth investments to firm-specific volatility. We define growth investments as net investments (capital expenditures less depreciation scaled by lagged net property, plant, and equipment) exceeding the median of 2.6%. We find a general negative relation between a firm's stock volatility in quarter t and a firm's net investment rate in quarter $t+1$, consistent with several prior empirical studies (Leahy and Whited, 1996; Guiso and Parigi, 1999; Bulan, 2005; Bloom, Bond and Van Reenen, 2007). More interestingly, we find that the negative investment-volatility relation is a function of a firm's liquidity, exhibiting a U-shaped in a firm's cash holdings. Firms with intermediate levels of cash exhibit a greater negative sensitivity of investment to firm volatility than firms with low or high levels of cash exhibit. The figure also shows that the U-shape pattern is stronger when a firm is smaller. In Internet Appendix Figure C.7, we show similar results using alternative net-investment-rate thresholds for option exercises. In Internet Appendix Table C.4, we show that the convexity is statistically significant.

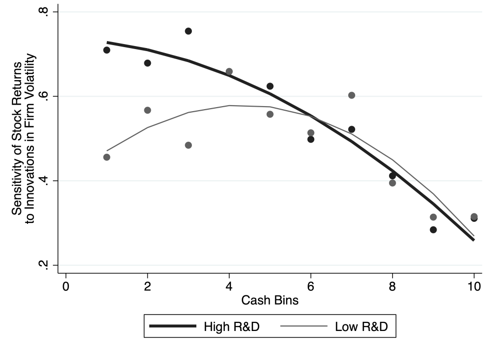
To examine the significance of the hump-shape shown in the figures more rigorously, we re-estimate equation (16) interacting ΔVOL with a firm's cash reserve (standardized within firm) and its square. Table 4 presents the results. Column (1) shows a significant positive coefficient on ΔVOL consistent with the positive return-volatility relation in Duffee (1995) and Grullon, Lyandres and Zhdanov (2012). Using the full sample, the quadratic term, $\Delta\text{VOL} \times \text{Cash}^2$ is not negative and statistically significant. However, in column (2), we interact the relations with a dummy *Large* that equals to one when a firm's size is in the top three quartiles of the firm's size distribution. Column (2) shows that the hump-shape relation is statistically significant for small firms as the relation is the coefficient on $\Delta\text{VOL} \times \text{Cash}^2$ is negative and statistically significant for small firms.

5. Testable Implications

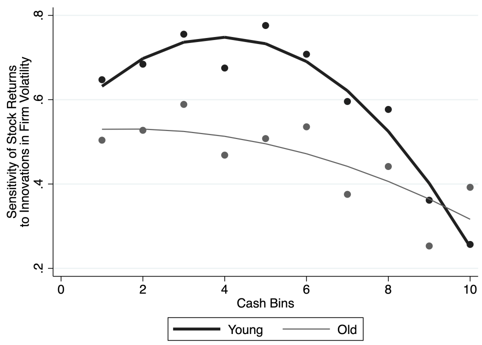
In this section, we vary the parameters in the baseline solution to the model depicted in Figure 1 and examine their implications for a firm's dynamics. These comparative



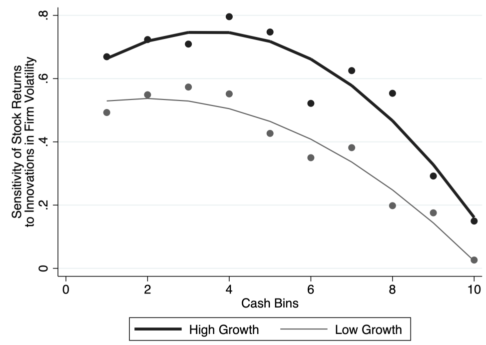
(a) Low vs High Capital (Levels)



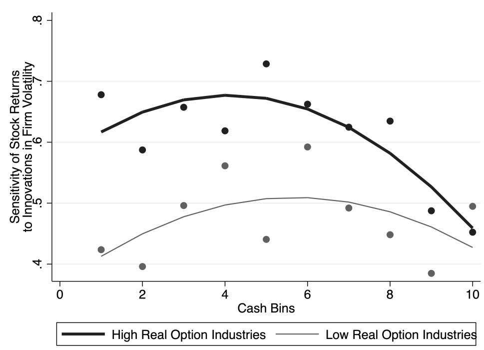
(b) High vs Low R&D Firms



(c) Young vs Old



(d) Low vs High Growth



(e) Low vs High Real Option Industries

Figure 6: **Option Values for Firms with More/Fewer Real Options and the Firm's Cash Reserve**

This Figure complements Figure 5 by initially forming two groups of firms based on proxies for a firm's real options instead of a firm's within-firm physical capital stock. Panel (a) splits the sample based on whether a firm's raw net property, plant, and equipment in quarter t exceeds the full panel median. (Note that we are not splitting on the firm's capital standardized within firm but rather the level of capital.) Panel (b) splits the sample based on a firm's R&D spending relative to lagged sales. Panel (c) splits the sample based on the time since a firm IPO'd, standardized within firm. Panel (d) splits the sample based on actual revenue growth from quarter $t+1$ to quarter $t+4$, standardized within firm. Panel (e) splits the sample based on whether a firm's industry has plenty of real options, as defined in Grullon, Lyandres and Zhdanov (2012), which are the Fama-French natural resource industries (27, 28, 30), high-tech industries (22, 32, 35, 36), and pharmaceutical and biotechnology industries (12, 13).

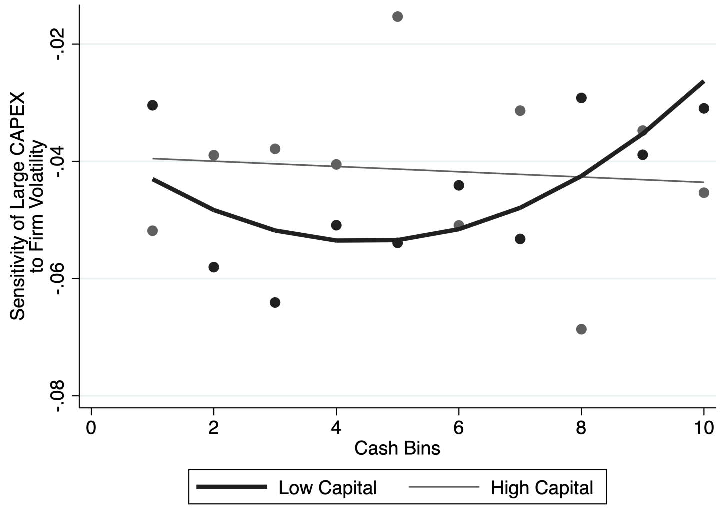


Figure 7: **Sensitivity of large investments to a firm's stock volatility by a firm's size and cash reserve**

We divide our firm-year sample into two groups based on a firm's net property, plant, and equipment (standardized within firm) at the end of year $t-1$. The "Low Capital" group includes all firm-year observations when a firm is below its average size. Then, for each of the two size groups, we form ten cash bins based on a firm's cash reserve (standardized within firm) at the end of year $t-1$. The Y-axis is the sensitivity of option exercises in year t to a firm's average daily stock volatility (standardized within firm) in year t. We classify option exercises as net investment rates above the sample median, which is 2.6% of net property, plant, and equipment. The X-axis denotes the ordinal values of the cash bins. When estimating the exercise-volatility sensitivity, we regress an indicator for option exercise in year t on a firm's stock volatility in year t and control for year fixed effects.

statics provide additional predictions for future research.

5.1. *Effect of exercise costs on the minimum required financial flexibility for investing*

Internet Appendix Figure D.1 illustrates how the exercise region varies with the cost of exercising the real option. The cost rises from $\Phi = 0.02$ in (a) to $\Phi = 0.06$ in (b). First, a higher exercise cost predictably pushes the exercise boundary out — up and to the right in (k, c) space. The firm is only willing to exercise when its capital stock is higher and when it is less financially constrained. This shift occurs because the higher exercise cost reduces the net profitability of exercising so that the firm only finds exercising profitable when capital (and thus cash flows) is larger. Additionally, the firm requires a higher cash reserve at the time of exercise (an upward shift in the exercise boundary) because an increase in the exercise cost increases the probability of equity dilution after paying the exercise cost since there is a larger outflow of cash associated with investment. Second, the issuance target shifts down when the exercise cost increases. Intuitively, when the exercise cost is higher, the firm is further away from the exercise boundary and the incentive to raise proceeds to fund the exercise costs are weaker. Internet Appendix Figure C.8 shows theoretically and empirically that real options incentivize firms to issue larger amounts to help finance the exercise cost.

Internet Appendix Figure D.2 illustrates how the exercise region varies with the degree to which the real option increases a firm's productivity. The productivity rises from $\mu = 0.18$ to (a) $\tilde{\mu} = 0.19$ and to (b) $\tilde{\mu} = 0.20$. An increase in the expected cash flows gained from exercising the real option predictably shifts the exercise boundary inwards — down and to the left. Intuitively, exercising becomes more attractive at lower levels of capital because the expected cash flow increase is larger when the productivity rate ($\tilde{\mu} > \mu$) increases more, all else equal. The downward shift in the exercise boundary occurs because the larger increase in cash flows helps the firm economize over issuance costs to a greater extent after paying the exercise cost Φ. In other words, the risk of having to pay issuance costs after paying the exercise cost Φ is lower because the expected cash flows are larger. Thus, firms are more likely to exercise projects with larger increases in productivity, especially when cash balances are lower.

Internet Appendix Figure D.3 illustrates how the decision to exercise varies with issuance costs. The fixed component of issuance costs is $0.25 \times \lambda_f$ (or one-quarter of the default fixed cost in Table 1) in (a) and $4.0 \times \lambda_f$ in (b). The exercise boundary rises as fixed issuance costs increase. Intuitively, the higher fixed component of issuance costs increases the firm's precautionary savings motive. Thus, the firm requires a larger cash buffer prior to exercising and paying the issuance cost Φ. Relatedly, the issuance target rises to economize on the higher fixed issuance costs. By contrast, Internet Appendix Figure D.4 shows that increasing only the proportional component of issuance costs lowers the issuance target or the amount a firm is willing to issue. Intuitively, as the proportional costs increase relative to the fixed costs, the firm focuses more on economizing over the proportional costs rather than the fixed costs. Interestingly, Internet Appendix Figure D.5 shows that when the fixed and proportional costs decrease, firms will issue when cash is greater than zero because the cost of delaying the benefits of exercising are greater than the costs of paying the issuance costs.

Internet Appendix Figure D.6 illustrates how the exercise decision varies with the effect of the real option on the volatility of the cash flows. First, an increase in the volatility of cash flows after exercising the real option leads firms to require larger cash balances even prior to exercising. Visually, the exercise boundary shifts up. This demand for larger cash reserves ahead of exercising occurs because a higher volatility post exercising increases the probability of costly issuance and default. Equity holders internalize this increase in the marginal value of cash post exercise and thus hold more cash before exercising. In other words, equity holders anticipate their post-investment optimal cash policy. Second, the issuance target boundary increases even before exercising the real option when the option's volatility increases. As with the optimal cash reserve at the time of exercise, this upward shift in the issuance target occurs because equityholders internalize that higher volatility after exercising increases the probability of issuance and default.

Internet Appendix Figure D.7 illustrates how the exercise decision varies with a firm's leverage. The leverage varies from no debt ($b = 0.00$) in (a) to a coupon of $b = 0.03$ in (b). First, leverage increases the exercise boundary. That is the firm requires more cash before exercising for any given level of capital. Intuitively, higher cash balances reduce the likelihood of running out of cash and defaulting, which results in inefficient liquidation of the firm's physical capital, after paying the exercise cost. Second, the exercise region shifts to the right because for low levels of capital, the profitability is too low after exercising to justify the investment. Relatedly, with higher leverage, the default region expands, evidenced by the rightwards shift in the issuance target.

6. Conclusion

This paper examines how costly financing affects the timing of investment. Firms with the ability to delay investment may do so to improve the internal cash available to fund the investment (financial flexibility) or may wait for the cash flows to improve (real flexibility). To examine these two forms of flexibility, we solve a dynamic model of a firm with both a dynamical capital stock as a state variable and a cash reserve as a state variable. Because firms facing costly financing value flexibility, the value of a real option, the timing of its exercise, and firm dynamics are closely linked to the firms liquidity in addition to its physical capital stock. In doing so, this paper effectively integrates two classical literatures on corporate cash balances (Miller and Orr, 1966) and on real options (McDonald and Siegel, 1986) in a novel way.

Our model generates several new predictions we find support for in the data. (1) The value of the real option is hump-shaped in cash, especially when a firm is smaller because the demand for financial flexibility is higher. (2) Exercising activity is U-shaped in cash, especially when a firm is smaller. (3) The optimal cash reserve at the time of exercise scaled by capital is declining and convex in capital. The empirical support we provide extends the limited empirical literature on real options.

Appendix A. Proofs

Lemma 1. *Before exercising the option, equity issuance is only optimal when $c = 0$ or at the same time as option exercise.*

Proof. We define an equivalent control problem using

$$\tilde{c}_t = e^{-(r-\lambda_c)t} c_t, \quad \tilde{D}_t = \int_0^t e^{-(r-\lambda_c)s} dD_s, \quad \tilde{I}_t = \int_0^t e^{-(r-\lambda_c)s} dI_s.$$

Then,

$$d\tilde{c}_t = e^{-(r-\lambda_c)t} dY_t - e^{-(r-\lambda_c)t}(b + i_t + g(k_t, i_t))dt - d\tilde{D}_t + d\tilde{I}_t,$$

and the optimization problem is

$$\begin{aligned} V(k,c) = \sup_{i \geq 0, \tilde{D}, \{\sigma_j, \tilde{I}_j\}, \nu} \mathbb{E}\Bigg[\int_0^{\tau,\nu} e^{-\lambda_c t} d\tilde{D}_t - \sum_{\sigma_j \geq 0} e^{-\lambda_c \sigma_j} \left(\tilde{I}_j + \lambda_p \tilde{I}_j + e^{-(r-\lambda_c)\sigma_j} \lambda_f \right) \\ + 1_{\{\nu < \tau\}} G(k_\nu, \tilde{c}_\nu) \\ + 1_{\{\tau < \nu\}} e^{-r(\tau - t)} (\ell k_\tau - b/r_{\text{debt}})_+ \Bigg], \end{aligned}$$

where τ is defined equivalently for $\tilde{c}$ as for c.

In this formulation, issuance can be delayed at a discount. To make things clear, we fix an investment strategy i and a dividend strategy D, index $\tilde{c}^\xi$ by an issuance strategy $\xi = \{\sigma_j, \tilde{I}_j\}_{j \in \mathbb{N}}$ with at least one issuance time say σ_i at which $\tilde{c}^\xi_{\sigma_i} > 0$ and $\sigma_i < \nu$. Let $\tilde{I}_i$ be the corresponding issuance amount. Let ξ^- be another issuance strategy that omits this issuance, but keep the rest of the issuance strategy as ξ. Finally, construct a third strategy ξ' like ξ^- but with an additional issuance of size $\tilde{I}_i$ at a time $\sigma' = \nu \wedge \inf\{t > \sigma_i : \tilde{c}^{\xi^-}_t < 0\}$.[10] Note that for the same (Y, i, D), the increment of $\tilde{c}^\xi$ and $\tilde{c}^{\xi'}$ are the same as between σ_i and σ'. With the same issuance size, we have $\tilde{c}^\xi_{\sigma'} = \tilde{c}^{\xi'}_{\sigma'}$, so the continuation values must coincide after σ', because the strategies are identical after σ'_i. Moreover, dividends and issuance until σ' have been identical, with one exception, for which ξ' has resulted in a larger discounting factor and a smaller discounted fixed cost.

We therefore conclude that the original strategy is dominated by the one issuing equity later: either only at $c = 0$ or simultaneously as exercising the option, i.e., at time ν. As this is true for any strategy, we may consider only strategies that issue equity when $c = 0$ or at the option exercise time. □

Let $\mathcal{O} = (0, \infty) \times (0, k_{\max})$. We restrict ourselves to the setting in which there is a maximal investment rate $i_{\max} < \infty$. It then follows from Lemma 1 that (10) and (11) can

[10] If $\tilde{c}^{\xi^-}_t$ fell below zero due to a lump sum dividend payout, we can balance out the dividend payout and the issuance to obtain the same result in the next step. If there are multiple issuances in ξ between σ_i and σ', we omit all of them in ξ^- and issue the sum of all missed size at σ' in ξ'.

be rewritten as

$$0 = \min\Bigg\{rV - \sup_{i_{\max}\geq i\geq 0}\Bigg\{\Big[i-\delta k\Big]\partial_k V + \Big[(r-\lambda_c)c + k^\alpha\mu - b - i - g(k,i)\Big]\partial_c V + \frac{1}{2}k^{2\alpha}\sigma^2\partial^2_{cc}V\Bigg\},$$
$$\partial_c V - 1, V(k,c) - \sup_{I\geq 0}\Big[G(k,c+I-\Phi) - I - \lambda(I)\Big], V(k,c) - G(k,c-\Phi)\Bigg\}, \quad c \geq \Phi, \tag{A.1}$$

$$0 = \min\Bigg\{rV - \sup_{i_{\max}\geq i\geq 0}\Bigg\{\Big[i-\delta k\Big]\partial_k V + \Big[(r-\lambda_c)c + k^\alpha\mu - b - i - g(k,i)\Big]\partial_c V + \frac{1}{2}k^{2\alpha}\sigma^2\partial^2_{cc}V\Bigg\},$$
$$\partial_c V - 1, V(k,c) - \sup_{I\geq \Phi - c}\Big[G(k,c+I-\Phi) - I - \lambda(I)\Big]\Bigg\}, \qquad c \in (0,\Phi), \tag{A.2}$$

with the boundary conditions

$$\begin{aligned} V(0,c) &= c && \text{at } k = 0,\\ V(k,0) &= \min\Bigg\{(\ell k - b/r_{\text{debt}})_+, \sup_{I\geq 0}\Big[V(k,I) - I - \lambda(I)\Big]\Bigg\} && \text{at } c = 0. \end{aligned} \tag{A.3}$$

Before we prove comparison for V and the convergence of the numeric algorithm, we establish the following technical lemma.

Lemma 2. *The function*

$$(k,c) \mapsto \sup_{I\geq(\Phi-c)_+}\Big[G(k,c+I-\Phi) - I - \lambda(I)\Big]$$

is continuous.

Proof. It will be convenient to consider the equivalent function

$$(k,c) \mapsto \sup_{c'\geq(c-\Phi)_+}\Big[G(k,c') - (c'+\Phi-c) - \lambda(c'+\Phi-c)\Big].$$

We note that the function G is uniquely pinned down (and continuous) by the proofs in Ali Kakhbod, Max Reppen, Tarik Umar and Hao Xing (2021). Because the function is the supremum of continuous functions, it is lower semi-continuous. It thus remains to show that it is upper semi-continuous. Let $\{(k_n, c_n)\}_{n\in\mathbb{N}}$ be a sequence of points converging to (k,c). Because $\lambda_p > 0$ and $G(k,c')$ grows like c' for large c', the optimizers c'_n exist and are uniformly bounded in n. Hence, there exists a convergent subsequence, which we index by m to distinguish, such that $\lim_{m\to\infty} c'_m = \hat{c}$. Because $c'_m \geq (c_m - \Phi)_+$, we have

$\hat{c} = \lim_{m\to\infty} c'_m \geq \lim_{m\to\infty}(c_m - \Phi)_+ = (c-\Phi)_+$. Therefore

$$\begin{aligned}
\sup_{c'\geq(c-\Phi)_+} &\Big[G(k,c') - (c'+\Phi-c) - \lambda(c'+\Phi-c)\Big] \\
&\geq G(k,\hat{c}) - (\hat{c}+\Phi-c) - \lambda(\hat{c}+\Phi-c) \\
&= \lim_{m\to\infty}\Big[G(k_m,c'_m) - (c'_m+\Phi-c_m) - \lambda(c'_m+\Phi-c_m)\Big] \\
&= \lim_{m\to\infty}\sup_{c'\geq(c_m-\Phi)_+}\Big[G(k_m,c') - (c'+\Phi-c_m) - \lambda(c'+\Phi-c_m)\Big],
\end{aligned}$$

where the second equality follows from the optimality of c'_m. Because this is true for any sequence, we conclude that the function is upper semi-continuous and thus also continuous. □

Theorem 3. *Let u and v be, respectively, possibly discontinuous viscosity sub- and supersolutions to* (A.1) *and* (A.2) *with the above boundary conditions in* (A.3) *(possibly with $\leq$ and $\geq$ inequalities, respectively). Assume further that u and v are both of linear growth in c and polynomial growth in k, i.e., they take values in $[c, c+p(k)]$ for some polynomial p. Then, $u \leq v$ everywhere in $\mathcal{O}$.*

Proof. Suppose there exists a point at which $u > v$. Fix some $\eta > 0$ and consider a maximizing sequence $(c_n, k_n)_{n\geq1}$ to $\sup_{\mathcal{O}} e^{-\eta k}(u-v) > 0$. By the growth condition, k_n is bounded by some k^*, where k^* depends only on η. Now, for any $\zeta > 0$ small enough, there exists a point $(\bar{k},\bar{c})$ such that $e^{-\eta\bar{k}}(u-v)(\bar{k},\bar{c}) = \delta_\zeta \geq \sup_{\mathcal{O}} e^{-\eta k}(u-v) - \zeta > 0$. We emphasize that $\bar{k}$ remains bounded, irrespective of ζ. In particular, for any η, $\delta_\zeta/(\zeta+\sqrt{\zeta})$ can be chosen arbitrarily large.

We begin by showing that if such a point lies on the boundary $c = 0$, then there is another with the same property on the interior. Consider points $(\bar{k},\bar{c})$ such that $\bar{c} = 0$. Then, depending on whether $u(\bar{k},0) \leq \bar{k} - \ell/r_{\text{debt}}$ or $u(\bar{k},0) \leq \mathcal{I}u$, we have

$$(u-v)(\bar{k},0) \leq \ell\bar{k} - b/r_{\text{debt}} - \max\{\ell\bar{k} - b/r_{\text{debt}}, \mathcal{I}v\} \leq 0$$

or

$$(u-v)(\bar{k},0) \leq \mathcal{I}u - \max\{\ell\bar{k} - b/r_{\text{debt}}, \mathcal{I}v\} \leq \sup_{I>0}\Big[u(\bar{k},I) - v(\bar{k},I)\Big].$$

The first case contradicts $\delta_\zeta > 0$, and the second shows that another a point with the same properties exists in the interior. Similarly, for $\bar{k} = 0$, we also get $(u-v)(\bar{c},0) \leq \bar{c} - \bar{c} = 0$. Hence, without loss of generality, we may assume $(\bar{k},\bar{c})$ lies away from $c = 0$ and $k = 0$.

Define for $\gamma > 0$

$$\Psi^{\epsilon,\gamma}(k,c,\ell,d) = (1-\gamma)e^{-\eta k}u(\ell,c) - e^{-\eta\ell}v(k,d) - \beta(c-\bar{c})^4 - \frac{1}{2\epsilon}\Big((c-d)^2 + (k-\ell)^2\Big) \quad \text{in } \mathcal{O}\times\mathcal{O}.$$

Clearly, for $\gamma > 0$ small enough,

$$\sup_{\mathcal{O}\times\mathcal{O}} \Psi^{\epsilon,\gamma} \geq \Psi^{\epsilon,\gamma}(\bar{k},\bar{c},\bar{k},\bar{c}) = e^{-\eta\bar{k}}\Big((1-\gamma)u(\bar{k},\bar{c}) - v(\bar{k},\bar{c})\Big) > \delta_\zeta.$$

In particular, for any $\gamma > 0$ and $\eta > 0$, $\Psi^{\epsilon,\gamma}$ has a maximizer $(k_{\epsilon,\gamma}, c_{\epsilon,\gamma}, \ell_{\epsilon,\gamma}, d_{\epsilon,\gamma})$, because of the growth conditions on u and v. Moreover, the growth conditions give an upper bound for this maximizer, depending only on γ and η. Therefore, $(k_{\epsilon,\gamma}, c_{\epsilon,\gamma}, \ell_{\epsilon,\gamma}, d_{\epsilon,\gamma})$ converges along a subsequence as $\epsilon \to 0$. From here on, let us only consider ϵ along this subsequence. Because the lower bound at the maximum above is independent of ϵ,

$$0 < \delta_\zeta < \liminf_{\epsilon\to 0} \Psi^{\epsilon,\gamma}(k_{\epsilon,\gamma}, c_{\epsilon,\gamma}, \ell_{\epsilon,\gamma}, d_{\epsilon,\gamma}),$$

which implies

$$\limsup_{\epsilon\to 0} \frac{1}{2\epsilon}\Big((c_{\epsilon,\gamma} - d_{\epsilon,\gamma})^2 + (k_{\epsilon,\gamma} - \ell_{\epsilon,\gamma})^2\Big) < \infty,$$

so $(k_{\epsilon,\gamma}, c_{\epsilon,\gamma}), (\ell_{\epsilon,\gamma}, d_{\epsilon,\gamma}) \to (k_\gamma, c_\gamma)$. Note that $k_\gamma \leq k^*$, again because of the growth condition.

Rearranging terms and letting $\epsilon \to 0$,

$$\begin{aligned}
\lim_{\epsilon\to 0} \beta(c_{\epsilon,\gamma} - \bar{c})^4 + \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\Big((c_{\epsilon,\gamma} - d_{\epsilon,\gamma})^2 + (k_{\epsilon,\gamma} - \ell_{\epsilon,\gamma})^2\Big)& \\
\leq \limsup_{\epsilon\to 0} e^{-\eta k_{\epsilon,\gamma}}(1-\gamma)u(k_{\epsilon,\gamma}, c_{\epsilon,\gamma}) - e^{-\eta \ell_{\epsilon,\gamma}} v(\ell_{\epsilon,\gamma}, d_{\epsilon,\gamma}) - \delta_\zeta& \\
\leq e^{-\eta k_\gamma}\Big((1-\gamma)u(k_\gamma, c_\gamma) - v(k_\gamma, c_\gamma)\Big) - \delta_\zeta& \\
\leq \zeta.&
\end{aligned}$$

That is,

$$\lim_{\epsilon\to 0} \beta(c_{\epsilon,\gamma} - \bar{c})^4 + \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\Big((c_{\epsilon,\gamma} - d_{\epsilon,\gamma})^2 + (k_{\epsilon,\gamma} - \ell_{\epsilon,\gamma})^2\Big) \leq \zeta. \tag{A.4}$$

As β may be taken arbitrarily large, we ensure that $\zeta < \beta\bar{c}^4$, so that $c_\gamma > 0$.

If $k_\gamma = 0$, we directly obtain $u(c_\gamma, 0) \leq c_\gamma \leq v(c_\gamma, 0)$, which is a contradiction. Hence, (k_γ, c_γ) must lie in the interior, and so will $(k_{\epsilon,\gamma}, c_{\epsilon,\gamma})$ and $(\ell_{\epsilon,\gamma}, d_{\epsilon,\gamma})$ for sufficiently small ϵ.

Because the maxima are attained in interior points, we proceed to use Ishii's lemma, from which we obtain $(p^u_\gamma, X) \in \bar{J}^{2,+}(e^{-\eta k_{\epsilon,\gamma}}(1-\gamma)u(k_{\epsilon,\gamma}, c_{\epsilon,\gamma}))$ and $(p^v_\gamma, Y) \in \bar{J}^{2,-}(e^{-\eta \ell_{\epsilon,\gamma}} v(\ell_{\epsilon,\gamma}, d_{\epsilon,\gamma}))$ (Crandall, Ishii and Lions, 1992, Theorem 3.2), satisfying

$$p^u_\gamma = (p^u_c, p^u_k) = (p^v_c + 4\beta(c_{\epsilon,\gamma} - \bar{c})^3, p^v_k), \quad p^v_\gamma = (p^v_c, p^v_k) = \left(\frac{c_{\epsilon,\gamma} - d_{\epsilon,\gamma}}{\epsilon}, \frac{k_{\epsilon,\gamma} - \ell_{\epsilon,\gamma}}{\epsilon}\right)$$

and

$$k^{2\alpha}_{\epsilon,\gamma} X - \ell^{2\alpha}_{\epsilon,\gamma} Y \leq k^{2\alpha}_{\epsilon,\gamma} 12\beta(c_{\epsilon,\gamma} - \bar{c})^2 + \frac{(k^\alpha_{\epsilon,\gamma} - \ell^\alpha_{\epsilon,\gamma})^2}{\epsilon} + o(1),$$

where $o(1)$ denotes a term that converges to 0 as $\epsilon \to 0$.

We will be splitting into cases, depending on what part of the equation (A.1) (or (A.2)) is satisfied. Because the equations differ only in one term, depending on whether $c \geq \Phi$, we assume (A.1) holds until we treat the one part that differs.

Because u is a subsolution, $\tilde{u} = (1-\gamma)e^{-\eta k}u$ satisfies

$$\begin{aligned}
0 \geq \min\Bigg\{ & r\tilde{u} - \sup_{i\in[0,i_{\max}]}\Bigg(\Big[i - \delta_\zeta k_{\epsilon,\gamma}\Big](\eta\tilde{u} + \partial_k\tilde{u}) \\
&\qquad + \Big[(r-\lambda_c)c_{\epsilon,\gamma} + k_{\epsilon,\gamma}^{\alpha}\mu - b - i - g(k_{\epsilon,\gamma}, i)\Big]\partial_c\tilde{u} \\
&\qquad\qquad + \frac{1}{2}k_{\epsilon,\gamma}^{2\alpha}\sigma^2\partial_{cc}^2\tilde{u}\Bigg), \\
& \tilde{u} - (1-\gamma)e^{-\eta k_{\epsilon,\gamma}} \sup_{I \geq \Phi - c_{\epsilon,\gamma}}\Big[G(k_{\epsilon,\gamma}, c_{\epsilon,\gamma} + I - \Phi) - I - \lambda(I)\Big], \\
& \tilde{u} - (1-\gamma)e^{-\eta k_{\epsilon,\gamma}} G(k_{\epsilon,\gamma}, c_{\epsilon,\gamma} - \Phi), \\
& \partial_c\tilde{u} - (1-\gamma)e^{-\eta k_{\epsilon,\gamma}}\Bigg\}.
\end{aligned} \tag{A.5}$$

Similarly, $\tilde{v} = e^{-\eta k}v$ satisfies

$$\begin{aligned}
0 \leq \min\Bigg\{ & r\tilde{v} - \sup_{i\in[0,i_{\max}]}\Bigg(\Big[i - \delta_\zeta \ell_{\epsilon,\gamma}\Big](\eta\tilde{v} + \partial_k\tilde{v}) \\
&\qquad + \Big[(r-\lambda_c)d_{\epsilon,\gamma} + \ell_{\epsilon,\gamma}^{\alpha}\mu - b - i - g(\ell_{\epsilon,\gamma}, i)\Big]\partial_c\tilde{v} \\
&\qquad\qquad + \frac{1}{2}\ell_{\epsilon,\gamma}^{2\alpha}\sigma^2\partial_{cc}^2\tilde{v}\Bigg), \\
& \tilde{v} - e^{-\eta \ell_{\epsilon,\gamma}} \sup_{I \geq \Phi - d_{\epsilon,\gamma}}\Big[G(\ell_{\epsilon,\gamma}, d_{\epsilon,\gamma} + I - \Phi) - I - \lambda(I)\Big], \\
& \tilde{v} - e^{-\eta \ell_{\epsilon,\gamma}} G(\ell_{\epsilon,\gamma}, d_{\epsilon,\gamma} - \Phi), \\
& \partial_c\tilde{v} - e^{-\eta \ell_{\epsilon,\gamma}}\Bigg\}.
\end{aligned} \tag{A.6}$$

We split into four cases (in reverse order), depending on which expression is smallest in Equation (A.5). We begin with the simple case of

$$p_c^u \leq (1-\gamma)e^{-\eta k_{\epsilon,\gamma}}.$$

Subtracting the two equations (A.5) and (A.6) thus gives

$$4\beta(c_{\epsilon,\gamma} - \bar{c})^3 = p_c^u - p_c^v \leq (e^{-\eta k_{\epsilon,\gamma}} - e^{-\eta \ell_{\epsilon,\gamma}}) - \gamma e^{-\eta k_{\epsilon,\gamma}}.$$

Letting $\epsilon \to 0$ in the last inequality,

$$4\beta(c_\gamma - \bar{c})^3 \leq -\gamma e^{-\eta k_\gamma},$$

which contradicts with Equation (A.4) because ζ can be chosen arbitrarily small, independently of k^*.

For the option exercise condition $\tilde{u} - (1-\gamma)e^{-\eta k_{\epsilon,\gamma}} G(k_{\epsilon,\gamma}, c_{\epsilon,\gamma} - \Phi) \leq 0$, we differentiate depending on the sign of $c_\gamma - \Phi$. If $c_\gamma < \Phi$, then also $c_{\epsilon,\gamma} < \Phi$ for small ϵ, so the option cannot be exercised without enough cash reserve. If $c_\gamma > \Phi$ then both $c_{\epsilon,\gamma} \geq \Phi$ and $d_{\epsilon,\gamma} \geq \Phi$ for small ϵ, and subtracting $\tilde{u} - (1-\gamma)e^{-\eta k_{\epsilon,\gamma}} G(k_{\epsilon,\gamma}, c_{\epsilon,\gamma} - \Phi)$ and $\tilde{v} - e^{-\eta \ell_{\epsilon,\gamma}} G(\ell_{\epsilon,\gamma}, d_{\epsilon,\gamma} - \Phi)$ yields

$$\begin{aligned}
\tilde{u}(k_{\epsilon,\gamma}, c_{\epsilon,\gamma}) - \tilde{v}(\ell_{\epsilon,\gamma}, d_{\epsilon,\gamma}) \leq{} & (e^{-\eta k_{\epsilon,\gamma}} G(k_{\epsilon,\gamma}, c_{\epsilon,\gamma} - \Phi) - e^{-\eta \ell_{\epsilon,\gamma}} G(\ell_{\epsilon,\gamma}, d_{\epsilon,\gamma} - \Phi)) \\
& - \gamma e^{-\eta k_{\epsilon,\gamma}} G(k_{\epsilon,\gamma}, c_{\epsilon,\gamma} - \Phi).
\end{aligned}$$

Thus, by continuity of G,

$$\delta_\zeta \leq e^{-\eta k_\gamma}((1-\gamma)u - v)(k_\gamma, c_\gamma) \leq \limsup_{\epsilon \to 0} (\tilde{u}(k_{\epsilon,\gamma}, c_{\epsilon,\gamma}) - \tilde{v}(\ell_{\epsilon,\gamma}, d_{\epsilon,\gamma})) \leq 0.$$

This implies that $\delta_\zeta \leq 0$, which is a contradiction. Consider finally $c_\gamma = \Phi$ with $c_{\epsilon,\gamma} \geq \Phi$ and $d_{\epsilon,\gamma} < \Phi$ so that $\tilde{u}$ satisfies (A.5) but $\tilde{v}$ does not satisfy the corresponding condition in (A.6). In this case, we use lower semi-continuity of $\tilde{v}$ to get

$$\liminf_{\epsilon \to 0} \tilde{v}(\ell_{\epsilon,\gamma}, d_{\epsilon,\gamma}) \geq \tilde{v}(k_\gamma, c_\gamma) \geq e^{-\eta k_\gamma} G(k_\gamma, c_\gamma - \Phi),$$

as $\tilde{v}$ satisfies (A.6) in $(k_\gamma, c_\gamma) = (k_\gamma, \Phi)$, and we can follow the same steps as above.

After observing the continuity established in Lemma 2, the case involving the issuance term follows the same steps as the case $c_{\epsilon,\gamma} \geq \Phi$ and $d_{\epsilon,\gamma} \geq \Phi$ above.

In the final case, we subtract the equations and get

$$\begin{aligned}
r(\tilde{u} - \tilde{v}) \leq{} & \sup_{i \in [0, i_{\max}]} \Big\{ \Big[i - \delta_\zeta k_{\epsilon,\gamma} \Big] (\eta \tilde{u}(k_{\epsilon,\gamma}, c_{\epsilon,\gamma}) + p_k^u) \\
& + \Big[(r - \lambda_c)c + k_{\epsilon,\gamma}^\alpha \mu - b - i - g(k_{\epsilon,\gamma}, i) \Big] (p_c^v + 4\beta(c_{\epsilon,\gamma} - \bar{c})^3) + \frac{1}{2} k_{\epsilon,\gamma}^{2\alpha} \sigma^2 X \\
& - \Big[i - \delta_\zeta \ell_{\epsilon,\gamma} \Big] (\eta \tilde{v}(\ell_{\epsilon,\gamma}, d_{\epsilon,\gamma}) + p_k^v) \\
& - \Big[(r - \lambda_c)c + \ell_{\epsilon,\gamma}^\alpha \mu - b - i - g(\ell_{\epsilon,\gamma}, i) \Big] p_c^v - \frac{1}{2} \ell_{\epsilon,\gamma}^{2\alpha} \sigma^2 Y \Big\} \\
\leq{} & \sup_{i \in [0, i_{\max}]} \Big\{ i\eta (\tilde{u}(k_{\epsilon,\gamma}, c_{\epsilon,\gamma}) - \tilde{v}(\ell_{\epsilon,\gamma}, d_{\epsilon,\gamma})) \\
& + \Big[(r - \lambda_c)c + k_{\epsilon,\gamma}^\alpha \mu - b - i - g(k_{\epsilon,\gamma}, i) \Big] 4\beta(c_{\epsilon,\gamma} - \bar{c})^3 \\
& - \delta_\zeta (\ell_{\epsilon,\gamma} - k_{\epsilon,\gamma}) p_k^u + \Big[(k_{\epsilon,\gamma}^\alpha - \ell_{\epsilon,\gamma}^\alpha)\mu - (g(k_{\epsilon,\gamma}, i) - g(\ell_{\epsilon,\gamma}, i)) \Big] p_c^v \\
& + 6 k_{\epsilon,\gamma}^{2\alpha} \sigma^2 \beta (c_{\epsilon,\gamma} - \bar{c})^2 + \frac{(k_{\epsilon,\gamma}^\alpha - \ell_{\epsilon,\gamma}^\alpha)^2}{\epsilon} \Big\} + o(1).
\end{aligned}$$

Let $\eta < (r-\Delta)/i_{\max}$ for $\Delta \in (0, r)$. Then, taking lim sup as $\epsilon \to 0$, and using that $g(\cdot, i)$ and $k \mapsto k^\alpha$ are Lipschitz in the neighborhood of (k_γ, c_γ), i.e.,

$$|g(k_{\epsilon,\gamma}, i) - g(\ell_{\epsilon,\gamma}, i)| + \mu|k^\alpha_{\epsilon,\gamma} - \ell^\alpha_{\epsilon,\gamma}| \leq R|k_{\epsilon,\gamma} - \ell_{\epsilon,\gamma}|,$$

we get

$$\begin{aligned} &\limsup_{\epsilon\to 0} \Delta(\tilde{u}(k_{\epsilon,\gamma}, c_{\epsilon,\gamma}) - \tilde{v}(\ell_{\epsilon,\gamma}, d_{\epsilon,\gamma})) \\ &\qquad \leq \lim_{\epsilon\to 0}\left[(\delta_\zeta + R^2)\frac{(k_{\epsilon,\gamma} - \ell_{\epsilon,\gamma})^2}{\epsilon} + R\frac{(c_{\epsilon,\gamma} - d_{\epsilon,\gamma})}{\sqrt{\epsilon}}\frac{(k_{\epsilon,\gamma} - \ell_{\epsilon,\gamma})}{\sqrt{\epsilon}}\right. \\ &\qquad\qquad\qquad\qquad \left. + R'(|c_{\epsilon,\gamma} - \bar{c}|^2 + |c_{\epsilon,\gamma} - \bar{c}|^3) + o(1)\right], \end{aligned}$$

for some constant R', depending on k^* (i.e., η), $i_{\max}$, β, and the model parameters. In other words, R' is independent of ζ. By Equation (A.4), the right-hand side is bounded by $R''(\zeta + \sqrt{\zeta})$, for some constant $R'' > 0$ that is also independent of ζ. Finally, because $\Delta > 0$,

$$\delta_\zeta \leq e^{-\eta k_\gamma}((1-\gamma)u - v)(k_\gamma, c_\gamma) \leq \limsup_{\epsilon\to 0}(\tilde{u}(k_{\epsilon,\gamma}, c_{\epsilon,\gamma}) - \tilde{v}(\ell_{\epsilon,\gamma}, d_{\epsilon,\gamma})) \leq \frac{R''}{\Delta}(\zeta + \sqrt{\zeta}),$$

which is a contradiction because $\delta_\zeta/(\zeta + \sqrt{\zeta})$ can be chosen arbitrarily large. Hence, there cannot exist a point (k, c) such that $(u - v)(k, c) > 0$. □

To use the comparison result above, we must show that the value function satisfies the growth condition. The lower bound is trivial, by construction, so it remains to show the upper bound.

Lemma 4. *The value function $V(k, c)$ is bounded by $c + M + k$ for some constant M.*

Proof. We may assume this is true for G, as its proof follows the same but simpler steps (because there is no option execution). In fact, until the last step, the proof would be identical by replacing V by G below.

Consider the following optimization problem:

$$\begin{aligned} V^c_R(k, c) &= c + V_R(k) \\ &= c + \sup_{\tau, i\geq 0, \nu} \mathbb{E}\left[\int_0^\tau e^{-rt}\Big(k^\alpha\mu - b - i_t - g(k_t, i_t)\Big)dt\right. \\ &\qquad\qquad + 1_{\{\tau<\nu\}}e^{-r(\tau-t)}(\ell k_\tau - b/r_{\text{debt}})_+ \\ &\qquad\qquad \left. + 1_{\{\nu<\tau\}}e^{-r(\nu-t)}(G_R(k_\nu) - \Phi)\right], \end{aligned}$$

where G_R would have been defined similarly but without option execution. This is the optimization problem for a firm that is subject to neither external financing costs nor a cash liquidity premium, and that can invest and choose its default time optimally, i.e., a real option. This value function V_R dominates the firm value in Equation (9), i.e., $V(k,c) \leq V_R^c(k,c)$ for any (k,c). The equation for $V_R(k)$ is given by dynamic programming:

$$0 = \min\{rV_R - \sup_{i\geq 0}((i-\delta k)V_R' + k^\alpha \mu - b - i - g(k,i)), V_R - (\ell k - b/r_{\text{debt}})_+, V_R - G_R - \Phi\}. \tag{A.7}$$

We assume that this equation satisfies the comparison principle, which can be proven as in Theorem 3.

We show that the function $v : k \mapsto M + k$ is a supersolution to (A.7), so by the comparison principle,

$$V(k,c) \leq c + V_R(k) \leq c + k + M.$$

Plugging in v into (A.7), we get

$$rM + rk + \delta k - k^\alpha \mu + b,$$

which is minimized at

$$((r+\delta)/\mu\alpha)^{1/(\alpha-1)},$$

with the minimum value

$$(r+\delta)(1-1/\alpha)(\mu\alpha/(r+\delta))^{1/(1-\alpha)} + rM + b.$$

For M large enough, this minimum is always positive.

We also have that $v - (\ell k - b/r_{\text{debt}}) > 0$ for all $M \geq 0$. It follows that v satisfies the two first viscosity supersolution conditions of Equation (A.7). We are done if $v \geq G_R - \Phi$. However, because this step is not necessary in the construction of a supersolution for G_R, the proof so far shows that $G_R \leq M + k$ for some M. With this M, clearly $v \geq G_R > G_R - \Phi$, so v indeed a viscosity supersolution and $V(k,c) \leq c + k + M$.

□

Corollary 5. *The value function pair (V,G) is the (continuous) unique solution to the HJB equation(s).*

Another consequence of the comparison result in Theorem 3 is the convergence of the numerical scheme outlined below (see Barles and Souganidis (1991)).

Corollary 6. *Numerical solutions converge to the value function as the discretization gets finer.*

The HJB equation(s) for (V,G) are solved in a square domain $[0,c_{\max}] \times [0,k_{\max}]$ via policy iteration, which produces value functions $V(k,c)$, $G(k,c)$, and investment policy function $i(k,c)$ in addition to the regions of dividend payouts, equity issuance, and option exercise. The singular structure is approximated as in (Reppen, Jean-Charles and Soner, 2020, Section 4), which also describes the policy iteration algorithm, and the impulse control issuance as in (Reppen, Jean-Charles and Soner, 2020, Section 6.1.2).

In addition to (A.3), the boundary conditions where $c = c_{\max}$ and $k_{\max}$ are given by

$$
\begin{aligned}
0 &= \partial_c V - 1 && \text{at } c = c_{\max} \\
0 &= \min \Big\{ rV + \delta k \partial_k V - \Big[rc + k^{\alpha}\mu - b \Big] \partial_c V - \frac{1}{2} k^{2\alpha} \sigma^2 \partial^2_{cc} V, \\
&\qquad \partial_c V - 1, \;\; \mathcal{I}V, \\
&\qquad V - G(k, c - \Phi) \Big\} && \text{at } k = k_{\max}
\end{aligned}
$$

At the corners, the c-conditions are used.

References

Altınkılıç, Oya, and Robert S. Hansen. 2000. "Are There Economies of Scale in Underwriting Fees? Evidence of Rising External Financing Costs." *The Review of Financial Studies*, 13(1): 191–218.

Anderson, Ronald W., and Andrew Carverhill. 2012. "Corporate Liquidity and Capital Structure." *The Review of Financial Studies*, 25(3): 797–837.

Ang, Andrew, Robert J. Hodrick, Yuhang Xing, and Xiaoyan Zhang. 2006. "The Cross-Section of Volatility and Expected Returns." *The Journal of Finance*, 61(1): 259–299.

Ang, Andrew, Robert J. Hodrick, Yuhang Xing, and Xiaoyan Zhang. 2009. "High idiosyncratic volatility and low returns: International and further U.S. evidence." *Journal of Financial Economics*, 91(1): 1–23.

Back, Kerry, Ali Kakhbod, and Hao Xing. 2021. "Multistage Financing: Milestone Bonuses or Deferred Compensation." *Working paper*.

Barles, Guy, and Panagiotis E. Souganidis. 1991. "Convergence of approximation schemes for fully nonlinear second order equations." *Asymptotic Analysis*, 4(3): 271–283.

Basu, Susanto, and John G. Fernald. 1997. "Returns to Scale in U.S. Production: Estimates and Implications." *Journal of Political Economy*, 105(2): 249–283.

Bertola, Guiseppe, and Ricardo J. Caballero. 1994. "Irreversibility and Aggregate Investment." *The Review of Economic Studies*, 61(2): 223–246.

Black, Fischer, and Myron Scholes. 1973. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, 81(3): 637–654.

Bloom, Nick, Stephen Bond, and John Van Reenen. 2007. "Uncertainty and Investment Dynamics." *The Review of Economic Studies*, 74(2): 391–415.

Bolton, Patrick, Hui Chen, and Neng Wang. 2011. "A Unified Theory of Tobin's q, Corporate Investment, Financing, and Risk Management." *The Journal of Finance*, 66(5): 1545–1578.

Bolton, Patrick, Neng Wang, and Jinqiang Yang. 2019. "Investment under uncertainty with financial constraints." *Journal of Economic Theory*, 184: 104912.

Boot, Arnoud, and Vladimir Vladimirov. 2019. "(Non-)Precautionary Cash Hoarding and the Evolution of Growth Firms." *Management Science*, 65(11): 5290–5307.

Boyle, Glenn W., and Graeme A. Guthrie. 2003. "Investment, Uncertainty, and Liquidity." *The Journal of Finance*, 58(5): 2143–2166.

Bulan, Laarni, Christopher Mayer, and C. Tsuriel Somerville. 2009. "Irreversible investment, real options, and competition: Evidence from real estate development." *Journal of Urban Economics*, 65(3): 237–251.

Bulan, Laarni T. 2005. "Real options, irreversible investment and firm uncertainty: New evidence from U.S. firms." *Review of Financial Economics*, 14(3): 255–279. Real Options SI.

Caballero, Ricardo J. 1991. "On the Sign of the Investment-Uncertainty Relationship." *The American Economic Review*, 81(1): 279–288.

Caballero, Ricardo J. 1999. "Chapter 12 Aggregate investment." In . Vol. 1 of *Handbook of Macroeconomics*, 813–862. Elsevier.

Caprio, Lorenzo, Mara Faccio, and John J. McConnell. 2011. "Sheltering Corporate Assets from Political Extraction." *The Journal of Law, Economics, and Organization*, 29(2): 332–354.

Cooper, Russell, John Haltiwanger, and Laura Power. 1999. "Machine Replacement and the Business Cycle: Lumps and Bumps." *American Economic Review*, 89(4): 921–946.

Crandall, Michael G, Hitoshi Ishii, and Pierre-Louis Lions. 1992. "User's guide to viscosity solutions of second order partial differential equations." *Bulletin of the American mathematical society*, 27(1): 1–67.

Cunningham, Christopher R. 2006. "House price uncertainty, timing of development, and vacant land prices: Evidence for real options in Seattle." *Journal of Urban Economics*, 59(1): 1–31.

Dai, Min, Xavier Giroud, Wei Jiang, and Neng Wang. 2021. "A Q theory of internal capital markets." *NBER Working Paper 27931*.

Décamps, Jean-Paul, and Stéphane Villeneuve. 2007. "Optimal dividend policy and growth option." *Finance and Stochastics*, 11(1): 3–27.

Décamps, Jean-Paul, and Stéphane Villeneuve. 2013. "Optimal Investment under liquidity constraints." In *Real Options, Ambiguity, Risk and Insurance*. 3–19. IOS Press.

Dittmar, Amy, and Anil Shivdasani. 2003. "Divestitures and Divisional Investment Policies." *The Journal of Finance*, 58(6): 2711–2744.

Dittmar, Amy, and Jan Mahrt-Smith. 2007. "Corporate governance and the value of cash holdings." *Journal of Financial Economics*, 83(3): 599–634.

Dixit, Avinash K., and Robert S. Pindyck. 1994. *Investment under Uncertainty. Economics Books*, Princeton University Press.

Doms, Mark, and Timothy Dunne. 1998. "Capital Adjustment Patterns in Manufacturing Plants." *Review of Economic Dynamics*, 1(2): 409–429.

Duffee, Gregory R. 1995. "Stock returns and volatility a firm-level analysis." *Journal of Financial Economics*, 37(3): 399–420.

Décaire, Paul H, Erik P Gilje, and Jérôme P Taillard. 2020. "Real Option Exercise: Empirical Evidence." *The Review of Financial Studies*, 33(7): 3250–3306.

Décamps, Jean-Paul, and Stéphane Villeneuve. 1994. "Optimal dividend policy and growth option." *Finance and Stochastics*, 11: 3–27.

Décamps, Jean-Paul, Thomas Mariotti, Jean-Charles Rochet, and Stéphane Villeneuve. 2011. "Free Cash Flow, Issuance Costs, and Stock Prices." *The Journal of Finance*, 66(5): 1501–1544.

Eberly, Janice, Sergio Rebelo, and Nicolas Vincent. 2009. "Investment and value: A neoclassical benchmark,." *Working Paper*.

Faulkender, Michael, and Rong Wang. 2006. "Corporate Financial Policy and the Value of Cash." *The Journal of Finance*, 61(4): 1957–1990.

Fazzari, Steven, R. Glenn Hubbard, and Bruce Petersen. 1988. "Investment, Financing Decisions, and Tax Policy." *The American Economic Review*, 78(2): 200–205.

Froot, Kenneth A., David S. Scharfstein, and Jeremy C. Stein. 1993. "Risk Management: Coordinating Corporate Investment and Financing Policies." *The Journal of Finance*, 48(5): 1629–1658.

Gao, Huasheng, Jarrad Harford, and Kai Li. 2013. "Determinants of corporate cash policy: Insights from private firms." *Journal of Financial Economics*, 109(3): 623–639.

Girgis, Nadia Makary. 1968. "Optimal Cash Balance Levels." *Management Science*, 15(3): 130–140.

Gomes, João F. 2001. "Financing Investment." *American Economic Review*, 91(5): 1263–1285.

Gomes, João F., and Lukas Schmid. 2021. "Equilibrium Asset Pricing with Leverage and Default." *The Journal of Finance*, 76(2): 977–1018.

Graham, John R. 2000. "How Big Are the Tax Benefits of Debt?" *The Journal of Finance*, 55(5): 1901–1941.

Graham, John R, and Campbell R Harvey. 2001. "The theory and practice of corporate finance: evidence from the field." *Journal of Financial Economics*, 60(2): 187–243.

Grenadier, Steven R., and Andrey Malenko. 2010. "A Bayesian Approach to Real Options: The Case of Distinguishing between Temporary and Permanent Shocks." *The Journal of Finance*, 65(5): 1949–1986.

Grenadier, Steven R., and Andrey Malenko. 2011. "Real Options Signaling Games with Applications to Corporate Finance." *The Review of Financial Studies*, 24(12): 3993–4036.

Grenadier, Steven R., Andrey Malenko, and Nadya Malenko. 2016. "Timing Decisions in Organizations: Communication and Authority in a Dynamic Environment." *American Economic Review*, 106(9): 2552–81.

Grullon, Gustavo, and David L. Ikenberry. 2021. "Excess Capacity, Marginal q and Corporate Investment." *Working Paper*.

Grullon, Gustavo, Evgeny Lyandres, and Alexei Zhdanov. 2012. "Real Options, Volatility, and Stock Returns." *The Journal of Finance*, 67(4): 1499–1537.

Guiso, Luigi, and Giuseppe Parigi. 1999. "Investment and Demand Uncertainty." *The Quarterly Journal of Economics*, 114(1): 185–227.

Harford, Jarrad. 1999. "Corporate Cash Reserves and Acquisitions." *The Journal of Finance*, 54(6): 1969–1997.

Harford, Jarrad, Sattar A. Mansi, and William F. Maxwell. 2008. "Corporate governance and firm cash holdings in the US." *Journal of Financial Economics*, 87(3): 535–555.

Hennessy, Christopher A., and Toni M. Whited. 2005. "Debt Dynamics." *The Journal of Finance*, 60(3): 1129–1165.

Hennessy, Christopher A., and Toni M. Whited. 2007. "How Costly Is External Financing? Evidence from a Structural Estimation." *The Journal of Finance*, 62(4): 1705–1745.

He, Zhiguo, and Konstantin Milbradt. 2014. "Endogenous Liquidity and Defaultable Bonds." *Econometrica*, 82(4): 1443–1508.

Hugonnier, Julien, Semyon Malamud, and Erwan Morellec. 2015. "Capital Supply Uncertainty, Cash Holdings, and Investment." *The Review of Financial Studies*, 28(2): 391–445.

Jensen, Michael C. 1986. "Agency Costs of Free Cash Flow, Corporate Finance, and Takeovers." *The American Economic Review*, 76(2): 323–329.

Kakhbod, Ali, Max Reppen, Tarik Umar, and Hao Xing. 2021. "The Cash-Cap Model: A Two-State Model of Firm Dynamics." *Working Paper*.

Kaplan, Steven N., and Luigi Zingales. 1997. "Do Investment-Cash Flow Sensitivities Provide Useful Measures of Financing Constraints?" *The Quarterly Journal of Economics*, 112(1): 169–215.

Kellogg, Ryan. 2014. "The Effect of Uncertainty on Investment: Evidence from Texas Oil Drilling." *American Economic Review*, 104(6): 1698–1734.

Kim, Chang-Soo, David C. Mauer, and Ann E. Sherman. 1998. "The Determinants of Corporate Liquidity: Theory and Evidence." *Journal of Financial and Quantitative Analysis*, 33(3): 335–359.

Lambrecht, Bart M. 2017. "Real options in finance." *Journal of Banking & Finance*, 81: 166–171.

Leahy, John V., and Toni M. Whited. 1996. "The Effect of Uncertainty on Investment: Some Stylized Facts." *Journal of Money, Credit and Banking*, 28(1): 64–83.

Lee, Inmoo, Scott Lochhead, Jay Ritter, and Quanshui Zhao. 1996. "THE COSTS OF RAISING CAPITAL." *Journal of Financial Research*, 19(1): 59–74.

Leland, Hayne E. 1994. "Corporate Debt Value, Bond Covenants, and Optimal Capital Structure." *The Journal of Finance*, 49(4): 1213–1252.

Mauer, David C., and Alexander J. Triantis. 1994. "Interactions of Corporate Financing and Investment Decisions: A Dynamic Framework." *The Journal of Finance*, 49(4): 1253–1277.

McDonald, Robert, and Daniel Siegel. 1986. "The Value of Waiting to Invest." *The Quarterly Journal of Economics*, 101(4): 707–727.

Merton, Robert C. 1973. "Theory of Rational Option Pricing." *The Bell Journal of Economics and Management Science*, 4(1): 141–183.

Miao, Jianjun, and Neng Wang. 2007. "Investment, consumption, and hedging under incomplete markets." *Journal of Financial Economics*, 86(3): 608–642.

Miller, Merton H., and Daniel Orr. 1966. "A Model of the Demand for Money by Firms*." *The Quarterly Journal of Economics*, 80(3): 413–435.

Moel, Alberto, and Peter Tufano. 2002. "When Are Real Options Exercised? An Empirical Study of Mine Closings." *The Review of Financial Studies*, 15(1): 35–64.

Paddock, James L., Daniel R. Siegel, and James L. Smith. 1988. "Option Valuation of Claims on Real Assets: The Case of Offshore Petroleum Leases*." *The Quarterly Journal of Economics*, 103(3): 479–508.

Peters, Ryan H., and Lucian A. Taylor. 2017. "Intangible capital and the investment-q relation." *Journal of Financial Economics*, 123(2): 251–272.

Pinkowitz, Lee, René Stulz, and Rohan Williamson. 2006. "Does the Contribution of Corporate Cash Holdings and Dividends to Firm Value Depend on Governance? A Cross-country Analysis." *The Journal of Finance*, 61(6): 2725–2751.

Quigg, Laura. 1993. "Empirical Testing of Real Option-Pricing Models." *The Journal of Finance*, 48(2): 621–640.

Reppen, Max A., Rochet. Jean-Charles, and Mete H. Soner. 2020. "Optimal dividend policies with random profitability." *Mathematical Finance*, 30(1): 228–259.

Riddick, Leigh A., and Toni M. Whited. 2009. "The Corporate Propensity to Save." *The Journal of Finance*, 64(4): 1729–1766.

Vath, Vathana Ly, Huyen Pham, and Stéphane Villeneuve. 2008. "A mixed singular/switching control problem for a dividend policy with reversible technology investment." *The Annals of applied probability*, 18(3): 1164–1200.

Whited, Toni M. 1992. "Debt, Liquidity Constraints, and Corporate Investment: Evidence from Panel Data." *The Journal of Finance*, 47(4): 1425–1460.

Table 1: Model Parameters

Parameter	Name	Values	Comments
r	Interest rate	6%	In line with long term average yield to maturity on 30 year U.S. Treasuries.
λ_c	Cash holding cost, liquidity premium	1%	In line with Bolton, Chen and Wang (2011); Bolton, Wang and Yang (2019).
μ	Expected productivity shock	0.18	In line with the estimates of Eberly, Rebelo and Vincent (2009) for large U.S. firms.
$\tilde{\mu}$	Expected productivity shock after exercising the real option	0.19	A 0.01 increase relative to μ.
σ	Volatility of productivity shock	0.09	In line with the estimates of Eberly, Rebelo and Vincent (2009) for large U.S. firms.
$\tilde{\sigma}$	Volatility of productivity shock after exercising the real option	0.09	Same as σ.
θ	Degree of adjustment cost	1.5	See Whited (1992).
δ	Depreciation rate	10.07%	In line with the estimates of Eberly, Rebelo and Vincent (2009) for large U.S. firms.
b	Long term debt coupon rate	0.02	Together with r_{debt} below yields a long term debt face value 0.22.
λ_p	Variable issuance cost	6%	In line with the estimates of Altınkılıç and Hansen (2000).
λ_f	Fixed issuance cost	0.05	λ_f is chosen so that the fixed issuance cost for firms with average size in our model is at the same magnitude as the fixed cost rate in Bolton, Chen and Wang (2011) multiplied by our average size.
α	Curvature of production function. When $\alpha < 1$, then diminishing returns to scale	0.7	$\alpha = 0.75$ in Riddick and Whited (2009) and $\alpha = 0.627$ with std 0.219 for the full sample of firms in Hennessy and Whited (2007).
ℓ	Recovery rate in liquidation of capital	90%	The choice of ℓ is consistent with Hennessy and Whited (2007), where the recovery rate is estimated to be 0.896 for the full sample of firms.
r_{debt}	Cost of financing for long-term debt	9%	r_{debt} is chosen as $1.5r$. Results are insensitive to this parameter, because firms are left with no proceed after the bondholder is repaid after liquidation in most of numeric experiments.
Φ	Exercise Cost	0.03	Φ is the one-time cost of exercising the real option. After paying Φ, the productivity process permanently changes to $\tilde{\mu}$ and $\tilde{\sigma}$.

Table 2: Summary Statistics Variables winsorized at the 1% level.

Panel A: Compustat Characteristics						
	Obs.	Mean	Std. Dev.	P25	P50	P75
Property, Plant & Equipment (Net)	93628	683.65	2233.05	20.29	67.3	312.39
Cash and Short-Term Investments	93628	195.4	591.34	4.19	21.73	104.29
Cash-to-Capital (%)	93628	104.36	226.08	6.86	22.86	83.05
Net Investment Rate (%)	93628	4.52	24.64	-4.86	2.69	13.08
Capital Expenditures	93628	117.06	362.47	4.16	14.28	59.7
Depreciation	93628	90.21	267.52	3.65	11.94	49.8

Panel B: CRSP Characteristics						
	Obs.	Mean	Std. Dev.	P25	P50	P75
VOL	1856565	3.37	2.42	1.76	2.68	4.16
ΔVOL	1856565	0	1.81	-.75	-.02	.72
Market Beta	1856565	.82	.39	.54	.81	1.07
Book-to-Market	1856565	.75	.69	.3	.55	.94
Market Equity	1856565	11.89	2.24	10.23	11.74	13.44
Cumulative Past Six-Months Returns	1856565	4.01	38.91	-19.43	-.26	20.5
Turnover	1856565	1.14	1.54	.22	.58	1.42

Table 3: H1: The minimum cash-to-capital ratio for exercise is declining and convex in capital

The outcome is the cash-to-capital ratio at the end of year t for firm i (standardized within firm). The sample includes each firm-year t immediately preceding a firm-year $t+1$ in which a firm exercises a real option. To proxy for real option activity, we require the net investment rates in year $t+1$ to exceed the sample median of 2.6% in Panel A and the 75th percentile of 13.1% in Panel B. *Capital* is a firm's property, plant, and equipment net of depreciation at the end of year t standardized within firm. We standardize capital within firm so that the quadratic form of capital captures the distance from a firm's average size rather than an average across firms in the sample. Columns (2) and (3) split the sample after and on or before the median year of 1992, respectively. Columns (4) to (10) restrict the sample to industries based on the SIC-1 identifier. We control for year fixed effects. Standard errors are clustered by year. *, **, and *** indicate significance at the 10%, 5%, and 1% level, respectively.

Panel A: Net investment rates in year $t+1$ must exceed 50th percentile (2.6%)

	Cash (t) / Capital (t)									
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Capital (t)	-0.25***	-0.23***	-0.27***	-0.31***	-0.24***	-0.18***	-0.28***	-0.29***	-0.30***	-0.32***
	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.02)	(0.03)
Capital^2 (t)	0.04***	0.05***	0.05***	0.04**	0.07***	0.01	0.08***	0.07***	0.04**	0.12***
	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.02)	(0.04)
Specification	All	Yr≤92	Yr>'92	SIC1=1	SIC1=2	SIC1=3	SIC1=4	SIC1=5	SIC1=7	SIC1=8
Quarter FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
% Adjusted R^2	6.84	5.10	8.49	8.63	7.12	7.05	8.04	8.20	9.35	8.92
% Within R^2	4.38	3.64	5.18	6.42	4.17	2.25	6.01	5.87	6.54	7.46
Observations	41164	20653	20511	3890	8857	13615	2933	6667	3678	1357

Panel B: Net investment rates in year $t+1$ must exceed 75th percentile (13.1%)

	Cash (t) / Capital (t)									
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Capital (t)	-0.28***	-0.24***	-0.32***	-0.33***	-0.28***	-0.20***	-0.32***	-0.30***	-0.33***	-0.40***
	(0.01)	(0.01)	(0.02)	(0.03)	(0.03)	(0.02)	(0.03)	(0.02)	(0.03)	(0.06)
Capital^2 (t)	0.03***	0.05***	0.03***	0.07**	0.06**	-0.02	0.08**	0.10***	0.01	0.15**
	(0.01)	(0.02)	(0.01)	(0.03)	(0.02)	(0.02)	(0.04)	(0.02)	(0.03)	(0.06)
Specification	All	Yr≤92	Yr>'92	SIC1=1	SIC1=2	SIC1=3	SIC1=4	SIC1=5	SIC1=7	SIC1=8
Quarter FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
% Adjusted R^2	6.42	4.87	8.12	8.77	6.98	5.64	9.93	7.92	8.39	9.87
% Within R^2	4.46	3.26	5.88	6.33	4.52	2.08	6.37	5.95	6.02	10.25
Observations	19762	10251	9511	2307	3632	6314	1317	3444	2050	637

Table 4: H2: The value of a firm's real options are hump-shaped in cash, and more so when the firm is small

The outcome variable is firm i's stock return in month t. The main explanatory variable is ΔVOL the change of the stock volatility of firm i in month t. We interact changes in firm volatility with a firm's cash reserve (standardized within firm) and its square. We control for firm i's market beta, book-to-market ratio, market equity, cumulative past six month returns, and turnover. *Large* is an indicator that equals to one if a firm's property, plant, and equipment net of depreciation and standardized within firm is in the top three quartiles of the sample. We include year-month fixed effects and cluster standard errors by year-month. *, **, and *** indicate significance at the 10%, 5%, and 1% level, respectively.

	Stock Returns (i,t)	
	(1)	(2)
ΔVOL (i,t)	0.52***	0.64***
	(0.05)	(0.05)
Cash (i,t)	-0.54***	-0.67***
	(0.03)	(0.03)
ΔVOL (i,t) × Cash (i,t)	-0.11***	-0.12***
	(0.02)	(0.02)
Cash^2 (i,t)	0.09***	0.08***
	(0.01)	(0.02)
ΔVOL (i,t) × Cash^2 (i,t)	0.00	-0.03**
	(0.01)	(0.01)
Market Beta (i)	-0.06	-0.16
	(0.23)	(0.23)
Book-to-Market (i,t)	0.65***	0.73***
	(0.07)	(0.07)
Market Equity (i,t)	0.00	0.02
	(0.05)	(0.05)
Cumulative Past Returns (i,t-8,t-2)	0.01**	0.00*
	(0.00)	(0.00)
Turnover (i,t)	0.80***	0.80***
	(0.06)	(0.06)
Large=1		-0.79***
		(0.05)
Large=1 × ΔVOL (i,t)		-0.25***
		(0.03)
Large=1 × Cash (i,t)		0.54***
		(0.04)
Large=1 × ΔVOL (i,t) × Cash (i,t)		0.05*
		(0.03)
Large=1 × Cash^2 (i,t)		-0.08***
		(0.02)
Large=1 × ΔVOL (i,t) × Cash^2 (i,t)		0.04***
		(0.02)
Constant	0.08	0.32
	(0.42)	(0.42)
Specification	All	All
Year-Month FE	Yes	Yes
% Adjusted R^2	16.24	16.34
Observations	1842770	1842770

Internet Appendix to
Real Options and Financial Flexibility

This Internet Appendix contains supplementary theoretical and empirical work.

1. Additional Baseline Model Output
   (a) Figure B.1 provides a heat map for the investment rates in Figure 1
2. Appendix C - Additional Empirical Work
   (a) Table C.1 presents the COMPUSTAT sample selection criteria
   (b) Table C.2 presents the CRSP sample selection criteria
   (c) Table C.3 describes the SIC-1 categories used for robustness in the main tables
   (d) Robustness for **H1**
      i. Figure C.1b repeats Figure 2b using acquisition spending to proxy for real option exercises
      ii. Figure C.2 repeats Figure 2 using alternative thresholds for large net investments
      iii. Figure C.3 repeats Figure 2 adding intangible capital
   (e) Robustness for **H2**
      i. Figure C.4 repeats Figure 5b using Fama-MacBeth regression specifications
      ii. Figure C.5 repeats Figure 5b splitting on a firm's cash-to-capital ratio rather than cash reserve
      iii. Figure C.6 repeats Figure 5b using tangible and intangible capital
      iv. Figure C.7 repeats Figure 7 using alternative thresholds for large net investments
   (f) Figure C.8 shows that firms with real options have more cash immediately after issuance
3. Appendix D - Comparative statics for Figure 1 provide several additional testable predictions
   (a) Figure D.1 varies the exercise cost Φ in Figure 1
   (b) Figure D.2 varies the productivity μ in Figure 1
   (c) Figure D.3 varies the fixed issuance cost λ_f
   (d) Figure D.6 shows how a firm's choices vary with the effect of the real option on a firm's cash flow volatility
   (e) Figure D.4 shows how a firm's choices vary with the proportional issuance costs
   (f) Figure D.5 lowers both the fixed and proportional issuance costs to 0.25x base values and shows that firms may issue equity before running out of cash
   (g) Figure D.7 shows how a firm's choices vary with leverage

Appendix B. Additional Baseline Model Output

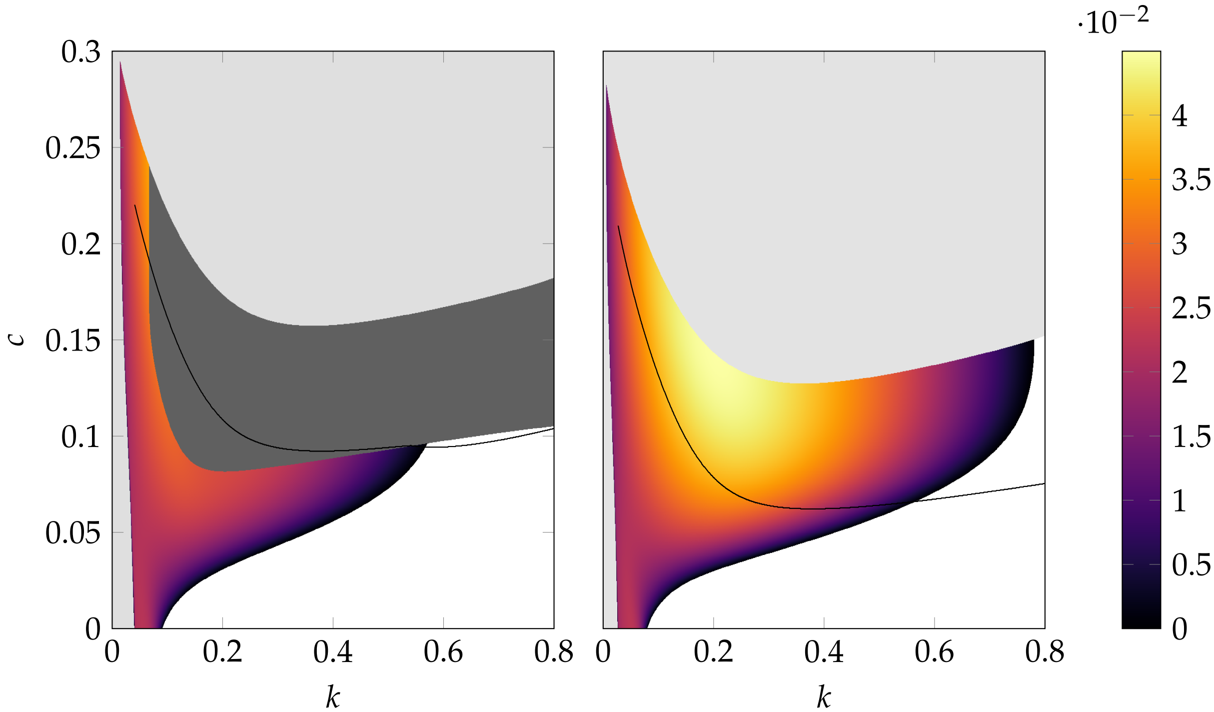


Figure B.1: **Investment Heat Map**

This figure provides a heat map of the optimal investment amount i before (left figure) and after (right figure) the firm exercises the real option. Note that the legend always indicates a positive investment rate because we are showing the investment rate i not the net investment rate $i - \delta k$. The legend on the far right indicates the intensity of the investment i. Because of diminishing returns to scale, investment declines as firm size k (capital) increases. Also, because of the marginal value of cash, investment increases with cash c holding capital k fixed. Parameters used are summarized in Table 1.

Appendix C. Empirical Appendix

Table C.1: Compustat Sample Selection
This table presents the criteria used to prepare the firm-year dataset.

Criteria	Obs. Lost	Obs. Remaining
COMPUSTAT, 1970 – 2020		536,801
Less:		
Pre-IPO Data	(30,156)	506,645
Firms headquartered outside of USA	(90,843)	415,802
Firms incorporated outside of USA	(5,308)	410,494
Financials (SIC-1==6)	(138,644)	271,850
Utilities (SIC-2==49)	(17,555)	254,295
Public Administration (SIC-1==9)	(5,055)	249,240
Missing cash and cash equivalents	(9,114)	240,126
PP&E less than $5M or missing PP&E	(89,072)	149,830
Negative cash and cash equivalents	(13)	149,817
Less than $1M in sales	(1,426)	148,391
Require merge with CRSP data	(32,233)	116,158
Drop if stock volatility missing	(12)	116,146
Drop if change in volatility missing	(5,798)	110,348
Drop if net investment missing	(12,577)	97,771
Singleton Firms	(1,069)	96,702
SIC-4 industries with one firm	(3,074)	93,628
Final sample, 1971 - 2020 (8,006 firms)		93,628

Table C.2: CRSP Sample Selection
This table presents the criteria used to prepare the stock-month dataset used to measure the value of a firm's real options.

Criteria	Obs. Lost	Obs. Remaining
CRSP, 1960-Jan — 2020-Dec		4,396,301
Less:		
Data prior to July, 1963 (Fama-French Factors start then)	(61,193)	4,335,108
Utilities (SIC-2==49)	(129,975)	4,205,133
Financials (SIC-1==6)	(1,213,282)	2,991,851
Public Administration (SIC-1==9)	(90,884)	2,900,967
Keep one record when duplicate PERMNO-month	(12,279)	2,888,688
Does not merge with daily CRSP file	(101,655)	2,787,033
Missing volatility	(1,999)	2,785,034
Missing month-to-month change in volatility	(23,112)	2,761,922
Lacks six months of past returns	(173,087)	2,588,835
Missing turnover	(199,377)	2,389,458
Keep records with annual lagged COMPUSTAT data	(389,623)	1,999,835
Missing book-to-market ratio	(127,234)	1,872,601
Missing market equity in prior month	(12,965)	1,859,636
Missing property, plant & equipment	(2,971)	1,856,665
Missing cash & equivalents	(100)	1,856,565
Final sample, 1964-Apr — 2020-Dec (15,221 firms)		1,856,565

Table C.3: SIC Code Reference

SIC First Digit	Category Description
1	Mining & Construction
2	Manufacturing (Food, Apparel, Furniture, Chemicals, Petroleum)
3	Manufacturing (Rubber, Leather, Stone, Metals, Transportation Equipment)
4	Transportation and Public Utilities
5	Wholesale and Retail Trade
7	Services (Hotel, Personal, Business, Auto Repair)
8	Services (Health, Legal, Educational, Social, Museums)

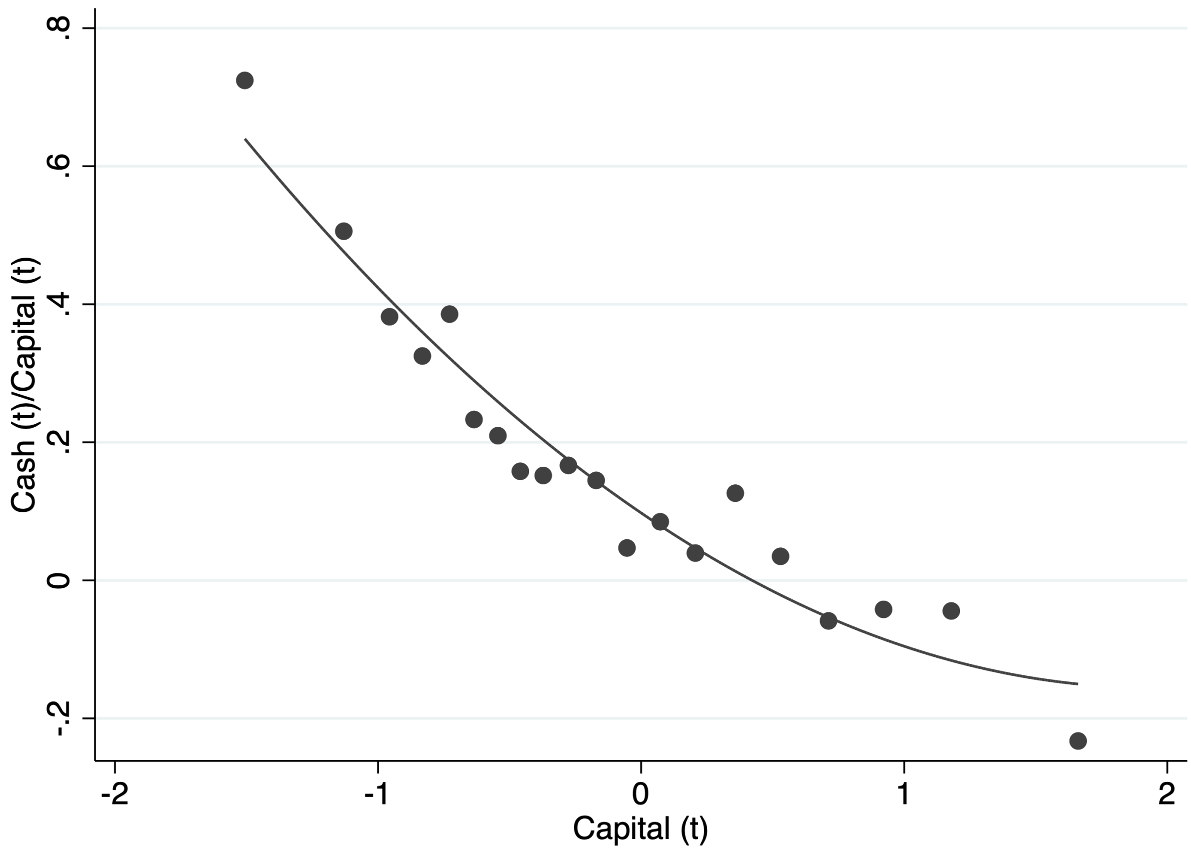
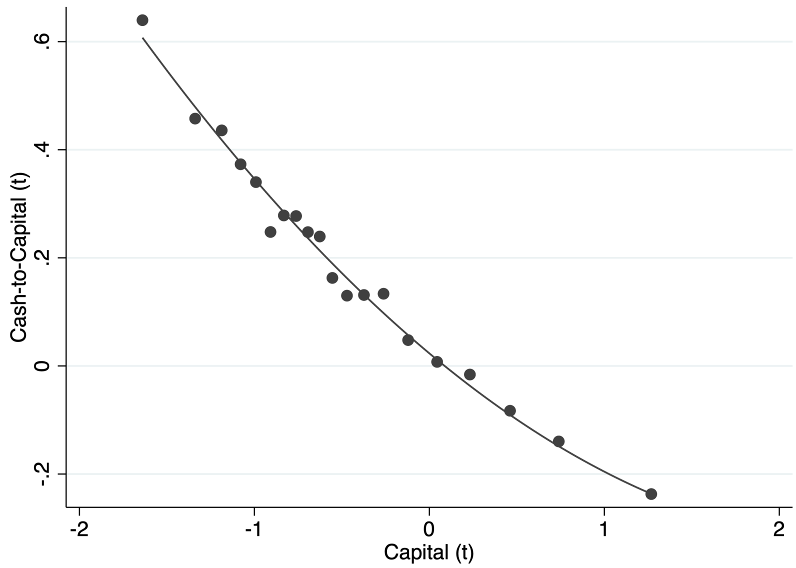
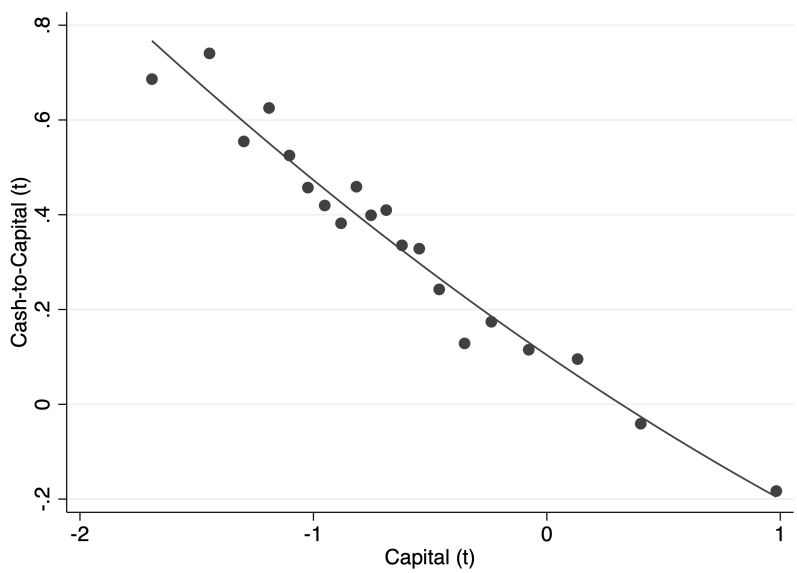


Figure C.1: **Repeats Figure 2 using acquisition spending to proxy for real option exercises**

The Y-axis is a firm's cash-to-capital ratio at the end of quarter t (standardized within firm). The X-axis is a firm's physical and intangible capital at the end of quarter t (standardized within firm). We limit the sample to quarter immediately preceding a quarter $t+1$ with acquisition spending greater than 5% of a firm's capital. Controls for industry trends.



(a) Restricting the sample to firm years preceding firm years with net investment rates above the 75th percentile (13.1%).



(b) Restricting the sample to firm years preceding firm years with net investment rates above the 90th percentile (28.8%).

Figure C.2: **Varying the net-investment-rate threshold for option exercises in Figure 2**

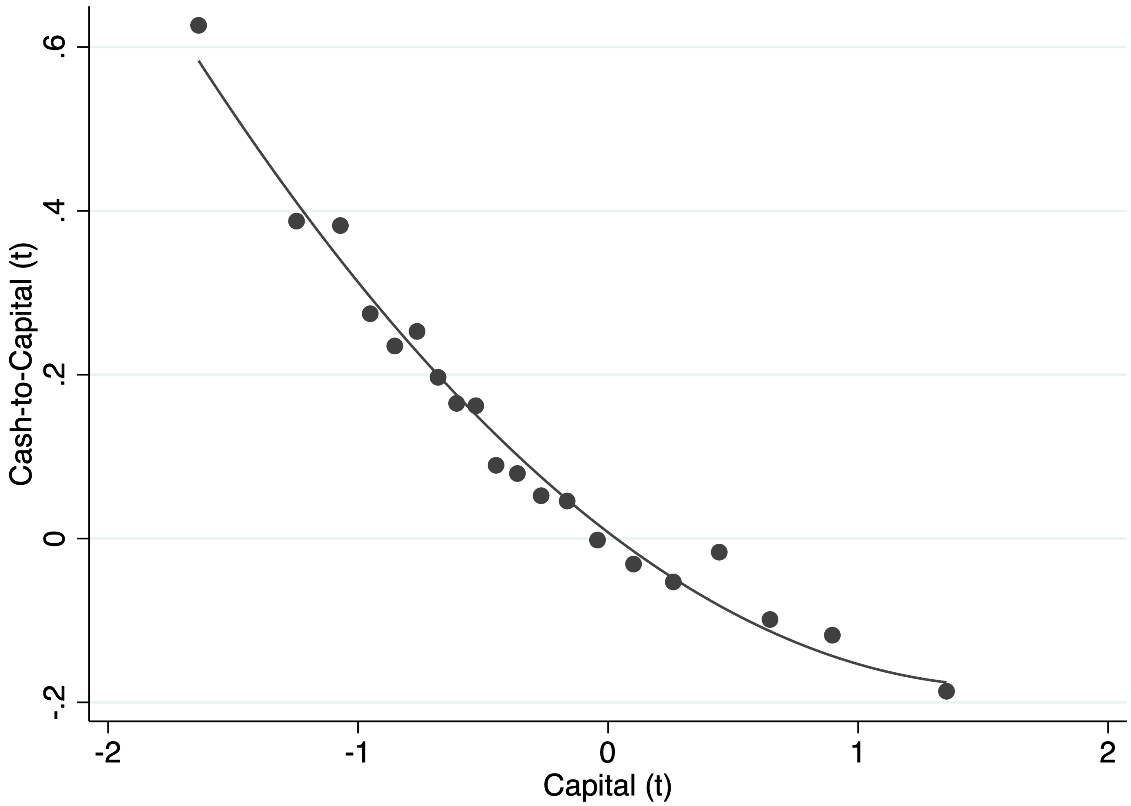


Figure C.3: **Repeating Figure 2 adding intangible capital**

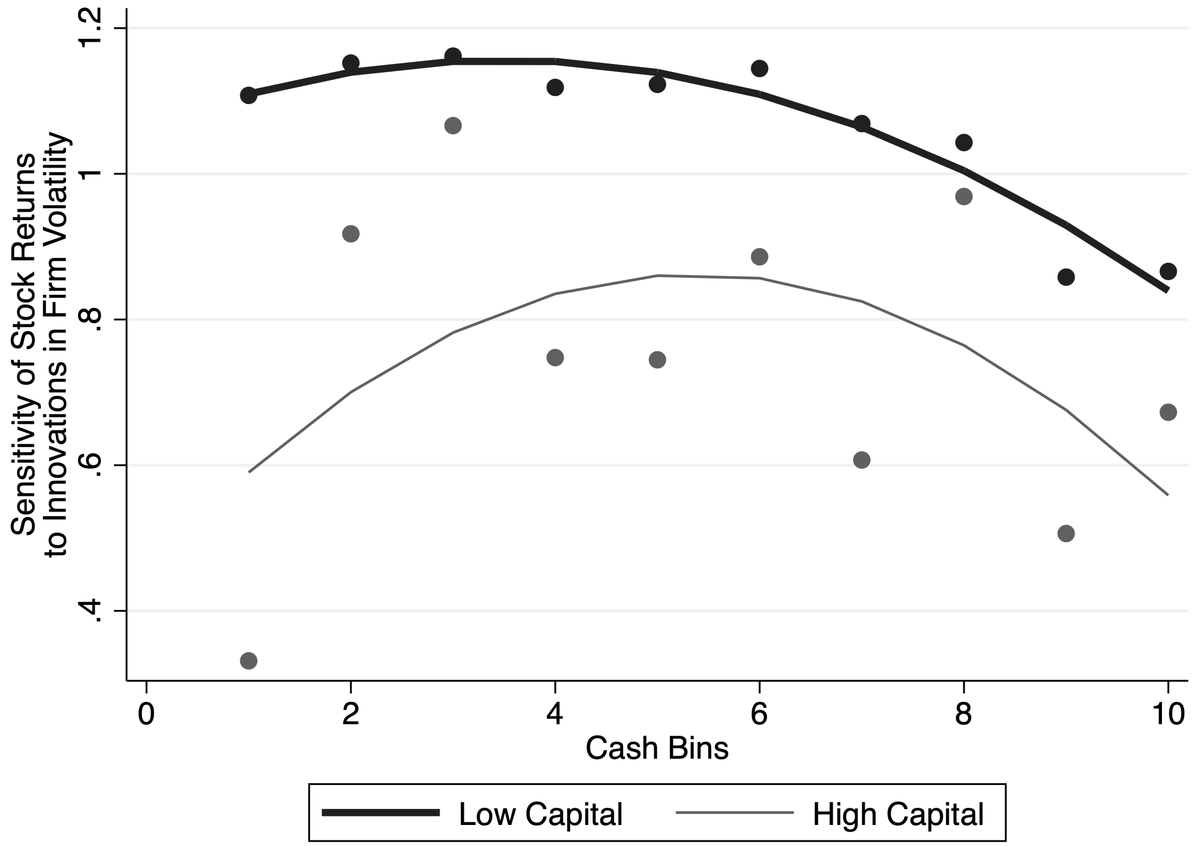


Figure C.4: **Robustness of Figure 5b to a Fama-MacBeth specification**

This figure repeats Figure 5 using the Fama-MacBeth regressions to estimate the sensitivity of a firm's stock returns to innovations in firm-specific volatility. We bin our full firm-month sample into two groups based on a firm's net property, plant, and equipment (standardized within firm) in quarter *t*. The "Low Capital" group captures when a firm is small or in the first quartile of its size distribution. Then, for each size group, we form ten cash bins based on a firm cash reserve at the end of quarter *t* standardized within firm. The Y-axis is the sensitivity of firm stock returns to innovations in firm-specific stock volatility calculated for each bin per Grullon, Lyandres and Zhdanov (2012). The X-axis denotes the ordinal values of the cash bins.

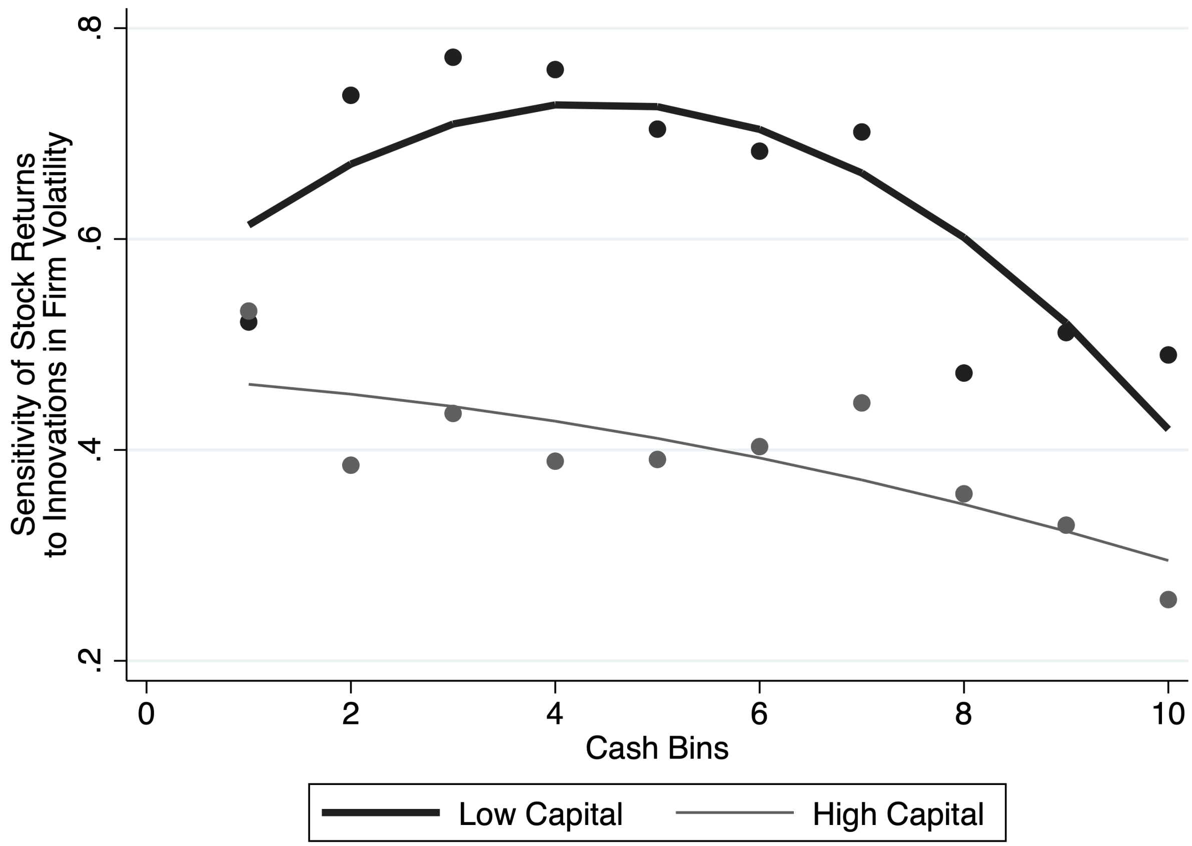


Figure C.5: **Robustness of Figure 5b to binning firms based on cash-to-capital (standardized within firm) instead of cash (standardized within firm)**

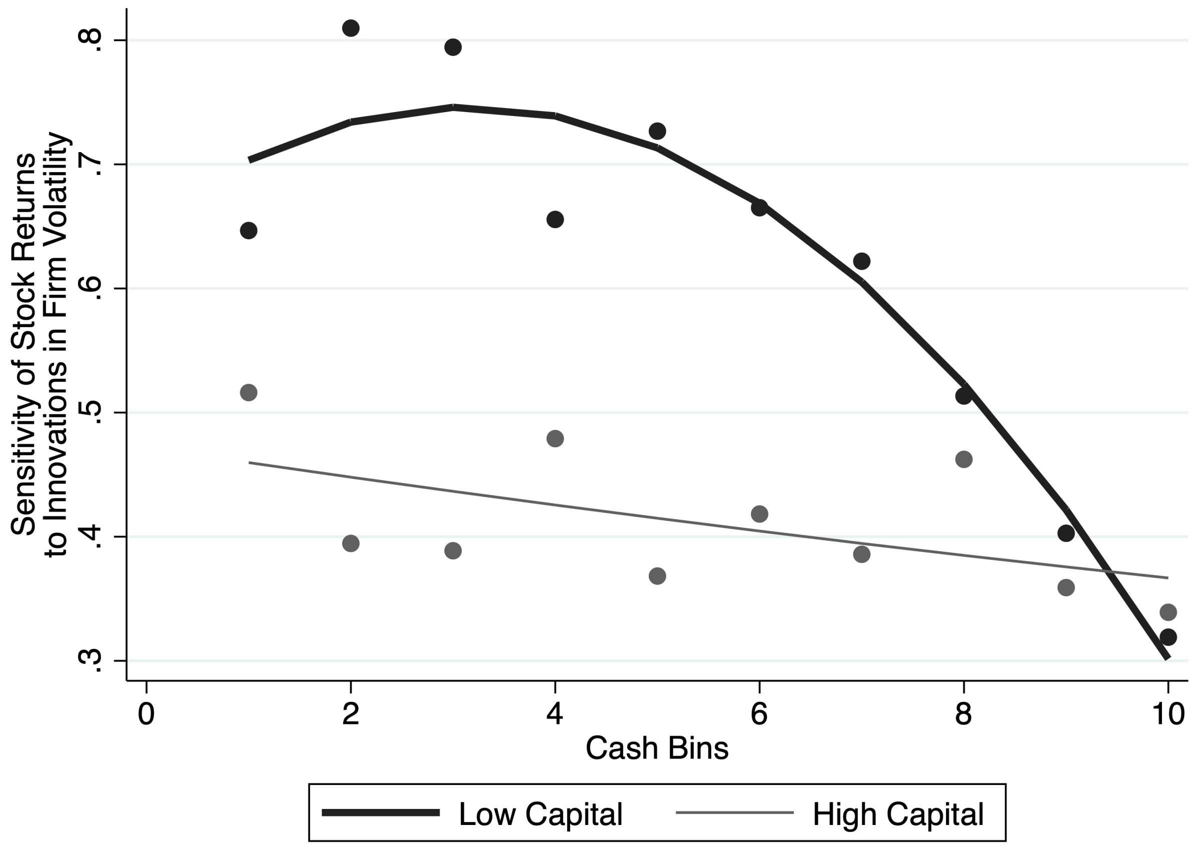


Figure C.6: **Robustness of Figure 5b to adding intangible capital**
When determining a firm's size, we use a firm's physical property, plant, and equipment net of depreciation and a firm's intangible capital stock as calculated by Peters and Taylor (2017).

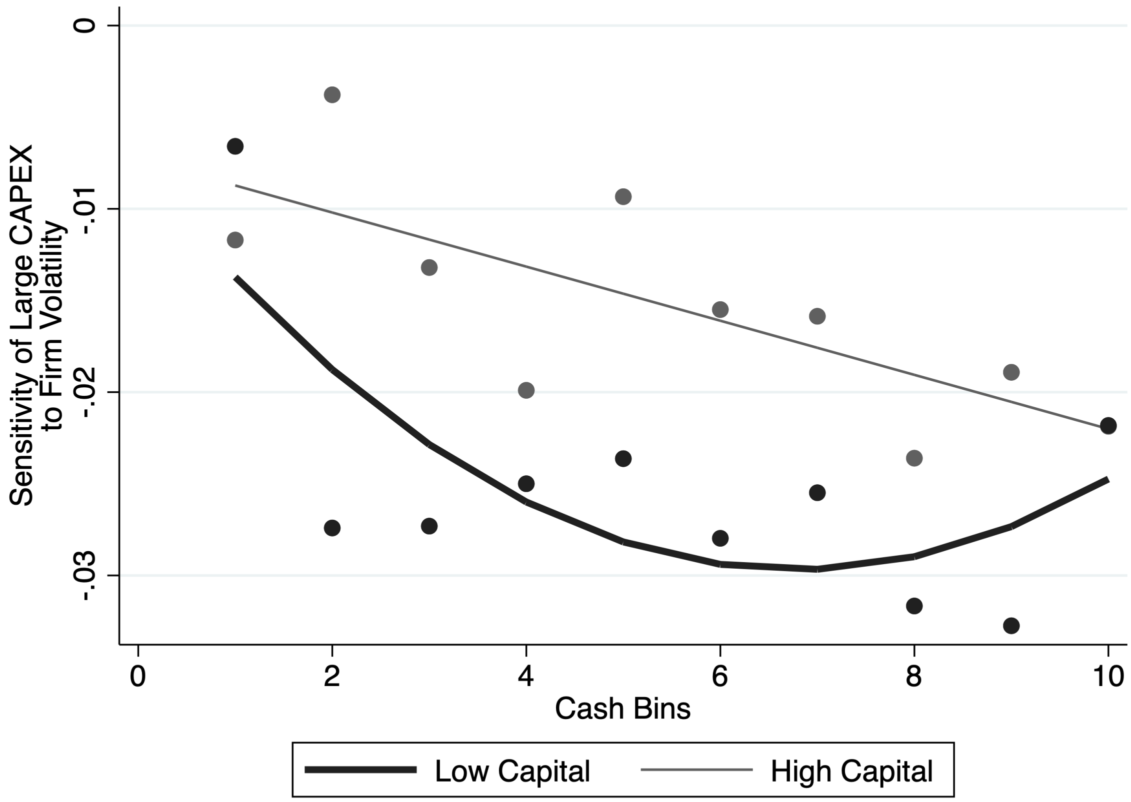
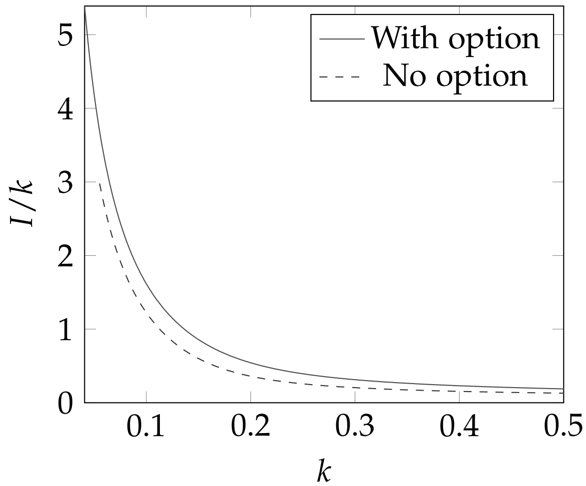


Figure C.7: **Varying the Threshold for Large Investment in Figure 7**
We repeat Figure 7 defining option exercises as net investment rates above the 75th percentile of 13.1%.

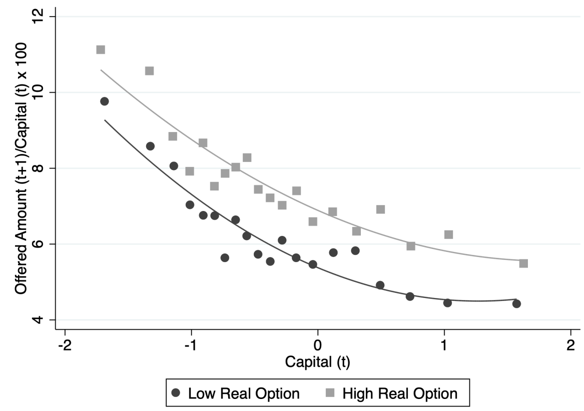
Table C.4: H2: A firm's exercising activity is U-shaped in cash

The outcome is an indicator that equals to one if the firm exercises real options in year t. We proxy for real option exercises with an indicator variable that equals one when a firm's net investment rate exceeds the sample median of 2.6%. A firm's net investment rate is calculated as capital expenditures less depreciation in year t all scaled by capital at the end of year $t-1$. The main explanatory variable is VOL, the average daily stock volatility of firm i in year t. We interact a firm's stock volatility with a firm's lagged cash reserve (standardized within firm) and its square. We control for year fixed effects. *Large* is an indicator that equals to one if a firm's property, plant, and equipment net of depreciation is above the firm's sample mean. Standard errors are clustered by year. *, **, and *** indicate significance at the 10%, 5%, and 1% level, respectively.

	$\mathbb{1}$(Large Net Investment) (i,t+1)	
	(1)	(2)
VOL (i,t)	-0.049***	-0.052***
	(0.006)	(0.007)
Cash (i,t-1)	0.002	0.001
	(0.003)	(0.003)
VOL (i,t) × Cash (i,t-1)	0.000	-0.001
	(0.004)	(0.005)
L.Cash^2 (i,t)	0.007***	0.003
	(0.002)	(0.003)
VOL (i,t) × L.Cash^2 (i,t)	0.002	0.003
	(0.002)	(0.003)
Large		-0.127***
		(0.008)
Large × VOL (i,t)		0.006
		(0.006)
Large × Cash (i,t-1)		0.048***
		(0.005)
Large × VOL (i,t) × Cash (i,t-1)		0.001
		(0.006)
Large × Cash^2 (i,t-1)		-0.010***
		(0.003)
Large × VOL (i,t) × Cash^2 (i,t-1)		-0.003
		(0.003)
Constant	0.481***	0.546***
	(0.002)	(0.004)
Specification	All	All
Quarter FE	Yes	Yes
% Adjusted R^2	10.81	12.55
Observations	84486	84486



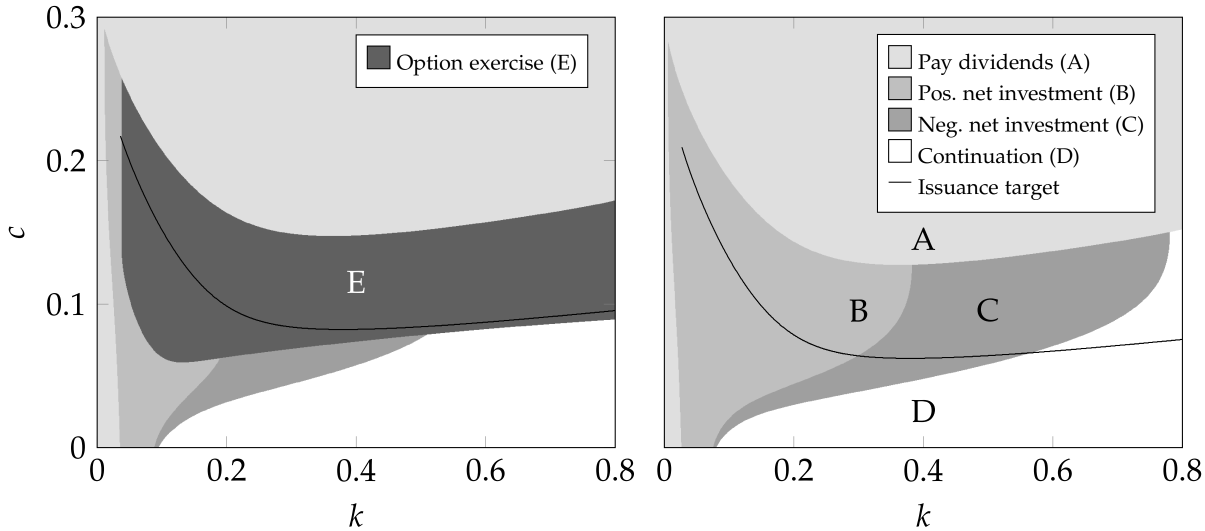
(a) Predicted. The solid red line represents the predicted optimal issuance amount when the firm has a real option. By contrast, the dashed line represents the predicted optimal issuance amount for a firm similar in all respects except without a real option. Evidently, a firm with a real option is predicted to issue more equity conditional on issuance. Intuitively, the firm with a real option raises funds to finance the fixed exercise cost and to have enough precautionary savings after paying the exercise cost.



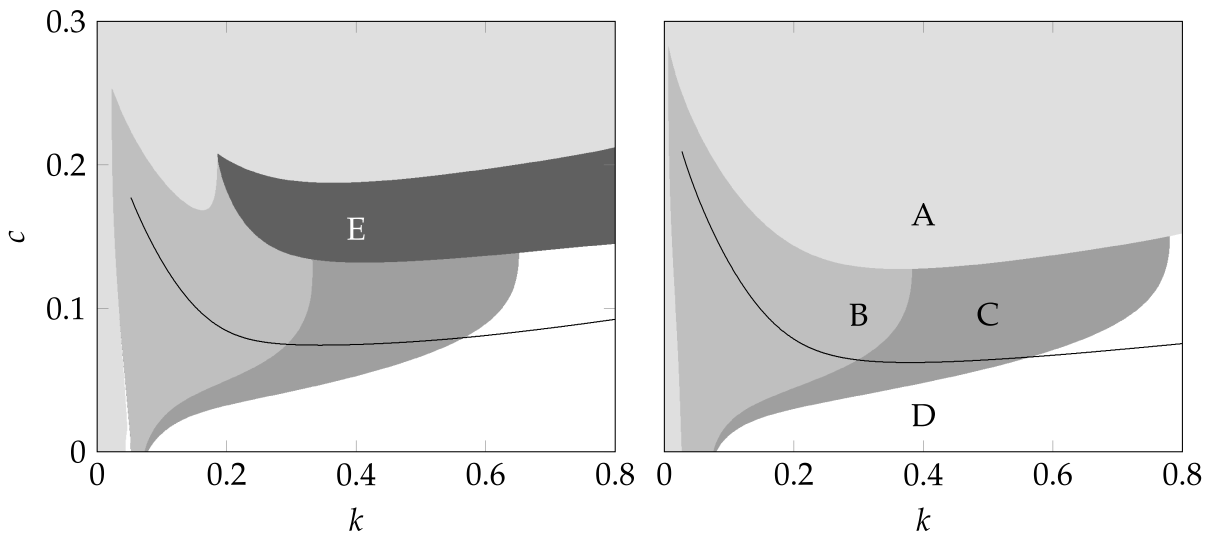
(b) Actual. The offered amount is the value of stock sold in quarter $t+1$ scaled by capital at the end of quarter t. The horizontal axis shows a firm's total capital (tangible and intangible) standardized within firm. The sample is all offerings in quarter $t+1$ with proceeds greater than or equal to 1% of capital stock at the end of quarter t. We split the sample for firms with low and high real options. To do so, we calculate the sensitivity of a stock's returns to the volatility in the returns and compare the stocks with a sensitivity above and below the median sensitivity. The figure clearly shows that for firms with more real options, the amount offered is higher.

Figure C.8: **Predicted vs actual issuance amounts when firms have more/less real options**

Appendix D. Additional Testable Predictions



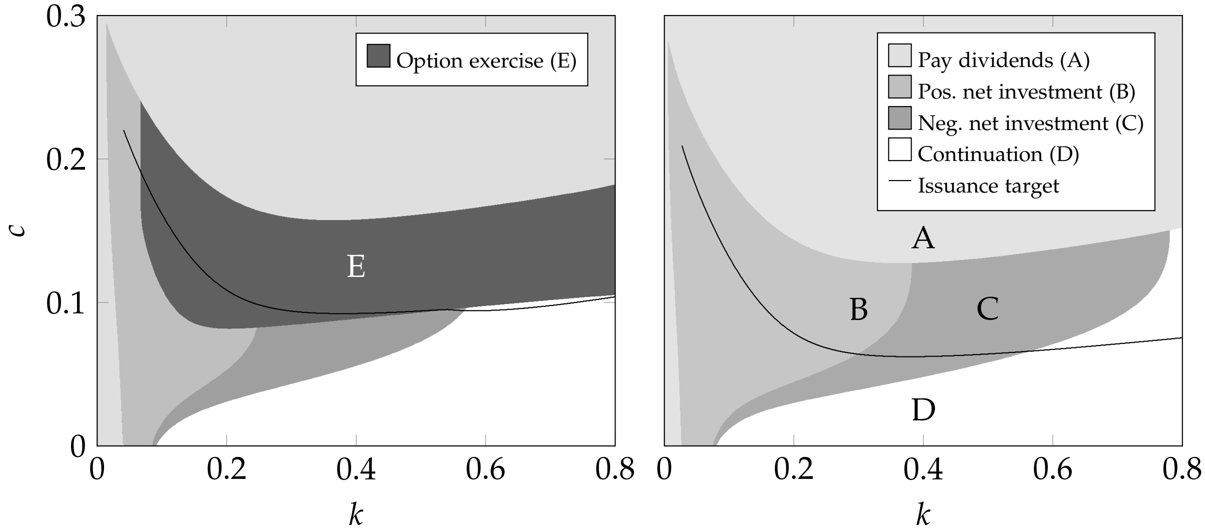
(a) $\Phi = 0.02$



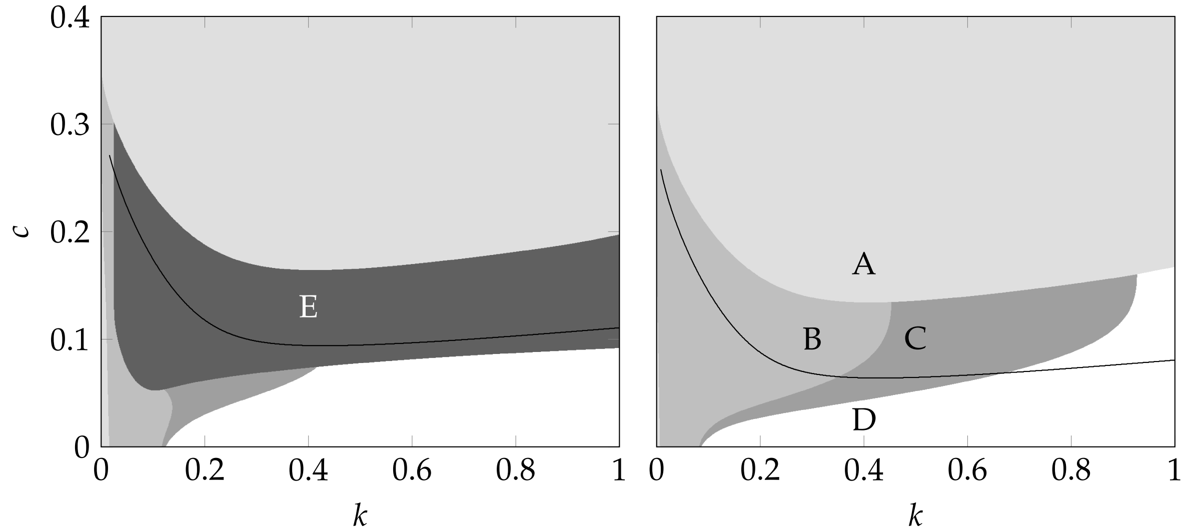
(b) $\Phi = 0.06$

Figure D.1: **Varying the exercise cost of the real option**

Parameters used are summarized in Table 1.



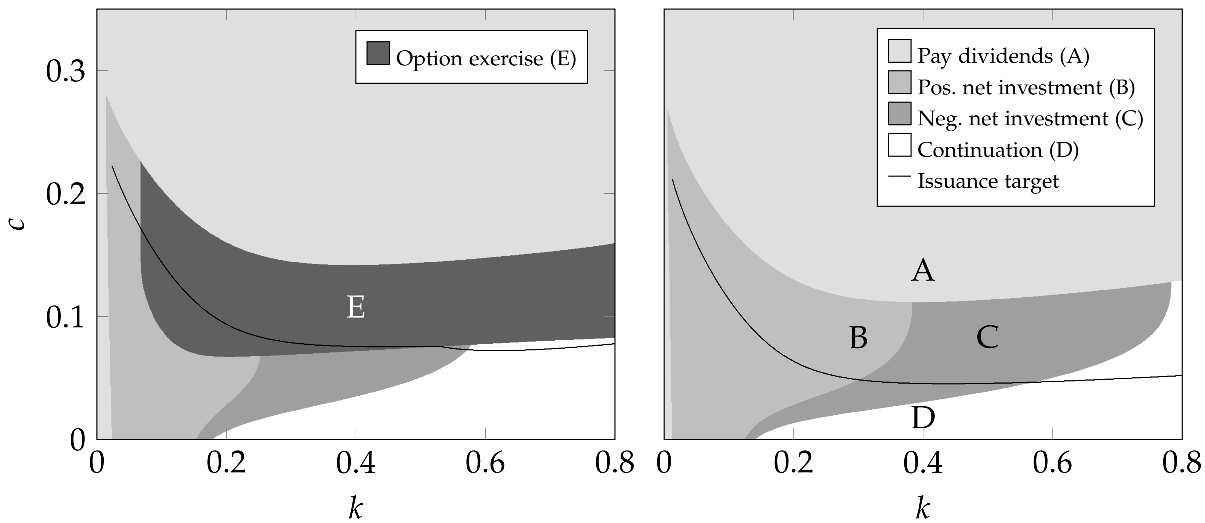
(a) $\tilde{\mu} = 0.19$



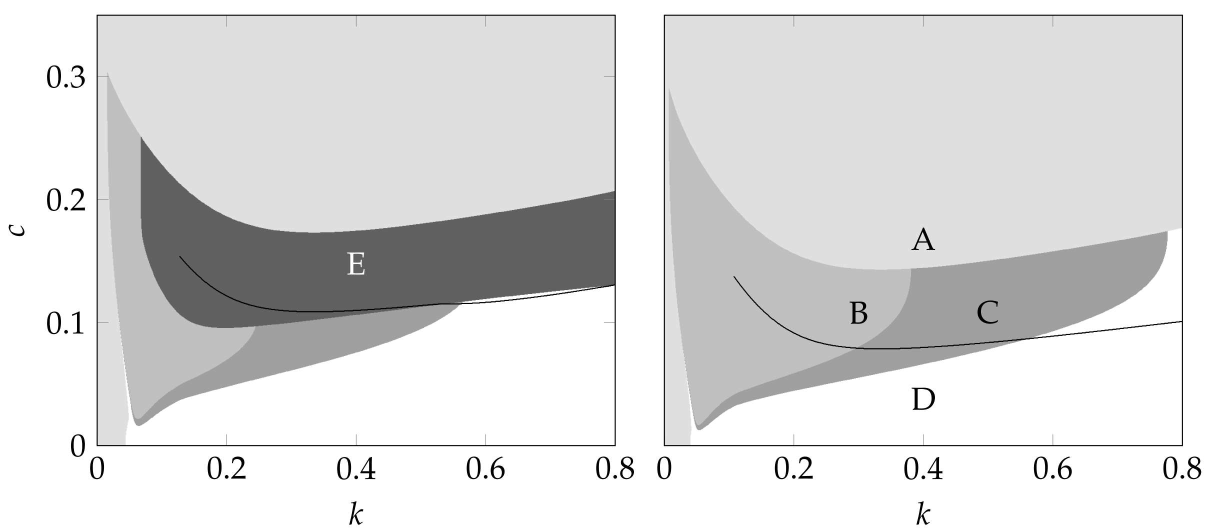
(b) $\tilde{\mu} = 0.20$

Figure D.2: **Varying productivity of the real option**

Parameters used are summarized in Table 1.



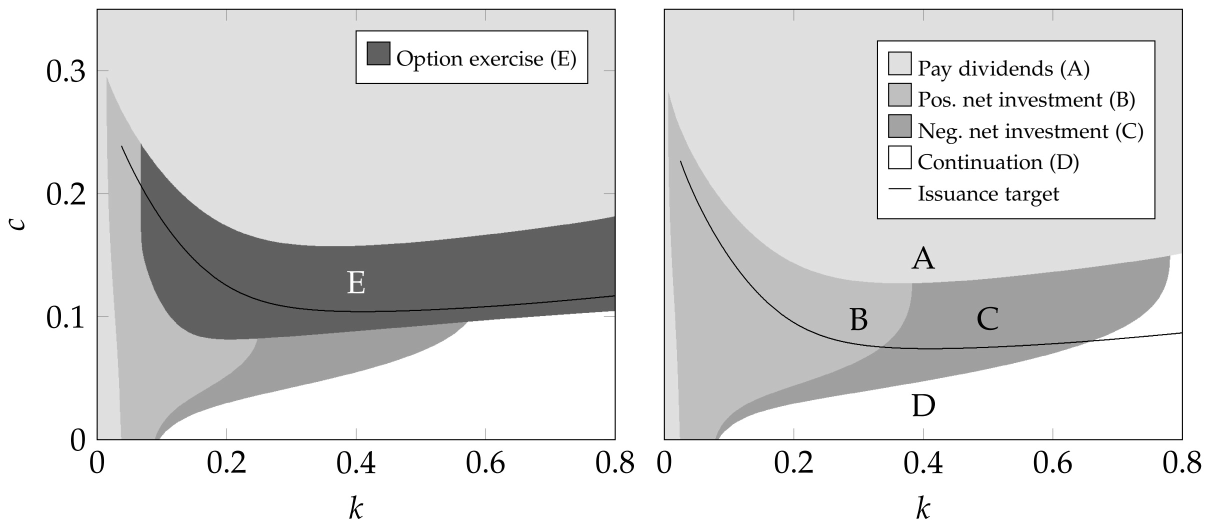
(a) Before and after with $\lambda_f = 0.25\times$



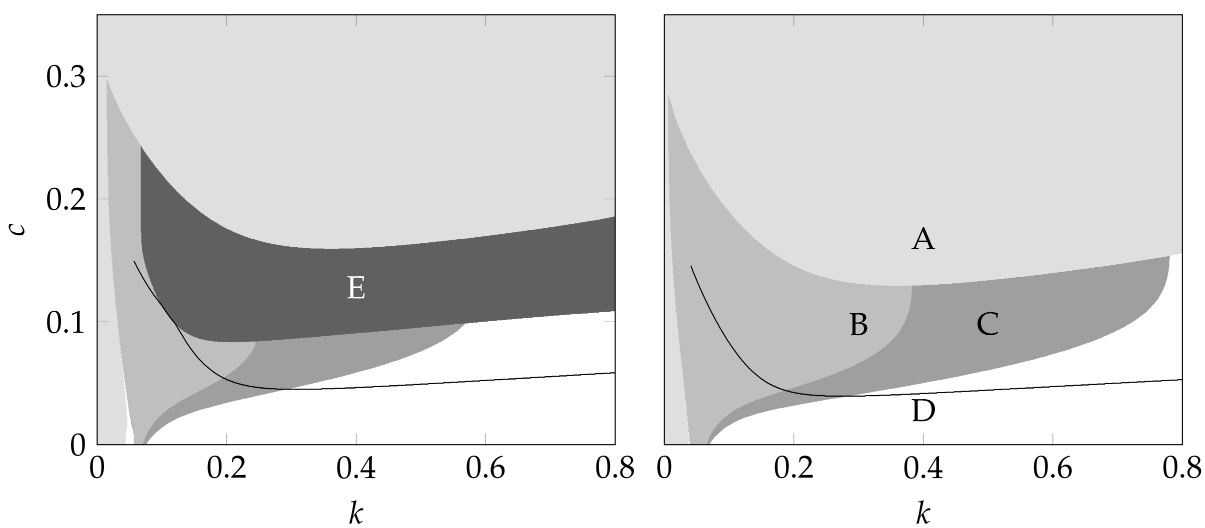
(b) Before and after with $\lambda_f = 4.0\times$

Figure D.3: **Varying Fixed Issuance Costs**

Parameters used are summarized in Table 1.



(a) Before and after with $\lambda_p = 0.5\times$



(b) Before and after with $\lambda_p = 4.0\times$

Figure D.4: **Varying Proportional Issuance Costs**

Parameters used are summarized in Table 1.

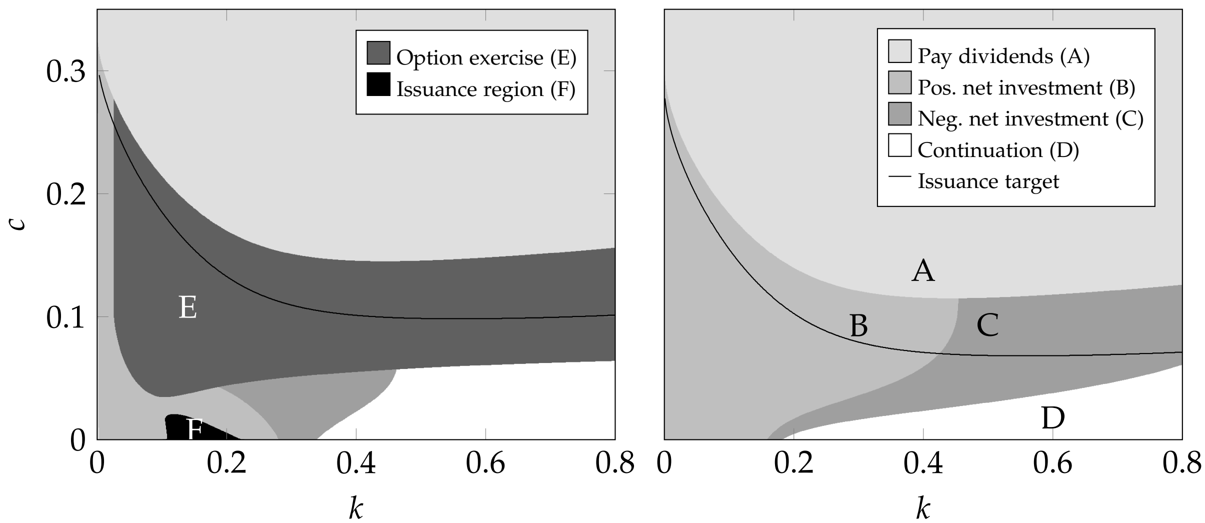
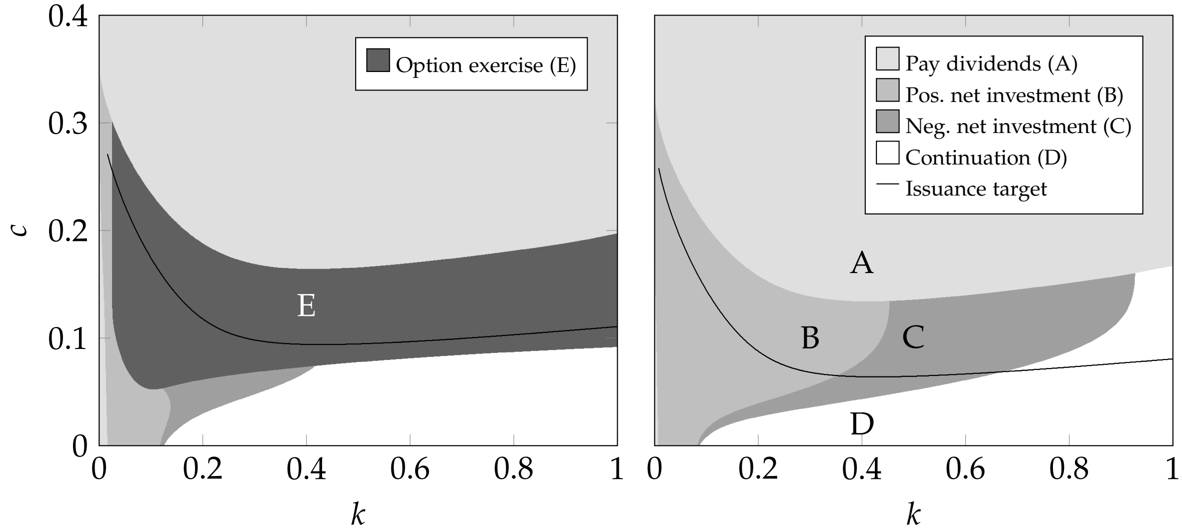


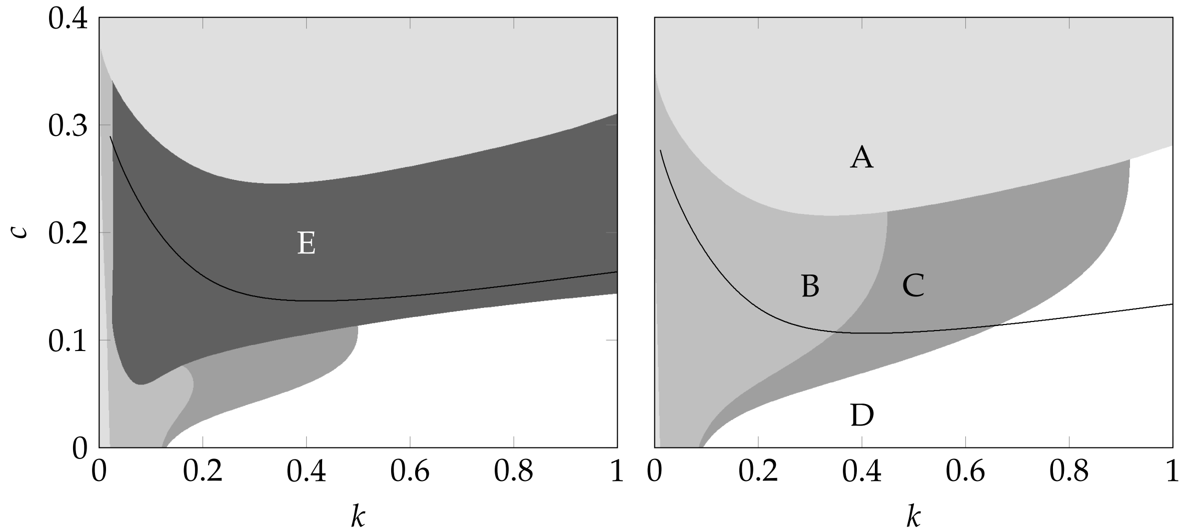
Figure D.5: **The exercise region when λ_f and λ_p are 0.25$\times$ base values and $\tilde{\mu} = 0.2$**

Parameters used are summarized in Table 1.

Figure D.5 lowers both the fixed λ_f and proportional costs λ_p. Interestingly, for sufficiently small issuance costs, a new issuance region emerges. Firms may choose to issue even when the cash reserve is still positive. Thus, when the (k, c) touches the new issuance region, it becomes optimal for the firm to issue equity up to the issuance target and then pay the exercise cost Φ simultaneously. In the figure, the costs are 0.25$\times$ their baseline values, and $\tilde{\mu} = 0.2$ to emphasize the effect. The effect still appears for the baseline $\tilde{\mu}$ value, but requires lower costs to be as clearly visible. Intuitively, when k is not too low and c is low, the likelihood of paying the issuance cost is already high, so the benefits of postponing issuance becomes less attractive relative to immediate issuing and exercising to receive the improved cash flow from the option. That this new region emerges when issuance costs are lower indicates the importance of considering such costs when examining a firm's decision to exercise a real option.



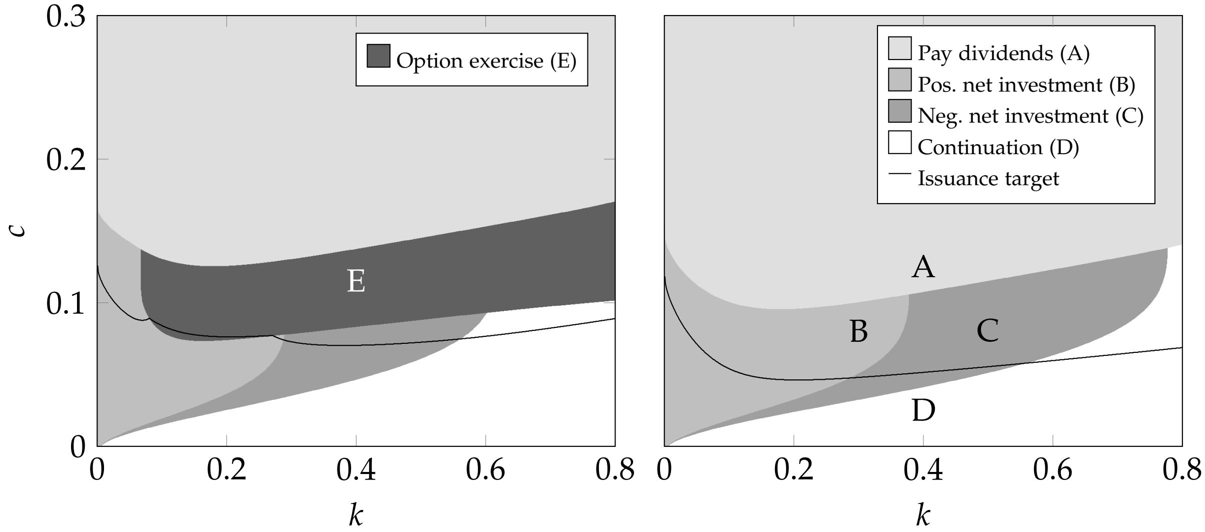
(a) $\tilde{\sigma} = 0.09$. Note that the baseline volatility of the firm before exercise is also $\sigma = 0.09$. Thus, in this figure, exercising the real option increases only productivity from $\mu = 0.18$ to $\tilde{\mu} = 0.20$



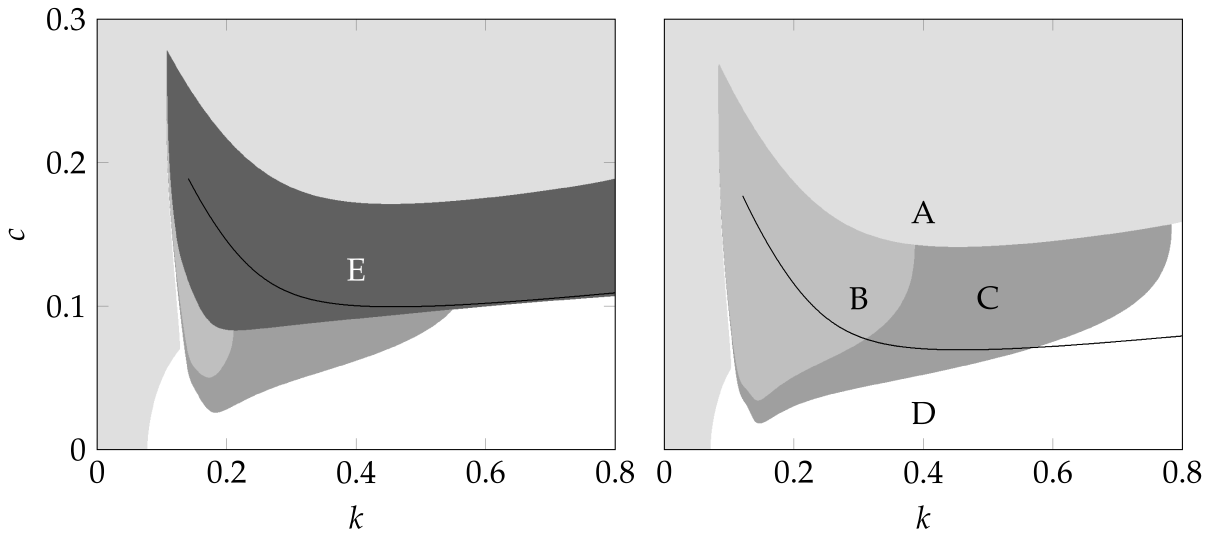
(b) $\tilde{\sigma} = 0.13$. Note that the baseline volatility of the firm before exercise remains $\sigma = 0.09$. Thus, in this figure, exercising the real option increases the volatility of the cash flows from $\sigma = 0.09$ to $\tilde{\sigma} = 0.13$ and the productivity from $\mu = 0.18$ to $\tilde{\mu} = 0.20$.

Figure D.6: **Varying volatility of the real option**

Except where noted differently, the parameters used are summarized in Table 1.



(a) $b = 0.00$



(b) $b = 0.03$

Figure D.7: **Varying firm leverage**

Parameters used are summarized in Table 1.