Index Option Returns and Systemic Equity Risk

Weiping Li,[1]

Tim Krehbiel, corresponding author,[2]

February 10, 2016

[1]University Chair Professor, Southwest Jiaotong University, Chengdu, Sichuan Providence, 610031, P. R. China and Watson Faculty Fellow of Finance and Professor of Mathematics, Oklahoma State University, College of Arts and Sciences, Stillwater, OK-74078, Tel: 405–744–5852, Fax: 405–744–8275, Email: ⟨w.li@okstate.edu⟩, Internet: ⟨http://www.math.okstate.edu/∼ wli/⟩.

[2]Watson Family Chair in Financial Risk Management and MSQFE Director, Department of Finance, Oklahoma State University, Spears School of Business, Stillwater, OK-74078, Tel: 405–744– 8660, Fax: 405–744–5180, Email: ⟨tim.krehbiel@okstate.edu⟩, Internet: ⟨http://spears.okstate.edu/∼tlk⟩.

Abstract

In an environment characterized by stochastic variances and correlations, we demonstrate through construction of the equilibrium index option value from constituent components, that the generalized PDE identifies the stochastic elements differentially affecting index option prices relative to prices of aggregated constituent stock options. A unified treatment of the generalized partial differential system for index and constituent stock options illustrates that nonlinear interactive terms emanating from stochastic correlation affect index option price and return essentially different from contributions to the aggregated risks of the constituent stock options. Our study contributes to the growing evidence of priced correlation risk in markets for index and constituent stock options.

Theorem 1. illustrates the pricing differential, while Proposition 1 illustrates that the pricing differential produces a quantifiable metric of the measure of the nonlinear interactive terms. The quantifiable metric is constructed from the difference between the model free implied variance of the index and a weighted aggregate of the model free implied variances of the constituent stocks. Proposition 2 identifies that index variance risk premium includes additional significant contributions from the nonlinear interactive risks not present in the aggregated returns of the constituent stocks. The nonlinear interactive risks produce a wedge between the instantaneous expected excess index and aggregated stock option returns.

Keywords: Stochastic Correlation; Stock Index Option, Expected excess option return

JEL Classification: G13

1. Introduction

In an environment characterized by stochastic variances and correlations, expected index and index option returns are not simple aggregations of constituent stock and constituent stock option returns. Through the total variance risk premium, expected index return depends on nonlinear interactive risk premiums not present in the aggregation of constituent stock returns. These same nonlinear interactive risk premiums drive a wedge between expected index option excess return and aggregated constituent expected stock option excess returns. The difference between index implied variance and aggregated constituent stock implied variances is the quantification of the measure of the nonlinear interactive risk premium's contribution to the total variance risk premium.

Our paper is consistent with the findings of Cao and Huang(2008) who identify that index option returns cannot be replicated through combinations of the index and a portfolio of equity options. In the stochastic environment in which we develop the system of equilibrium equity and index option pricing relationships (Theorem 1), replication of index option returns would require the underlying index, a portfolio of constituent stock options and knowledge of the state of the correlation array. Given full knowledge of the off diagonal elements of the stochastic correlation array, it still may not be possible to replicate the index option if the correlation array is not positive definite. In proposition 1 of this paper we develop the difference between the index implied variance and aggregated constituent stock implied variances as an indicator of systemic risk which we define as the erosion of diversification benefits corresponding to a not positive definite state of the correlation array.

This study contributes to the branch of index option pricing literature seeking to explain differential pricing of index options vis-á-vis constituent stock options and index returns vis-á-vis constituent stock returns. We demonstrate that stochastic correlation among equity returns is a potential source of two current puzzles in the Finance literature; the differential in index option returns relative to the returns of the constituent company's equity options; and of the index's return relative to the aggregated returns of the constituent stocks. We consider the consequences of stochastic correlation among equity returns, in the context of a stock index and its constituent stocks; the constituent stock options and the index option. In an environment characterized by stochastic variances and correlations among the constituent stock returns, important cross product terms attributable to stochastic correlations produce a wedge between the aggregated expected returns of the constituent stocks and the expected return of the index. In like manner, these same cross product terms drive a wedge between expected index option return and aggregated expected stock option returns.

The current literature includes many studies addressing index, index option and stock option returns. Constantinides, Jackwerth and Savov (2013) conclude that a crisis related jump in the market index or index volatility explains the cross section of index option returns. In a sequence of articles, Driessen, Maenhout and Vilkov (2009, 2013) examine evidence in support of priced correlation risk, first in the context of index option and aggregated constituent stock option implied variances, then in the context of a specific stochastic correlation model. They produce evidence of a large negative correlation risk premium through examination of the discrepancy between implied index and realized correlation. Our evaluation of the difference between the DowJones implied variance and aggregated constituent stock implied variances is consistent with statistically significant contributions to the index variance risk premium from nonlinear interactive risk premiums emanating from stochastic correlation.

While we are not the first to consider uncertainty in the form of Brownian motions for stock prices, volatilities and correlations; see Driessen, Maenhout and Vilkov (2009, 2013), to our knowledge, we are the first to emphasize that index option prices are affected differentially relative to prices of individual stock options by interaction terms resulting from stochastic return correlations. In the proposed environment, we derive the system of equilibrium partial differential equations which we use as the basis to illustrate the differential between index option expected return and aggregated stock option expected return.

The literature contains numerous studies addressing the observed differential in index and stock option pricing. Coval and Shumway (2001) conjecture that factors in addition to the underlying index contribute to the expected return of index options, reaching the conclusion that in addition to the underlying index, changes in volatility, jumps in volatility and market liquidity all materially affect index option expected returns. Bollen and Whaley (2004) conclude that systematic difference in the implied volatility function of index options and individual stock options is due to price pressure emanating from alternate sources of excess demand. Bakshi, Kapadia and Madan (2003) also document differential pricing of individual equity options vis-á-vis index options, they argue the joint presence of risk aversion and kurtotic physical return distributions generates greater absolute value of negative skewness in the risk neutral index return distribution relative to the risk neutral distributions of individual stock returns.

Almost as numerous are the studies addressing the apparent excess returns earned by short positions in out-of-the-money index put options. Broadie et al (2009) conclude that large average returns for OTM index puts are consistent with benchmark returns computed from either the Black-Scholes or Heston stochastic volatility models. To simulate paths of the underlying index they calibrate an affine-jump diffusion to sample index returns. Their simulation methodology addresses the non-normality of option returns, limited sample size, and non-zero expected option return. They demonstrate that average returns, CAPM alphas, and Sharp ratios for deep OTM put options are not statistically different than what is expected based on benchmark option pricing models. In contrast, Chambers et al (2014) conclude that index put options exhibit statistically significant risk premiums relative to model benchmarks.

Our study addresses the puzzling pricing differential and index option excess returns by extending Heston's stochastic volatility model (1993) to a system of constituent stocks linked through a stochastic correlation array. We make three significant contributions.

First, Proposition 2 characterizes contributions to expected index return from the risk premium attributable to the instantaneous change of index variance. This risk premium is shown to include interactive risk premia among the stock risks and correlation risks which are lacking in the variance risk premium of the individual stocks.

Theorem 1 in conjunction with Proposition 2 indicates that the contribution to expected excess index option return from index option elasticity and index variance risk premium includes additional interactive and logarithmic risk premia arising from stochastic correlation that are identical to the risk premia that differentiate expected excess index return from aggregated expected stock excess return.

Second, it is shown in Proposition 1 that under the risk neutral measure, the expectation of the interactive and stochastic variance terms which differentially affect index option prices is calculated by the difference between the model free implied variance of index options and the weighted average model free implied variance of component stock options. A test of the time series properties of this difference is consistent with statistically significant incremental contributions of stochastic correlation and interactive terms to the price of index options vis-a-vis individual stock options.

Third, invertibility of the stochastic correlation matrix is linked to the magnitude of this difference. We label the difference in implied variances a systemic risk indicator. In a period where the systemic risk indicator (value typically negative) approaches zero from below the benefit of diversification diminishes as return correlations approach unity.

In this paper, we treat the index as a system of individual stocks characterized by stochastic stock return variances and correlations. In Theorem 1, we show that there is a generalized partial differential equation system that unifies the option pricing theorems for both individual stocks and the index as a weighted average of individual stocks. The partial differential equation system, clearly identifies the differential role of interactive terms and stochastic correlations for index option and component stock option pricing and hedging. Thus individual option prices contain a term that correlates with all other individual stocks, and new terms from both the first and second order option sensitivities (Greeks) for the variances from each individual stock. On the other hand, the index option price is impacted by stochastic correlations through a term with weighted correlations among all individual stocks, and the first and second order option sensitivities for the variance of the index.

Note that Driessen et al (2009) state on page 1378 that a risk-based explanation for the contrast between index and individual options requires that aggregated individual processes be exposed to a risk factor that is lacking from the individual processes. Our results in understanding the difference between index and individual stock options identifies exactly these risk-based quantities from interactive cross-risk premia.

We analyze the role of risk premiums for the index in Proposition 2, and show that the index risk premium reflects all risk premiums from individual stocks, all risk premium from each interaction between different stocks and more interactive risk premia among stock risks and correlation risks. Those interactive risk premia are the nonlinear contributions for the risk premium of the index option. The instantaneous excess index option return can be expressed explicitly from Theorem 1 and Proposition 2 to have the nonlinear cross terms. Our results (Proposition 2), in this paper not only offer the answer to Driessen et al (2009)'s puzzle, but also specify the precise risk factors which produce the difference between expected index and individual option returns. The precise risk factors are the correlation and interactive risk premia that are lacking from the individual processes. The correlation and interactive risk premia fill the gap between expected index and individual option returns. We show empirically the statistical significance of the contribution from correlation and interactive terms using the model-free implied variance estimation method. The index option always has a valuable hedge against correlation increases and insures against the interactive risks due to loss in diversification benefits. Our results in this paper offer a novel insight on the source of various volatility risk premia in the index options from the vantage point of the equilibrium partial differential equation system.

Our model specifies a stock index as a system of stochastically correlated component Ito processes with each processes' instantaneous variance itself an Ito process. It is the term for both first and second order partial derivatives of the index variance that contain many additional interactive variance contributions, illustrated in Proposition 2, that differentiates the index option price and return from component stock option prices and returns. The generalized Black-Scholes-Merton partial differential system (Theorem 1) is under-determined in the correlation terms. Thus it is technically difficult to extract the correlations from the option price and return of the index and of the component stocks[1].

Furthermore, we define a systemic risk indicator for the index and its components. The systemic risk indicator is the basis for empirical tests of Proposition 2 which provide evidence consistent with statistically significant differential contributions of stochastic correlations and interactive terms to index option and component stock option returns. Our statistical test of the contribution of correlation and interactive terms to the index's variance risk premium is based on the difference between the model free implied volatility of the index and the weighted average of the model free implied volatilities of the component stocks. We present evidence of the statistical significance of this difference for options written on the Dow Jones Average and options written on the component stocks of the average. The difference between index model free implied volatility and the weighted average of component stock model free implied volatilities is linked to the positive definiteness of the stochastic correlation matrix and is proposed as an indicator of systemic risk.

Proposition 1 demonstrates that the difference between the model-free implied variance of the index and the sum of weighted model-free implied variances of component stocks is exactly given by the risk neutral expectation of the weighted stochastic correlations. Hence the model-free implied

[1] This is the major difficulty to verify the precise contributions for each individual term (11), (12) and (13) empirically. We will develop a different method to extract these terms in a future study.

variance of the index is equivalent to a weighted sum of the component stock's model-free implied variances if and only if the expectation of all stochastic correlations and interactive terms vanishes. Using the model-free implied variance estimation method of Britten-Jones and Neuberger (2000), Jiang and Tian (2005) and a sample of Dow Jones Index options and options on component stocks, we test the significance of the risk neutral expectation of weighted stochastic correlations. The data strongly rejects the null hypothesis that the difference between model free implied index variance and weighted model free implied component variances is on average zero.

Driessen et al (2009) show empirically that correlation risk is priced in the sense that assets (index options) that pay off well when market-wide correlations are greater than expected earn negative excess returns. Their analysis poses a challenge for index option pricing. In the authors' words: " By considering individual options on all index components, our analysis emphasizes that a challenge in option pricing concerns explaining the difference between expected index and individual option returns. This is challenging since the index process is the weighted average of the individual processes." Under the stochastic assumptions of their model, we demonstrate that both the level of return variance and the instantaneous change of return variance contribute to expected index return through separate but related risk premiums. The contribution to expected return of the instantaneous change of return variance emanates from specifying the index as a correlated system of component stocks where each component's price and instantaneous variance evolve as Ito processes. These stochastic assumptions produce interactive risk premia from the product rule in the stochastic calculus that resolve the seeming puzzle; (see Appendix C). We show that the interactive terms contributing to the total variance risk premium of the index have important implications for pricing and hedging of options.

It has long been recognized that empirical implications of the Black-Scholes Merton model are inconsistent with observed option market prices. Models relaxing the original stochastic specification through incorporation of stochastic volatility and/or jumps have been suggested as an improvement upon the original BSM assumptions. Heston (1993) provides a closed form model for the value of an option written on an underlying stochastic process which incorporates a square-root stochastic volatility process. Duffie et al (2000) generalize this approach to a broader class of affine models which can be expanded to specify a double-jump process for both the equity price and its stochastic variance. Bakshi, Cao and Chen (1997) evaluate a set of candidate index option pricing models with respect to the criteria of internal consistency of implied parameters, out-of-sample pricing and hedging performance. Their study, applied to $S\&P$ 500 index options, found that each of the models produces significantly different implied volatility estimates across the range of strike prices thereby failing the test of internal consistency. However the stochastic volatility specification produced the greatest improvement relative to the BSM model for out-of-sample pricing and hedge performance. Adding jumps to this model further improved hedging performance for short dated options. The single asset case considered in these studies is addressed in Corollary 1.

Recently other authors have begun to explore the role of stochastic correlation in financial applications. Buraschi et al (2010) explore the importance of stochastic correlations in a portfolio choice model incorporating both stochastic volatilities and correlations modeled as continuous time Wishart diffusion processes. In this environment optimal portfolios include positions hedging against stochastic volatility and stochastic correlation. Studies of international portfolio diversification have long recognized a role for stochastic correlation. Longin and Solnik (1995) reject the null hypothesis of constant correlation of return among international equity markets. Ball and Torous (2000) found that the estimated correlation structure is dependent on economic policies, the level of capital market integration and relative business cycle conditions. They conclude that ignoring stochastic correlation can easily imply erroneous portfolio choice and risk management decisions.

The rest of the paper is organized as follows. In Section 2, we first show that index and constituent stock option prices are governed by a generalized Black-Scholes-Merton differential system. Some special cases for the generalized Black-Scholes-Merton differential system are investigated. In

Section 3, we show the necessary and sufficient condition on the nontrivial correlation risks as the difference between the model-free implied variance of index returns and the weighted model-free implied variance of component stock returns. In Section 4, we verify empirically the nontrivial correlation risks from Proposition 1. In Section 5, we classify market variance risks and list all non-trivial contributions to risk premia, as well as identify the idiosyncratic risk and risks from jumps. We conclude in Section 6.

2. Equity Risks and Returns

Driessen, Maenhout and Vilkov (2009) indicated that it is challenging in option pricing to understand the difference between expected index and individual option returns. " This is challenging since the index process is the weighted average of the individual processes." Examining the generalized Black-Scholes-Merton differential system for the index and constituent options, we resolve this challenging puzzle with stochastic correlation risks. The stochastic correlation risks link various index option sensitivities (Greeks) and constituent stock option sensitivities.

Suppose that $d\sigma_i(t) = \mu_{i,\sigma}(t)dt + \nu_{i,\sigma}(t)dB_{\sigma_i}(t)$. Each stock's instantaneous variance $\sigma_i^2(t)$ follows an Ito process and with respect to the risk-neutral measure Q,

$$d(\sigma_i^2(t)) = E^Q[d(\sigma_i^2(t))] + \zeta_i(\sigma_i)dB_{\sigma_i}(t), \tag{1}$$

where $E^Q[d(\sigma_i^2(t))] = 2\sigma_i(t)\mu_{i,\sigma}(t) + \nu_{i,\sigma}^2(t)$ and $\zeta_i(\sigma_i) = 2\sigma_i(t)\nu_{i,\sigma}(t)$ follows from the Ito lemma, $E^Q[dB_j(t)dB_{\sigma_i}(t)] = 0$ and$E^Q[dB_{\sigma_i}(t)dB_{\sigma_j}(t)] = 0$. Since $B_j(t)$ is the Brownian motion driving the stock $(j, S_j(t))$, the stochastic volatility $\sigma_i(t)$ drives by another Ito process which is independent from the Brownian motion $B_i(t)$. Also with correlation $\rho_{ij}(t)$ assumption to be an Ito process, we assume that $E^Q[dB_{\sigma_i}(t)dB_{\sigma_j}(t)] = 0$.

Let C_α be the option price for the corresponding stock (α, S_α) with $\alpha \in \{1, 2, \cdots, n, I\}$. We consider the the index $S_I(t)$ and each individual stock $S_i(t)$ as underlying assets. The index $S_I(t)$ is given by a weighted sum of stocks, $S_I(t) = \sum_{i=1}^n w_i S_i(t)$, where w_i is constant for simplicity[2]. There are n copies of driving forces $B_i(t)$ for each individual stock and n copies of the stochastic variances $\sigma_i^2(t)$ for each individual stock. We consider the system is correlated to each other from one stock to another, hence there are $\frac{n(n-1)}{2}$ copies of correlations $\rho_{ij}(t)$ which are Ito processes as well.

We assume the market evolves as under those Ito processes with the following assumptions:

1. Assume that $\frac{dS_i(t)}{S_i(t)} = E^Q[\frac{dS_i(t)}{S_i(t)}] + \sigma_i(t)dB_i(t)$ for $i \in \{1, 2, 3, \cdots, n\}$.

2. Assume that $d\sigma_i(t) = \mu_{i,\sigma}(t)dt + \nu_{i,\sigma}(t)dB_{\sigma_i}(t)$. The volatility $\sigma_i^2(t)$ is an Ito process, denoted by $d(\sigma_i^2(t)) = E^Q[d(\sigma_i^2(t))] + \zeta_i(\sigma_i)dB_{\sigma_i}(t)$ for $i \in \{1, 2, 3, \cdots, n\}$, where $E^Q[d(\sigma_i^2(t))] = 2\sigma_i(t)\mu_{i,\sigma}(t) + \nu_{i,\sigma}^2(t)$ and $\zeta_i(\sigma_i) = 2\sigma_i(t)\nu_{i,\sigma}(t)$.

3. Assume that the instantaneous correlation between Brownian motions $B_i(t)$ and $B_j(t)$ for stocks $(i, S_i(t))$ and $(j, S_j(t))$ satisfies $E^Q[dB_i(t)dB_j(t)] = \rho_{ij}(t)dt$ and the covariance matrix $(\sigma_i(t)\sigma_j(t)\rho_{ij}(t))_{1\leq i,j\leq n}$ is positive-definite for almost all t.

4. Assume that the instantaneous correlation between Brownian motions $B_{\sigma_i}(t)$ and $B_{\sigma_j}(t)$ for variances $(i, \sigma_i^2(t))$ and $(j, \sigma_j^2(t))$ satisfies $E^Q[dB_{\sigma_i}(t)dB_{\sigma_j}(t)] = 0$ for $i \neq j$.

5. Assume that the instantaneous correlation between Brownian motions $B_{\sigma_i}(t)$ and $B_j(t)$ for a variance $(i, \sigma_i^2(t))$ and a stock $(j, S_j(t))$ satisfies $E^Q[dB_j(t)dB_{\sigma_i}(t)] = 0$ for $i \neq j$ and [3] $E^Q[dB_i(t)dB_{\sigma_i}(t)] = \rho_i(t)dt$.

Theorem 1. *Under the above assumptions, we have the* **generalized Black-Scholes-Merton**

[2] w_i depends on fluctuating outstanding shares or on the stock price levels can be treated similarly, see the footnote in Appendix A.

[3] $E^Q[dB_i(t)dB_{\sigma_i}(t)] \neq 0$ reflects the volatiiity "leverage effect" from empirical analysis that the volatility of a stock increases while the stock produces a negative return.

partial differential system, *for* $\alpha \in \{1, 2, \cdots, n\}$,

$$rC_\alpha dt = \frac{\partial C_\alpha}{\partial t}dt + \sum_{i=1}^{n} S_i(t)\frac{\partial C_\alpha}{\partial S_i(t)} \cdot E^Q[\frac{dS_i(t)}{S_i(t)}] + \sum_{i=1}^{n} \frac{\partial C_\alpha}{\partial \sigma_i^2(t)} E^Q[d(\sigma_i^2(t))]$$

$$+\frac{1}{2}\sum_{i,j=1}^{n}\left(S_i(t)S_j(t)\frac{\partial^2 C_\alpha}{\partial S_i(t)\partial S_j(t)} \cdot (\sigma_i(t)\sigma_j(t)\rho_{ij}(t)dt)\right)$$

$$+\frac{1}{2}\sum_{i=1}^{n}\left(S_i(t)\sigma_i(t)\frac{\partial C_\alpha^2}{\partial S_i(t)\partial \sigma_i^2(t)}(\zeta_i(\sigma_i)\rho_i(t)) + \zeta_i^2(\sigma_i)\frac{\partial^2 C_\alpha}{\partial \sigma_i^2(t)\partial \sigma_i^2(t)}\right)dt,$$

and the option of the index follows the differential relation

$$rC_I(t)dt = \frac{\partial C_I}{\partial t}dt + \frac{\partial C_I}{\partial S_I} \cdot \sum_{i=1}^{n} w_i S_i E^Q[\frac{dS_i(t)}{S_i(t)}] + \frac{\partial C_I}{\partial \sigma_I^2} E^Q[d\sigma_I^2(t)]$$

$$+\frac{1}{2}\frac{\partial^2 C_I}{\partial S_I^2}\cdot\sum_{i,j=1}^{n}(w_i w_j S_i(t)S_j(t)(\sigma_i(t)\sigma_j(t)\rho_{ij}(t)dt)) + \frac{\partial^2 C_I}{\partial S_I \partial \sigma_I^2}E^Q[dS_I(t)d\sigma_I^2(t)] + \frac{1}{2}\frac{\partial^2 C_I}{\partial \sigma_I^2(t)\partial \sigma_I^2(t)}E^Q[d\sigma_I^2(t)d\sigma_I^2(t)],$$

with $C_\alpha(T) = \max(0, S_\alpha(T) - K_\alpha)$ *and* $C_I(T) = \max(0, S_I(T) - K_I)$ *for* $\alpha \in \{1, 2, \cdots, n\}$[4], *where* $E^Q[d\sigma_I^2(t)]$ *is given in Proposition 2, and* K_α *and* K_I *are the strike prices for individual stock and the index respectively.*

The instantaneous expected excess option returns of the stock $(\alpha, S_\alpha(t))$ *is given by*

$$\frac{1}{dt}E_t^P\left[\frac{dC_\alpha}{C_\alpha} - rdt\right] = -\sum_{i=1}^{n}\Omega_i^\alpha(t) \cdot (E^Q - E^P)[\frac{dS_i(t)}{S_i(t)}] - \sum_{i=1}^{n}\frac{1}{C_\alpha}\frac{\partial C_\alpha}{\partial \sigma_i^2(t)}(E^Q - E^P)[d(\sigma_i^2(t))],$$

where $\Omega_i^\alpha(t) = \frac{S_i(t)}{C_\alpha}\frac{\partial C_\alpha}{\partial S_i(t)}$ *is the elasticity of the call option of the underlying stock* $(\alpha, S_\alpha(t))$ *with respect to* $S_i(t)$ *for* $\alpha \in \{1, 2, \cdots, n\}$, *and the instantaneous expected excess option returns of the*

[4]The boundary condition for American options will be replaced by free-boundary problems. Define the stopping time $\tau_{b_\alpha} = \inf\{0 \le s \le T : S_\alpha(t) \le b_\alpha(t)\}$ for the optimal stopping boundary $b_\alpha : [0, T] \to R$. The free boundary condition for American options is given by (1) $C_\alpha(t, x) = (S_\alpha - K_\alpha)^+$ for $x = b_\alpha(t)$, (2) $\frac{C_\alpha(t,x)}{\partial x}(t, x) = -1$ for $x = b_\alpha(t)$ smooth fit, (3) $C_\alpha(t, x) > (S_\alpha - K_\alpha)^+$ in the continuation region $\{(t, x) \in [0, T] \times R_+ | x > b_\alpha(t)\}$ and (4) $C_\alpha(t, x) = (S_\alpha - K_\alpha)^+$ in the stopping region $\{(t, x) \in [0, T] \times R_+ | x < b_\alpha(t)\}$ for $\alpha = 1, 2 \cdots n, I$. See Myneni (1992) for the survey and details).

index is given by

$$\frac{1}{dt}E_t^P\left[\frac{dC_I}{C_I}-rdt\right]=-\Omega_I^I(t)(E^Q-E^P)[\frac{dS_I(t)}{S_I(t)}]-\frac{1}{C_I}\frac{\partial C_I}{\partial\sigma_I^2(t)}(E^Q-E^P)[d\sigma_I^2(t)],$$

where $\Omega_I^I(t)=\frac{S_I(t)}{C_I(t)}\frac{\partial C_I(t)}{\partial S_I(t)}$ *is the elasticity of the index option with respect to the index* $S_I(t)$*, and* $(E^Q-E^P)[d\sigma_I^2(t)]$ *is the risk premium of the index variance given in Proposition 2.*

The proof of Theorem 1 is given in Appendix A. Theorem 1 is the first result to study the whole market system under the correlation assumption and the stochastic volatilities for each individual stock price. It gives a consistency $n+1$ differential relations among the options prices $C_\alpha, \alpha\in\{1,2,\cdots,n,I\}$ depending upon those Ito processes for both individual stock, volatility and the correlations. Theorem 1 also provides analytical evaluations of the whole market system on expected returns. It ties closely with the risk premium of the index variance. For each individual option C_α, the essential point is that own stochastic volatility and stochastic correlations are integrated in the equilibrium pricing relationship.Under the market assumptions (4) and (5), the contributions from $\frac{\partial^2 C_\alpha}{\partial\sigma_i^2(t)\partial\sigma_j^2(t)}d(\sigma_i^2(t))d(\sigma_j^2(t))$ and $\frac{\partial^2 C_\alpha}{\partial\sigma_i^2(t)\partial S_j(t)}d(\sigma_i^2(t))dS_j(t)$ vanish under the risk-neutral measure for $i\neq j$. Hence the correlation $\rho_{ij}(t)$ only contributes in the term from the Assumption (3). It is important to notice the difference between the index and the individual stock option relations. For the index option pricing relation, the correlation $\rho_{ij}(t)$ not only contributes in the term $\frac{1}{2}\frac{\partial^2 C_I}{\partial S_I^2}\cdot\sum_{i,j=1}^n w_iw_jS_i(t)S_j(t)\left[\sigma_i(t)\sigma_j(t)\rho_{ij}(t)dt\right]$, but also nontrivially in the term $\frac{\partial C_I}{\partial\sigma_I^2}E^Q[d\sigma_I^2(t)]$ from Proposition 2 and the last term $\frac{\partial^2 C_I}{\partial\sigma_I^2(t)\partial\sigma_I^2(t)}\cdot E^Q[d\sigma_I^2(t)\cdot d\sigma_I^2(t)]$. By Proposition 2, there are other terms in $d\sigma_I^2(t)$ which are not from $\sum_{i=1}^n w_i^2 d\sigma_i^2(t)$. In another words $E^Q[(d\sigma_I^2(t)-\sum_{i=1}^n w_i^2 d\sigma_i^2(t))]$ can be nonzero due to the correlation in (8), (9) and (10) and its nonlinear interactive stochastic volatilities with stochastic correlation in (11), (12) and (13).. Even there are trivial interactive risk and the interactive risk premia, we have nontrivial contribution $E_t^Q[d\rho_{ij}(t)]$ from the basic assumption that the correlations $\rho_{ij}(t)$ are Ito processes. Furthermore the second order option sensitivities for the variance of the index $\frac{1}{2}\frac{\partial^2 C_I}{\partial\sigma_I^2(t)\partial\sigma_I^2(t)}E^Q[d\sigma_I^2(t)d\sigma_I^2(t)]$ contribute more than the second order option sensitivities for the individual stock variance $\frac{1}{2}\sum_{i=1}^n \zeta_i^2(\sigma_i)\frac{\partial^2 C_\alpha}{\partial\sigma_i^2(t)\partial\sigma_i^2(t)}dt$ due to those interactive risks by Proposition 2. These are two different avenues through which stochastic correlations affect the index option pricing different from their affect for the individual option pricing. Based on these, we answer the challenging question proposed by Driessen, Maenhout and Vilkov (2009) on the difference between expected index and individual option returns in the option pricing models in Theorem 1 from the difference $E^Q[d\sigma_I^2] - \sum_{i=1}^n w_i^2 E^Q[d\sigma_i^2] \neq 0$.

The instantaneous expected excess option return of the individual stock $(\alpha, S_\alpha(t))$ not only can be evaluated analytically, but also has new information from the consideration of a whole market system. The elasticity of the call option of an individual stock with respect to the other underlying stock in the same index system contributes in the option returns ($\Omega_i^\alpha(t)$ for $i \neq \alpha$). This new contribution is not studied before, For stochastic variances, the contribution of the risk premium of the same underlying variance is considered in (B6) of Broadie et al (2009). For a whole market system perspective point of view, the relative contributions of risk premiums should also play a role in the excess expected returns ($\frac{\partial \ln C_\alpha}{\partial\sigma_i^2(t)}(E^Q - E^P)[d(\sigma_i^2(t))]$ for $i \neq \alpha$).

The instantaneous expected excess option return of the index $S_I(t)$ explicitly represents the linear factor contributions from the elasticity of the call option of the index with respect to the index, and the risk premium of the index variance $(E^Q - E^P)[d\sigma_I^2(t)]$. It is the risk premium of the index variance given in Proposition 2 that contributes nonlinear interactive risk premium due to the correlation risk. Constantinides et al (2013) show empirically that a large number of additional factors do not succeed in pricing the option returns. The major contribution of the whole market system construction shows analytically that the instantaneous expected excess option return of the index are highly volatile depending on every constituent stock return variance, and nonlinear contributions due to the interactive cross-terms in the risk premium of the index variance (see also discussions after Proposition 2) form the correlation risk.

The quadratic covariance form $\sum_{1\le i,j\le n}\sigma_i(t)\sigma_j(t)\rho_{ij}(t)$ attributes the second order option sensitivities to both the index and individual stock in the generalized Black-Scholes-Merton partial differential system in Theorem 1. Note that $\sigma_w(t) = (w_1\sigma_1(t), \cdots, w_n\phi_n(t))$ as $n \times 1$ vector and

$$\sum_{1\le i,j\le n}\sigma_i(t)\sigma_j(t)\rho_{ij}(t) = \sigma_w(t)^T\Sigma_S\sigma_w(t),$$

for the symmetric correlation matrix $\Sigma_S = (\rho_{ij}(t))_{1\le i,j\le n}$ and A^T stands for the transpose of A. There is an orthogonal matrix $O_{n\times n}(t)(O^{-1}_{n\times n}(t) = O^T_{n\times n}(t))$ such that

$$O_{n\times n}(t)\Sigma_S O^T_{n\times n}(t) = diag(\lambda_1(t), \cdots, \lambda_n(t)),$$

where $\lambda_i(t)$ is positive for almost all t on a diagonal matrix. Let $\sigma^0_w(t) = O_{n\times n}(t)\sigma_w(t) = (\psi_1(t), \cdots, \psi_n(t))$. We obtain that

$$\begin{aligned}
\sigma_w(t)^T\Sigma_S\sigma_w(t) &= \sigma_w(t)^T O^T_{n\times n}(t)(O_{n\times n}(t)\Sigma_S O^T_{n\times n}(t))O_{n\times n}(t)\sigma_w(t) \\
&= (O_{n\times n}(t)\sigma_w(t))^T diag(\lambda_1(t), \cdots, \lambda_n(t))(O_{n\times n}(t)\sigma_w(t)) \\
&= \lambda_1(t)\psi_1^2(t) + \lambda_2(t)\psi_2^2(t) + \cdots + \lambda_n(t)\psi_n^2(t).
\end{aligned}$$

If $\sigma_w(t)^T\Sigma_S\sigma_w(t) \le 0$, then there must exist at least one $\lambda_i(t) \le 0$ and the correlation matrix Σ_S becomes no longer positive definite on this time t in a measure zero set. The maximum value of $MFIV_{DJX} - MFIV_{WAVG}$ reported in Table 3 is positive, indicating for at least one date in the sample, the benefit of diversification vanished due to non positive definiteness of Σ_S. The systemic risk indicator from the Proposition 2 consistently appears in Theorem 1 for the generalized Black-Scholes-Merton differential system. In general, even though some $\lambda_i(t) \le 0$, the total sum $\sum_{i=1}^n \lambda_i(t)\psi_i^2(t)$ may remain to be positive. This is exactly what we observe in Figure 4, even though $MFIV_{DJX} - MFIV_{WAVG} < 0$, it can be the case that Σ_S is not invertible. Hence the single factor to detect the degeneracy of the positivity of the correlation matrix Σ_S, is the determinant of Σ_S

$$\det \Sigma_S = \lambda_1(t) \cdots \lambda_n(t).$$

From the positivity of Σ_S for almost all t, the determinant $\det \Sigma_S > 0$ holds for almost all t. As long as $\det \Sigma_S \leq 0$ at any moment t (again only measure zero set of such times), we can surely detect the correlation matrix being degenerate. This reflects the systemic risk indicator $\det \Sigma_S$ as a single parameter. Table 4 demonstrates $MFIV_{DJX} - MFIV_{WAVG}$ increases towards zero in periods of market turbulence, implying that Σ_S is not invertible during these periods. that shows that $\sigma_w(t)^T \Sigma_S \sigma_w(t) \leq 0$ can be verified from the model-free implied volatilities for most cases. Thus we find out that the difference between the index model-free implied volatility and weighted average of component stock model-free implied volatilities can serve as *a systemic risk indicator*. Hence we define the term $MFIV_{DJX} - MFIV_{WAVG}$ as **a Systemic Risk Indicator**. The covariance matrices are essential for risk management, asset pricing, proprietary trading and portfolio managements. Assuming non-stochastic ρ, time series data will yield a positive definite correlation matrix, and the puzzle posed by Driessen, Maenhout and Vilkov (2009) remains. In a system with stochastic ρ, the DMV puzzle is attributable to the stochastic correlation affect on component and index option prices. In such a system the stochastic covariance matrix may not be positive definite at all times. Some methods by using a form of local average and/or the historical information can be adapted to approach the matrix Σ_S, but determining its positive definiteness is another difficult issue. In this paper we focus on a systemic risk indicator as the term $MFIV_{DJX} - MFIV_{WAVG}$.

When the index is considered as a single individual stock $n = 1$, there are no constituent stock return correlations, hence we obtain Hestons (1993) stochastic volatility model.

Corollary 1. *Assume the stock $S(t)$ is an Ito process ($\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)dB(t)$) and $\sigma^2(t)$ is an Ito process ($d\sigma^2(t) = \mu_\sigma(t)dt + \zeta(\sigma)dB_\sigma(t)$). Assume that $E^Q[dB(t)dB_\sigma(t)] = \rho(t)dt$. Then the generlized Black-Scholes-Merton differential equation for the option price $C = C(t, S(t), \sigma^2(t))$ is* *given by*

$$rC = \frac{\partial C}{\partial t} + \mu(t)\cdot S(t)\frac{\partial C}{\partial S(t)} + \mu_\sigma(t)\frac{\partial C}{\partial \sigma^2(t)} + \frac{\sigma^2(t)}{2}S^2(t)\frac{\partial^2 C}{\partial S^2(t)} + S(t)\sigma(t)\zeta(\sigma)\rho(t)\frac{\partial^2 C}{\partial S\partial\sigma^2(t)} + \frac{\zeta^2(\sigma)}{2}\frac{\partial^2 C}{\partial\sigma^2(t)\partial\sigma^2(t),} \quad (2)$$

with the terminal condition $C(T) = \max(0, S(T) - K)$ *for the strike price* K *and the maturity date* T*. The instantaneous expected stock option excess return is given by*

$$\frac{1}{dt}E^P\left[\frac{dC}{C} - rdt\right] = \Omega(t)E^P[\sigma(t)dB(t)] + \frac{1}{C}\frac{\partial C}{\partial\sigma^2}E^P[\zeta(\sigma)dB_\sigma(t)],$$

where $\Omega(t) = \frac{S(t)}{C}\frac{\partial C}{\partial S(t)}$ *is the elasticity of the option for the underlying stock* $S(t)$*.*

Equation (2) measures new hedges from the additional stochastic volatility, it provides the new tau-hedge for the stochastic volatility $\frac{\partial C}{\partial\sigma^2(t)}$. The hedge for the volatility $\frac{\partial C}{\partial\sigma^2(t)}$ is totally different from the Vega hedging in the usual Black-Scholes-Merton equation. In the usual Black-Scholes-Merton equation, one assumes that the sensitivity for the covariance is constant and derives the Black-Scholes-Merton formula.[5] From the Black-Scholes-Merton formula, one indicates the volatility as a parameter and computes the first derivative from the formula with respect to the covariance parameter. Our extension is taking the stochastic volatility into the consideration at the beginning. We adapt the stochastic calculus to evaluate the instantaneous change of the option pricing under the assumption that the volatility follows an Ito process. These new contribution $\mu_\sigma(t)\frac{\partial C}{\partial\sigma^2(t)}$ in the equation (2) may lead to simplification of the earlier treatment for the separate implied volatility methods.

Notice that the term $\frac{\partial^2 C}{\partial S(t)\partial\sigma^2(t)}dS(t)d\sigma^2(t)$ contributes into the option price due to the assump-

[5]Latane and Rendelman (1976) first introduced the implied volatility that empirically makes the Black-Scholes-Merton formula fit market prices of options $C^{BS}(S_t, t, K, T, \hat{\sigma}) = \tilde{C}_t$ for the implied volatility $\hat{\sigma}$. In the derivation of the Black-Scholes-Merton model it is assumed that the diffusion coefficient of the driving Brownian motion is a *constant*. This notion is only validated for an extremely restricted class of models. Renault and Touzi (1996) introduced the hedging volatility that equates the Black-Scholes delta with the delta of the real market $\frac{\partial C^{BS}(S_t,t,K,T,\hat{\sigma}_h)}{\partial S_t} = \frac{\partial\tilde{C}_t}{\partial S_t}$. Again the hedging volatility provides systematic bias in this approximation. Canina and Figleswski (1993) shows that the implied volatility based predictors do contains a substantial amount of information on future volatility and are better than times series based method, at the same time, most researches conclude that the implied volatility is a biased predictor.

tion $E^Q[dB(t)dB_\sigma(t)] = \rho(t)dt$ from the stochastic calculus. But the last term $\frac{\sigma^2(t)}{2}S^2(t)\frac{\partial^2 C}{\partial S^2(t)}$ in our extension also plays a different role from the usual Gamma hedging. Since $\sigma^2(t)$ follows an Ito process, the Gamma contribution in the option pricing model is effected by the variance $\sigma^2(t)$ which is no longer constant. Hence the second order Greek on the stochastic volatility $\frac{\zeta^2(t)}{2}\frac{\partial^2 C}{\partial\sigma^2(t)\partial\sigma^2(t)}$ plays an nontrivial role in the hedging with respect to the term $\sigma^2(t)$.

When the index consists of four individual stocks, that is $n = 4$, there are six correlation terms in the covariance matrix. The covariance matrix is given by $(\sigma_i(t)\sigma_i(t)\rho_{ij}(t))_{1\le i,j\le 4} =$

$$\begin{pmatrix} \sigma_1^2(t) & \sigma_1(t)\sigma_2(t)\rho_{12}(t) & \sigma_1(t)\sigma_3(t)\rho_{13}(t) & \sigma_1(t)\sigma_4(t)\rho_{14}(t) \\ \sigma_1(t)\sigma_2(t)\rho_{12}(t) & \sigma_2^2(t) & \sigma_2(t)\sigma_3(t)\rho_{23}(t) & \sigma_2(t)\sigma_4(t)\rho_{24}(t) \\ \sigma_1(t)\sigma_3(t)\rho_{13}(t) & \sigma_2(t)\sigma_3(t)\rho_{23}(t) & \sigma_3^2(t) & \sigma_3(t)\sigma_4(t)\rho_{34}(t) \\ \sigma_1(t)\sigma_4(t)\rho_{14}(t) & \sigma_2(t)\sigma_4(t)\rho_{24}(t) & \sigma_3(t)\sigma_4(t)\rho_{34}(t) & \sigma_4^2(t) \end{pmatrix}$$

is positive-definite for almost all t. We have the **generalized Black-Scholes-Merton partial differential system**, for $\alpha \in \{1, 2, \cdots, 4\}$,

$$rC_\alpha dt = \frac{\partial C_\alpha}{\partial t}dt + \sum_{i=1}^{4} S_i(t)\frac{\partial C_\alpha}{\partial S_i(t)} \cdot E^Q[\frac{dS_i(t)}{S_i(t)}] + \sum_{i=1}^{4}\frac{\partial C_\alpha}{\partial \sigma_i^2(t)}E^Q[d(\sigma_i^2(t))]$$

$$+\frac{1}{2}\sum_{i,j=1}^{4}\left(S_i(t)S_j(t)\frac{\partial^2 C_\alpha}{\partial S_i(t)\partial S_j(t)} \cdot (\sigma_i(t)\sigma_j(t)\rho_{ij}(t)dt)\right)$$

$$+\frac{1}{2}\sum_{i=1}^{4}\left(S_i(t)\sigma_i(t)\frac{\partial C_\alpha^2}{\partial S_i(t)\partial\sigma_i^2(t)}(\zeta_i(\sigma_i)\rho_i(t)) + \zeta_i^2(\sigma_i)\frac{\partial^2 C_\alpha}{\partial\sigma_i^2(t)\partial\sigma_i^2(t)}\right)dt,$$

and the option of the index follows the differential relation

$$rC_I(t)dt = \frac{\partial C_I}{\partial t}dt + \frac{\partial C_I}{\partial S_I} \cdot \sum_{i=1}^{4} w_i S_i E^Q[\frac{dS_i(t)}{S_i(t)}] + \frac{\partial C_I}{\partial \sigma_I^2}E^Q[d\sigma_I^2(t)]$$

$$+\frac{1}{2}\frac{\partial^2 C_I}{\partial S_I^2}\cdot\sum_{i,j=1}^{4} w_i w_j S_i(t)S_j(t)\left[\sigma_i(t)\sigma_j(t)\rho_{ij}(t)dt\right]+\frac{\partial^2 C_I}{\partial S_I \partial \sigma_I^2}E^Q[dS_I(t)d\sigma_I^2(t)]+\frac{1}{2}\frac{\partial^2 C_I}{\partial\sigma_I^2(t)\partial\sigma_I^2(t)}E^Q[d\sigma_I^2(t)d\sigma_I^2(t)],$$

There are five equations for the option pricing for the index system, but six correlation terms $\{\rho_{12}(t), \rho_{13}(t), \rho_{14}(t), \rho_{23}(t), \rho_{24}(t), \rho_{34}(t)\}$. Although there is a constraint for the covariance matrix $(\sigma_i(t)\sigma_i(t)\rho_{ij}(t))_{1\le i,j\le 4}$ to be positive-definite almost surely, we have an underdetermined system for the correlation coefficients. The positive-definite condition is only part of consistency requirements. The consistent or inconsistent conditions for this underdetermined system would be very interesting to study in Finance to further understand the option pricings for the index and its components.

When the index consists of two or three individual stocks, that is $n = 2, 3$, the generalized Black-Scholes-Merton partial differential system is overdetermined system since there are more equations than the number of correlation terms. Determining the correlation coefficients would be a challenge problem in empirical analysis.

3. Implied Price Processes and Stochastic Volatility

Britten-Jones and Neuberger (2000) develop a methodology to extract model free implied volatility directly from option prices. Their methodology is also valid under the jump-diffusion assumption see Jiang and Tian (2005).

Let $\sigma_\alpha^2(t;\tau)$ be the implied volatility under the risk-neutral measure Q for the return on asset S_α over a τ-maturity, where $\alpha \in \{1, 2, \cdots, n, I\}$. The price S_α must be a positive martingale and satisfy $\frac{dS_\alpha}{S_\alpha} = \sigma_\alpha(t)dW^Q$, where W^Q is the Brownian motion under the measure Q. By the formula (13) of Britten-Jones and Neuberger (2000), we have

$$\sigma_\alpha^2(t;\tau) = E^Q\left[\int_t^{t+\tau}\phi_\alpha^2(s)ds\right], \tag{3}$$

where $\phi_\alpha = \frac{dS_\alpha}{S_\alpha}$, and the relation between σ and ϕ is given in the formula (13) of Britten-Jones and Neuberger (2000).. We denote the price of a t-maturity call option on asset α with strike price K_α at time t by $C_\alpha(K_\alpha, t)$. The main result of Britten-Jones and Neuberger (2000) in the formula (14) with $t_2 = t$ and $S_\alpha(t) = S_t$ is given by

$$\sigma^2_{MF,\alpha}(t) = 2\int_0^\infty \frac{C_\alpha(K_\alpha, t) - (S_\alpha(0) - K_\alpha)^+}{K_\alpha^2} dK_\alpha, \qquad \alpha \in \{1, 2, \cdots, n, I\}. \tag{4}$$

The realized return variance of asset α from t to $t+\tau$ is given by $RV_\alpha(t;\tau) = \int_t^{t+\tau} \phi_\alpha^2(s)ds$, and the model-free implied variance is given by

$$MFIV_\alpha(\tau) = \sigma^2_{MF,\alpha}(\tau) = E^Q\left[RV_\alpha(t;\tau)\right].$$

Proposition 1. *The difference between the index's model-free implied variance and the sum of weighted of model-free implied variances of component stocks is the measure of the correlation risks.*

$$\sigma_I^2(t;\tau) - \sum_{i=1}^n w_i^2 \sigma_i^2(t,\tau) = \frac{1}{2}\sum_{i\neq j} w_i w_j E^Q\left[\int_t^{t+\tau} \phi_i(s)\phi_j(s)\rho_{ij}(s)ds\right].$$

Furthermore, we have

$$\int_0^\infty \frac{C_I(K_I, t) - (S_I(0) - K_I)^+}{K_I^2} dK_I - \sum_{i=1}^n w_i^2 \int_0^\infty \frac{C_i(K_i, t) - (S_i(0) - K_i)^+}{K_i^2} dK_i$$

$$= \frac{1}{2}\sum_{i\neq j} w_i w_j E^Q\left[\int_0^t \phi_i(s)\phi_j(s)\rho_{ij}(s)ds\right].$$

The proof of Proposition 1 is given in Appendix B. Our Proposition 1 precisely indicates that the model-free implied variance of the index is the same model-free implied variance of individual stock if and only if all correlation risks

$$E^Q\left[\int_t^{t+\tau} \phi_i(s)\phi_j(s)\rho_{ij}(s)ds\right] = 0$$

for all $1 \leq i, j \leq n$ and $i \neq j$. This uniquely characterizes the role of correlation risks in understanding the index variance risk premium and the stochastic volatilities of each individual stock. This completes a challenging question for the relation between the risk of the index and the risks of stocks under the risk-neutral measure. Proposition 1 provides a precise relation on the empirical finding among the correlation risks, the index variance risk premium and the sum of weighted variances of individual stocks under the risk neutral measure given by Driessen, Maenhout and Vilkov (2009).

For each individual asset $S_i(t)$, the strike K_i may not be the same. Proposition 1 gives the exact measure for the stochastic correlations and interactive terms through the observations of the index option price and the individual option prices with different strike prices $K_\alpha(\alpha \in \{1, 2, \cdots, n, I\})$. The formula (5) in Britten-Jones and Neuberger (2000) is incorrect, where they also assumed all the strike prices is the same constant.

4. Empirical Results

The Dow Jones Industrial Average is the oldest continuous U.S. equity market index. It provides a price weighted barometer of U.S. equity market conditions. The Chicago Board Options Exchange began trading index options written on the Dow in 1997. We use daily observations of closing best bid and offer quotations for the Dow Jones Industrial Average index options (DJX) and for options written on the average's components for all components included in the average from October 1997 until October 2009. DJX options are European style, cash settled and based on 1/100th of the current value of the average. They expire on the Saturday following the 3^{rd} Friday of the expiration month with last day of trading typically on the preceding Thursday.

All put /call options written on the DJX and component stock's were obtained from OptionMetrics. Options with days till expiration $14 \leq \tau \leq 60$ were retained. Initial screens further eliminated options with zero; trading volume, open interest and bid price and those options for which bid ask midpoints violated European option pricing boundaries. Out-of-the money options were identified based on option's delta from the Black-Scholes model. Retained in the sample were out-of-the-money call options with delta $0.15 \leq \Delta_C \leq 0.50$ and out-of-the-money put options $-0.50 \leq \Delta_P \leq -0.05$, we selected the option maturity which produced the greatest number of out-of-the money puts and calls where the remaining data presented multiple maturities for a given trade day,

The Dow Jones Industrial Average is a price weighted index calculated from the share prices of thirty component companies. Throughout the sample period there were sixteen changes to the roster of component companies. The majority of changes to the roster's component companies occurred due to mergers and acquisitions or in response to fundamental economic trends. One roster change deserves mention as the financial condition of the component companies impacts the sample of option prices for component firms. On June 1, 2009 General Motors filed for bankruptcy while a component of the average. General Motors and CitiGroup were removed from the average on June 8th, 2009. Closing option quotes are available for GM only through August 25, 2008 and the screens applied in this study eliminate GM options from the sample after August 14th, 2008. The resulting sample, October 6, 1997 through August 14, 2008 contains 12,792 call and 29,883 put DJX option quotes from 2,727 trading days.

Share price and dividend histories of the component stocks were obtained from the CRSP database. OptionMetrics was the source for continuous compounded index dividend yield and interest rates.

Table 1 and Figure 1 summarize evidence on the relationship between model free implied variance and realized variance for DJX options. All empirical results are based on the square root of variance. Results for DJX options are consistent with the those reported by Driessen, Maenhout, Vilkov (2009) for S&P 100 index options. For DJX options, model free implied volatility is on average 323 basis points greater than realized index volatility. The null hypothesis of equal average model free implied volatility and average realized volatility is rejected for DJX options based on Newey and West (1987) heteroskedasticity and autocorrelation consistent standard errors with 9 lags. Driessen, Maenhout, Vilkov (2009) report an average difference between S&P 100 index model free implied and realized volatilities of 389 basis points.

Table 1 and Figure 2 summarize evidence on the relationship between price weighted cross-sectional averages of component stock's model free implied and realized volatility. In contrast to the evidence for index options, the price weighted average of component stock's realized volatility is on average 142 basis points greater than the price weighted average of component stock's model free implied volatility. This difference is also significantly different from zero based on the t-test of Newey and West (1987). Driessen, Maenhout, Vilkov (2009) report qualitatively similar results for the equally weighted averages of S&P 100 component stock's realized and model free implied volatilities.

Proposition 1 links the difference between index and price weighted average of component stock's model free implied volatilities to the risk neutral expectation of correlation risks. Table 2 presents a direct test of the statistical significance of the average difference between model free implied volatility from DJX options and the model free implied volatility from component stock options. The time series average difference is -796 basis points and statistically significant based on Newey and West (1987) heteroskedasticity and autocorrelation consistent standard errors with 9 lags. This result demonstrates that correlation risks plays a significant role in explaining difference between the expected index option and component stock option returns.

In Figure 4, we show the time series of the model-free implied standard deviations calculated from index options, Index and a price weighted cross sectional average (WAVG) over our 10/1997 to 8/2008 OptionMetrics sample. The contemporaneous difference between the index model-free implied volatility and weighted average of component stock model-free implied volatilities can be interpreted as a systemic risk indicator. During periods of market stress where correlation among asset returns systematically increases towards +1.0, the benefit of diversification inherent in the model free implied index volatility is diminished. During such periods the spread between these two model free implied volatilities should approach zero. The systemic indicator, which signals degeneracy of the stochastic correlation matrix, occurs when the difference between model-free implied volatility of the index (DJIA) and the weighted average of model-free implied volatilities of the component stocks increases to zero. The difference is typically negative as the weighted average of implied volatilities of the component stocks is greater than the model-free implied volatility of the index.

Figure 4 tracks the value of the systemic risk indicator on the left axis and the value of squared log change of the DJIA on the right. The circled area is a puzzle. In that time period there are large daily log changes of the index, accompanying these large log changes of the index the systemic index is very far below zero. Table 4 presents dates in the sample when the systemic risk indicator is greater than -0.01.

The single factor to detect the degeneracy of the stochastic correlation matrix is the determinant of the correlation matrix. We have discussed these in Section 2 after Theorem 1. In the following, we evaluate the Systemic Risk Indicator with respect to logarithmic changes in the Dow Jones Industrial Average.

Evaluation of Systemic Risk Indicator

The remainder of this section presents an evaluation of the Systemic Risk Indicator with respect to logarithmic changes in the Dow Jones Industrial Average. The Systemic Indicator formed from the contemporaneous difference between index model free implied volatility and price weighted average of component stock's model free implied volatilities signals market stress or erosion of the benefit of diversification when the spread between the contemporaneous index and weighted average model free implied volatilities approaches zero. Table 5 presents the descriptive statistics of the Systemic Risk Indicator for the entire sample and for subperiods.

As expected the mean value of the Systemic Indicator is significantly less than zero at standard confidence levels. Which implies that on an average day throughout the sample period the volatility of the index with the inherent benefit of diversification is less than the price weighted average of the component stock's implied volatilities.

For the purpose of evaluating the Systemic Indicator and market volatility a figure is presented for each sub period . Within each subperiod the series of daily index squared logarithmic change is plotted with the contemporaneous Systemic Indicator values. A horizontal line is plotted at the subperiod's mean value of the Systemic Indicator. During the October 1997 through December 1999 subperiod, two bursts of volatility are evident. The first associated with the market experience during the Asian currency crisis of September and October of 1997, the second, the market turbulence during late summer and early October 1998. Clearly both periods are accompanied by levels of the indicator greater than the subperiod average; implying diminished benefit of diversification was a significant contributor to realized volatility during these episodes.

During the January 2001 through December 2002 subperiod, three bursts of volatility are evident in Figure 6. The first in March 2001, the second, following the events of September 11, 2001 and the third during August through October 2002. As evidenced by the March 2001 experience not all periods of extreme volatility are accompanied by denigration of the benefit of diversification. Throughout the market turbulence of March 2001 the Systemic Indicator remains below the subperiod average. Figure 7 presents the January 2003 through December 2004 subperiod. A single episode of significant market turbulence during first quarter 2003 is evident. Like the period of market turbulence throughout third quarter 2002 the Systemic indicator is greater than subperiod average throughout the first quarter of 2003.

The final subperiod January 2007 through August 14, 2008 is presented in Figure 8. It is interesting to note the average benefit of diversification during this subperiod is smaller relative to other subperiods and for the sample as a whole. The recent financial market events are characterized by periods of market turbulence accompanied by above subperiod average values of the Systemic Indicator.

5. All Risks and Risk Premiums

In this section we develop the contribution to index expected return due to index variance risk premia (level and instantaneous change) and from component stock variance and correlation risk premia. We also identify the difference between the contribution to index expected return from total variance risk and the idiosyncratic risk for the index and consider the variance risk premia for the index in the presence of jumps.

5.1 Index variance risk premia

As studied in Driessen et al (2009), index return variance can be decomposed into component stock return variances. However in the presence of stochastic correlations a complete treatment of risks must account for contributions to the total variance risk premium due to both the level and the instantaneous change of correlations. The emphasis of correlation risk has also been discussed extensively in Câmara, Krehbiel and Li (2011) from a different perspective.

Suppose that the constituent companies of the stock index have the price of stock $S_i(t), i = 1, 2, \cdots, n$ and each stock $i, S_i(t)$ follow an Ito process with instantaneous variance $\sigma_i^2(t)$ which itself follows an Ito process. The instantaneous correlation between Brownian processes $B_i(t)$ and $B_j(t)$

that drive stocks $(i, S_i(t))$ and $(j, S_j(t))$ is[6]

$$E_t^Q[dB_i(t)dB_j(t)] = \rho_{ij}(t)dt. \tag{5}$$

We assume that $\rho_{ij}(t)$ is also an Ito process for $1 \leq i \neq j \leq n$, and the covariance matrix $(\sigma_i(t)\sigma_j(t)\rho_{ij}(t))_{1\leq i,j\leq n}$ is positive-definite matrix for almost all t[7]. For an index weighted w_i on the i-th stock S_i, the instantaneous index variance $\sigma_I^2(t)$ at time t is given by

$$\begin{aligned} \sigma_I^2(t) &= \sigma_w^T \Sigma_s \sigma_w = \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_i(t)\sigma_j(t)\rho_{ij}(t) \\ &= \sum_{i=1}^n w_i^2 \sigma_i^2(t) + \sum_{i=1}^n \sum_{j\neq i}^n w_i w_j \sigma_i(t)\sigma_j(t)\rho_{ij}(t), \end{aligned} \tag{6}$$

where $\sigma_w = (w_1\sigma_1(t), \cdots, w_n\sigma_n(t))$ is the weighted vector, and $\Sigma_s = (\rho_{ij}(t))_{1\leq i,j\leq n}$ is the correlation matrix of the the stock index. The correlation matrix Σ_s is symmetric and stochastic. The diagonalization of the symmetric covariance matrix $(\sigma_i(t)\sigma_j(t)\rho_{ij}(t))_{1\leq i,j\leq n}$ has positive eigenvalues for almost all t.

The risk premium attributable to the level of the index's return variance is, directly from (2),

$$E_t^Q[\sigma_I^2(t)] - E_t^P[\sigma_I^2(t)] = \sum_{i=1}^n w_i^2 [E_t^Q - E_t^P](\sigma_i^2(t))) + \sum_{i\neq j} w_i w_j [E_t^Q - E_t^P](\sigma_i(t)\sigma_j(t)\rho_{ij}(t)), \tag{7}$$

where Q is an risk neutral measure (P is the physical measure) and $[E_t^Q - E_t^P](X) = E_t^Q[X] - E_t^P[X]$. The instantaneous change in index variance is dependent upon the shocks to component stock variance $\sigma_i^2(t)$, and to correlations $\rho_{ij}(t)$. The risk premium attributable to instantaneous change in index variance inclusive of compensation for market risk if variance shocks are correlated with market risk is given by $E_t^Q[d\sigma_I^2(t)] - E_t^P[d\sigma_I^2(t)]$.

[6]Assumptions on $S_i(t), \sigma_i^2(t)$ and correlations $\rho_{ij}(t)$ are same as in Section 2, Assumption (1), (2) and (3).

[7]The stochastic correlation $\rho_{ij}(t)$ satisfies both an Ito process and the positive-definite constraint for the covariance matrix $(\sigma_i(t)\sigma_j(t)\rho_{ij}(t))_{1\leq i,j\leq n}$. This forces the stochastic correlation satisfies $-1 \leq \rho_{ij}(t) \leq 1$ almost surely for almost all t.

Proposition 2. *The expectation of the instantaneous change in index variance under the risk neutral measure is given by*

$$E_t^Q[d\sigma_I^2] = \sum_{i=1}^n w_i^2 E_t^Q[d\sigma_i^2(t)] + \sum_{i=1}^n \sum_{j\neq i}^n w_i w_j E_t^Q(\frac{d\sigma_i(t)}{\sigma_i(t)} + \frac{d\sigma_j(t)}{\sigma_j(t)} + \frac{d\rho_{ij}(t)}{\rho_{ij}(t)}$$

$$+\frac{d\sigma_i(t)}{\sigma_i(t)} \cdot \frac{d\sigma_j(t)}{\sigma_j(t)} + \frac{d\sigma_j(t)}{\sigma_j(t)} \cdot \frac{d\rho_{ij}(t)}{\rho_{ij}(t)} + \frac{d\rho_{ij}(t)}{\rho_{ij}(t)} \cdot \frac{d\sigma_i(t)}{\sigma_i(t)})(\sigma_i(t)\sigma_j(t)\rho_{ij}(t)),$$

and the risk premium attributable to the instantaneous change in index variance with respect to the actual measure P is given by

$$E_t^Q[d\sigma_I^2(t)] - E_t^P[d\sigma_I^2(t)] = \sum_{i=1}^n w_i^2 [E_t^Q - E_t^P](d\sigma_i^2(t)) + \sum_{i=1}^n \sum_{j\neq i}^n w_i w_j [E_t^Q - E_t^P](\frac{d\sigma_i(t)}{\sigma_i(t)} + \frac{d\sigma_j(t)}{\sigma_j(t)} + \frac{d\rho_{ij}(t)}{\rho_{ij}(t)}$$

$$+\frac{d\sigma_i(t)}{\sigma_i(t)} \cdot \frac{d\sigma_j(t)}{\sigma_j(t)} + \frac{d\sigma_j(t)}{\sigma_j(t)} \cdot \frac{d\rho_{ij}(t)}{\rho_{ij}(t)} + \frac{d\rho_{ij}(t)}{\rho_{ij}(t)} \cdot \frac{d\sigma_i(t)}{\sigma_i(t)})(\sigma_i(t)\sigma_j(t)\rho_{ij}(t)).$$

The proof of Proposition 2 is given in Appendix C.

The instantaneous index variance risk premium reflects all instantaneous component stock's variance risk premia $[E_t^Q - E_t^P](d\sigma_i^2(t))$, and the (logarithmic) underlying risk premia

$$[E_t^Q - E_t^P](\frac{d\sigma_i(t)}{\sigma_i(t)}(\sigma_i(t)\sigma_j(t)\rho_{ij}(t))) = [E_t^Q - E_t^P](d\sigma_i(t)\sigma_j(t)\rho_{ij}(t))), \tag{8}$$

$$[E_t^Q - E_t^P](\frac{d\sigma_j(t)}{\sigma_j(t)}(\sigma_i(t)\sigma_j(t)\rho_{ij}(t))) = [E_t^Q - E_t^P](\sigma_i(t)d\sigma_j(t)\rho_{ij}(t))) \tag{9}$$

from each interaction between stocks i and j, as well as their correlation

$$[E_t^Q - E_t^P](\frac{d\rho_{ij}(t)}{\rho_{ij}(t)}(\sigma_i(t)\sigma_j(t)\rho_{ij}(t)) = [E_t^Q - E_t^P](\sigma_i(t)\sigma_j(t)d\rho_{ij}(t)), \tag{10}$$

plus more interactive risk premia among the stocks risk and correlation risk

$$[E_t^Q - E_t^P](\frac{d\sigma_i(t)}{\sigma_i(t)} \cdot \frac{d\sigma_j(t)}{\sigma_j(t)}(\sigma_i(t)\sigma_j(t)\rho_{ij}(t)) = [E_t^Q - E_t^P](d\sigma_i(t) \cdot d\sigma_j(t) \cdot \rho_{ij}(t)), \tag{11}$$

$$[E_t^Q - E_t^P](\frac{d\sigma_j(t)}{\sigma_j(t)} \cdot \frac{d\rho_{ij}(t)}{\rho_{ij}(t)}(\sigma_i(t)\sigma_j(t)\rho_{ij}(t)) = [E_t^Q - E_t^P](\sigma_i(t) \cdot d\sigma_j(t) \cdot d\rho_{ij}(t)), \tag{12}$$

$$[E_t^Q - E_t^P](\frac{d\rho_{ij}(t)}{\rho_{ij}(t)} \cdot \frac{d\sigma_i(t)}{\sigma_i(t)}(\sigma_i(t)\sigma_j(t)\rho_{ij}(t)) = [E_t^Q - E_t^P](d\sigma_i(t) \cdot \sigma_j(t) \cdot d\rho_{ij}(t)). \tag{13}$$

Note that those risk premia from (8), (9) and (10) are the only contributions considered by Driessen et al 2009) in (3) of page 1382. But the crucial feature of the Ito process and the Ito product rule indicates that there are non-trivial contributions from risk premia in (11), (12) and (13). These interactive risk premium identify exactly the role of correlation risk in the risk premium due to instantaneous change in index variance.

Example on the nontrivial interactive risk and the interactive risk premia

We now specify the instantaneous variance $\sigma_i(t)$ and the instantaneous correlation $\rho_{ij}(t)$ under the risk neutral measure. (i) Assume that the instantaneous variance $\sigma_i(t)$ is an Ito process: $d\sigma_i(t) = E_t^Q[d\sigma_i(t)] + \beta_{\sigma_i}(t)dB_{\sigma_i}(t)$, with $E_t^Q[dB_j(t)dB_{\sigma_i}(t)] = 0$ and $E_t^Q[dB_{\sigma_i}(t)dB_{\sigma_j}(t)] = 0$. (ii) Assume that the instantaneous correlation $\rho_{ij}(t)$ is an Ito process, for $1 \leq i \neq j \leq n$,

$$d\rho_{ij}(t) = E_t^Q[d\rho_{ij}(t)] + \rho_{ij}^0(t)dB_0(t) + \sum_{k=1}^{n} \rho_{ij}^k(t)dB_{\sigma_k}(t),$$

with $E_t^Q[dB_0(t)dB_j(t)] = 0, E_t^Q[dB_0(t)dB_{\sigma_i}(t)] = 0$. The term attributed to interactive risk in $d\sigma_I^2(t)$ is given by

$$\beta_{\sigma_i}(t)\beta_{\sigma_j}(t)dB_{\sigma_i}(t)dB_{\sigma_j}(t) \cdot \rho_{ij}(t) + \sigma_i(t)\beta_{\sigma_j}(t)\rho_{ij}^j(t)dt + \sigma_j(t)\beta_{\sigma_i}(t)\rho_{ij}^i(t)dt;$$

and the terms (11), (12) and (13) attributed to the interactive risk premia in $[E_t^Q - E_t^P](d\sigma_I^2(t))$ is given by

$$[E_t^Q - E_t^P](\sigma_i(t)\beta_{\sigma_j}(t)\rho_{ij}^j(t) + \sigma_j(t)\beta_{\sigma_i}(t)\rho_{ij}^i(t))dt.$$

Example on the trivial interactive risk and the interactive risk premia

(a) Assume that the instantaneous variance $\sigma_i(t)$ is an Ito process: $d\sigma_i(t) = E_t^Q[d\sigma_i(t)] + \beta_{\sigma_i}(t)dB_{\sigma_i}(t)$, with $E_t^Q[dB_j(t)dB_{\sigma_i}(t)] = 0$ and $E_t^Q[dB_{\sigma_i}(t)dB_{\sigma_j}(t)] = 0$. (b) Assume that the instantaneous correlation $\rho_{ij}(t)$ is an Ito process, for $1 \le i \ne j \le n$,

$$d\rho_{ij}(t) = E_t^Q[d\rho_{ij}(t)] + \rho_{ij}^0(t)dB_0(t),$$

with $E_t^Q[dB_0(t)dB_j(t)] = 0, E_t^Q[dB_0(t)dB_{\sigma_i}(t)] = 0$. The term attributed to interactive risk in $d\sigma_I^2(t)$ and the terms (11), (12) and (13) attributed to the interactive risk premia in $[E_t^Q - E_t^P](d\sigma_I^2(t))$ are identically zero. Hence the total interactive risk in $d\sigma_I^2(t)$ and the total interactive risk premia in $[E_t^Q - E_t^P](d\sigma_I^2(t))$ are both trivial.

5.2 Idiosyncratic Risk

Index idiosyncratic risk due to the weighted average of component stock variances has a positive relation with the expected return of the index. Goyal and Santa-Clara (2003) investigated the link between idiosyncratic risk and index variance risk and find a significant positive relation between the average stock variance (largely idiosyncratic) and the return on the index. We compare the complete index total variance with correctly incorporating stochastic correlation and idiosyncratic risk due to the weighted average of component stock variances.

The idiosyncratic risk $V_I(t)$ at time t is given by

$$V_I(t) = \frac{1}{n}\sum_{i=1}^{n} w_i^2\sigma_i^2(t). \tag{14}$$

The index total variance is then the sum of the idiosyncratic variance and stochastic correlation terms: $\sigma_I(t) = V_I(t) + \sum_i \sum_j w_i w_j \sigma_i(t)\sigma_j(t)\rho_{ij}(t)$. Thus we have the index idiosyncratic risk premium given by

$$E_t^Q[V_I(t)] - E_t^P[V_I(t)] = \frac{1}{n}\sum_{i=1}^{n} w_i^2 [E_t^Q - E_t^P](d\sigma_i^2(t)). \tag{15}$$

Proposition 3. *The difference between the index total variance risk premium and the idiosyncratic risk premium for the index is given by*

$$\frac{n-1}{n}\sum_{i=1}^{n} w_i^2[E_t^Q - E_t^P](d\sigma_i^2(t)) + \sum_{i=1}^{n}\sum_{j\neq i}^{n} w_i w_j [E_t^Q - E_t^P]\Big(\frac{d\sigma_i(t)}{\sigma_i(t)} + \frac{d\sigma_j(t)}{\sigma_j(t)} + \frac{d\rho_{ij}(t)}{\rho_{ij}(t)}$$

$$+\frac{d\sigma_i(t)}{\sigma_i(t)}\cdot\frac{d\sigma_j(t)}{\sigma_j(t)} + \frac{d\sigma_j(t)}{\sigma_j(t)}\cdot\frac{d\rho_{ij}(t)}{\rho_{ij}(t)} + \frac{d\rho_{ij}(t)}{\rho_{ij}(t)}\cdot\frac{d\sigma_i(t)}{\sigma_i(t)}\Big)(\sigma_i(t)\sigma_j(t)\rho_{ij}(t)).$$

The proof of Proposition 3 follows from Proposition 2 and equation (15). This result extends results of Goyal and Santa-Clara (2003), and indicates that the correlation risk may dominate the difference, hence the significant positive relation between the return of the index and the correlation risk may be bigger than the positive relation between the average stock variance (largely idiosyncratic) and the return on the index.

5.3 Risks from Jumps

Assume the jump process for each individual stock i follows from $Y_i = \sum_{k=1}^{N(t)} Y_{ik}$, where $Y_{ik} \sim N(\alpha, \gamma)$ and $N(t)$, Y_{ik} and $B_i(t)$ are independent. The mean and covariance matrix of the underlying random variables $\mathbf{B}(T)$ and Y_i is given by

$$\begin{bmatrix} \mathbf{B}(t) \\ Y_i(t) \end{bmatrix} \sim N\left(\begin{bmatrix} \mathbf{0} \\ \alpha \end{bmatrix}, \begin{bmatrix} \Sigma_s t & (\gamma\rho_{s_i y}\sqrt{t})_{1\leq i\leq n} \\ & \gamma^2 \end{bmatrix}\right),$$

where Σ_s is the covariance matrix of $\mathbf{B}(t)$, $(\gamma\rho_{s_i y}\sqrt{t})_{1\le i\le n} = (\gamma\rho_{s_1 y}\sqrt{t}, \cdots, \gamma\rho_{s_n y}\sqrt{t})$ and the covariance matrix is a symmetric matrix. The instantaneous correlation between Brownian process $B_i(t)$ and Y_j that drive stock is $E_t[dB_i(t)\Delta Y_j] = \sigma_i(t)\gamma\rho_{s_i y_j}(t)\sqrt{dt}$, where $\rho_{s_i y_j}(t)$ is assumed to be an Ito process. For an index weighted w_i on the i-th stock S_i, the instantaneous index variance $\sigma_{IJ}^2(t)$ with jump diffusions at time t is given by

$$\begin{aligned}
\sigma_{IJ}^2(t) &= \sigma_w^T \Sigma_s \sigma_w + \sigma_w^T \cdot (\rho_{sy}) \cdot (\gamma\mathbf{1}) + [N(t)]\gamma^2 \\
&= \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_i(t)\sigma_j(t)\rho_{ij}(t) + \sum_{i=1}^n \gamma w_i \sigma_i(t)\rho_{s_i y_j}(t) + [N(t)]\gamma^2 \\
&= \sum_{i=1}^n w_i^2 \sigma_i^2(t) + \sum_{i=1}^n \sum_{j\ne i}^n w_i w_j \sigma_i(t)\sigma_j(t)\rho_{ij}(t) \\
&\quad + \sum_{i=1}^n \gamma w_i \sigma_i(t)\rho_{s_i y_i}(t) + \sum_{i=1}^n \sum_{j\ne i}^n \gamma w_i \sigma_i(t)\rho_{s_i y_j}(t) + [N(t)]\gamma^2
\end{aligned}$$

where $\sigma_w = (w_1\sigma_1(t), \cdots, w_n\sigma_n(t))$ is the weighted vector and $(\rho_{sy}) = (\rho_{s_i y_j})_{1\le i,j\le n}$, $[N(t)]$ is the random process for counting the number of jumps that occur at or before time t, and $(\gamma\mathbf{1}) = (\gamma, \cdots, \gamma) \in \mathbf{R}^n$.

Without the jump diffusions, the index variance $\sigma_{IJ}^2(t)$ is reduced into the market variance risk $\sigma_I^2(t)$. The risk from jumps in the expression

$$[N(t)]\gamma^2 + \sum_{i=1}^n \gamma w_i \sigma_i(t)\rho_{s_i y_i}(t) + \sum_{i=1}^n \sum_{j\ne i}^n \gamma w_i \sigma_i(t)\rho_{s_i y_j}(t)$$

may dominant the market variance risk in $\sigma_{IJ}^2(t)$, with the pure jump risk $[N(t)]\gamma^2$ and the same correlation risk $\gamma w_i \sigma_i(t)\rho_{s_i y_j}(t)$ between the jump diffusion and the Brownian motions that drives the stocks. Now an index option is constructed largely to exposure the index-wide correlation risk and the jump risk as well as their correlations.

Proposition 4. *The total risk variance premium for the index with jumps is given by*

$$E_t^Q[d\sigma_{IJ}^2(t)] - E_t^P[d\sigma_{IJ}^2(t)] = E_t^Q[d\sigma_I^2(t)] - E_t^P[d\sigma_I^2(t)] + E_t^Q[d\sigma_J^2(t)] - E_t^P[d\sigma_J^2(t)],$$

where $E_t^Q[d\sigma_I^2(t)] - E_t^P[d\sigma_I^2(t)]$ *is the risk variance premium for the index without jumps in Proposition 1, and* $E_t^Q[d\sigma_J^2(t)] - E_t^P[d\sigma_J^2(t)]$ *is the risk variance premium for jumps and their correlations,*

$$E_t^Q[d\sigma_J^2(t)] - E_t^P[d\sigma_J^2(t)] = [E_t^Q - E_t^P](\Delta N(t)\gamma^2)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma w_i[E_t^Q - E_t^P](\frac{d\sigma_i(t)}{\sigma_i(t)} + \frac{\rho_{s_iy_j}(t)}{\rho_{s_iy_j}(t)} + \frac{d\sigma_i(t)}{\sigma_i(t)} \cdot \frac{\rho_{s_iy_j}(t)}{\rho_{s_iy_j}(t)})(\sigma_i(t)\rho_{s_iy_j}(t)).$$

The proof follows the same method in Appendix C. The total variance risk premium for the index with jumps is the variance risk premium for the index without jumps plus the variance risk premium due purely jumps and their correlations. The cross-section with respect to the actual measure P for the jump risks raises challenges for empirical analysis. Markets are incomplete and the stochastic discount factor is not identified in the presence of jumps. Identifying the average jump size is another major challenge to implement in any jump-diffusion model. Yan (2010) argues the average jump size of the stochastic discount factor is positive, and proposes the slope of implied volatility smile to proxy the jump risk.

6. Conclusion

Stochastic correlations are important risk factors in financial markets, and the evidence is consistent with the risks due to stochastic correlations being priced by the market in the sense that index options pay off well when market-wide correlations are higher than expected earn negative excess returns. Our empirical evidence from the model-free implied variance of index options suggests that the measure of the correlation risks are indeed statistically significant. The data strongly rejects the null hypothesis that the difference between model-free implied index variance and weighted model-free implied component variances is drawn from a distribution with zero median. By using Britten-Jones and Neuberger (2000) model-free implied volatility method, we characterize the role of correlation risks uniquely in terms of the model-free implied variance of the index and the model-free implied variances of component stocks. We show both theoretically and empirically that the measure of the correlation risks is precisely the difference between the model-free implied variance of the market risk and the model-free implied variance of individual stock risk.

There are growing evidence that individual option prices and returns behave differently empirically, and Driessen et al (2009) add further evidence that individual options do not embed a negative variance risk premium nor earn economically significant returns in excess of a one-factor model, but the index option does embed a negative variance risk premium and behaves differently. Constantinides et al (2013) show that the leverage-adjusted returns on S&P 500 Index options strongly reject the predictions of the Black-Scholes-Merton model. We establish a generalized Black-Scholes-Merton partial differential system to encode all the information about the total index variance and each component variance, as well as the expected excess returns of the index and individual stock options. Thus individual option prices consists of a term that correlates with all individual stocks, and new terms from both the first and second order Greeks on the variances from each individual stock. This is essentially different from the Black-Scholes-Merton model with implied volatility. On the other hand, the index option price consists of a term with weighted correlations among all individual stocks, and the first and second order Greeks on the variance of the market. It is the variance of the market that contains many new interactive variance contributions to distinguish the index option price and return from individual option prices and returns.

Models relaxing the original stochastic specification through incorporation of stochastic volatility and/or jumps have been suggested as an improvement upon the original BSM assumptions. Each of the those models produces significantly different implied volatility estimates across the range of strike prices failing the test of internal consistency, when applied to $S\&P$ 500 index options. Our generalized Black-Scholes-Merton partial differential systems provide a mathematically complete description of the dynamic asset pricings for both the index and its components. The generalized system reflects correlation contributions for the individual stock and interactive risk premia among the stocks for the index. The similar common term on the correlations with all individual stocks present a systematic risk indicator as we show empirically. We find that exactly the same term also provides the measure of the correlation risks for the system. The systematic indicator is zero when the covariance matrix degenerates to a non-positive definite matrix for only a measure zero set of times.

We study the index total variance risk premium, and obtain a mathematically complete structure for the index total variance risk premium. We find that an index's total variance risk premium is due not only to changes in index's components' variance and changes in components' return correlations, but is also due to important interactions between components' variances, changes in components' variances, correlations among components' returns and changes in correlations. Further, we identify the measure on the difference between the market variance risk and the idiosyncratic risk for the index. Even when the jump processes are assumed for each individual stock, we get the precise measure on the total risk variance premium for the index and specify the role of the risk variance premium for jumps and their correlations.

Appendix A

The stock market consists of n stocks. Each stock price $S_i(t)$ satisfies the following stochastic differential equation

$$\frac{dS_i(t)}{S_i(t)} - E^Q[\frac{dS_i(t)}{S_i(t)}] = \sigma_i(t)dB_i(t), \tag{16}$$

for $i = 1, 2, \cdots, n$. Thus we have

$$\begin{aligned}
\frac{dS_i(t)}{S_i(t)} \cdot \frac{dS_j(t)}{S_j(t)} &= (E^Q[\frac{dS_i(t)}{S_i(t)}] + \sigma_i(t)dB_i(t)) \cdot (E^Q[\frac{dS_j(t)}{S_j(t)}] + \sigma_j(t)dB_j(t)) \\
&= E^Q[\frac{dS_i(t)}{S_i(t)}] \cdot E^Q[\frac{dS_j(t)}{S_j(t)}] + E^Q[\frac{dS_i(t)}{S_i(t)}] \cdot \sigma_j(t)dB_j(t) \\
&\quad + \sigma_i(t)dB_i(t) \cdot E^Q[\frac{dS_j(t)}{S_j(t)}] + \sigma_i(t)\sigma_j(t) \cdot dB_i(t)dB_j(t).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
E^Q[\frac{dS_i(t)}{S_i(t)} \cdot \frac{dS_j(t)}{S_j(t)}] &= E^Q[\frac{dS_i(t)}{S_i(t)}] \cdot E^Q[\frac{dS_j(t)}{S_j(t)}] + E^Q[\sigma_i(t)\sigma_j(t) \cdot dB_i(t)dB_j(t)] \\
&= E^Q[\frac{dS_i(t)}{S_i(t)}] \cdot E^Q[\frac{dS_j(t)}{S_j(t)}] + \sigma_i^2(t)\sigma_j^2(t)\rho_{ij}(t)dt,
\end{aligned}$$

where the last equality follows from the instantaneous correlation assumption between the Brownian motion $B_i(t)$ and $B_j(t)$ in (5). Thus we get the following from the above identities,

$$\frac{dS_i(t)}{S_i(t)} \cdot \frac{dS_j(t)}{S_j(t)} - E^Q[\frac{dS_i(t)}{S_i(t)} \cdot \frac{dS_j(t)}{S_j(t)}] \tag{17}$$

$$= E^Q[\frac{dS_i(t)}{S_i(t)}] \cdot \sigma_j(t)dB_j(t) + \sigma_i(t)dB_i(t) \cdot E^Q[\frac{dS_j(t)}{S_j(t)}] + \sigma_i(t)\sigma_j(t) \cdot dB_i(t)dB_j(t) - \sigma_i^2(t)\sigma_j^2(t)\rho_{ij}(t)dt.$$

Let $C_\alpha(t, S_1(t), \cdots, S_n(t), \sigma_1^2(t), \cdots, \sigma_n^2(t))$ be the option price for the stock $(\alpha, S_\alpha(t))$, where the correlation risk $\rho_{ij}(t)$ links the option price C_α with other stock values as well as the stochastic volatilities $\sigma_1^2(t), \cdots, \sigma_n^2(t)$ for $\alpha \in \{1, 2, \cdots, n\}$. Let $X_i(t) = S_i(t)$ and $Y_i(t) = \sigma_i^2(t)$ for $i =$

$1, 2, \cdots, n$. Both $X_i(t)$ and $Y_i(t)$ are Ito processes. Now we apply the Ito-Doeblin formula for multiple processes, for $\alpha \in \{1, 2, \cdots, n\}$,

$$dC_\alpha(t, S_1(t), \cdots, S_n(t), \sigma_1^2(t), \cdots, \sigma_n^2(t)) \tag{18}$$

$$= \frac{\partial C_\alpha}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial C_\alpha}{\partial S_i(t)} dS_i(t) + \sum_{i=1}^{n} \frac{\partial C_\alpha}{\partial \sigma_i^2(t)} d(\sigma_i^2(t))$$

$$+\frac{1}{2} \sum_{i,j=1}^{n} \left(\frac{\partial^2 C_\alpha}{\partial S_i(t) \partial S_j(t)} dS_i(t) dS_j(t) + \frac{\partial^2 C_\alpha}{\partial S_i(t) \partial \sigma_j^2(t)} dS_i(t) d\sigma_j^2(t) + \frac{\partial^2 C_\alpha}{\partial \sigma_i^2(t) \partial \sigma_j^2(t)} d\sigma_i^2(t) d\sigma_j^2(t) \right).$$

By (16),

$$\begin{aligned} \frac{\partial C_\alpha}{\partial S_i(t)} dS_i(t) &= S_i(t) \frac{\partial C_\alpha}{\partial S_i(t)} \frac{dS_i(t)}{S_i(t)} \\ &= S_i(t) \frac{\partial C_\alpha}{\partial S_i(t)} \left[E^Q[\frac{dS_i(t)}{S_i(t)}] + \sigma_i(t) dB_i(t) \right]. \end{aligned}$$

Hence we have

$$E^Q[\frac{\partial C_\alpha}{\partial S_i(t)} dS_i(t)] = S_i(t) \frac{\partial C_\alpha}{\partial S_i(t)} \cdot E^Q[\frac{dS_i(t)}{S_i(t)}]. \tag{19}$$

By (1), we have

$$\frac{\partial C_\alpha}{\partial \sigma_i^2(t)} d(\sigma_i^2(t)) = \frac{\partial C_\alpha}{\partial \sigma_i^2(t)} \left(E^Q[d(\sigma_i^2(t))] + \zeta_i(\sigma_i) dB_{\sigma_i}(t) \right),$$

$$E^Q[\frac{\partial C_\alpha}{\partial \sigma_i^2(t)} d(\sigma_i^2(t))] = \frac{\partial C_\alpha}{\partial \sigma_i^2(t)} E^Q[d(\sigma_i^2(t))]. \tag{20}$$

Similarly, by (16) and (1),

$$E^Q[\frac{\partial^2 C_\alpha}{\partial S_i(t) \partial S_j(t)} dS_i(t) dS_j] = S_i(t) S_j(t) \frac{\partial^2 C_\alpha}{\partial S_i(t) \partial S_j(t)} \cdot E^Q[\frac{dS_i(t)}{S_i(t)} \frac{dS_j(t)}{S_j(t)}]. \tag{21}$$

$$E^Q[\frac{\partial^2 C_\alpha}{\partial S_i(t) \partial \sigma_j^2(t)} dS_i(t) d\sigma_j^2(t)] = S_i(t) \frac{\partial^2 C_\alpha}{\partial S_i(t) \partial \sigma_j^2(t)} E^Q[\frac{dS_i(t)}{S_i(t)}] \cdot E^Q[d\sigma_j^2(t)]. \tag{22}$$

$$E^Q[\frac{\partial^2 C_\alpha}{\partial\sigma_i^2(t)\partial\sigma_j^2(t)}d\sigma_i^2(t)d\sigma_j^2(t)] = \frac{\partial^2 C_\alpha}{\partial\sigma_i^2(t)\partial\sigma_j^2(t)}E^Q[d\sigma_i^2(t)]E^Q[d\sigma_j^2(t)], \tag{23}$$

where the identity follows from (1) and the assumption $E^Q[dB_{\sigma_i}(t)\cdot dB_{\sigma_j}(t)] = 0$ for $i \neq j$. By the Ito process property, both $E^Q[\frac{dS_i(t)}{S_i(t)}]\cdot E^Q[d\sigma_j^2(t)]$ and $E^Q[d\sigma_i^2(t)]E^Q[d\sigma_j^2(t)]$ equal to zero due to $dt \cdot dt = 0$ for $i \neq j$.

By taking the expectation with respect to the risk-neutral measure on (18) and (19), (20), (21), (22) and (23), we obtain the following **generalized Black-Scholes-Merton partial differential system**,

$$\begin{aligned} rC_\alpha dt &= \frac{\partial C_\alpha}{\partial t}dt + \sum_{i=1}^n S_i(t)\frac{\partial C_\alpha}{\partial S_i(t)}\cdot E^Q[\frac{dS_i(t)}{S_i(t)}] + \sum_{i=1}^n \frac{\partial C_\alpha}{\partial\sigma_i^2(t)}E^Q[d(\sigma_i^2(t))] \\ &+ \frac{1}{2}\sum_{i,j=1}^n\left(S_i(t)S_j(t)\frac{\partial^2 C_\alpha}{\partial S_i(t)\partial S_j(t)}\cdot(\sigma_i(t)\sigma_j(t)\rho_{ij}(t))\right)dt \\ &+ \frac{1}{2}\sum_{i=1}^n\left(S_i(t)\sigma_i(t)\frac{\partial C_\alpha^2}{\partial S_i(t)\partial\sigma_i^2(t)}(\zeta_i(\sigma_i)\rho_i(t)) + \zeta_i^2(\sigma_i)\frac{\partial^2 C_\alpha}{\partial\sigma_i^2(t)\partial\sigma_i^2(t)}\right)dt, \end{aligned}$$

for $\alpha \in \{1, 2, \cdots, n\}$.

The instantaneous expected excess option returns of the stock $(\alpha, S_\alpha(t))$ is given by

$$\begin{aligned} \frac{1}{dt}E_t^P\left[\frac{dC_\alpha}{C_\alpha} - rdt\right] &= \frac{1}{dt}(E_t^P - E_t^Q)\left[\frac{dC_\alpha}{C_\alpha} - rdt\right] + \frac{1}{dt}E_t^Q\left[\frac{dC_\alpha}{C_\alpha} - rdt\right] \\ &= \sum_{i=1}^n \frac{S_i(t)}{C_\alpha}\frac{\partial C_\alpha}{\partial S_i(t)}\cdot(E^P - E^Q)[\frac{dS_i(t)}{S_i(t)}] + \sum_{i=1}^n \frac{\partial C_\alpha}{C_\alpha\partial\sigma_i^2(t)}(E^P - E^Q)[d(\sigma_i^2(t))] \\ &= \sum_{i=1}^n \Omega_i^\alpha(t)\cdot(E^P - E^Q)[\frac{dS_i(t)}{S_i(t)}] + \sum_{i=1}^n \frac{\partial \ln C_\alpha}{\partial\sigma_i^2(t)}(E^P - E^Q)[d(\sigma_i^2(t))] \end{aligned}$$

where the measurement of the instantaneous expected excess option return follows from the definition with respect to the physical measure (see also Broadie, et al (2009) and Cao and Huang (2007)), the second equality follows from the proof in the above generalized BSM partial differential system and $\frac{dS_i(t)}{S_i(t)}d\sigma_j^2(t) = 0, d\sigma_i^2(t)d\sigma_j^2(t) = 0$, $\Omega_i^\alpha(t) = \frac{S_i(t)}{C_\alpha}\frac{\partial C_\alpha}{\partial S_i(t)}$ is the elasticity of the call option of the underlying stock $(\alpha, S_\alpha(t))$ with respect to $S_i(t)$ for $\alpha \in \{1, 2, \cdots, n\}$, the last from the calculations based on our assumptions.

For the index $S_I = \sum_{i=1}^n w_i S_i(t)$ with $\sigma_I^2(t) = \sum_{i,j=1}^n w_i w_j \sigma_i(t)\sigma_j(t)\rho_{ij}(t)$, the option price $C_I(t, S_I, \sigma_I^2(t))$ with stochastic market variance satisfies

$$dC_I = \frac{\partial C_I}{\partial t}dt + \frac{\partial C_I}{\partial S_I}dS_I(t) + \frac{\partial C_I}{\partial \sigma_I^2}d\sigma_I^2(t) \tag{24}$$

$$+\frac{1}{2}\frac{\partial^2 C_I}{\partial S_I^2}dS_I(t)dS_I(t) + \frac{\partial^2 C_I}{\partial S_I \partial \sigma_I^2}dS_I(t)d\sigma_I^2(t) + \frac{1}{2}\frac{\partial^2 C_I}{\partial \sigma_I^2(t)\partial \sigma_I^2(t)}d\sigma_I^2(t)\cdot d\sigma_I^2(t).$$

Now we compute the price of the market index in terms of each individual stock value and their correlations. For simplicity, we assume that the weights $\{w_i\}_{1\le i\le n}$ are constant.

$$\begin{aligned} dS_I &= d(\sum_{i=1}^n w_i S_i(t)) \\ &= \sum_{i=1}^n w_i S_i(t)\frac{dS_i(t)}{S_i(t)} \\ &= \sum_{i=1}^n w_i S_i(t)\left(E^Q[\frac{dS_i(t)}{S_i(t)}] + \sigma_i(t)dB_i(t)\right) \end{aligned}$$

Hence we obtain[8]

$$E^Q[dS_I(t)] = \sum_{i=1}^n w_i S_i E^Q[\frac{dS_i(t)}{S_i(t)}]. \tag{25}$$

Similarly, we have

$$E^Q[dS_I(t)dS_I(t)] = \sum_{i,j=1}^n w_i w_j S_i(t)S_j(t)\left[E^Q[\frac{dS_i(t)}{S_i(t)}]\cdot E^Q[\frac{dS_j(t)}{S_j(t)}] + \sigma_i^2(t)\sigma_j^2(t)\rho_{ij}(t)dt\right]. \tag{26}$$

[8]For the stock indices with weights that depend on the number of outstanding shares or on the price level $S_i(t)$, we have

$$dS_I = \sum_{i=1}^n w_i S_i(t)(\frac{dw_i}{w_i} + \frac{dS_i(t)}{S_i(t)} + (\frac{dw_i}{w_i}\cdot\frac{dS_i(t)}{S_i(t)}).$$

The rest computation follows exactly by replacing dS_I for the fluctuated weights in the stock indices.

Thus we have the Black-Scholes-Merton differential equation for the index

$$
\begin{aligned}
rC_I(t)dt &= E^Q[dC_I(t)] \\
&= \frac{\partial C_I}{\partial t}dt + E^Q[\frac{\partial C_I}{\partial S_I}dS_I(t) + \frac{\partial C_I}{\partial \sigma_I^2}d\sigma_I^2(t)] \\
&\quad + E^Q[\frac{1}{2}\frac{\partial^2 C_I}{\partial S_I^2}dS_I(t)dS_I(t) + \frac{\partial^2 C_I}{\partial S_I \partial \sigma_I^2}dS_I(t)d\sigma_I^2(t) + \frac{1}{2}\frac{\partial^2 C_I}{\partial \sigma_I^2(t)\partial \sigma_I^2(t)}d\sigma_I^2(t)d\sigma_I^2(t)] \\
&= \frac{\partial C_I}{\partial t}dt + \frac{\partial C_I}{\partial S_I}E^Q[dS_I(t)] + \frac{\partial C_I}{\partial \sigma_I^2}E^Q[d\sigma_I^2(t)] + \frac{1}{2}\frac{\partial^2 C_I}{\partial S_I^2}E^Q[dS_I(t)dS_I(t)] \\
&\quad + \frac{\partial^2 C_I}{\partial S_I \partial \sigma_I^2}E^Q[dS_I(t)d\sigma_I^2(t)] + \frac{1}{2}\frac{\partial^2 C_I}{\partial \sigma_I^2(t)\partial \sigma_I^2(t)}E^Q[d\sigma_I^2(t)d\sigma_I^2(t)] \\
&= \frac{\partial C_I}{\partial t}dt + \frac{\partial C_I}{\partial S_I}\cdot\sum_{i=1}^{n} w_i S_i E^Q[\frac{dS_i(t)}{S_i(t)}] + \frac{\partial C_I}{\partial \sigma_I^2}E^Q[d\sigma_I^2(t)] \\
&\quad + \frac{1}{2}\frac{\partial^2 C_I}{\partial S_I^2}\cdot\sum_{i,j=1}^{n} w_i w_j S_i(t) S_j(t)\,[\sigma_i(t)\sigma_j(t)\rho_{ij}(t)dt] \\
&\quad + \frac{\partial^2 C_I}{\partial S_I \partial \sigma_I^2}E^Q[dS_I(t)d\sigma_I^2(t)] + \frac{1}{2}\frac{\partial^2 C_I}{\partial \sigma_I^2(t)\partial \sigma_I^2(t)}E^Q[d\sigma_I^2(t)d\sigma_I^2(t)],
\end{aligned}
$$

where the last equality follows from the assumptions (3)–(5).

The instantaneous expected excess option returns of the index $S_I(t)$ is given by

$$
\frac{1}{dt}E_t^P\left[\frac{dC_I}{C_I} - rdt\right] = \Omega_I^I(t)(E^P - E^Q)[\frac{dS_I(t)}{S_I(t)}] - \frac{\partial \ln C_I}{\partial \sigma_I^2(t)}(E^Q - E^P)[d\sigma_I^2(t)],
$$

where $\Omega_I^I(t) = \frac{S_I(t)}{C_I(t)}\frac{\partial C_I(t)}{\partial S_I(t)}$ is the elasticity of the index option with respect to the index $S_I(t)$, and $(E^Q - E^P)[d\sigma_I^2(t)]$ is the risk premium of the index variance given in Proposition 2. Therefore Theorem 1 follows.

Appendix B

By the realized return variance of the index $RV_I(t,\tau) = \int_t^{t+\tau} \phi_I^2(s)ds$,

$$\begin{aligned}
\sigma_I^2(t;\tau) &= E_t^Q\left[\int_t^{t+\tau} \phi_I^2(s)ds\right] \\
&= E_t^Q\left[\int_t^{t+\tau} \sum_{1\le i,j\le n} w_i w_j \phi_i(s)\phi_j(s)\rho_{ij}(s)ds\right] \\
&= \sum_{i=1}^n w_i^2 E_t^Q\left[\int_t^{t+\tau} \phi_i^2(s)ds\right] \\
&\quad + \sum_{i=1}^n \sum_{j\ne i} w_i w_j E_t^Q\left[\int_t^{t+\tau} \phi_i(s)\phi_j(s)\rho_{ij}(s)ds\right].
\end{aligned}$$

Therefore we have the difference between the model-free implied variance of the market risk and the model-free implied variance of individual stock risk is given by

$$\sigma_I^2(t;\tau) - \sum_{i=1}^n w_i^2 E_t^Q\left[\int_t^{t+\tau} \phi_i^2(s)ds\right]$$

and the quantity for the difference is exactly the correlation risks,

$$\sum_{i=1}^n \sum_{j\ne i} w_i w_j E_t^Q\left[\int_t^{t+\tau} \phi_i(s)\phi_j(s)\rho_{ij}(s)ds\right].$$

On the other hand, we have, by (4),

$$\sigma_{MF,I}^2(t) = 2\int_0^\infty \frac{C_I(K_I,t) - (S_I(0) - K_I)^+}{K_I^2} dK_I;$$

$$\sigma_{MF,i}^2(t) = 2\int_0^\infty \frac{C_i(K_i,t) - (S_i(0) - K_i)^+}{K_i^2} dK_i.$$

Hence the identity in Proposition 4 follows.

Appendix C

For Ito processes $u(t)$ and $v(t)$, we have the Ito product rule

$$d(u(t)v(t)) = du(t) \cdot v(t) + u(t) \cdot dv(t) + du(t) \cdot dv(t).$$

Hence we have, for $u(t)$ and $v(t)$ almost surely nonzero, the Ito product rule

$$d(u(t)v(t)) = (\frac{du(t)}{u(t)} + \frac{dv(t)}{v(t)} + \frac{du(t)}{u(t)} \cdot \frac{dv(t)}{v(t)})u(t)v(t).$$

Thus $d\sigma_i^2(t) = 2\sigma_i(t)d\sigma_i(t) + d\sigma_i(t) \cdot d\sigma_i(t)$. So one obtains that $d\sigma_i^2(t) = 2\sigma_i(t)d\sigma_i(t)$ or $d\sigma_i(t) \cdot d\sigma_i(t) = 0$ if and only if $\sigma_i(t)$ is not an Ito process. Similarly,

$$\begin{aligned}
d(u_1u_2u_3) &= d(u_1 \cdot (u_2u_3)) \\
&= du_1 \cdot (u_2u_3) + u_1 d(u_2u_3) + du_1 \cdot d(u_2u_3) \\
&= du_1 \cdot (u_2u_3) + u_1 \cdot (du_2 \cdot u_3 + u_2 du_3 + du_2 \cdot du_3) \\
&\quad + du_1 \cdot (du_2 \cdot u_3 + u_2 du_3 + du_2 \cdot du_3) \\
&= du_1(u_2u_3) + du_2(u_1u_3) + du_3(u_1u_2) + du_1du_2(u_3) + du_2du_3(u_1) + du_3du_1(u_2),
\end{aligned}$$

where $du_1 \cdot du_2 \cdot du_3 = 0$ for both Ito processes u_1, u_2 and u_3. Hence we obtain the generalization of Ito product formula

$$d(u_1u_2u_3) = du_1 \cdot (u_2u_3) + du_2 \cdot (u_1u_3) + du_3 \cdot (u_1u_2) + du_1 \cdot du_2 \cdot (u_3) + du_2 \cdot du_3 \cdot (u_1) + du_3 \cdot du_1 \cdot (u_2).$$

For u_1, u_2, u_3 almost surely nonzero, the generalized Ito product formula is given by

$$d(u_1u_2u_3) = (\frac{du_1}{u_1} + \frac{du_2}{u_2} + \frac{du_3}{u_3} + \frac{du_1}{u_1} \cdot \frac{du_2}{u_2} + \frac{du_2}{u_2} \cdot \frac{du_3}{u_3} + \frac{du_3}{u_3} \cdot \frac{du_1}{u_1})(u_1u_2u_3).$$

Therefore for $\sigma_i(t), \sigma_j(t)$ and $\rho_{ij}(t)$ almost surely nonzero we have $d(\sigma_i(t)\sigma_j(t)\rho_{ij}(t)) =$

$$(\frac{d\sigma_i(t)}{\sigma_i(t)} + \frac{d\sigma_j(t)}{\sigma_j(t)} + \frac{d\rho_{ij}(t)}{\rho_{ij}(t)} + \frac{d\sigma_i(t)}{\sigma_i(t)} \cdot \frac{d\sigma_j(t)}{\sigma_j(t)} + \frac{d\sigma_j(t)}{\sigma_j(t)} \cdot \frac{d\rho_{ij}(t)}{\rho_{ij}(t)} + \frac{d\rho_{ij}(t)}{\rho_{ij}(t)} \cdot \frac{d\sigma_i(t)}{\sigma_i(t)})(\sigma_i(t)\sigma_j(t)\rho_{ij}(t)).$$

Note that $d(\sigma_i(t)\sigma_j(t)\rho_{ij}(t)) = \frac{\sigma_j(t)}{\sigma_i(t)}\rho_{ij}(t)d\sigma_i^2(t) + \sigma_i(t)\sigma_j(t)d\rho_{ij}(t)$ is given in Driessen, Maenhout and Vilkov (2009) (3) on page 1382. The term $(\frac{d\sigma_i(t)}{\sigma_i(t)} \cdot \frac{d\sigma_j(t)}{\sigma_j(t)} + \frac{d\sigma_j(t)}{\sigma_j(t)} \cdot \frac{d\rho_{ij}(t)}{\rho_{ij}(t)} + \frac{d\rho_{ij}(t)}{\rho_{ij}(t)} \cdot \frac{d\sigma_i(t)}{\sigma_i(t)})(\sigma_i(t)\sigma_j(t)\rho_{ij}(t))$ vanishes if and only if both $\sigma_i(t), \sigma_j(t)$ and $\rho_{ij}(t)$ are not Ito processes with only drifting factors. Hence the formula Driessen, Maenhout and Vilkov (2009) (3) in the total index variance risk premium is not correct for the assumption that $\sigma_i(t)$ and $\rho_{ij}(t)$ are Ito processes with nonzero diffusion terms.

$$\begin{aligned}
d\sigma_I^2(t) &= d\left(\sum_{i=1}^n w_i^2\sigma_i^2(t) + \sum_{i=1}^n\sum_{j\neq i}^n w_iw_j\sigma_i(t)\sigma_j(t)\rho_{ij}(t)\right) &(27)\\
&= \sum_{i=1}^n w_i^2 d\sigma_i^2(t) + \sum_{i=1}^n\sum_{j\neq i}^n w_iw_j d(\sigma_i(t)\sigma_j(t)\rho_{ij}(t)) &(28)\\
&= \sum_{i=1}^n w_i^2 d\sigma_i^2(t) + \sum_{i=1}^n\sum_{j\neq i}^n w_iw_j(\frac{d\sigma_i(t)}{\sigma_i(t)} + \frac{d\sigma_j(t)}{\sigma_j(t)} + \frac{d\rho_{ij}(t)}{\rho_{ij}(t)} &(29)\\
&\quad +\frac{d\sigma_i(t)}{\sigma_i(t)} \cdot \frac{d\sigma_j(t)}{\sigma_j(t)} + \frac{d\sigma_j(t)}{\sigma_j(t)} \cdot \frac{d\rho_{ij}(t)}{\rho_{ij}(t)} + \frac{d\rho_{ij}(t)}{\rho_{ij}(t)} \cdot \frac{d\sigma_i(t)}{\sigma_i(t)})(\sigma_i(t)\sigma_j(t)\rho_{ij}(t)). &(30)
\end{aligned}$$

$$E^Q[d\sigma_I^2] = \sum_{i=1}^n w_i^2 E^Q[d\sigma_i^2(t)] + \sum_{i=1}^n\sum_{j\neq i}^n w_iw_j E^Q(\frac{d\sigma_i(t)}{\sigma_i(t)} + \frac{d\sigma_j(t)}{\sigma_j(t)} + \frac{d\rho_{ij}(t)}{\rho_{ij}(t)} \quad (31)$$

$$+\frac{d\sigma_i(t)}{\sigma_i(t)} \cdot \frac{d\sigma_j(t)}{\sigma_j(t)} + \frac{d\sigma_j(t)}{\sigma_j(t)} \cdot \frac{d\rho_{ij}(t)}{\rho_{ij}(t)} + \frac{d\rho_{ij}(t)}{\rho_{ij}(t)} \cdot \frac{d\sigma_i(t)}{\sigma_i(t)})(\sigma_i(t)\sigma_j(t)\rho_{ij}(t)).$$

Hence Proposition 1 follows.

Assume that $d\sigma_i(t) = E_t^Q[d\sigma_i(t)] + \beta_{\sigma_i}(t)dB_{\sigma_i}(t)$. Then we have

$$d(\sigma_i^2(t)) = 2d\sigma_i(t) \cdot \sigma_i(t) + d\sigma_i(t) \cdot d\sigma_i(t)$$

$$= 2\sigma_i(t)E_t^Q[d\sigma_i(t)] + \beta_{\sigma_i}^2(t)dt + 2\sigma_i(t)\beta_{\sigma_i}(t)dB_{\sigma_i}(t)$$

$$= E_t^Q[d(\sigma_i^2(t))] + \zeta_i(\sigma_i)dB_{\sigma_i}(t),$$

where $E_t^Q[d(\sigma_i^2(t))] = 2\sigma_i(t)E_t^Q[d\sigma_i(t)] + \beta_{\sigma_i}^2(t)dt$ and $\zeta_i(\sigma_i) = 2\sigma_i(t)\beta_{\sigma_i}(t)$. Under the assumptions, We have the following identities.

$$d\sigma_i(t) \cdot \sigma_j(t)\rho_{ij}(t) = [E_t^Q[d\sigma_i(t)] + \beta_{\sigma_i}(t)dB_{\sigma_i}(t)] \cdot \sigma_j(t)\rho_{ij}(t); \tag{32}$$

$$\sigma_i(t)\sigma_j(t) \cdot d\rho_{ij}(t) = \sigma_i(t)\sigma_j(t) \cdot [E_t^Q[d\rho_{ij}(t)] + \rho_{ij}^0 dB_0(t) + \sum_{k=1}^n \rho_{ij}^k dB_{\sigma_k}(t)]; \tag{33}$$

$$d\sigma_i(t) \cdot d\sigma_j(t) \cdot \rho_{ij}(t) = \beta_{\sigma_i}(t)\beta_{\sigma_j}(t)dB_{\sigma_i}(t)dB_{\sigma_j}(t) \cdot \rho_{ij}(t); \tag{34}$$

where $E_t^Q[d\sigma_i(t)]E_t^Q[d\sigma_j(t)] = 0, E_t^Q[d\sigma_i(t)]dB_{\sigma_j}(t) = 0$ from the stochastic calculus,

$$\sigma_i(t) \cdot d\sigma_j(t) \cdot d\rho_{ij}(t) = \sigma_i(t)\beta_{\sigma_j}(t)\rho_{ij}^j(t)dt. \tag{35}$$

By using (32), (33), (34), (35) and the previous calculation, we obtain

$$d\sigma_I^2(t) = \sum_{i=1}^n w_i^2 d\sigma_i^2(t) + \sum_{1\le i\neq j\le n} w_i w_j([E_t^Q[d\sigma_i(t)] + \beta_{\sigma_i}(t)dB_{\sigma_i}(t)] \cdot \sigma_j(t)\rho_{ij}(t)+ \tag{36}$$

$$[E_t^Q[d\sigma_j(t)] + \beta_{\sigma_j}(t)dB_{\sigma_j}(t)] \cdot \sigma_i(t)\rho_{ij}(t) + \sigma_i(t)\sigma_j(t) \cdot [E_t^Q[d\rho_{ij}(t)] + \rho_{ij}^0 dB_0(t) + \sum_{k=1}^n \rho_{ij}^k dB_{\sigma_k}(t)]+$$

$$\beta_{\sigma_i}(t)\beta_{\sigma_j}(t)dB_{\sigma_i}(t)dB_{\sigma_j}(t) \cdot \rho_{ij}(t) + \sigma_i(t)\beta_{\sigma_j}(t)\rho_{ij}^j(t)dt + \sigma_j(t)\beta_{\sigma_i}(t)\rho_{ij}^i(t)dt).$$

Therefore the expectation of the instantaneous change in index variance under the risk neutral measure is given by

$$E_t^Q[d\sigma_I^2(t)] = \sum_{i=1}^n w_i^2 E_t^Q[d\sigma_i^2(t)] + \sum_{i=1}\sum_{j\neq i} w_i w_j (E_t^Q[d\sigma_i(t)] \cdot E_t^Q[\sigma_j(t)\rho_{ij}(t)]+ \tag{37}$$

$$E_t^Q[d\sigma_j(t)] \cdot E_t^Q[\sigma_i(t)\rho_{ij}(t)] + E_t^Q[\sigma_i(t)\sigma_j(t)]E_t^Q[d\rho_{ij}(t)] + E_t^Q[\sigma_i(t)\beta_{\sigma_j}(t)\rho_{ij}^j(t) + \sigma_j(t)\beta_{\sigma_i}(t)\rho_{ij}^i(t)]dt).$$

References

Ball, C. and W. Torus (1985) “On jumps in common stock prices and their impact on call option pricing.” *Journal of Finance*, 40, 155-173.

Bakshi, G., C. Cao, and Z. Chen (1997) “Empirical performance of alternative option pricing models.” *Journal of Finance*, 52, 2003-2049.

Bakshi, G., Kapadia, N., and D. Madan (2003) “ Stock Return Characteristics, Skew Laws, and Differential Pricing of Individual Equity Options,” *Review of Financial Studies,* 16 (1), 101–143.

Black, F., and M. Scholes (1973) “The pricing of options and corporate liabilities.” *Journal of Political Economy*, 81, 637-659.

Britten-Jones and Neuberger (2000), “ Option Prices, Implied Price Processes and Stochastic Volatility”, *Journal of Finanace*, 55, 839-866.

Broadie, M., Chernov, M., and Johannes, M., (2009). “Understanding Index Option Returns”, *Review of Financial Studies 22*, 4403–4529.

Buraschi, A., Porchia, P., and Trojani, F., (2010) “Correlation Risk and Optimal Portfolio Choice”, *Journal of Finance*, 65, 393–420.

Camara, A, Krehbiel, T and Li, W (2011), “Expected returns, risk premia, and volatility surfaces implicit in option market prices”. *Journal of Banking and Finance*, 35, 215-230.

Cao. C., and Huang, J., (2008) Determinants of S&P 500 index option returns, *Review of Derivatives Research 10*, 1–38.

Canina, L., and Figlewski, S. (1993) “The informational content of implied volatility”, *Review of Financial Studies*, 6, 659-681.

Chambers,D.R.,M.Foy, J.Liebner,Q.Lu (2014) ”Index Option Returns Still Puzzling”. *The Review of Financial Studies*, 27,1915-1928.

Constantinides, G.M., Jackwerth, J. C., and Perrakis (2009) Mispricing of S&P 500 index option. *Review of Financial Studies 22*, 1247–1277.

Constantinides, G.M., Jackwerth, J. C., and Savov, A., (2013), "The puzzle of Index Option Returns", *Review of Financial Studies*, forthcoming.

Driessen, J., P. Maenhout, G. Vilkov. (2009) "The price of correlation risk: Evidence from equity options." *Journal of Finance*, 64, 1377-1406.

Driessen, Joost and Maenhout, Pascal J. and Vilkov, Grigory, Option-Implied Correlations and the Price of Correlation Risk (July 26, 2013). Advanced Risk & Portfolio Management Paper. Available at SSRN.

Duffie, D., J. Pan, and K. Singleton (2000) "Transform analysis and asset pricing for affine jump diffusions." *Econometrica*, 68, 1343-1376.

Goyal, A. and Santa-Clara, P (2003) "Idiosyncratic Risk Matters!", *The Journal of Finance*, 58, 975-1007.

Heston, S., (1993) "A closed-form solution for options with stochastic volatility with applications to Bond and Currency options", *Review of Financial Studies*, 6, 327-343.

Jiang and Tian (2005), "The Model-free implied volatility and its information content". *Review of Financial Studied*, 18, 1305-1342.

Jones, C. S., (2006) A nonlinear factor analysis of S&P 500 index option returns, *Journal of Finance 61*, 2325–2363.

Latané, Henery A., and Rendelman, Richard. Jr. (1976) "Standard deviations of stock price ratios implied in option prices." *Journal of Finance*, 31, 369-381.

Merton, R. (1976) "Option pricing when underlying stock returns are discontinuous." *Journal of Financial Economics*, 3, 125-144.

Myneni, R., (1992) The pricing of the American option, *The Annals of Applied Probability*, 2, 1–23.

Renault, E., and Touzi, N. (1996), "Option Hedging and Implicit Volatilities", *Mathematical Finance*, 6, 279-302.

Yan, Shu., (2011) "Jump risk, stock returns and slope of implied volatility smile". *Journal of*

Financial Economics, 99, 216-233.

Tables and Figures:

Table 1
Variance Risk Premia in Index and Component Options

The table reports the time-series average of realized and model-free implied standard deviations for Dow Jones Industrial Average options and for component options on the stocks in the Dow average over the 10/6/1997 to 8/14/2008 sample period. For component options the standard deviations are price weighted cross-sectional averages, WAVG, across all component stocks. Realized standard deviation, RVOL, is calculated from squared daily returns over a 30-day window. The model-free implied standard deviation, MFIV, is calculated from a cross-section (across strikes) of options, using the methodology of Britten-Jones and Neuberger (2000) and Jiang and Tan (2005). The data on option prices are from OptionMetrics and standard deviations are expressed in annual terms. The p-values, based on Newey and West (1987) heteroskedasticity and autocorrelation consistent standard errors with 9 lags, are for the null hypothesis that implied and realized standard deviation are on average equal.

	Index	WAVG
RVOL	0.1676	0.2938
MFIV	0.1999	0.2795
RVOL-MFIV	-0.0323	0.0142
H_0 : RVOL-MFIV=0	2.5523e-28	1.1308e-05

Figure 1
Implied versus realized volatility for index options

The figure presents the time series of the square root of the model-free implied index variance and of the square root of the realized variance over our 10/1997 to 8/2008 OptionMetrics sample. The model-free implied index variance is calculated from a cross-section (across strikes)of options on the Dow Jones Industrial Average, using the methodology of Britten-Jones and Neuberger (2000) and Jiang and Tian (2005). Realized standard deviation, RVOL, is calculated from squared daily returns over a 30-day window. Standard deviations are expressed in annual terms.

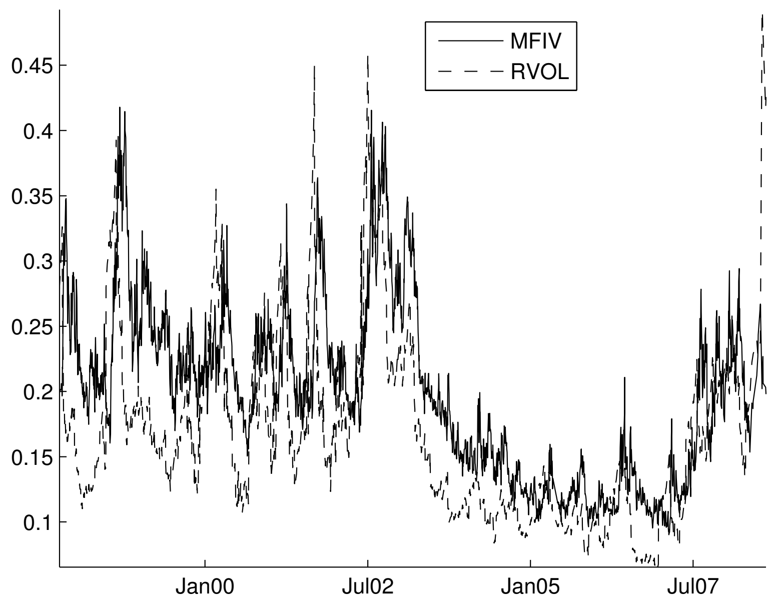

Figure 2
Cross-sectional average of implied and realized volatility for component options.

The figure presents the time series of the (price weighted) cross-sectional average, WAVG, of the square root of the model-free implied component stock variance and of the square root of the realized component stock variance over our 10/1997 to 8/2008 OptionMetrics sample. For each component stock in the Dow index, the model-free implied index variance is calculated from a cross-section (across strikes)of options, using the methodology of Britten-Jones and Neuberger (2000) and Jiang and Tian (2005). Realized standard deviation, RVOL, is calculated from squared daily CRSP stock returns over a 30-day window. Standard deviations are expressed in annual terms. Because of changes in the roster of the component stocks, a total of 41 component stocks are considered over the 12-year sample period.

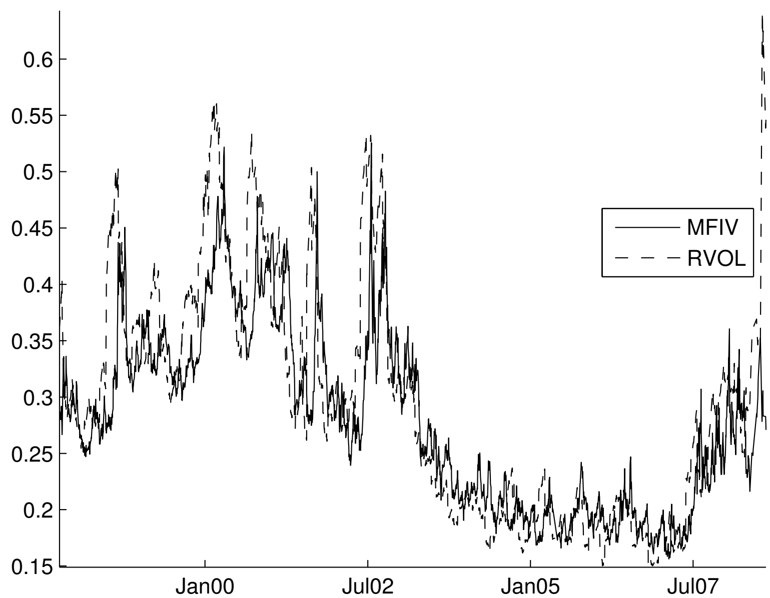

Table 2
Model-free Implied Standard Deviations
Index and Weighted Average of Components

The table reports the time-series average of the model-free implied standard deviations for Dow Jones Industrial Average options and for component options on the stocks in the Dow average over the 10/6/1997 to 8/14/2008 sample period. For component options the standard deviations are price weighted cross-sectional averages, WAVG, across all component stocks. The model-free implied standard deviation, MFIV, is calculated from a cross-section (across strikes) of options, using the methodology of Britten-Jones and Neuberger (2000) and Jiang and Tan (2005). The data on option prices are from OptionMetrics and standard deviations are expressed in annual terms. The p-values, based on Newey and West (1987) heteroskedasticity and autocorrelation consistent standard errors with 9 lags, are for the null hypothesis that model-free implied standard deviations are on average equal.

	MFIV
Index	0.1999
WAVG	0.2795
Index -WAVG	-0.0796
H_0 : WAVG-Index=0	3.474e-289

Figure 3
Model-free Implied Standard Deviations Index and Weighted Average of Components

The figure presents the time series of the model-free implied standard deviations calculated from index options, Index, and a price weighted cross sectional average, WAVG, over our 10/1997 to 8/2008 OptionMetrics sample. For the index and each component stock in the Dow index, the model-free implied variance is calculated from a cross-section (across strikes)of options, using the methodology of Britten-Jones and Neuberger (2000) and Jiang and Tian (2005). Standard deviations are expressed in annual terms.

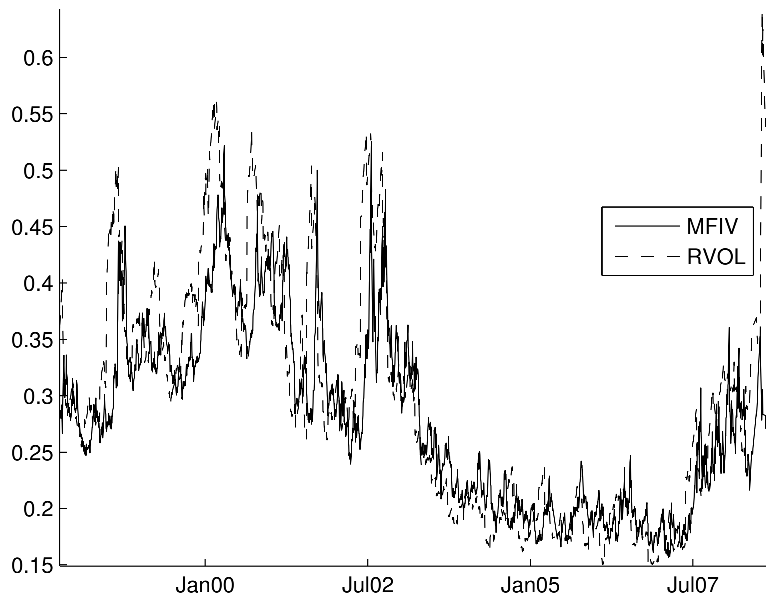

Table 3
Systemic Risk Indicator: $MFIV_{DJX}$ **-** $MFIV_{WAVG}$

The table reports the time-series descriptive statistics of the difference between model free implied volatility for Dow Jones Industrial Average options and for component options on the stocks in the Dow average over the 10/6/1997 to 8/14/2008 sample period. For component options the standard deviations are price weighted cross-sectional averages, WAVG, across all component stocks. The model-free implied standard deviation, MFIV, is calculated from a cross-section (across strikes) of options, using the methodology of Britten-Jones and Neuberger (2000) and Jiang and Tan (2005). The data on option prices are from OptionMetrics and standard deviations are expressed in annual terms.

	Indicator
Maximum	0.0329
Median	-0.0684
Minimum	-0.2577
Standard Deviation	0.0449
N	2591

Figure 4
Systemic Risk Indicator and DJX Squared Log Change

The figure presents the time series of the systemic risk indicator and squared daily log change in the Dow Jones Average, over our 10/1997 to 8/2008 OptionMetrics sample. The circled area identifies a period of significant market turbulence indicated by the large squared logarithmic change in DJX **not** accompanied by a corresponding erosion of the benefits of diversification the latter indicated by contemporaneous values of Systemic Risk Indicator significantly less than zero.

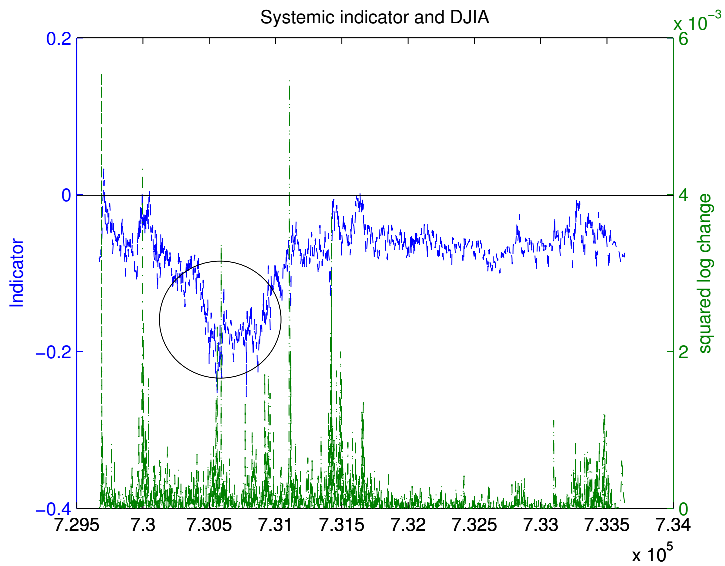

Table 4
Systemic Risk Indicator: $MFIV_{DJX}$ **-** $MFIV_{WAVG} >= -0.01$

The table includes dates during sample period where the Systemic Risk Indicator approaches zero, indicative of erosion of the benefit of diversification.

04-Nov-1997	27-Aug-1998	02-Aug-2002	22-Jan-2003	15-Aug-2007
05-Nov-1997	28-Aug-1998	09-Aug-2002	24-Jan-2003	17-Aug-2007
06-Nov-1997	16-Oct-1998	12-Aug-2002	29-Jan-2003	10-Sep-2007
07-Nov-1997	19-Oct-1998		31-Jan-2003	
10-Nov-1997	20-Oct-1998		03-Feb-2003	
11-Nov-1997	21-Oct-1998		04-Feb-2003	
12-Nov-1997	27-Oct-1998		05-Feb-2003	
13-Nov-1997	28-Oct-1998		06-Feb-2003	
14-Nov-1997			07-Feb-2003	
24-Nov-1997			10-Feb-2003	
25-Nov-1997			21-Feb-2003	
			24-Feb-2003	
			25-Feb-2003	
			07-Mar-2003	
			17-Mar-2003	

Table 5
Systemic Risk Indicator Descriptive Statistics

The table presents Systemic Risk Indicator descriptive statistics for the entire sample and four sub-periods chosen to be of approximately equal number of observations. Positive values of the Systemic Risk Indicator occur during subperiods A and C.

	Entire	A	B	C	D
max	0.032889	0.032889	-0.0055028	0.0035315	-0.0059791
median	-0.068436	-0.077204	-0.077698	-0.058537	-0.051995
mean	-0.079655	-0.074036	-0.087377	-0.055617	-0.052447
min	-0.25766	-0.17794	-0.22942	-0.084117	-0.097139
std	0.044909	0.035685	0.043009	0.017274	0.0177

Entire: sample	10/06/97 to 08/14/08	N = 2591
A	10/06/97 to 12/29/99	N = 549
B	12/29/00 to 12/31/02	N = 490
C	01/02/03 to 12/31/04	N = 499
D	01/03/07 to 08/14/08	N = 343

Figure 5
Systemic Risk Indicator and DJIA Squared Log Change 10/06/97 to 12/29/99

The figure illustrates market turbulence during fourth quarter 1997 and third quarter 1998 occurs contemporaneously with an increase in the Systemic Risk Indicator indicative of deterioration in the benefits of diversification during these periods.

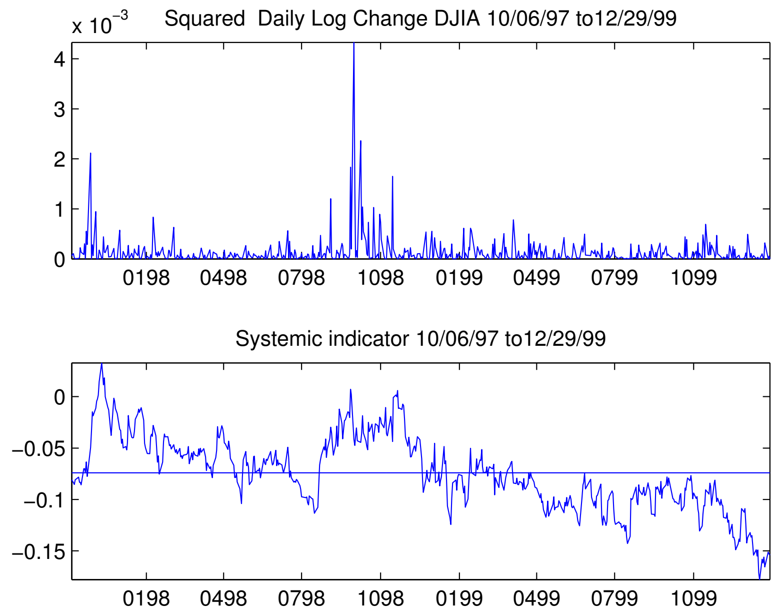

Figure 6
Systemic Indicator and DJIA Squared Log Change 12/29/00 to 12/31/02

The figure illustrates that market turbulence during third quarter 2002 was accompanied by deterioration in the benefits of diversification as indicated by the increase in the Systemic Risk Indicator during this period.

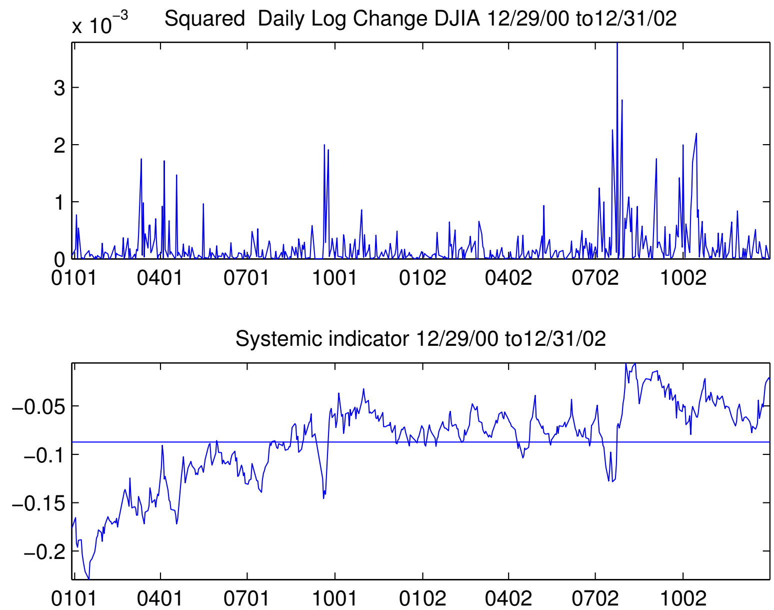

Figure 7
Systemic Indicator and DJIA Squared Log Change 01/02/03 to 12/31/04

The figure illustrates that market turbulence during first quarter 2003 was accompanied by deterioration in the benefits of diversification as indicated by the increase in the Systemic Risk Indicator during this period.

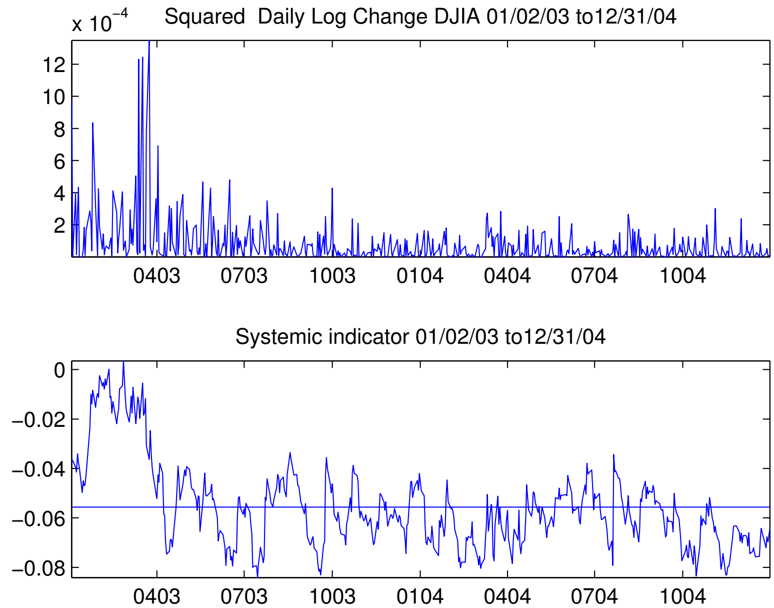

Figure 8
Systemic Indicator and DJIA Squared Log Change 01/03/07 to 08/14/08

The figure illustrates the performance of the Systemic Risk Indicator through the onset of the financial crisis. Interestingly the Indicator did not attain positive values in this period as it did during previous periods of market turbulence.