Effectiveness of linear extrapolation in model-free implied moment estimation*

Geul Lee†

School of Banking and Finance, University of New South Wales

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Abstract

This study proposes a new methodology with which the effectiveness of linear extrapolation (LE) of Jiang and Tian (2005) for the implied moment estimators of Bakshi et al. (2003) can be evaluated, even when the true moments are unknown. Using S&P 500 index options data and truncation sensitivity functions, this study suggests that although LE is effective for all three estimators, the implied skewness and kurtosis estimators can remain sensitive to truncation, i.e., the complete unavailability of option prices for the outermost part of deep-out-of-the-money or deep-in-the-money region of strike price domain, even when LE is employed to mitigate truncation.

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†Address: Level 3, East Wing, Business School Building, UNSW Australia, Sydney, 2052 NSW Australia. Phone: (61) 02 9385 5867. Fax: (61) 02 9385 6347. E-mail address: geul.lee@unsw.edu.au

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Abstract

This study proposes a new methodology with which the effectiveness of linear extrapolation (LE) of Jiang and Tian (2005) for the implied moment estimators of Bakshi et al. (2003) can be evaluated, even when the true moments are unknown. Using S&P 500 index options data and truncation sensitivity functions, this study suggests that although LE is effective for all three estimators, the implied skewness and kurtosis estimators can remain sensitive to truncation, i.e., the complete unavailability of option prices for the outermost part of deep-out-of-the-money or deep-in-the-money region of strike price domain, even when LE is employed to mitigate truncation.

1. Introduction

Although the question of how unobservable deep-out-of-the-money (DOTM) option prices can be inferred is raised and answered in several studies, linear extrapolation (LE) has attained its dominant popularity as a DOTM option price approximation method for model-free implied moment estimation in recent literature because of its minor effect on the model-freeness of the implied moment estimators. Recently, with the increased popularity of the implied moment estimators of Bakshi et al. (2003), the use of LE has been extended and is mostly in conjunction with the implied skewness and kurtosis estimators of Bakshi et al. (2003) (e.g., Buss and Vilkov, 2012; Chang et al., 2012; Chang et al., 2013; DeMiguel et al., 2013; Neumann and Skiadopoulos, 2013).

LE was first introduced by Jiang and Tian (2005) to reduce the estimation error due to truncation, i.e., the complete unavailability of option prices for the outermost part of DOTM or deep-in-the-money (DITM) region of strike price domain, of the model-free implied volatility estimator of Britten-Jones and Neuberger (2000). Truncation induces an estimation bias of the model-free implied moment estimator because one is required to conduct integrations of weighted option price with respect to strike price from zero to positive infinity in order to estimate implied moments.[1] Jiang and Tian (2005) call the estimation error due to truncation as a "truncation error" because such an error is caused by option prices that look as if they are "truncated."

LE is conducted initially through the assumption that the Black-Scholes implied volatility curve is flat beyond the minimum and maximum strike prices for which the option price is observable, followed by the extension of the curve on the basis of such an assumption. The option prices beyond the minimum and maximum strike price can then be inferred from this extended curve, and the integrations can be done with respect to strike price from zero to positive infinity with the newly obtained option prices. Given this definition, LE can be regarded as a zeroth-order approximation of Black-Scholes implied volatility. As the order suggests, LE is a rough approximation method. However, Jiang and Tian (2005) show that LE renders the estimation error smaller than in cases of no treatment. From this case, LE can be conjectured to have at least a positive effect on reducing the truncation error of the implied volatility estimator.

However, two questions should be answered before applying LE, especially for implied skewness or kurtosis estimation. First, how effective is LE? In other words, is it adequately effective to make the truncation error negligible? The severity of truncation is different across markets and over time, so the truncation error can generate noise in the estimate if it is incompletely reduced. Hence, although determining whether LE can alleviate truncation error is by itself valuable, checking if LE can alleviate the truncation error up to the level where the error becomes insignificant is also needed. Second, can LE be applied to higher moment estimators, i.e., the implied skewness and kurtosis estimators of Bakshi et al. (2003)? Although Jiang and Tian (2005) show the effectiveness of LE and derives the upper bound of truncation error for the implied volatility estimator of Britten-Jones and Neuberger (2000), neither the effectiveness nor the upper bound can be accepted as it is when LE is employed for another estimator. Furthermore, given that skewness and kurtosis are more closely related to the shape of the tail

[1] Weighted call price needs to be integrated for the implied volatility estimator of Britten-Jones and Neuberger (2000), whereas weighted OTM price needs to be integrated for the implied moment estimators of Bakshi et al. (2003).

density than volatility is, the truncation error is likely larger for these higher moment estimators. Therefore, the effectiveness of LE should be assessed separately when it is used for the higher moment estimators of Bakshi et al. (2003).

This study proposes a measure with which the effectiveness of LE can be assessed for any set of out-of-the-money (OTM) European option prices from which a differentiable Black-Scholes implied volatility function of log-moneyness can be estimated. A significant advantage of this measure is that it can be applied to option prices that are observed from markets. In most related studies, option prices are generated using option pricing models to demonstrate the truncation error (e.g., Dennis and Mayhew, 2002; Jiang and Tian, 2005; Dennis and Mayhew, 2009) because the true level of the implied moment is required to calculate the truncation error. In contrast, the problem of the unavailability of the true implied moment is circumvented in this study through the use of a different approach to measure the effectiveness of LE.

If the truncation error is reduced effectively so that the effect of truncation on the implied moment estimate is negligible, the estimate should not change significantly after a marginal change in the width of the integration domain. In other words, if the implied moment estimate varies significantly when the moment is re-estimated after an inclusion or exclusion of a few option prices at the end of the integration domain while LE is consistently applied, it can be assumed that the truncation error has not been fully reduced. Hence, the sensitivity of the implied moment estimator to a marginal change in the integration domain width, i.e., the truncation sensitivity of the implied moment estimator, can be used to assess the effectiveness of the truncation treatment method. This study shows that the sensitivity can be formulated for the estimators of Bakshi et al. (2003) regardless of whether LE is applied. Measuring the effectiveness of LE is therefore possible through a comparison of the sensitivity of implied moment estimators with and without the application of LE.

The following results are obtained through an empirical analysis of S&P 500 index options data. First, all three implied moment estimators of Bakshi et al. (2003) tend to become less sensitive to truncation after LE. This result implies that LE is indeed effective for all three estimators. With this result, LE can be conjectured to also reduce the truncation error of model-free implied moment estimators other than the implied volatility estimator of Britten-

Jones and Neuberger (2000). Second, the implied volatility estimator of Bakshi et al. (2003) is found to be considerably insensitive to truncation when applied to S&P 500 index options data, regardless of whether LE is applied. Finally, implied skewness and kurtosis estimators are found to be relatively sensitive to truncation even when LE is employed, especially when the estimation heavily relies on the option prices that are approximated by LE.

Although the overall result suggests that LE is effective for all implied moment estimators of Bakshi et al. (2003), the truncation error can also be too large to be regarded as negligible even after LE when implied skewness or kurtosis is estimated. This result implies that a supplementary truncation treatment may be needed along with LE when a higher implied moment is estimated. A possible supplementary method that is easy to employ is an additional data filtration, which removes observations with too high truncation sensitivity with LE. With this, the truncation sensitivity function is by itself found to be an instrument for truncation treatment.

The rest of this paper is organized as follows. Section 2 explains how the truncation sensitivity of implied moment estimators can be formulated, for both cases with and without LE applied. Section 3 describes the data used in this study. Section 4 demonstrates how the sensitivity function defined in Section 2 can be applied to option prices. Section 5 reports the results of empirical analysis on S&P 500 index options data. Section 6 concludes this study.

2. Truncation Sensitivity Function

This section first revisits the basic definitions of implied moment estimators in Bakshi et al. (2003) and then demonstrates how the sensitivity of estimators to a marginal change in option price availability can be formulated. Section 2.1 briefly describes the implied moment estimators and explains how they can be rearranged in terms of log-moneyness. Section 2.2 shows how sensitivity can be defined. Section 2.3 explains how truncation sensitivity should be measured when LE is applied.

2.1 Implied moment estimators

2.1.1 Definition

To estimate the skewness and kurtosis of implied risk-neutral log-return density, Bakshi et al. (2003) first introduce volatility contract V, cubic contract W, and quartic contract X whose payoffs at maturity are the second, third, and fourth power of the holding period log-return, respectively.[2] Using OTM option prices, Bakshi et al. (2003) construct the fair value of these contracts under the risk-neutral measure as

$$\begin{aligned} V(t,\tau) = & \int_{S(t)}^{\infty} \frac{2\left(1 - \ln\left[\frac{K}{S(t)}\right]\right)}{K^2} C(t,\tau;K)dK \\ & + \int_{0}^{S(t)} \frac{2\left(1 + \ln\left[\frac{S(t)}{K}\right]\right)}{K^2} P(t,\tau;K)dK, \end{aligned} \tag{1}$$

$$\begin{aligned} W(t,\tau) = & \int_{S(t)}^{\infty} \frac{6\ln\left[\frac{K}{S(t)}\right] - 3\left(\ln\left[\frac{K}{S(t)}\right]\right)^2}{K^2} C(t,\tau;K)dK \\ & - \int_{0}^{S(t)} \frac{6\ln\left[\frac{S(t)}{K}\right] + 3\left(\ln\left[\frac{S(t)}{K}\right]\right)^2}{K^2} P(t,\tau;K)dK, \end{aligned} \tag{2}$$

$$\begin{aligned} X(t,\tau) = & \int_{S(t)}^{\infty} \frac{12\left(\ln\left[\frac{K}{S(t)}\right]\right)^2 - 4\left(\ln\left[\frac{K}{S(t)}\right]\right)^3}{K^2} C(t,\tau;K)dK \\ & + \int_{0}^{S(t)} \frac{12\left(\ln\left[\frac{S(t)}{K}\right]\right)^2 + 4\left(\ln\left[\frac{S(t)}{K}\right]\right)^3}{K^2} P(t,\tau;K)dK, \end{aligned} \tag{3}$$

where $C(t,\tau;K)$ and $P(t,\tau;K)$ denote the price of call and put options with strike price K and time to maturity τ at day t, respectively. Then, the τ-period implied risk-neutral volatility, skewness, and kurtosis can be derived as

$$\text{VOL}(t,\tau) = \left[e^{r\tau}V(t,\tau) - \mu^2(t,\tau)\right]^{1/2}, \tag{4}$$

$$\text{SKEW}(t,\tau) = \frac{e^{r\tau}W(t,\tau) - 3\mu(t,\tau)e^{r\tau}V(t,\tau) + 2\mu^3(t,\tau)}{\left[e^{r\tau}V(t,\tau) - \mu^2(t,\tau)\right]^{3/2}}, \tag{5}$$

[2]Because skewness and kurtosis are defined as standardized skewness and kurtosis, volatility can also be estimated from the denominator of either the skewness or kurtosis estimator.

$$\text{KURT}(t,\tau) = \frac{e^{r\tau}X(t,\tau) - 4\mu(t,\tau)e^{r\tau}W(t,\tau) + 6e^{r\tau}\mu^2(t,\tau)V(t,\tau) - 3\mu^4(t,\tau)}{[e^{r\tau}V(t,\tau) - \mu^2(t,\tau)]^2}, \tag{6}$$

where

$$\mu(t,\tau) = e^{r\tau} - 1 - \frac{e^{r\tau}}{2}V(t,\tau) - \frac{e^{r\tau}}{6}W(t,\tau) - \frac{e^{r\tau}}{24}X(t,\tau). \tag{7}$$

2.1.2 Rearrangement

If the parameters t and τ are removed for brevity, and λ denotes the log-moneyness $\ln(K/S)$, Equations (1)–(3) can be rearranged as

$$V = \int_0^{\infty} \frac{2\,(1-\lambda)}{Se^{\lambda}} C(\lambda) d\lambda + \int_{-\infty}^{0} \frac{2\,(1-\lambda)}{Se^{\lambda}} P(\lambda) d\lambda, \tag{8}$$

$$W = \int_0^{\infty} \frac{6\lambda - 3\lambda^2}{Se^{\lambda}} C(\lambda) d\lambda + \int_{-\infty}^{0} \frac{6\lambda - 3\lambda^2}{Se^{\lambda}} P(\lambda) d\lambda, \tag{9}$$

$$X = \int_0^{\infty} \frac{12\lambda^2 - 4\lambda^3}{Se^{\lambda}} C(\lambda) d\lambda + \int_{-\infty}^{0} \frac{12\lambda^2 - 4\lambda^3}{Se^{\lambda}} P(\lambda) d\lambda. \tag{10}$$

With Equations (8)–(10), option prices are considered in a log-moneyness domain of integration, not a strike price one. Not only does this rearrangement simplify the definitions and allow the domain of integration to be more closely related to the domain of risk-neutral density function, but this also enables the easy definition of the minimum and maximum values of the integration domain as simple univariate functions, as will be shown below.

2.2 Definition of sensitivity

Now, suppose that the minimum and maximum values of the log-moneyness domain are finite, i.e., a truncation does exist, and the endpoint values are derived from differentiable univariate functions $f(\alpha)$ and $g(\alpha)$, which satisfy the conditions $f(\alpha) \leq 0$ and $g(\alpha) \geq 0$ for all α, respectively. If this is the case, the fair value estimates can be defined as univariate functions $\widehat{V}(\alpha), \widehat{W}(\alpha)$, and $\widehat{X}(\alpha)$ as follows:

$$\widehat{V}(\alpha) = \int_0^{g(\alpha)} \frac{2(1-\lambda)}{Se^{\lambda}} C(\lambda) d\lambda + \int_{f(\alpha)}^{0} \frac{2(1-\lambda)}{Se^{\lambda}} P(\lambda) d\lambda, \tag{11}$$

$$\widehat{W}(\alpha) = \int_0^{g(\alpha)} \frac{6\lambda - 3\lambda^2}{Se^{\lambda}} C(\lambda) d\lambda + \int_{f(\alpha)}^{0} \frac{6\lambda - 3\lambda^2}{Se^{\lambda}} P(\lambda) d\lambda, \tag{12}$$

$$\widehat{X}(\alpha) = \int_0^{g(\alpha)} \frac{12\lambda^2 - 4\lambda^3}{Se^{\lambda}} C(\lambda) d\lambda + \int_{f(\alpha)}^{0} \frac{12\lambda^2 - 4\lambda^3}{Se^{\lambda}} P(\lambda) d\lambda. \tag{13}$$

Then, the implied volatility, skewness, and kurtosis estimates can also be obtained as univariate functions:

$$\widehat{\mathrm{VOL}}(\alpha) = [e^{r\tau}\widehat{V}(\alpha) - \widehat{\mu}^2(\alpha)]^{1/2}, \tag{14}$$

$$\widehat{\mathrm{SKEW}}(\alpha) = \frac{e^{r\tau}\widehat{W}(\alpha) - 3\widehat{\mu}(\alpha)e^{r\tau}\widehat{V}(\alpha) + 2\widehat{\mu}^3(\alpha)}{[e^{r\tau}\widehat{V}(\alpha) - \widehat{\mu}^2(\alpha)]^{3/2}}, \tag{15}$$

$$\widehat{\mathrm{KURT}}(\alpha) = \frac{e^{r\tau}\widehat{X}(\alpha) - 4\widehat{\mu}(\alpha)e^{r\tau}\widehat{W}(\alpha) + 6e^{r\tau}\widehat{\mu}^2(\alpha)\widehat{V}(\alpha) - 3\widehat{\mu}^4(\alpha)}{[e^{r\tau}\widehat{V}(\alpha) - \widehat{\mu}^2(\alpha)]^2}, \tag{16}$$

where

$$\widehat{\mu}(\alpha) = e^{r\tau} - 1 - \frac{e^{r\tau}}{2}\widehat{V}(\alpha) - \frac{e^{r\tau}}{6}\widehat{W}(\alpha) - \frac{e^{r\tau}}{24}\widehat{X}(\alpha). \tag{17}$$

Given Equations (11)–(16), the derivative of $\widehat{\mathrm{VOL}}(\alpha)$, $\widehat{\mathrm{SKEW}}(\alpha)$, and $\widehat{\mathrm{KURT}}(\alpha)$ with respect to α can be formulated as

$$\widehat{\mathrm{VOL}}'(\alpha) = \frac{1}{2}[e^{r\tau}\widehat{V}(\alpha) - \widehat{\mu}^2(\alpha)]^{-1/2}\left(e^{r\tau}\widehat{V}'(\alpha) - 2\widehat{\mu}(\alpha)\widehat{\mu}'(\alpha)\right), \tag{18}$$

$$\widehat{\mathrm{SKEW}}'(\alpha) = \frac{\widehat{\Theta}_S(\alpha)\widehat{\Gamma}'_S(\alpha) - \widehat{\Gamma_S}(\alpha)\widehat{\Theta}'_S(\alpha)}{\widehat{\Theta}^2_S(\alpha)}, \tag{19}$$

$$\widehat{\mathrm{KURT}}'(\alpha) = \frac{\widehat{\Theta}_K(\alpha)\widehat{\Gamma}'_K(\alpha) - \widehat{\Gamma}_K(\alpha)\widehat{\Theta}'_K(\alpha)}{\widehat{\Theta}^2_K(\alpha)}, \tag{20}$$

where $\widehat{\Gamma}_S(\alpha)$ and $\widehat{\Gamma}_K(\alpha)$ denote the numerator of $\widehat{\mathrm{SKEW}}(\alpha)$ and $\widehat{\mathrm{KURT}}(\alpha)$, respectively, and $\widehat{\Theta}_S(\alpha)$ and $\widehat{\Theta}_K(\alpha)$ denote the denominator of $\widehat{\mathrm{SKEW}}(\alpha)$ and $\widehat{\mathrm{KURT}}(\alpha)$, respectively. Given Equations (11)–(16), the derivatives in Equations (18)–(20) can be obtained as

$$\widehat{\Gamma}'_S(\alpha) = e^{r\tau}\widehat{W}'(\alpha) - 3e^{r\tau}\left(\widehat{V}(\alpha)\widehat{\mu}'(\alpha) + \widehat{\mu}(\alpha)\widehat{V}'(\alpha)\right) + 6\widehat{\mu}^2(\alpha)\widehat{\mu}'(\alpha), \tag{21}$$

$$\widehat{\Theta}'_S(\alpha) = \frac{3}{2}[e^{r\tau}\widehat{V}(\alpha) - \widehat{\mu}^2(\alpha)]^{1/2}\left(e^{r\tau}\widehat{V}'(\alpha) - 2\widehat{\mu}(\alpha)\widehat{\mu}'(\alpha)\right), \tag{22}$$

$$\widehat{\Gamma}'_K(\alpha) = e^{r\tau}\widehat{X}'(\alpha) - 4e^{r\tau}\left(\widehat{W}(\alpha)\widehat{\mu}'(\alpha) + \widehat{\mu}(\alpha)\widehat{W}'(\alpha)\right)$$
$$+ 6e^{r\tau}\left(2\widehat{\mu}(\alpha)\widehat{V}(\alpha)\widehat{\mu}'(\alpha) + \widehat{\mu}^2(\alpha)\widehat{V}'(\alpha)\right) - 12\widehat{\mu}^3(\alpha)\widehat{\mu}'(\alpha), \tag{23}$$
$$\widehat{\Theta}'_K(\alpha) = 2[e^{r\tau}\widehat{V}(\alpha) - \widehat{\mu}^2(\alpha)]\left(e^{r\tau}\widehat{V}'(\alpha) - 2\widehat{\mu}(\alpha)\widehat{\mu}'(\alpha)\right), \tag{24}$$
$$\widehat{V}'(\alpha) = g'(\alpha)\frac{2(1-g(\alpha))}{Se^{g(\alpha)}}C(g(\alpha)) - f'(\alpha)\frac{2(1-f(\alpha))}{Se^{f(\alpha)}}P(f(\alpha)), \tag{25}$$
$$\widehat{W}'(\alpha) = g'(\alpha)\frac{6g(\alpha)-3g^2(\alpha)}{Se^{g(\alpha)}}C(g(\alpha)) - f'(\alpha)\frac{6f(\alpha)-3f^2(\alpha)}{Se^{f(\alpha)}}P(f(\alpha)), \tag{26}$$
$$\widehat{X}'(\alpha) = g'(\alpha)\frac{12g^2(\alpha)-4g^3(\alpha)}{Se^{g(\alpha)}}C(g(\alpha)) - f'(\alpha)\frac{12f^2(\alpha)-4f^3(\alpha)}{Se^{f(\alpha)}}P(f(\alpha)), \tag{27}$$
$$\widehat{\mu}'(\alpha) = -e^{r\tau}\left(\frac{1}{2}\cdot\widehat{V}'(\alpha) + \frac{1}{6}\cdot\widehat{W}'(\alpha) + \frac{1}{24}\cdot\widehat{X}'(\alpha)\right). \tag{28}$$

2.3 Truncation sensitivity with LE

This subsection derives the sensitivity of the implied moment estimate to a change in option price availability when LE is applied. Because extremely DOTM options have an infinitesimal value and, therefore, excluding them have little effect on moment estimates, extrapolation is generally done only up to fixed limit points that are far enough from the at-the-money point.[3] This approach is also used in this study, and extrapolation is assumed to be done up to the points in which log-moneyness is equal to finite and fixed limit values $\lambda_{\min}$ and $\lambda_{\max}$, respectively. Both $\lambda_{\min}$ and $\lambda_{\max}$ are presumed to be significantly different from zero so that both $f(\alpha)$ and $g(\alpha)$ do not exceed the corresponding limit values. Now, suppose that the fair value of moment-related contracts V, W, and X is estimated with the use of a set of option prices that cover the log-moneyness domain $[f(\alpha), g(\alpha)]$, as in Equations (11)–(13), and the level of Black-Scholes implied volatility in this domain is defined by a function $\sigma_{\text{BS}}(\lambda)$ which is differentiable with respect to log-moneyness λ. If the two endpoint Black-Scholes implied volatilities $\sigma_{\text{BS}}(f(\alpha))$ and $\sigma_{\text{BS}}(g(\alpha))$ are extrapolated up to the points at which the log-moneyness is equal to $\lambda_{\min}$ and $\lambda_{\max}$, respectively, the fair value estimates for the volatility, cubic, and quartic contracts in Equations (11)–(13) are replaced with the new estimates $\widetilde{V}(\alpha)$, $\widetilde{W}(\alpha)$, and $\widetilde{X}(\alpha)$, which can be

[3] Buss and Vilkov (2012) generate option prices up to the points in which the moneyness is 1/3 and 3, respectively. On the other hand, Neumann and Skiadopoulos (2013) choose the points where the option delta is 0.01 and 0.99, respectively.

obtained as follows:

$$\widetilde{V}(\alpha) = \widehat{V}(\alpha) + \int_{g(\alpha)}^{\lambda_{\max}} \frac{2(1-\lambda)}{Se^{\lambda}} \widetilde{C}(\sigma_{\text{BS}}(g(\alpha)), \lambda) d\lambda$$
$$+ \int_{\lambda_{\min}}^{f(\alpha)} \frac{2(1-\lambda)}{Se^{\lambda}} \widetilde{P}(\sigma_{\text{BS}}(f(\alpha)), \lambda) d\lambda, \tag{29}$$

$$\widetilde{W}(\alpha) = \widehat{W}(\alpha) + \int_{g(\alpha)}^{\lambda_{\max}} \frac{6\lambda - 3\lambda^2}{Se^{\lambda}} \widetilde{C}(\sigma_{\text{BS}}(g(\alpha)), \lambda) d\lambda$$
$$+ \int_{\lambda_{\min}}^{f(\alpha)} \frac{6\lambda - 3\lambda^2}{Se^{\lambda}} \widetilde{P}(\sigma_{\text{BS}}(f(\alpha)), \lambda) d\lambda, \tag{30}$$

$$\widetilde{X}(\alpha) = \widehat{X}(\alpha) + \int_{g(\alpha)}^{\lambda_{\max}} \frac{12\lambda^2 - 4\lambda^3}{Se^{\lambda}} \widetilde{C}(\sigma_{\text{BS}}(g(\alpha)), \lambda) d\lambda$$
$$+ \int_{\lambda_{\min}}^{f(\alpha)} \frac{12\lambda^2 - 4\lambda^3}{Se^{\lambda}} \widetilde{P}(\sigma_{\text{BS}}(f(\alpha)), \lambda) d\lambda, \tag{31}$$

where

$$\widetilde{C}(\sigma, \lambda) = S(N(d_1(\sigma, \lambda)) - e^{-\lambda r \tau} N(d_2(\sigma, \lambda))), \tag{32}$$

$$\widetilde{P}(\sigma, \lambda) = S(e^{-\lambda r \tau} N(-d_2(\sigma, \lambda)) - N(-d_1(\sigma, \lambda))), \tag{33}$$

$$d_1(\sigma, \lambda) = \frac{-\lambda + (r + 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}, \tag{34}$$

$$d_2(\sigma, \lambda) = d_1(\sigma, \lambda) - \sigma\sqrt{\tau}, \tag{35}$$

and $N(\cdot)$ denotes the standard normal cumulative distribution function. With this, the following proposition shows how the derivatives $\widetilde{V}'(\alpha)$, $\widetilde{W}'(\alpha)$, and $\widetilde{X}'(\alpha)$ can be formulated:

Proposition 1. If (1) the level of Black-Scholes implied volatility in log-moneyness domain $[f(\alpha), g(\alpha)]$ is defined by a function $\sigma_{\text{BS}}(\lambda)$ which is differentiable with respect to log-moneyness λ, (2) the two Black-Scholes implied volatilities $\sigma_{\text{BS}}(f(\alpha))$ and $\sigma_{\text{BS}}(g(\alpha))$ are linearly extrapolated up to the point in which log-moneyness is equal to fixed limit values $\lambda_{\min}$ and $\lambda_{\max}$, respectively, and (3) the value of $\lambda_{\min}$ and $\lambda_{\max}$ is significantly different from zero so that both $f(\alpha)$ and $g(\alpha)$ are not supposed to exceed those limit values, then the derivatives $\widetilde{V}'(\alpha)$, $\widetilde{W}'(\alpha)$, and $\widetilde{X}'(\alpha)$ can be formulated as

$$\widetilde{V}'(\alpha) = \gamma_0(\alpha) \int_{g(\alpha)}^{\lambda_{\max}} \left(2(1-\lambda) \exp\left[\gamma_1(\alpha)\lambda^2 + \gamma_2(\alpha)\lambda + \gamma_3(\alpha)\right]\right) d\lambda$$
$$+ \delta_0(\alpha) \int_{\lambda_{\min}}^{f(\alpha)} \left(2(1-\lambda) \exp\left[\delta_1(\alpha)\lambda^2 + \delta_2(\alpha)\lambda + \delta_3(\alpha)\right]\right) d\lambda, \tag{36}$$
$$\widetilde{W}'(\alpha) = \gamma_0(\alpha) \int_{g(\alpha)}^{\lambda_{\max}} \left((6\lambda - 3\lambda^2) \exp\left[\gamma_1(\alpha)\lambda^2 + \gamma_2(\alpha)\lambda + \gamma_3(\alpha)\right]\right) d\lambda$$
$$+ \delta_0(\alpha) \int_{\lambda_{\min}}^{f(\alpha)} \left((6\lambda - 3\lambda^2) \exp\left[\delta_1(\alpha)\lambda^2 + \delta_2(\alpha)\lambda + \delta_3(\alpha)\right]\right) d\lambda, \tag{37}$$
$$\widetilde{X}'(\alpha) = \gamma_0(\alpha) \int_{g(\alpha)}^{\lambda_{\max}} \left((12\lambda^2 - 4\lambda^3) \exp\left[\gamma_1(\alpha)\lambda^2 + \gamma_2(\alpha)\lambda + \gamma_3(\alpha)\right]\right) d\lambda$$
$$+ \delta_0(\alpha) \int_{\lambda_{\min}}^{f(\alpha)} \left((12\lambda^2 - 4\lambda^3) \exp\left[\delta_1(\alpha)\lambda^2 + \delta_2(\alpha)\lambda + \delta_3(\alpha)\right]\right) d\lambda, \tag{38}$$

where

$$\gamma_0(\alpha) = \frac{\sqrt{\tau}}{\sqrt{2\pi}} \cdot \sigma'_{\text{BS}}(g(\alpha)) \cdot g'(\alpha), \tag{39}$$
$$\gamma_1(\alpha) = -\frac{1}{2\sigma_{\text{BS}}(g(\alpha))^2\tau}, \tag{40}$$
$$\gamma_2(\alpha) = \frac{r}{\sigma_{\text{BS}}(g(\alpha))^2} - \frac{1}{2}, \tag{41}$$
$$\gamma_3(\alpha) = -\frac{(r + 0.5\sigma_{\text{BS}}(g(\alpha))^2)^2\tau}{2\sigma_{\text{BS}}(g(\alpha))^2}, \tag{42}$$
$$\delta_0(\alpha) = \frac{\sqrt{\tau}}{\sqrt{2\pi}} \cdot \sigma'_{\text{BS}}(f(\alpha)) \cdot f'(\alpha), \tag{43}$$
$$\delta_1(\alpha) = -\frac{1}{2\sigma_{\text{BS}}(f(\alpha))^2\tau}, \tag{44}$$
$$\delta_2(\alpha) = \frac{r}{\sigma_{\text{BS}}(f(\alpha))^2} - \frac{1}{2}, \tag{45}$$
$$\delta_3(\alpha) = -\frac{(r + 0.5\sigma_{\text{BS}}(f(\alpha))^2)^2\tau}{2\sigma_{\text{BS}}(f(\alpha))^2}, \tag{46}$$

$\sigma'_{\text{BS}}(\lambda)$ is the derivative of Black-Scholes implied volatility function $\sigma_{\text{BS}}(\lambda)$ with respect to log-moneyness λ, r is the risk-free rate, and τ is the time to maturity.

Proof. See Appendix A. □

3. Data

The daily data on S&P 500 index options used in this paper span the 11-year time period from January 2000 to December 2010. Closing OTM option quotes, the index level, risk-free rate yield curve, and the implied dividend rate data are collected from OptionMetrics via Wharton Research Data Services. The closing option price is approximated as the midpoint between the closing bid and the ask quotes. The risk-free rate for a specific time to maturity is approximated by linear interpolation of the two closest maturities on the yield curve. The implied dividend rate is employed to estimate the dividend rate $q(t_0, T)$ for day t_0 and maturity date T as

$$q(t_0, T) = \left[\prod_{i=1}^{n}(1 + q^*(t_i))\right]^{1/n} - 1, \tag{47}$$

where n is the number of implied dividend rate observations available for the days between t_0 and T, and $q^*(t_i)$ is the implied dividend rate on day t_i, which is between t_0 and T. This dividend rate is then used to derive the dividend-adjusted index level. After data collection, inadequate observations are removed with a set of data filters, which are briefly described in Appendix B. Table 1 reports some summary statistics of the option price sample used in this study.

4. Methodology

This section demonstrates how the truncation sensitivity functions introduced in Section 2 can be employed on option prices data. Section 4.1 explains how the Black-Scholes implied volatility curve is constructed in this study to obtain a dense set of option prices, as well as a differentiable Black-Scholes implied volatility function with respect to log-moneyness. In Section 4.2, the empirical procedure of truncation sensitivity estimation is described. Section 4.3 shows how the value of truncation sensitivity can be interpreted and used to assess the effectiveness of LE.

4.1 Construction of Black-Scholes implied volatility curve

An implied volatility curve is required for each maturity for which truncation sensitivity is measured, in order to obtain an adequate number of option prices from which a continuum of option prices can be approximated, as well as to estimate a differentiable implied volatility function. As done in Jiang and Tian (2005), maturity is first fixed to avoid the telescoping problem, which is pointed out by Christensen et al. (2002).[4] Black-Scholes implied volatilities are first collected from all available OTM option prices to fix the maturity. Next, a bicubic spline function is estimated with the use of the implied volatility observations. For the regions where some of the observations required for estimation are not available because of a difference in minimum or maximum strike price between different maturities, LE is applied to approximate the missing observation. With the estimated bicubic spline function, the implied volatility levels are then approximated at some fixed maturities for the strike prices for which at least one observation exists on that day. When only the exactly monthly maturities are considered, consecutive daily observations can be obtained only for the maturities of two, three, and four months during the sample period, due to data filtration and liquidity issues. Given this limitation, implied volatility curves only for the maturities of two and four months are examined.

After the maturity is fixed, the implied volatility curves are constructed for each maturity. First, the implied volatility levels are additionally approximated for the minimum and maximum values of the strike price domain with the use of the bicubic spline function. If the minimum and maximum strike prices are not observable for a maturity, they are linearly approximated with the use of the corresponding endpoint strike prices for the two closest maturities for which the endpoints are observable. After this additional approximation, all implied volatility values that are located beyond the minimum or maximum strike price are discarded. Not only does this reflect the level of truncation observed in the market, but this also removes the effect of LE that is conducted while fixing the maturity.

Next, piecewise quadratic function is used to estimate the shape of the implied volatility curve, in accordance with the approach of Broadie et al. (2007). Similar to the approach of Broadie et al. (2007), the following function is fitted:

$$\sigma_{BS}(\lambda) = \mathbf{1}_{\lambda \leq 0}[a_2\lambda^2 + a_1\lambda + a_0] + \mathbf{1}_{\lambda > 0}[b_2\lambda^2 + a_1\lambda + a_0] + \varepsilon, \tag{48}$$

[4] Christensen et al. (2002) point out that the time period between the present and maturity dates is overlapping for option samples in different days if they expire on the same day.

where $\sigma_{BS}(\lambda)$ is the level of Black-Scholes volatility at log-moneyness λ, and $\mathbf{1}_C$ is an indicator function whose value is one when condition C holds and zero otherwise. The piecewise function is defined as a function of log-moneyness λ to obtain the derivative $\sigma'_{BS}(\lambda)$. Although the implied volatility curve can also be estimated with the use of the cubic spline function, which provides a perfect fit as in Jiang and Tian (2005), a curve estimated with the use of the cubic spline function can be winding severely, so that the derivative of the implied volatility function may not be approximated stably. Comparatively, the derivative of the piecewise quadratic function is more stable, although the function itself does not ensure a perfect fit. Because the derivative of the implied volatility function at endpoints is significantly important in measuring sensitivity when LE is applied, the piecewise quadratic function is chosen to estimate implied volatility curve.[5]

After the implied volatility curve is estimated, the curve is translated into OTM option prices for the strike prices between the minimum and maximum values of the strike price domain, with a strike price interval of 0.1. When LE is employed for implied moment or truncation sensitivity estimation, it is applied to this curve up to the points in which the strike prices are equal to $S(t)/3$ and $3S(t)$, respectively, where $S(t)$ is the dividend-adjusted index level on day t. This setting means that $\lambda_{\min}$ and $\lambda_{\max}$ in Equations (29)–(38) are set as $-\ln 3$ and $\ln 3$, respectively. The strike price interval is again set as 0.1 for LE.

4.2 Measuring truncation sensitivity

Regardless of whether LE is applied or not, the basic procedure of truncation sensitivity measurement is identical. Specifically, one first sets the endpoint functions $f(\alpha)$ and $g(\alpha)$, and calculates the contract fair value estimates and their derivatives with respect to α. Then, all the other required derivatives can be calculated correspondingly on the basis of the procedure shown in Section 2.2. The only difference between the truncation sensitivity estimation with and without LE is the method of obtaining the contract fair value estimates and their derivatives. This subsection therefore describes the procedure of truncation sensitivity measurement that

[5] An unreported analysis suggests that the empirical results are qualitatively equivalent even when the piecewise quadratic function is replaced with the cubic spline function, although the results become less statistically significant partially due to the more noisy sensitivity approximation result.

can be applied to both cases with and without LE, with some additional comments for the case with LE which is slightly more complicated.

Endpoint functions $f(\alpha)$ and $g(\alpha)$ need to be defined first to measure sensitivity. Given Equations (11)–(13), $f(\alpha)$ and $g(\alpha)$ must be defined in a way that a real number $\bar{\alpha}$ exists, for which $f(\bar{\alpha}) = \ln(K_{\min}/S)$ and $g(\bar{\alpha}) = \ln(K_{\max}/S)$, where $K_{\min}$ and $K_{\max}$ are the minimum and maximum strike prices of the integration domain, respectively, and S is the underlying price. A simple definition that is applicable to any observed data is $f(\alpha) = c\alpha$ and $g(\alpha) = \alpha$, where

$$c = \frac{\ln(K_{\min}/S)}{\ln(K_{\max}/S)}. \tag{49}$$

With this definition, two implicit assumptions are made. First, α is nonnegative, no option prices are available when $\alpha = 0$, and the option price availability increases as α becomes larger. Second, the log-moneyness ratio c is not changed by an increase or decrease in option price availability. The first assumption is acceptable because it makes the relationship between α and option price availability clearly defined. The second assumption is also reasonable because it ensures that option price availability increases at each side (call side or put side) as α becomes larger, while also satisfying the condition that the real number $\bar{\alpha}$ mentioned above exists. Given such assumptions, this simple definition is employed for empirical analysis.

After $f(\alpha)$ and $g(\alpha)$ are set, the contract fair value estimates and their derivatives at $\alpha = \bar{\alpha}$ need to be calculated. With the definition of $f(\alpha)$ and $g(\alpha)$ above, $\bar{\alpha} = \ln(K_{\max}/S)$. $\widehat{V}(\bar{\alpha})$, $\widehat{W}(\bar{\alpha})$, $\widehat{X}(\bar{\alpha})$, $\widehat{V}'(\bar{\alpha})$, $\widehat{W}'(\bar{\alpha})$, and $\widehat{X}'(\bar{\alpha})$ need to be estimated as in Section 2.2 when LE is not employed, whereas $\widetilde{V}(\bar{\alpha})$, $\widetilde{W}(\bar{\alpha})$, $\widetilde{X}(\bar{\alpha})$, $\widetilde{V}'(\bar{\alpha})$, $\widetilde{W}'(\bar{\alpha})$, and $\widetilde{X}'(\bar{\alpha})$ are required when LE is considered. The fair value estimates and derivatives without LE are relatively easier to obtain because they can be collected via the model-free implied moment estimation procedure of Bakshi et al. (2003). On the other hand, two extra subtleties exist when derivatives with LE are calculated. First, the derivative $\sigma'_{BS}(\lambda)$ of the Black-Scholes implied volatility function needs to be evaluated at $\lambda = f(\bar{\alpha})$ and $\lambda = g(\bar{\alpha})$. This evaluation can be done with the implied volatility curve that is approximated using a differentiable function, i.e., piecewise quadratic function, which is demonstrated in Section 4.1. Second, because Equations (36)–(38) are in the form of transcendental function, no analytic solution exists for them. Fortunately, they can still be evaluated numerically with the use of a computational software. *Wolfram Mathematica* is used in this study to evaluate $\widetilde{V}'(\bar{\alpha})$, $\widetilde{W}'(\bar{\alpha})$, and $\widetilde{X}'(\bar{\alpha})$. With all the fair value estimates and their derivatives with respect to α, now, the derivatives of the implied moment estimates with respect to α, i.e., the truncation sensitivity of the implied moment estimators, can be obtained. When LE is not considered, plugging in the value of the required variables into Equations (18)–(20) enables one to calculate the derivatives. It is also the case when LE is considered, but after replacing $\widehat{V}(\bar{\alpha})$, $\widehat{W}(\bar{\alpha})$, $\widehat{X}(\bar{\alpha})$, $\widehat{V}'(\bar{\alpha})$, $\widehat{W}'(\bar{\alpha})$, and $\widehat{X}'(\bar{\alpha})$ with $\widetilde{V}(\bar{\alpha})$, $\widetilde{W}(\bar{\alpha})$, $\widetilde{X}(\bar{\alpha})$, $\widetilde{V}'(\bar{\alpha})$, $\widetilde{W}'(\bar{\alpha})$, and $\widetilde{X}'(\bar{\alpha})$, respectively.

4.3 Interpretation of truncation sensitivity

Although truncation sensitivity provides information on how sensitive to truncation an implied moment estimator is, refining the information is still needed to assess the effectiveness of LE for two reasons. First, unless no evidence exists that sensitivity is constant or at least stable over the α-axis, the measured sensitivity should be regarded as local and therefore be used to approximate a change in estimate caused by a small change in option price availability. Second, because option prices are assigned for discrete strike prices, defining the change in option price availability in terms of strike price rather than log-moneyness may be more practical. Hence, the change in implied moment estimate caused by an increase in strike price domain length by five, which is equal to the size of a single strike price interval in the S&P 500 index options market for short maturities, is linearly approximated to assess the effectiveness of LE.

To approximate this change, one first needs to determine the size of increase in $\bar{\alpha}$ that is required to increase the strike price domain length by five. In other words, one needs to obtain the value of increment i, which satisfies

$$h(i) \equiv (Se^{g(\bar{\alpha}+i)} - Se^{f(\bar{\alpha}+i)}) - (Se^{g(\bar{\alpha})} - Se^{f(\bar{\alpha})}) - 5 = 0. \tag{50}$$

Given the definition of $f(\alpha)$ and $g(\alpha)$ in Section 4.2, the function $h(i)$ is monotonically increasing in the domain $\{i : -\bar{\alpha} \leq i \leq \infty\}$ for any nonnegative constant $\bar{\alpha}$ and S. Hence, a unique value of i satisfies the condition above, and this value can be approximated numerically by minimizing the absolute value of $h(i)$. Figure 1 illustrates the approximated value of i for the S&P 500 index options data. It is shown in Figure 1 that the value of i is considerably small for the entire sample period, and therefore it is reasonable to use linear approximation.

5. Empirical analysis

This section reports the results of the empirical analysis on the effectiveness of LE. The sensitivity function defined in Section 2 and the empirical methodology described in Section 4 are applied to estimate the truncation sensitivity of the implied volatility, skewness, and kurtosis estimators of Bakshi et al. (2003), when they are used with and without LE on S&P 500 index options data. Section 5.1 compares the implied moment estimation results with and without LE to demonstrate the effect of LE in outline. Section 5.2 compares the truncation sensitivity of the implied moment estimators with and without LE to investigate the effectiveness of LE.

5.1 Implied moment estimate with and without LE

Figure 2 illustrates the level of implied moment estimates during the sample period with and without the application of LE. A number of interesting points can be made from the figure. First, LE is shown to have a large influence on implied moment estimate when it is used with a high moment estimator. Determining the difference between Figures 2a and 2b visually is difficult, whereas the shape of the lines in Figure 2d is evidently different from that in Figure 2c, and the difference is even larger in scale between Figures 2e and 2f. Second, LE is shown to affect the implied skewness and kurtosis estimates significantly for a part of the sample period, whereas the effect is less significant for the other parts. Figures 2c–2f show that the implied skewness and kurtosis estimates are changed significantly after LE during the period from 2004 to 2007, whereas the change is less outstanding for the other periods. Finally, although a notable change in the average level and short-term dynamics of the implied moment estimates after LE can be observed, no significant change occurs in terms of the mid- and long-term trend of moment estimate fluctuation. If the noisy estimates are ignored in Figures 2c–2f, the lines in the corresponding pair of subfigures show a considerable similarity in shape.

5.2 Truncation sensitivity with and without LE

Table 2 reports the truncation sensitivity comparison result for the implied moment estimators with and without LE. The comparison is conducted by approximation of the nominal and percentage change in the estimates after an increase in strike price domain length by five, as explained in Section 4.3, followed by an investigation into whether a significant difference exists in the mean of the absolute nominal and percentage changes before and after LE. Panel A of Table 2 shows the comparison result for the absolute nominal change in estimate. Panel A depicts that the decrease in truncation sensitivity is statistically significant for all cases, except the implied kurtosis estimate for the maturity of two months. Furthermore, Panel B of Table 2, which reports the comparison result for the absolute percentage change in estimate, indicates that the moment estimate becomes less sensitive to a small change in option price availability for all moments and maturities. With these results, LE can be conjectured to be effective and makes the estimates less sensitive to a change in option price availability for all three implied moment estimators of Bakshi et al. (2003). Another notable finding is that the truncation sensitivity of the implied volatility estimator is extremely low regardless of the application of LE. This finding suggests that the implied volatility estimator of Bakshi et al. (2003) is considerably robust to truncation when applied to S&P 500 index options data.

However, when we focus on the estimates on which LE has a relatively strong influence, a contradictory finding can be obtained. Figure 3 illustrates the approximated time-series dynamics of the nominal change in estimate after a change in strike price domain length by five. In Figures 3c–3f, most of the estimates that are highly sensitive to truncation without LE become even more sensitive with LE. Furthermore, the result is again similar in Figure 4, in which the size of the absolute percentage change is employed, so that the effect of the implied moment level is controlled. Given that the highly truncation-sensitive estimates are mostly located in the time period from 2004 to 2007, for which LE is shown to have the largest influence in Section 5.1, LE can be conjectured to have a reverse effect when the implied moment estimate is mostly determined by the option prices that are generated by LE. This is possible because if the estimate is mostly determined by the generated option prices, the change in the endpoint implied volatility level due to a change in the integration domain width will have a strong effect on the implied moment estimate.

Overall, the empirical results suggest that LE is effective and makes the implied moment estimate less sensitive to truncation for all three estimators of Bakshi et al. (2003). However, LE might also have an adverse effect and make the estimate even more sensitive to truncation when the estimation relies on LE too heavily. Hence, a supplementary truncation treatment may be required when LE is used in conjunction with estimators that are more closely related to the DOTM option prices and thus might rely heavily on LE. A simple supplementary treatment is to remove observations with extremely high truncation sensitivity, so that the estimation is less affected by abnormal changes in the truncation error.

6. Conclusion

The model-free estimation of the moments of implied risk-neutral density has been a topic of recent interest. Since option prices are completely unavailable for the outermost part of the strike price domain as if some of the option prices have been truncated, the missing option prices must be inferred from the available option prices for model-free implied moment estimation, for which weighted option prices need to be integrated with respect to strike price from zero to positive infinity. LE has been a popular choice of truncation treatment for the model-free implied moment estimators owing to its simplicity and approximation-based approach, and it has been frequently used in combination with the implied skewness and kurtosis estimators of Bakshi et al. (2003) after the higher moment estimators became popular. Nevertheless, less attention has been devoted to the issues of how effectively LE can reduce truncation errors and whether LE can alleviate the truncation error of higher moment estimators.

This study addresses both of these issues and introduces an empirical methodology, i.e., measurement of truncation sensitivity, with which the effectiveness of LE can be assessed for the option prices that are observed from markets. If the truncation error becomes negligible when LE is applied, an estimate should not change significantly after a marginal change in the width of integration domain. In other words, if an estimate changes significantly after including or excluding option prices at the end of the integration domain, even when LE is consistently employed, the truncation error is determined to be not fully reduced by LE. Hence, the sensitivity of the implied moment estimators to a marginal change in the width of integration domain, i.e., truncation sensitivity, can be used to assess how effectively LE reduces the truncation error. Based on this idea, this study defines the truncation sensitivity functions for the implied volatility, skewness, and kurtosis estimators of Bakshi et al. (2003) with and without LE applied, and then employs these functions to approximate how the estimates will change after a small increase in the number of option prices available for S&P 500 index options market.

This study makes three contributions to the literature on model-free implied moment estimation. First, the methodology used in this study shows how a truncation treatment method for implied moment estimation can be assessed when the true value of implied moment is unknown, so that the truncation error cannot be calculated. Because truncation sensitivity estimation does not require the true value of implied moments, the methodology proposed by this study can be applied to any set of OTM European option prices from which a differentiable Black-Scholes implied volatility function of log-moneyness can be estimated, and is therefore practical. Second, the present study shows how the effectiveness of LE varies according to the type of implied moment estimator for which LE is employed. Although LE is frequently used in conjunction with implied moment estimators other than the implied volatility estimator of Britten-Jones and Neuberger (2000) for which LE is first introduced, no studies validating this extended use of LE for different implied moment estimators have been conducted. The current study fills this gap and empirically shows that LE is effective even when used in combination with the implied moment estimators of Bakshi et al. (2003). However, the results also suggest that the remaining truncation error is not small enough to be regarded as negligible when LE is used with the implied skewness or kurtosis estimator of Bakshi et al. (2003). Finally, the truncation sensitivity function, which is introduced in the present study to measure the effectiveness of LE, can also be used as an instrument for supplementary truncation treatment along with LE. Observations with a large truncation error can be detected by measurement of truncation sensitivity, so an additional data filter based on truncation sensitivity can be employed to reduce the effect of truncation on implied moment estimators.

Although this study points out an issue on the use of LE in combination with the implied skewness or kurtosis estimator of Bakshi et al. (2003), it does not invalidate LE or any argument in the work of Jiang and Tian (2005). On the contrary, the argument of Jiang and Tian (2005) that LE reduces the truncation error of implied volatility estimator is supported by the current study as it shows that such is also the case for the implied volatility estimator of Bakshi et al. (2003). In addition, this study also shows that LE makes the implied skewness and kurtosis estimators less sensitive to truncation in most cases. Hence, this study still suggests the use of LE in conjunction with model-free implied moment estimators, including those for higher moments, but only on the condition that the size of the remaining truncation error be properly controlled by supplementary methods when LE is employed for higher moment estimation. As suggested previously, a simple but effective supplementary method is to discard observations whose truncation sensitivity is abnormally high, so that estimation results are not distorted by abrupt changes in the truncation error.

Appendix A. Proof of Proposition 1

Proposition 1 is proved only for the volatility contract V. The same proof can be applied to the other contracts W and X as well. It is assumed that day t, time to maturity τ, and risk-free rate r are known and fixed, and the underlying asset pays no dividends.

Proof.

$$\begin{aligned}
\widetilde{V}'(\alpha) &= g'(\alpha)\frac{2(1-g(\alpha))}{Se^{g(\alpha)}}C(g(\alpha)) + f'(\alpha)\frac{2(1-f(\alpha))}{Se^{f(\alpha)}}P(f(\alpha)) \\
&\quad - g'(\alpha)\frac{2(1-g(\alpha))}{Se^{g(\alpha)}}\widetilde{C}(\sigma_{\mathrm{BS}}(g(\alpha)), g(\alpha)) - f'(\alpha)\frac{2(1-f(\alpha))}{Se^{f(\alpha)}}\widetilde{P}(\sigma_{\mathrm{BS}}(f(\alpha)), f(\alpha)) \\
&\quad + \int_{g(\alpha)}^{\lambda_{\max}} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \frac{\partial \widetilde{C}(\sigma_{\mathrm{BS}}(g(\alpha)), \lambda)}{\partial \alpha}\right) d\lambda + \int_{\lambda_{\min}}^{f(\alpha)} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \frac{\partial \widetilde{P}(\sigma_{\mathrm{BS}}(f(\alpha)), \lambda)}{\partial \alpha}\right) d\lambda \\
&= \int_{g(\alpha)}^{\lambda_{\max}} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \frac{\partial \widetilde{C}(\sigma_{\mathrm{BS}}(g(\alpha)), \lambda)}{\partial \alpha}\right) d\lambda + \int_{\lambda_{\min}}^{f(\alpha)} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \frac{\partial \widetilde{P}(\sigma_{\mathrm{BS}}(f(\alpha)), \lambda)}{\partial \alpha}\right) d\lambda \\
&= \int_{g(\alpha)}^{\lambda_{\max}} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \frac{\partial \widetilde{C}(\sigma_{\mathrm{BS}}(g(\alpha)), \lambda)}{\partial \sigma_{\mathrm{BS}}(g(\alpha))} \cdot \frac{d\sigma_{\mathrm{BS}}(g(\alpha))}{dg(\alpha)} \cdot \frac{dg(\alpha)}{d\alpha}\right) d\lambda \\
&\quad + \int_{\lambda_{\min}}^{f(\alpha)} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \frac{\partial \widetilde{P}(\sigma_{\mathrm{BS}}(f(\alpha)), \lambda)}{\partial \sigma_{\mathrm{BS}}(f(\alpha))} \cdot \frac{d\sigma_{\mathrm{BS}}(f(\alpha))}{df(\alpha)} \cdot \frac{df(\alpha)}{d\alpha}\right) d\lambda \\
&= \int_{g(\alpha)}^{\lambda_{\max}} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \nu(\sigma_{\mathrm{BS}}(g(\alpha)), \lambda) \cdot \sigma'_{\mathrm{BS}}(g(\alpha)) \cdot g'(\alpha)\right) d\lambda \\
&\quad + \int_{\lambda_{\min}}^{f(\alpha)} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \nu(\sigma_{\mathrm{BS}}(f(\alpha)), \lambda) \cdot \sigma'_{\mathrm{BS}}(f(\alpha)) \cdot f'(\alpha)\right) d\lambda, \qquad (51)
\end{aligned}$$

where

$$\widetilde{C}(\sigma, \lambda) = S(N(d_1(\sigma, \lambda)) - e^{-\lambda r\tau}N(d_2(\sigma, \lambda))), \qquad (52)$$

$$\widetilde{P}(\sigma, \lambda) = S(e^{-\lambda r\tau}N(-d_2(\sigma, \lambda)) - N(-d_1(\sigma, \lambda))), \qquad (53)$$

$$d_1(\sigma, \lambda) = \frac{-\lambda + (r + 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}, \qquad (54)$$

$$d_2(\sigma, \lambda) = d_1(\sigma, \lambda) - \sigma\sqrt{\tau}, \qquad (55)$$

$N(\cdot)$ denotes the standard normal cumulative distribution function, and $\nu(\sigma, \lambda)$ is the option vega for Black-Scholes implied volatility σ and log-moneyness λ. Given that

$$\nu(\sigma, \lambda) = Sn\left(d_1\left(\sigma, \lambda\right)\right)\sqrt{\tau}, \tag{56}$$

where

$$n(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \tag{57}$$

it can be obtained that

$$\nu(\sigma, \lambda) = \frac{S\sqrt{\tau}}{\sqrt{2\pi}} \exp\left[-\frac{\lambda^2 - 2\lambda(r + 0.5\sigma^2)\tau + (r + 0.5\sigma^2)^2\tau^2}{2\sigma^2\tau}\right], \tag{58}$$

and therefore the first integration term in Equation (51) can be rearranged as

$$\begin{aligned}
&\int_{g(\alpha)}^{\lambda_{\max}} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \nu(\sigma_{\mathrm{BS}}(g(\alpha)), \lambda) \cdot \sigma'_{\mathrm{BS}}(g(\alpha)) \cdot g'(\alpha)\right) d\lambda \\
&= \frac{1}{S} \cdot \sigma'_{\mathrm{BS}}(g(\alpha)) \cdot g'(\alpha) \cdot \int_{g(\alpha)}^{\lambda_{\max}} \left(2(1-\lambda)e^{-\lambda} \cdot \nu(\sigma_{\mathrm{BS}}(g(\alpha)), \lambda)\right) d\lambda \\
&= \frac{\sqrt{\tau}}{\sqrt{2\pi}} \cdot \sigma'_{\mathrm{BS}}(g(\alpha)) \cdot g'(\alpha) \\
&\quad \cdot \int_{g(\alpha)}^{\lambda_{\max}} \left(2(1-\lambda) \exp\left[-\lambda - \frac{\lambda^2 - 2\lambda(r + 0.5\sigma_{\mathrm{BS}}(g(\alpha))^2)\tau + (r + 0.5\sigma_{\mathrm{BS}}(g(\alpha))^2)^2\tau^2}{2\sigma_{\mathrm{BS}}(g(\alpha))^2\tau}\right]\right) d\lambda \\
&= \gamma_0(\alpha) \int_{g(\alpha)}^{\lambda_{\max}} \left(2(1-\lambda) \exp\left[\gamma_1(\alpha)\lambda^2 + \gamma_2(\alpha)\lambda + \gamma_3(\alpha)\right]\right) d\lambda,
\end{aligned} \tag{59}$$

where

$$\gamma_0(\alpha) = \frac{\sqrt{\tau}}{\sqrt{2\pi}} \cdot \sigma'_{\mathrm{BS}}(g(\alpha)) \cdot g'(\alpha), \tag{60}$$

$$\gamma_1(\alpha) = -\frac{1}{2\sigma_{\mathrm{BS}}(g(\alpha))^2\tau}, \tag{61}$$

$$\gamma_2(\alpha) = \frac{r}{\sigma_{\mathrm{BS}}(g(\alpha))^2} - \frac{1}{2}, \tag{62}$$

$$\gamma_3(\alpha) = -\frac{(r + 0.5\sigma_{\mathrm{BS}}(g(\alpha))^2)^2\tau}{2\sigma_{\mathrm{BS}}(g(\alpha))^2}. \tag{63}$$

Similarly, the second integration term in Equation (51) can be rearranged as

$$\int_{\lambda_{\min}}^{f(\alpha)} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \nu(\sigma_{\text{BS}}(f(\alpha)), \lambda) \cdot \sigma'_{\text{BS}}(f(\alpha)) \cdot f'(\alpha) \right) d\lambda$$
$$= \delta_0(\alpha) \int_{\lambda_{\min}}^{f(\alpha)} \left(2(1-\lambda) \exp\left[\delta_1(\alpha)\lambda^2 + \delta_2(\alpha)\lambda + \delta_3(\alpha) \right] \right) d\lambda, \tag{64}$$

where

$$\delta_0(\alpha) = \frac{\sqrt{\tau}}{\sqrt{2\pi}} \cdot \sigma'_{\text{BS}}(f(\alpha)) \cdot f'(\alpha), \tag{65}$$

$$\delta_1(\alpha) = -\frac{1}{2\sigma_{\text{BS}}(f(\alpha))^2 \tau}, \tag{66}$$

$$\delta_2(\alpha) = \frac{r}{\sigma_{\text{BS}}(f(\alpha))^2} - \frac{1}{2}, \tag{67}$$

$$\delta_3(\alpha) = -\frac{(r + 0.5\sigma_{\text{BS}}(f(\alpha))^2)^2 \tau}{2\sigma_{\text{BS}}(f(\alpha))^2}. \tag{68}$$

By combining Equations (51), (59), and (64), it can be found that

$$\widetilde{V}'(\alpha) = \gamma_0(\alpha) \int_{g(\alpha)}^{\lambda_{\max}} \left(2(1-\lambda) \exp\left[\gamma_1(\alpha)\lambda^2 + \gamma_2(\alpha)\lambda + \gamma_3(\alpha) \right] \right) d\lambda$$
$$+ \delta_0(\alpha) \int_{\lambda_{\min}}^{f(\alpha)} \left(2(1-\lambda) \exp\left[\delta_1(\alpha)\lambda^2 + \delta_2(\alpha)\lambda + \delta_3(\alpha) \right] \right) d\lambda, \tag{69}$$

which is identical to Equation (36). □

Appendix B. Options data filtration procedure

After collecting option quotes data, the following data filters are applied to construct the final sample:

- Observations with midpoints of bid and ask quotes less than 0.375 are removed, since these observations may not reflect the true option value given a relatively larger tick size.
- Observations are excluded if time to maturity is shorter than one week or longer than one year, since these observations may induce liquidity-related biases.
- Observations with bid-ask spread larger than mid-point price are discarded for two reasons. First, since bid quotes are already very low for DOTM options, a too large bid-ask spread may result in having an abnormally high ask quote. Second, since options vega is lower for DOTM observations, Black-Scholes implied volatility gap between bid and ask quotes can be extreme for those observations when bid-ask spread is large, thereby making the implied volatility of mid-point price less reliable.
- Observations are included only if the sum of daily trading volume for the corresponding maturity date is nonzero, in order to prevent liquidity-related biases. In other words, the entire daily observations for a maturity date are excluded if none of them are traded in that day.
- Observations are removed if there is any missing data entry, if the bid quote is zero or higher than the ask quote, or if the mid-point price is beyond the no-arbitrage bound.

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Table 1: Option price sample properties

This table presents summary statistics of filtered S&P 500 index options dataset used in this study. The dataset contains 676,574 observations and spans an eleven year time period from January 2000 to December 2010. Effective spread is defined as the spread between ask quote and mid-point price. The following data filters are applied: (1) Observations with midpoints of bid and ask quotes less than 0.375 are excluded; (2) Observations are discarded if time to maturity is shorter than one week or longer than one year; (3) Observations with bid-ask spread larger than mid-point price are removed; (4) Observations are included only if the sum of daily trading volume is nonzero for the corresponding maturity date; and (5) Observations are excluded if there is any missing data entry, if the bid quote is zero or higher than the ask quote, or if the mid-point price is beyond the no-arbitrage bound.

Option type	Moneyness ($m = K/S$)		# of trading days to expiration (τ) $5 \leq \tau < 42$	$42 \leq \tau < 126$	$126 \leq \tau \leq 252$	Total # of obs.
OTM puts	$m < 0.70$	Mean price	1.29	2.14	5.17	
		Mean effective spread	0.42	0.42	0.57	
		Mean BS implied vol.	0.6358	0.4644	0.3847	
		# of obs.	7,592	24,873	27,541	60,006
	$0.70 \leq m < 0.85$	Mean price	2.42	6.58	17.70	
		Mean effective spread	0.40	0.57	0.92	
		Mean BS implied vol.	0.4097	0.3247	0.2707	
		# of obs.	35,449	47,189	31,580	114,218
	$0.85 \leq m < 1.00$	Mean price	9.78	25.63	47.75	
		Mean effective spread	0.58	0.98	1.23	
		Mean BS implied vol.	0.2465	0.2390	0.2266	
		# of obs.	108,908	75,536	44,691	229,135
OTM calls	$1.00 < m < 1.15$	Mean price	9.77	21.95	49.28	
		Mean effective spread	0.59	0.90	1.21	
		Mean BS implied vol.	0.1875	0.1861	0.1866	
		# of obs.	80,673	73,663	42,739	197,075
	$1.15 \leq m < 1.30$	Mean price	2.49	3.98	12.69	
		Mean effective spread	0.57	0.55	0.82	
		Mean BS implied vol.	0.3303	0.2074	0.1800	
		# of obs.	6,791	21,343	25,948	54,082
	$m \geq 1.30$	Mean price	1.21	1.87	3.59	
		Mean effective spread	0.59	0.55	0.57	
		Mean BS implied vol.	0.4665	0.2843	0.2187	
		# of obs.	892	5,182	15,984	22,058

Table 2: Comparison of truncation sensitivity with and without LE

This table reports the truncation sensitivity comparison result for the implied volatility, skewness, and kurtosis estimators of Bakshi et al. (2003) with and without linear extrapolation (LE). The comparison is conducted by applying the estimators on S&P 500 index options dataset, and then approximating the nominal and percentage change in the estimates after an increase in strike price domain length by five. The dataset spans an eleven year time period from January 2000 to December 2010. Option prices are estimated based on the implied volatility curve for the maturities of two and four months, which are extracted from daily implied volatility surfaces. When LE is applied, implied volatility at minimum and maximum strike prices are extrapolated up to the points in which strike prices are $3/S(t)$ and $3S(t)$, respectively, where $S(t)$ is the dividend-adjusted index level on day t. The nominal change in estimate is approximated by first finding the value of increment i which minimizes the absolute value of function $h(t,\tau,i)$, which is defined as

$$h(t,\tau,i) = (S(t)e^{g(\bar{\alpha}(t,\tau)+i,t,\tau)} - Se^{f(\bar{\alpha}(t,\tau)+i,t,\tau)}) - (S(t)e^{g(\bar{\alpha}(t,\tau),t,\tau)} - Se^{f(\bar{\alpha}(t,\tau),t,\tau)}) - 5,$$

where $\bar{\alpha}(t,\tau) \equiv \ln[K_{\max}(t,\tau)/S(t)]$, $f(\alpha,t,\tau) = c(t,\tau)\alpha$, $g(\alpha,t,\tau) = \alpha$, $c(t,\tau) = \ln[K_{\min}(t,\tau)/S(t)]/\ln[K_{\max}(t,\tau)/S(t)]$, and $K_{\min}(t,\tau)$ and $K_{\max}(t,\tau)$ are the observed minimum and maximum strike prices for day t and maturity τ, respectively. Then, the nominal change for each (t,τ) is approximated as the truncation sensitivity multiplied by i. Finally, the percentage change is calculated by dividing the nominal change by the corresponding implied moment estimate. The truncation sensitivity of implied volatility estimate is multiplied by 10,000 for better visibility, because the implied volatility estimate is found to be significantly insensitive to a change in option price availability when the estimator is applied to the dataset used in this study. ** and * denote statistical significance at the 1% and 5% levels, respectively.

Estimate	Maturity	Application of LE	Mean	Standard deviation	Difference in mean
Panel A. Absolute nominal change					
Implied volatility estimate (Sensitivity×10,000)	2 months	Without LE	0.0016	0.0051	0.0006**
		With LE	0.0010	0.0034	(5.57)
	4 months	Without LE	0.0062	0.0253	0.0015*
		With LE	0.0047	0.0292	(2.01)
Implied skewness estimate	2 months	Without LE	0.0093	0.0061	0.0005*
		With LE	0.0088	0.0089	(2.31)
	4 months	Without LE	0.0065	0.0048	0.0009**
		With LE	0.0056	0.0067	(5.60)
Implied kurtosis estimate	2 months	Without LE	0.0851	0.0594	−0.0016
		With LE	0.0867	0.0939	(−0.75)
	4 months	Without LE	0.0525	0.0400	0.0042**
		With LE	0.0483	0.0621	(2.95)
Panel B. Absolute percentage change					
Implied volatility estimate (Sensitivity×10,000)	2 months	Without LE	0.0043	0.0086	0.0016**
		With LE	0.0026	0.0065	(7.98)
	4 months	Without LE	0.0165	0.0464	0.0047**
		With LE	0.0118	0.0548	(3.41)
Implied skewness estimate	2 months	Without LE	0.0058	0.0034	0.0012**
		With LE	0.0046	0.0036	(12.19)
	4 months	Without LE	0.0043	0.0043	0.0013**
		With LE	0.0030	0.0030	(13.08)
Implied kurtosis estimate	2 months	Without LE	0.0112	0.0048	0.0028**
		With LE	0.0084	0.0059	(19.29)
	4 months	Without LE	0.0073	0.0033	0.0025**
		With LE	0.0048	0.0039	(25.47)

Figure 1: Increment in $\bar{\alpha}(t,\tau)$ required to increase the strike domain length by five

This figure illustrates the level of increment in $\bar{\alpha}(t,\tau)$, i.e., the value of α with which $f(\alpha)$ and $g(\alpha)$ coincide with the observed minimum and maximum strike prices $K_{\min}$ and $K_{\max}$ for time t and maturity τ, respectively, that is required to increase the strike domain length by five for the S&P 500 index options dataset used in this study. The dataset spans a time period from January 2000 to December 2010. Option prices are estimated based on the implied volatility curve for the maturities of two and four months that are extracted from daily implied volatility surfaces. The value of required increment i is approximated by minimizing the absolute value of function $h(t,\tau,i)$ which is defined as

$$h(t,\tau,i) = (S(t)e^{g(\bar{\alpha}(t,\tau)+i,t,\tau)} - Se^{f(\bar{\alpha}(t,\tau)+i,t,\tau)}) - (S(t)e^{g(\bar{\alpha}(t,\tau),t,\tau)} - Se^{f(\bar{\alpha}(t,\tau),t,\tau)}) - 5,$$

where $S(t)$ is the dividend-adjusted underlying index level on day t, $\bar{\alpha}(t,\tau) \equiv \ln[K_{\max}(t,\tau)/S(t)]$, $f(\alpha,t,\tau) = c(t,\tau)\alpha$, $g(\alpha,t,\tau) = \alpha$, and $c(t,\tau) = \ln[K_{\min}(t,\tau)/S(t)]/\ln[K_{\max}(t,\tau)/S(t)]$.

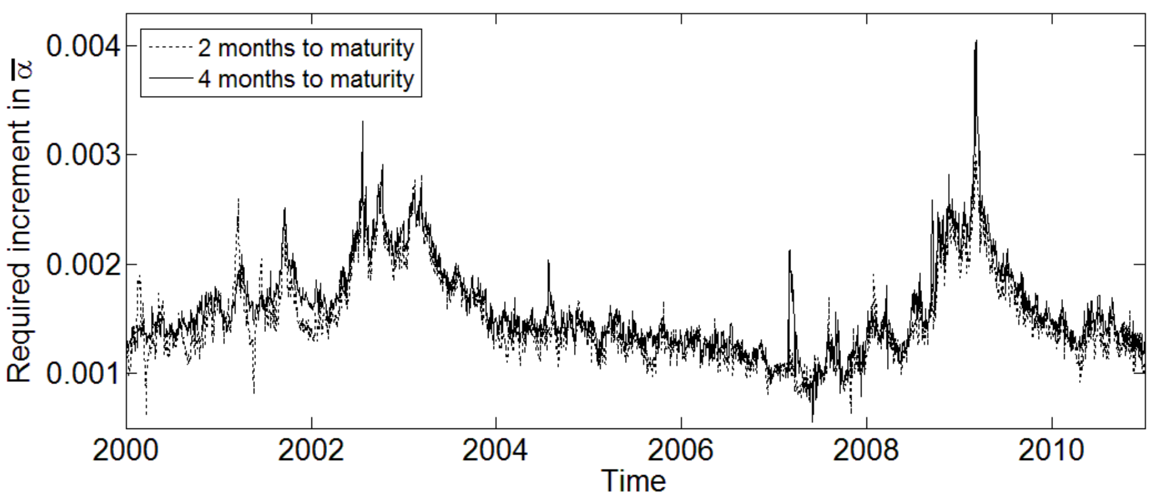
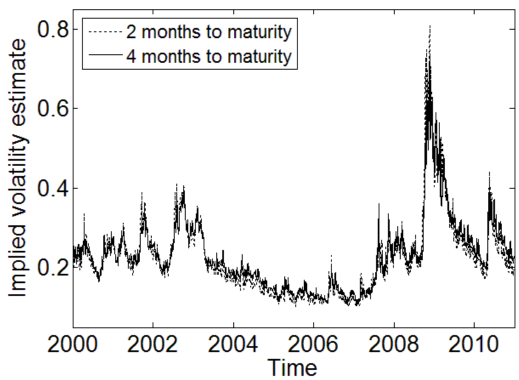
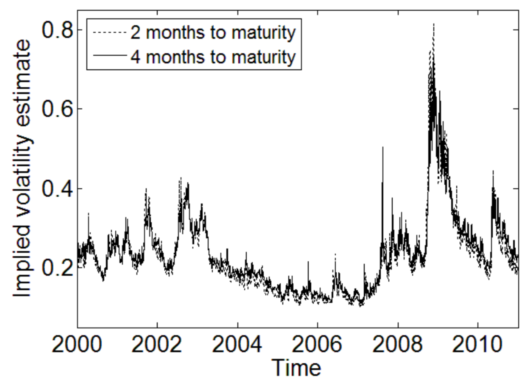


Figure 2: Implied moment estimate

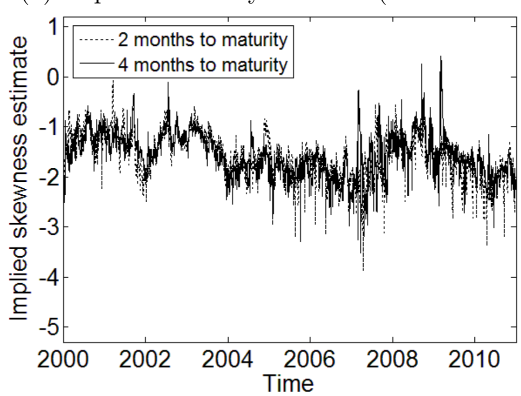
This figure illustrates the level of implied volatility, skewness, and kurtosis estimates with and without the application of linear extrapolation (LE). The S&P 500 index options dataset used spans a time period from January 2000 to December 2010. Option prices are estimated based on the implied volatility curve for the maturities of two and four months that are extracted from daily implied volatility surfaces. When LE is applied, the level of implied volatility at the minimum and maximum strike prices are extrapolated up to the points at which the strike prices are $3/S(t)$ and $3S(t)$, respectively, where $S(t)$ is the dividend-adjusted index level on day t.



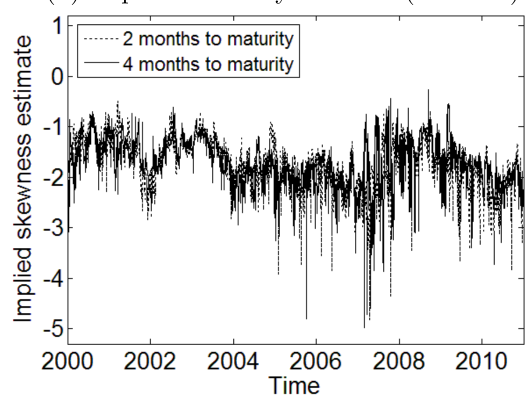
(a) Implied volatility estimate (without LE)



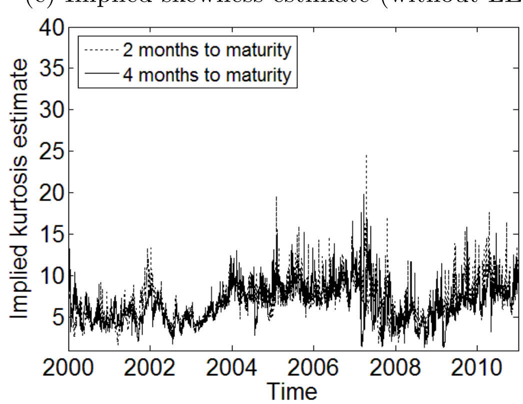
(b) Implied volatility estimate (with LE)



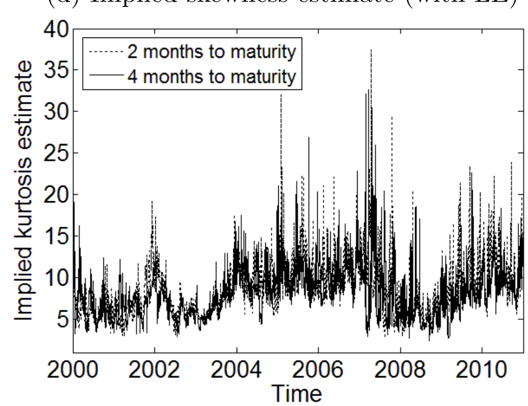
(c) Implied skewness estimate (without LE)



(d) Implied skewness estimate (with LE)



(e) Implied kurtosis estimate (without LE)



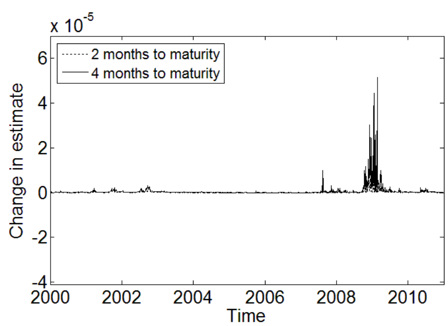
(f) Implied kurtosis estimate (with LE)

Figure 3: Change in moment estimate after an increase in strike price domain width

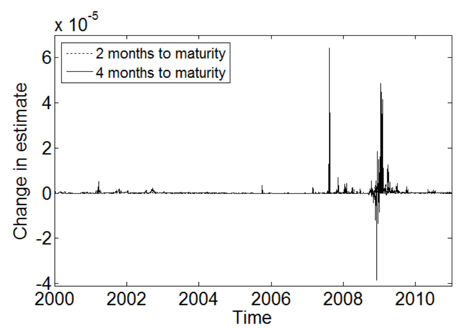
This figure reports the size of change in the implied volatility, skewness, and kurtosis estimates with and without linear extrapolation (LE), after an increase in strike price domain length by five. The S&P 500 index options dataset used here spans an eleven year time period from January 2000 to December 2010. Option prices are estimated based on the implied volatility curve for the maturities of two and four months, which are extracted from daily implied volatility surfaces. When LE is applied, implied volatility at minimum and maximum strike prices are extrapolated up to the points in which strike prices are $3/S(t)$ and $3S(t)$, respectively, where $S(t)$ is the dividend-adjusted index level on day t. Approximation of size of change in estimate is done by first finding the value of increment i which minimizes the absolute value of function $h(t,\tau,i)$, which is defined as

$$h(t,\tau,i) = (S(t)e^{g(\bar{\alpha}(t,\tau)+i,t,\tau)} - Se^{f(\bar{\alpha}(t,\tau)+i,t,\tau)}) - (S(t)e^{g(\bar{\alpha}(t,\tau),t,\tau)} - Se^{f(\bar{\alpha}(t,\tau),t,\tau)}) - 5,$$

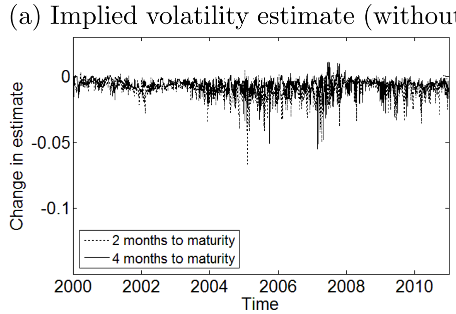
where $\bar{\alpha}(t,\tau) \equiv \ln[K_{\max}(t,\tau)/S(t)]$, $f(\alpha,t,\tau) = c(t,\tau)\alpha$, $g(\alpha,t,\tau) = \alpha$, $c(t,\tau) = \ln[K_{\min}(t,\tau)/S(t)]/\ln[K_{\max}(t,\tau)/S(t)]$, and $K_{\min}(t,\tau)$ and $K_{\max}(t,\tau)$ are the observed minimum and maximum strike prices for day t and maturity τ, respectively. Then, the size of change is approximated as the truncation sensitivity multiplied by i.



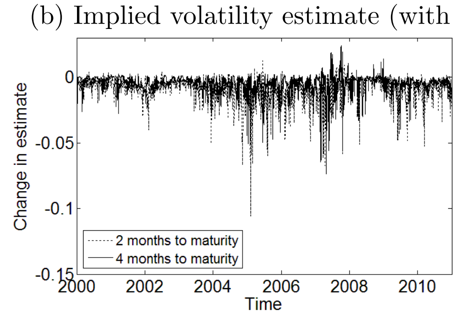
(a) Implied volatility estimate (without LE)



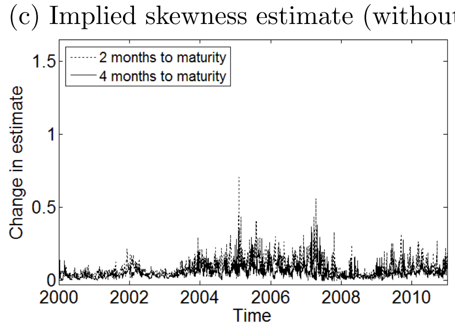
(b) Implied volatility estimate (with LE)



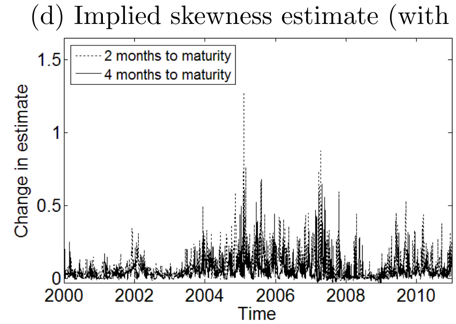
(c) Implied skewness estimate (without LE)



(d) Implied skewness estimate (with LE)



(e) Implied kurtosis estimate (without LE)



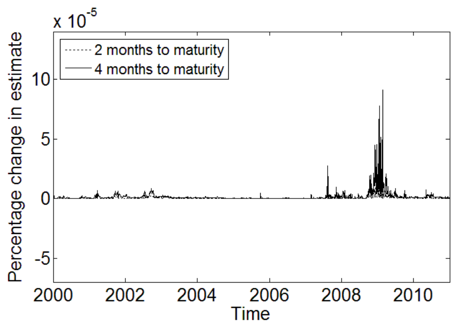
(f) Implied kurtosis estimate (with LE)

Figure 4: Percentage change in moment estimate after an increase in strike price domain width

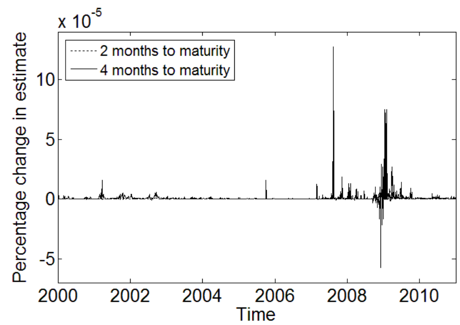
This figure reports the percentage change in the implied volatility, skewness, and kurtosis estimates with and without linear extrapolation (LE), after an increase in strike price domain length by five. The S&P 500 index options dataset used here spans an eleven year time period from January 2000 to December 2010. Option prices are estimated based on the implied volatility curve for the maturities of two and four months, which are extracted from daily implied volatility surfaces. When LE is applied, implied volatility at minimum and maximum strike prices are extrapolated up to the points in which strike prices are $3/S(t)$ and $3S(t)$, respectively, where $S(t)$ is the dividend-adjusted index level on day t. Rate of change in estimate is approximated by first finding the value of increment i which minimizes the absolute value of function $h(t, \tau, i)$, which is defined as

$$h(t, \tau, i) = (S(t)e^{g(\bar{\alpha}(t,\tau)+i,t,\tau)} - Se^{f(\bar{\alpha}(t,\tau)+i,t,\tau)}) - (S(t)e^{g(\bar{\alpha}(t,\tau),t,\tau)} - Se^{f(\bar{\alpha}(t,\tau),t,\tau)}) - 5,$$

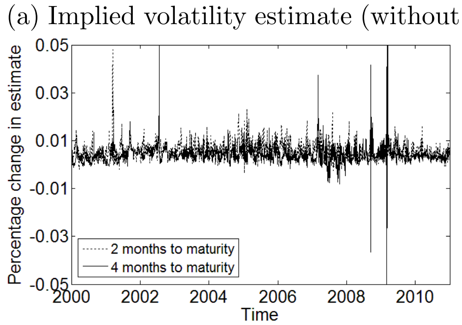
where $\bar{\alpha}(t, \tau) \equiv \ln[K_{\max}(t, \tau)/S(t)]$, $f(\alpha, t, \tau) = c(t, \tau)\alpha$, $g(\alpha, t, \tau) = \alpha$, $c(t, \tau) = \ln[K_{\min}(t, \tau)/S(t)]/\ln[K_{\max}(t, \tau)/S(t)]$, and $K_{\min}(t, \tau)$ and $K_{\max}(t, \tau)$ are the observed minimum and maximum strike prices for day t and maturity τ, respectively. Next, the size of change is approximated as the truncation sensitivity multiplied by i. Then, finally, the percentage change is calculated by dividing the size of change by the corresponding implied moment estimate.



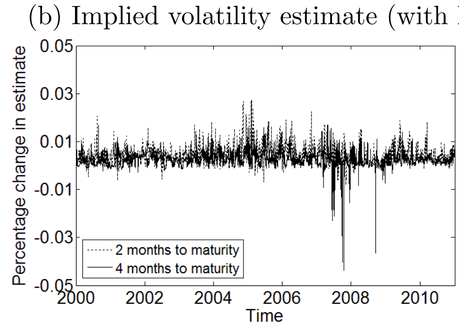
(a) Implied volatility estimate (without LE)



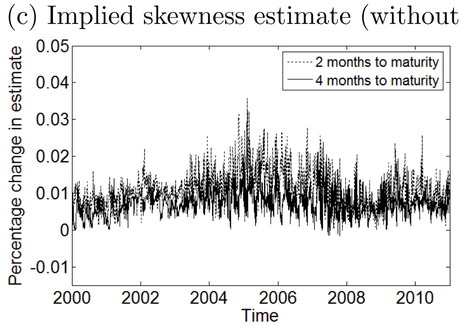
(b) Implied volatility estimate (with LE)



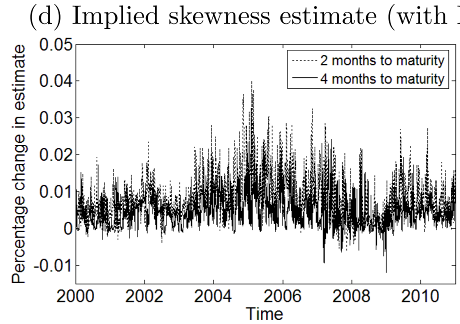
(c) Implied skewness estimate (without LE)



(d) Implied skewness estimate (with LE)



(e) Implied kurtosis estimate (without LE)



(f) Implied kurtosis estimate (with LE)