A General Equilibrium Approach to Pricing Volatility Risk

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Abstract

This paper provides a general equilibrium approach to pricing volatility. Existing models (e.g., ARCH/GARCH, stochastic volatility) take a statistical approach to estimating volatility, volatility indices (e.g., CBOE VIX) use a weighted combination of options, and utility based models assume a specific type of preferences (e.g. Bansal et al., 2014; Tauchen, 2014). In contrast we treat volatility as any other asset and price it using the Arrow and Debreu (1954) general equilibrium state pricing framework. Our results show that the general equilibrium volatility method developed in this paper provides superior forecasting ability for realized volatility and serves as an effective fear gauge. We demonstrate the flexibility and generality of our approach by pricing downside risk and upside opportunity. Finally, we show that the superior forecasting ability of our approach generates significant economic value through volatility timing.

Keywords: General equilibrium, volatility, state pricing, fear gauge

Classification codes: G12, G13, G17
1. Introduction

Volatility modelling has proceeded as a field separate from asset pricing. Statistical models, such as ARCH (Engle 1982; Engle 2002), GARCH (Bollerslev 1986), stochastic volatility (Barndorff-Nielsen 2002), and option prices (Fleming et al. 1995) are commonly used to estimate volatility, without reference to modern asset pricing theory. In this paper, we propose to price volatility using a general equilibrium asset-pricing framework. The advantage of such an approach is that volatility can be priced and measured in the most general setting available. The approach also allows us to extend measurement and pricing (such as downside risk pricing and upside opportunity pricing), which cannot be achieved with current approaches to volatility modelling.

This paper assumes a complete market setting where state prices are available for each time and state. State prices are derived from the Arrow and Debreu (1954) general equilibrium state pricing framework. Following Breeden and Litzenberger (1978), state prices are obtained for each time and state of the market that can be used to price all assets. We apply this state pricing approach to market volatility risk and are able to derive prices that are almost perfectly correlated with the CBOE Volatility Index (VIX), but are the result of a general equilibrium model. There are several advantages to treating volatility as any other asset and pricing it using the Arrow and Debreu (1954) approach. First, we treat volatility pricing the same as any other pricing exercise. Second, we do not have to assume a specific form of utility preferences to implement this approach. Third, the equilibrium price of volatility provides not only a more general approach that is very flexible and can also be used for individual securities, but also serves as a better predictor of future volatility and as an investor fear gauge.

Empirically, we generate market state prices from the S&P 500 index options and use them to price the 49 Fama and French industry volatility. The ex-ante industry volatility measure
constructed in this approach yields better forecasts of realized volatility than existing approaches and serves as a qualified investor fear gauge. We then demonstrate the flexibility and generality of our approach by pricing downside and upside opportunity, which complements the work of Bekaert and Engstrom (2017), Feunou et al. (2017), Kilic and Shaliastovich (2016) and Segal et al. (2015). We show that these new upside and downside volatility measures work well empirically. Finally, we analyze the economic value of volatility timing using the general equilibrium measures versus existing volatility measures. We show that the superior forecasting ability of our general equilibrium volatility measure has greater economic value for investors wishing to manage volatility.

Our paper is related to the recent literature on understanding volatility in a general equilibrium framework. As an extension of Bansal et al. (2004) and Bansal and Yaron (2004), Tauchen (2011) proposes a consumption-based general equilibrium model that assumes stochastic consumption. The model generates a two-factor structure for stock market volatility along with time-varying risk premiums on consumptions and volatility risk, and also the leverage effect. In the general equilibrium framework, Bansal et al. (2014) demonstrate that besides the cash flow risk and discount rate risk, volatility risk is an important and separate risk source that cannot be ignored. Different from this strand of literature, we impose no assumptions on consumption dynamics and rely only on state prices extracted from the options market. The focus of these existing studies is to explain the stylized facts (e.g., leverage effect, equity risk premium) in the market, while we aim to provide market participants with an easy and flexible tool to measure and manage the volatilities of asset portfolios or individual securities.

Our paper is also related to the prior literature on constructing volatility indices based on options. Britten-Jones and Neuberger (2000) use a replicating strategy to synthesize a variance
swap using options contracts, assuming continuity in the underlying asset price. Jiang and Tian (2005) build on a similar concept by incorporating a jump-diffusion stochastic volatility model. We adopt the general equilibrium approach and are able to construct the volatility indices for assets without options written on. Our paper also relates to recent efforts to disentangle the effects of upside and downside uncertainty on asset prices (Segal et al. 2015; Bekaert & Engstrom 2017) by proposing a new approach for volatility decomposition. One common approach for decomposing volatility into upside and downside components is to use a threshold to compute risk-neutral expectations of semi-variances (see e.g., Andersen and Bondarenko (2007); Feunou et al. (2017); Kilic and Shaliastovich (2015)). However, this approach still depends on the existence of traded options, while our approach does not.

The paper is organized as follows. Section 2 outlines volatility pricing using a general equilibrium model. Section 3 extends our approach to price downside risk and upside opportunity. Section 4 analyzes the economic value of our volatility measures in a volatility timing framework. Section 5 concludes.

2. Pricing volatility in a general equilibrium model

This section outlines the general equilibrium approach for pricing market volatility and shows how it can be extended to pricing industry volatility. Industry volatility prices are compared with existing approaches to forecasting realized volatility and evaluated as a gauge of investor fear.

2.1 Method
Under a state pricing approach, the value of any asset is the sum of the state prices multiplied by the payoff in each state. If, for example, we were to price the market portfolio $M$ which pays off $F_{ms}$ in each of $s$ states one period (set as 30 days in this paper) from now, the price is given by:

$$P_m = \sum_{s=1}^{s} \phi_{ms} F_{ms}$$  \hspace{1cm} (1)$$

For an arbitrage asset $i$, whose payoff $F_i$ depends on the level of the market, under the complete market setting, its price is given by:

$$P_i = \sum_{s=1}^{s} \phi_{ms} E[F_{is} | F_{ms}]$$  \hspace{1cm} (2)$$

If we were to take a linear projection\(^2\) of $F_i$ onto $F_M$, then we would obtain:

$$P_i = \sum_{s=1}^{s} \phi_{ms} [\alpha_i + \beta_i F_{ms}] \text{ or } P_i = \alpha_i + \beta_i P_m$$  \hspace{1cm} (3)$$

Since $\sum_{s=1}^{s} \phi_{ms}$ is the price of a risk-free asset with payoff of 1, $\alpha_i$ is the price of a riskless asset with payoff $\alpha_i$.

This is a relation that closely resembles the Sharpe-Lintner Capital Asset Pricing Model (Sharpe 1964); however, the derivation contains obvious differences. First, the market price of risk

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\(^1\) Breeden and Litzenberger (1978) argue that the market portfolio, as a proxy for aggregate consumption, is sufficient to represent the different states in the economy. We show in Appendix 1 how we obtain the state prices using market options, where the market is represented by S&P 500 index (SPX).

\(^2\) We could obtain this conditional expectation using more general non-linear methods as outlined in Friedman et al. (2001).
will vary over time as the state prices change. Second, the risk-free factor will be different for each asset $i$ depending on the magnitude of $\alpha_i$.\footnote{Appendix 2 provides a more detailed discussion of the relation between state pricing theory and the CAPM.}

We now consider pricing market volatility. Here, the payoff is the squared market return at each state. The price of market volatility under the Arrow and Debreu (1954) approach is given by:

\[
SVX_M^2 = \sum_{s=1}^{S} \phi_{ms} R_{ms}^2
\]

where $SVX_M$ is the state pricing volatility index for the market. It is the general equilibrium price of market volatility.

Compared to the calculation of the VIX, the $SVX_M$ formula offers a more straightforward approach.\footnote{For a more detailed discussion on how to construct $SVX_M$ and how it performs against other volatility measures in predicting future market volatility, see Liu and O'Neill (2015). The approach of using Arrow-Debreu securities to price squared returns has also been adopted by Brennan and Cao (1996) and Cao and Ou-Yang (2008).} To price volatility on an arbitrary asset, for example industry $I$, the approach above yields:

\[
SVX_I^2 = \sum_{s=1}^{S} \phi_{is} E[R_{is}^2 | R_{ms}^2]
\]

An assumption of a linear relation between individual asset return and market return (as in Eq. 3) would lead to a linear relation between individual asset return squared and market return squared conditional on the given market return, $R_{ms}$. Naturally, we have:
\[ SVX_i^2 = \sum_{s=1}^{S} \phi_{ms} [\alpha_s + \beta_s R_{ms}^2 ] \] (6)

or

\[ SVX_i^2 = \alpha_{rf} + \beta_i SVX_M^2 \] (7)

where \( \alpha_{rf} \) is the price of a riskless asset with payoff \( \alpha_i \), \( SVX_M^2 \) is as defined above in Eq. 4.

The details of the construction of \( SVX_i \) are presented in Appendix 3. We see that under the linear projection approach, the volatility price of any asset depends on the market price of volatility in a straightforward manner.

We compare two other volatility measures to our measure. The first is an ad-hoc industry volatility index using the widely available CBOE volatility index VIX. To achieve that, we simply replace \( SVX_M^2 \) by \( VIX_M^2 \):

\[ VIX_i^2 = \alpha_{rf} + \beta_i VIX_M^2 \] (8)

where \( VIX_M \) is the CBOE VIX.

The second measure is the historical volatility, \( HV_i \), which is the realized volatility in the previous year.

2.2 Data

To estimate state prices in the complete market setting, we obtain prices and implied volatilities of S&P 500 index options and S&P 500 index dividend yields from the Ivy DB US OptionMetrics, available through Wharton Research Data Services. The options data are available
on a daily basis from January 4, 1996 to April 29, 2016. Interest rates are taken from the Center for Research in Security Prices (CRSP) Zero Curve file. We apply a cubic spline to the interest rate term-structure data to match the length of the risk-free rate with the corresponding option maturity.

The 49 industry portfolios are obtained from Kenneth R. French’s Data Library. The daily return data are available from July 1927 to February 2017. However, our sample period is dictated by the availability of the S&P 500 options data in estimating the state prices, which are only available from January 4, 1996 to April 29, 2016. Given we need to estimate the betas in Eq. 7 using a fixed two-year rolling window, we examine daily industry returns since January 1994.

[Table 1 about here.]

Summary statistics of $SVX_i$ are reported in Table 1. The mean of the annualized $SVX_i$ varies across industries, ranging from 0.129 for the utility industry to 0.273 for the coal industry. Consistent with economic intuition, the “necessities” industries, such as Food, Soda, Beer, Smoke, and Utilities (as defined in Boudoukh et al. 1994) are insensitive to business cycles, and are the least volatile, on average.

[Figure 1 about here.]

$SVX_i$ for the 49 industry portfolios is illustrated in Figure 1. It is apparent that there is a strong positive correlation among the volatility indices. Figure 1 also reveals that there is a significant time variation in volatility for all industries in the analysis, showing an upward spike

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5 We are grateful for Kenneth French for supplying this data: [http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html](http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html).
in the 1998 Asian Financial Crisis and during the technology bubble of the early 2000’s. A peak occurs around the 2008 Global Financial Crisis.

2.3 Forecasting future 30-day realized volatility

The primary goal of a volatility index is to serve as a measure of the next 30-day expected volatility (CBOE 2009). In this section, we examine the information content of $SVX_t$ in predicting subsequent 30-day realized return volatility from January 1996 to April 2016. We regress the future 30-day close-to-close realized volatility on different volatility measures in the following models:

$$RV_{I,t,t+30} = \alpha_I + \beta_I SVX_{I,t} + \epsilon_{I,t,t+30}$$

(9)

where $I$ stands for one of the 49 industry portfolios, $RV_{I,t,t+30}$ denotes the annualized realized volatility over the next 30 days and it is defined as $RV_{I,t,t+30} = 100(\frac{365}{30} \sum_{i=1}^{22} R_i^2)^{1/2}$, where $R_i^2$ is the square of the daily portfolio return. We also run this regression for $VIX_t$ and $HV_t$, where $HV_I$ is the annualized realized volatility in the previous year. These measures are highly correlated on average, so we cannot include both in the same regression to determine which index is more statistically significant. Instead, we use the Mincer and Zarnowitz (1969) regression to compare the prediction performances of these three volatility measures.

To be an unbiased volatility predictor, we expect alphas to be not significantly different from 0 and betas to be not significantly different from 1. If $SVX_I$ is a better predictor than the
other measures, the forecasting regression using it as the predictor is expected to generate the highest model explanatory power, as expressed in adjusted R-squared.

We run the above regressions for each of 49 industry portfolios and report the arithmetic average of regression results in Table 2. The covariance matrix is computed according to Newey and West (1994) to correct for any potential serial correlations in the error terms.

The mean level of the $\beta$ of $SVX_i$ is 1.01, with an average standard error of 1.5%. As expected, $\beta$ of $SVX_i$ is significantly different from 0 at 1% level, and not significantly different from 1 at any levels. In comparison, $\beta$ of $VIX_i$ has a mean level of 0.918 with a standard error of 1.4%. It is significantly different from both 0 and 1 at 1% level. $\beta$ of $HV_i$ has the lowest mean level out of the three measures; namely, 0.677 with a standard error of 1.5%. On average, $\alpha$ of $SVX_i$ is 0.046, which is marginally significantly different from 0 at 5% level. The mean for $\alpha$ of $VIX_i$ is 0.04 and is not significantly different from 0 at 5% level. In comparison, the $\alpha$ of $HV_i$ has an average value of 0.072 and is significantly different from 0. $SVX_i$ provides higher adjusted R-squared than the regressions using $VIX_i$ and $HV_i$. On average, the model with $SVX_i$ as the predictor has a 1.4% higher adjusted R-squared than $VIX_i$ and 19% higher than $HV_i$. Therefore we conclude that the state price volatility index $SVX_i$ is a more efficient forecaster of future
realized volatility than its counterparts. This industry-level result reinforces Liu and O’Neill’s (2015) finding that state price volatility outperforms VIX and other predictors at the market level.

2.4 Fear gauge

It is well-documented that there is a negative correlation between the rate of change in volatility (e.g., CBOE VIX) and daily market returns (see for example, Carr & Wu 2009). As expected market volatility increases, investors demand a higher rate of return on stocks and prices fall, which ultimately leads to a drop in the current market return. Therefore, we study the contemporaneous relation between rates of change in various industry volatility measures and daily industry portfolio returns. In particular, we investigate whether these indices contain any fear information from the market state prices. Generally, a fall in an industry portfolio usually implies a rally in investor fear in the segment. Therefore, we expect to see negative betas in all volatility measures. A fear gauge, such as CBOE VIX (Whaley 2009), should respond more to negative changes in portfolio returns than positive changes. We are interested in testing whether industry state price volatility measures can capture this asymmetric fear gauge effect. We regress the daily changes of various measures against industry portfolio returns in the following forms:

$$
\Delta SVX_{t,i} = \alpha_t + \beta_{1,i} R_{t,i} + \beta_{2,i} R^-_{t,i} + \epsilon_{t,i}
$$

where $I$ stands for each of the 49 industry portfolios, $\Delta$ measures the daily changes, and $R_{t,i}$ is the daily industry portfolio return, $R^-_{t,i}$ is defined as $\min(R_{t,i}, 0)$.

[Table 3 about here.]

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6 In untabulated results, we also compare the forecasting abilities of ARCH and GARCH volatility measures with that of $SVX_t$. We find that $SVX_t$ outperforms these two measures in terms of adjusted R-squared.
We run the above regressions for each of 49 industry portfolios and report the arithmetic average of regression results in Table 3. The covariance matrix is computed according to Newey and West (1994) to correct for any potential serial correlations in the error terms. On average, $\beta_1$ is significantly less than 0 at the 1% level, implying that there is an inverse relation between contemporaneous changes in volatility indices and changes in portfolio returns. $\beta_2$ is also significantly less than 0 at the 1% level. These results show that the response to different swings in portfolio returns is strongly asymmetric, and are consistent with the findings of Whaley (2009) and Liu and O'Neill (2015). Besides statistical significance, the coefficients are also economically meaningful: on average, an increase in the industry return by 100 basis would result in 51.7 basis decrease in $SVX_I$ and an decrease in the industry return by 100 basis would lead to 73.8 basis increase in $SVX_I$.

3. Extensions

This section describes the flexibility and generality of the general equilibrium approach by modelling downside risk and upside opportunity. We show how these measures can be used to better forecast realized volatility, as well as downside risk and upside opportunity. We also consider how these measures can be used as an investor fear gauge.

3.1 Downside risk and upside opportunity in a general equilibrium model
In Section 2.1, we showed how to construct the industry state-price volatility index $SVX_I$ in a general equilibrium setting. Here, we use the general pricing approach to price downside risk and upside risk.  

Downside market risk is given by:

$$BEX_M = \sqrt{\sum_{s=1}^{S} \phi_{ms} R_{ms}^2 I_{R_{ms}<0}}$$ \hspace{1cm} (11)

where $I_{R_{ms}<0}$ is an indicator variable equal to 1 if $R_{ms}$ is less than zero.

Similarly upside market risk is given by:

$$BUX_M = \sqrt{\sum_{s=1}^{S} \phi_{ms} R_{ms}^2 I_{R_{ms}>0}}$$ \hspace{1cm} (12)

where $I_{R_{ms}>0}$ is an indicator variable equal to 1 if $R_{ms}$ is greater than zero.

Industry measures of $BEX$ and $BUX$ are obtained in an analogous manner to industry volatility $SVX_I$ (Eq. 6) and can be represented as:

$$BEX^2_I = \alpha_{Down,I} + \beta_{Down,I} BEX^2_M$$ \hspace{1cm} (13)

$$BUX^2_I = \alpha_{Up,I} + \beta_{Up,I} BUX^2_M$$ \hspace{1cm} (14)

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7 Recent literature highlights the importance of distinguishing downside and upside volatility risk. See Patton and Sheppard (2015), Bekaert and Engstrom (2017), and Segal et al. (2015).
To estimate the alphas and betas in Eq. 13 and Eq. 14, we use a linear least squares regression of squared daily industry returns on squared S&P 500 market returns. Specifically, we are interested in the alphas and betas in the following regressions:

\[ R_i^2 = \alpha_{\text{Down},i} + \beta_{\text{Down},i} R_{\text{M|Down}}^2 + \epsilon_i \quad (15) \]

\[ R_i^2 = \alpha_{\text{Up},i} + \beta_{\text{Up},i} R_{\text{M|Up}}^2 + \epsilon_i \quad (16) \]

where \( R_i^2 \) is the daily squared industry return, and \( R_{\text{M|Down}}^2 \) (\( R_{\text{M|Up}}^2 \)) is the market return squared conditional on whether the market has gone down (up) from the previous day, regardless of movement in the industry portfolio.\(^8\) The return is computed using the closing value at the end of day. We estimate each beta using a two-year fixed rolling window. That is, on the 505\(^{\text{th}}\) day, we use the past two years (504 trading days) of return squared to estimate the alphas and betas in the above regressions.

3.2 Industry state-price volatility that incorporates downside market volatility

Prior studies have shown that returns become more correlated in a bear market (e.g., Ang & Chen 2002; Campbell et al. 2002). As a result, we extend the basic formula for \( SVX_i \) by using an alternative linear projection that incorporates market downside movement:

\[ (SVX_i^D)^2 = \sum_{s=1}^{S} \phi_{ms} [\alpha_i + \beta_i R_{ms}^2 + \gamma_i R_{ms}^2 I_{R_{ms} < 0}] \quad (17) \]

Which can be solved to yield:

\(^8\) We also considered different definitions of conditional downturn return squared for the industry portfolio. Results are qualitatively the same when we measure the beta of the downturn industry return and the downturn market return. We selected the current definition because it is more meaningful for examining how the industry portfolio responds and contributes to a downturn in the whole market.
\[(SVX_I^p)^2 = \alpha_{rfI} + \beta_I SVX_M^2 + \gamma_I BEX_M^2 \tag{18}\]

where \(\alpha_{rfI}\) is the price of a riskless asset with payoff \(\alpha_I\), \(SVX_M^2\) is as defined above in Eq. 4. and \(I_{R_{ms}<0}\) is an indicator variable equal to 1 if \(R_{ms}\) is less than zero. In this definition, the volatility price of any asset depends on its sensitivity to the price of market volatility and (in addition) to the price of market downside volatility.

To estimate the alphas, betas, and gammas in Eq. 18, we use a linear least square regression of squared daily industry returns on squared daily S&P 500 market returns.

\[R_I^2 = \alpha_{Down,I} + \beta_{Down,I} R_M^2 + \gamma_{Down,I} R_{M|Down}^2 + \epsilon_I \tag{19}\]

where \(R_I^2\) is the daily squared industry return, and \(R_{M|Down}^2\) is the market return squared conditional on whether the market has gone down from the previous day, regardless of movement in the industry portfolio.

### 3.3 Forecasting future 30-day realized volatility

To extend the analysis described in Section 2.3, we examine the information content of \(SVX_I^p\) in predicting the future 30-day realized volatility in each industry portfolio. We regress the future 30-day close-to-close realized volatility on different volatility measures in the following models:

\[RV_{I,J,t+30} = \alpha_I + \beta_I SVX_I^p + \epsilon_{I,J,t+30} \tag{20}\]
Results are reported in Table 4 Panel A. The mean level of $\beta$ of $SVX_I^D$ is 1.003, with an average standard error of 1.5%. $\beta$ of $SVX_I^D$ is significantly different from 0 at 1% level, and not significantly different from 1 at any levels. Comparing to results in Table 2, we can see that $SVX_I^D$ outperforms other volatility candidates in terms of adjusted R-squared.

3.4 Forecasting future 30-day realized downside volatility

A typical volatility measure does not describe the proportion of upside gain versus downside threat. In this paper, we solve this problem by introducing a downside (upside) volatility index $BEX_I$ ($BUX_I$) for each industry portfolio as an ex-ante predictor of future downside (upside) volatility. It is important to note that, unlike the comparison with $SVX_I$ and $VIX_I$ in the previous section, we do not have a VIX benchmark per se because is not mathematically feasible to derive a downside VIX using market state prices. We regress the future 30-day close-to-close realized downside volatility in the following way:

$$RVD_{t,t+30} = \alpha_t + \beta_I BEX_{t,t+30}^2 + \epsilon_{t,t+30}$$  \hspace{1cm} (21)

where $I$ stands for each of the 49 industry portfolios, $RVD_{t,t+30}$ denotes the realized downside volatility over the next 30 days and it is defined as $RVD_{t,t+30} = 100\frac{365}{30} \sum_{i=1}^{22} R_{i,t+30} I_R R_{i,t+30}^{22}$.  

We expect alphas to be not significantly different from 0 and betas to be not significantly different from 1 if $BEX_I$ is an unbiased forecaster, and betas to be significantly different from 0. We run the above regression for each of 49 industry portfolios and report the arithmetic average of regression results in Panel B in Table 4. The covariance matrix is computed according to Newey and West (1994) to correct for any potential serial correlations in the error terms. The mean level
of $\beta$ of $BEX_1$ is 0.917, with an average standard error of 1.8%. $\beta$ of $BEX_1$ is significantly different from 0 and 1 at 1% level. On average, $\alpha$ of $BEX_1$ is 0.029 and is not significantly different from 0 at 5% level. The average adjusted R-squared of Eq. 21 is 34.7%. We show that $BEX_1$ is an efficient forecaster of future realized downside volatility.

3.5 Fear gauge

We further study the contemporaneous relation between rates of change in $SVX^D_I$ and daily industry portfolio returns. We are interested in testing whether the modified industry volatility measure can better capture the fear gauge. We regress the daily changes of $SVX^D_I$ against the industry portfolio returns in the following forms:

$$\Delta SVX^D_I = \alpha_I + \beta_{1,t} R_{t,i} + \beta_{2,t} R_{t,i}^- + \epsilon_{t,i}$$  \hspace{1cm} (22)

where $I$ stands for each of the 49 industry portfolios, $\Delta$ measures the daily changes, and $R_{t,i}$ is the daily industry portfolio return, $R_{t,i}^-$ is defined as $\min(R_{t,i},0)$. We perform a similar analysis for $\Delta BEX^D_I$ to see whether the downside volatility measure can serve as a qualified fear gauge or not.

[Table 5 about here.]

We run the above regressions for each of 49 industry portfolios and report the arithmetic average of regression results in Table 5. The covariance matrix is computed according to Newey and West (1994) to correct for any potential serial correlations in the error terms. On average, for both $\Delta SVX^D_I$ and $\Delta BEX^D_I$, all $\beta_i$ are significantly less than 0 at 1% level, implying there is an inverse relation between the contemporaneous changes of volatility indices and those of portfolio.
returns. For both $\Delta SVX^D_{t,t}$ and $\Delta BEX^D_{t,t}$, $\beta_2$ are also significantly less than 0 at 1% level. These results shows that the response to different swings in portfolio returns is strongly asymmetric. This is consistent with findings reported in Section 2.4. By comparing the adjusted R-squared between Table 5 and Table 2, we can see that incorporating the downside risk into the volatility index can enhance monitoring effectiveness (37.9% and 37.6% for $\Delta SVX^D_{t,t}$ and $\Delta BEX^D_{t,t}$ in Table 5, and 36.7% for $\Delta SVX_{t,t}$ in Table 2). The results confirm that measures incorporating downside risk are better measures of fear gauge.

3.6 Volatility forecasting: out of sample evidence

Besides the in-sample forecasting evidence, here we further compare the volatility predictability by examining the out-of-sample tests for four volatility predictors: $HV_t$, $VIX_t$, $SVX_t$, and $SVX^D_t$. Specifically, we use a rolling fixed window approach. Every day, each forecasting model is estimated with a fixed rolling window to obtain the one-month-ahead volatility forecast. To ensure robustness, we use a one-year window, a two-year window and a three-year window. The out-of-sample forecasting accuracy is judged by three criteria: root-mean-square error (RMSE), mean-absolute error (MAE) and mean-absolute-percentage error (MAPE).

Table 6 reports the estimation results for the average values of RMSE, MAE and MAPE across 49 industries in three different rolling windows. First, $HV_t$ performs the worst among four volatility measures, regardless of criterion or rolling window. For instance, in the one-year rolling window case, the average RMSE values for $VIX_t$, $SVX_t$, and $SVX^D_t$ are all around 0.083, while it is 0.100 for $HV_t$. Second, in almost all the scenarios, $SVX^D_t$ is the best predictor, with $SVX_t$ and $VIX_t$ being the second and third best predictors. Overall, the out-of-sample forecasting results reiterate the earlier in-sample test findings.
4. Economic value of volatility timing

This section investigates the economic value of using various predictors to forecast monthly industry volatility. First, consider an investor who allocates wealth between an industry portfolio and a risk-free asset using a volatility predictor to maximize utility gains. Similar to Fleming et al. (2001), Fleming et al. (2003), and Marquering and Verbeek (2004), we assume the investor maximizes the following utility based expression:

\[ U[E_t(R_{p,t+1}), \sigma_{p,t+1}^2] = E_t(R_{p,t+1}) - 0.5\gamma \sigma_{p,t+1}^2, \tag{23} \]

where \( E_t(R_{p,t+1}) \) and \( \sigma_{p,t+1}^2 \) respectively are the conditional mean and variance of the portfolio returns, and \( \gamma \) refers to the coefficient of the investor’s relative risk aversion. We set \( \gamma \) to a realistic estimate of 3, as suggested by Campbell and Thompson (2008) and Rapach et al. (2016). We also use values of \( \gamma \) of 4 and 5 for robustness and sensitivity analysis. The portfolio return is \( E_t(R_{p,t+1}) = r_{f,t+1} + w_t (E_t(R_{I,t+1}) - r_{f,t+1}) \), where \( w_t \) is the portfolio weight of industry portfolio \( I \), \( E_t(R_{I,t+1}) \) is the conditional expected return of the industry portfolio, and \( r_{f,t+1} \) denotes the risk-free rate, which is known ex-ante. The portfolio variance is \( \sigma_{p,t+1}^2 = w_t^2 \sigma_{I,t+1}^2 \), where \( \sigma_{I,t+1}^2 \) denotes the conditional variance of industry portfolio \( I \). The optimal weight for industry \( I \) is given by:

\[ w_t = \frac{E_t(R_{I,t+1}) - r_{f,t+1}}{\gamma \sigma_{I,t+1}^2}, \tag{24} \]

The current study focuses on monthly realized volatility forecasting, so we assume the portfolio is rebalanced monthly. We set the month \( t \) expected return as the historical mean using

---

\(^9\) Grossman (1976) and Admati (1985) show how negative exponential utility and normal distributions result in the above mean and variance maximization.
return data up to period $t$. The expected variance of portfolio $I$, $\sigma_{I,t+1}^2$, is based on the two-year rolling out-of-sample forecast using Eq. 11, with one of four different volatility measures ($HV_i$, $VIX_i$, $SVX_i$, and $SVX^D_i$) as the predictor. The accuracy of volatility forecasting determines the performance of this volatility timing strategy. Our benchmark strategy is the buy-and-hold strategy of the respective industry portfolios, $I$.

We adopt two criteria to compare the performance of different strategies. The first is the commonly used annualized Sharpe ratio. The second is the certainty equivalent return (CER) gain of a volatility timing strategy relative to that of a naïve buy-and-hold strategy:

$$CER = \left( R_p - 0.5\gamma \sigma_p^2 \right) - \left( R_{\text{naive}} - 0.5\gamma \sigma_{\text{naive}}^2 \right).$$

(25)

Intuitively, the CER gains of Eq. 25 are the incremental management fees that the investor is willing to pay to invest in the volatility timing strategies based on the volatility forecasts over the buy-and-hold strategy.

[Table 6 about here.]

Table 7 presents the average performance of 49 industries from January 1998 to April 2016. In Panel A, we assume there is no transaction cost. We first examine the basic case of $\gamma$ equal to 3. Here, in terms of Sharpe ratio, the volatility timing strategy based on historical volatility ($HV_i$) performs worse than the naïve buy-and-hold strategy (0.326 vs 0.335). However, the volatility timing strategies based on implied volatility series, that is, $VIX_i$, $SVX_i$, and $SVX^D_i$, generate higher Sharpe ratios than the buy-and-hold strategy. The highest Sharpe ratio is obtained by using $SVX^D_i$ as the volatility predictor (0.396). The results on CER gains reveal that all the volatility timing strategies outperform the buy-and-hold strategy. $SVX^D_i$ again performs best: the investor is
prepared to pay a hefty incremental annual management fee of 311 basis points bps) to have access
to predictive regression based on $SVX_i^{D}$, instead of the buy-and-hold strategy. In contrast, the
investor is only willing to pay 249 bps for the strategy using $HV_i$. When the investor is more risk-
averse, we find that the Sharpe ratios are similar to the basic case, but the management fees that
the investor is willing to pay to be involved with the volatility timing strategy using $SVX_i^{D}$ increase
from 331 bps (γ = 3) to 590 bps (γ = 4) and 866 bps (γ = 5). This result suggests that volatility
timing is especially important for risk-averse investors.

The volatility timing strategy requires monthly rebalancing, so its performance might be sensitive to transaction costs. With this in mind, we analyze the impact of transaction costs on our results. Following Bandi et al. (2008) and Nolte and Xu (2015), we define the transaction cost adjusted portfolio return as follows:

$$\bar{R}_{p,t+1} = R_{p,t+1} - \rho(1 + R_{p,t+1}) |\Delta w_{t+1}|,$$

(26)

where $\bar{R}_{p,t+1}$ is the transaction cost adjusted portfolio return, $R_{p,t+1}$ is the pre-adjusted portfolio return, $\rho$ is the transaction cost parameter, and is set to be 0.0025, and $\Delta w_{t+1}$ is the change of weight from month $t$ to month $t+1$.

Panel B of Table 7 presents the results for transaction cost adjusted performance. It clearly shows that, even when we account for transaction cost, volatility timing strategies based on $VIX_i$, $SVX_i$, and $SVX_i^{D}$, still largely outperform the buy-and-hold strategy and the volatility timing

---

10 Portfolio theory confirms that a change of weights has no impact on the Sharpe ratio when allocating assets between a portfolio and the risk-free rate.
strategy based on $HV_t$. Consistent with Panel A, $SVX^D_t$ generates the highest Sharpe ratio (0.367) and CER gain (269 bps) for $\gamma = 3$, and again, economic values are larger when the investor is more risk-averse (CER gains of 544 bps for $\gamma = 4$ and 829 bps for $\gamma = 5$).

In summary, this section uses a volatility timing strategy to show that the strong forecasting abilities of the industry volatility indices ($VIX_I$, $SVX_I$, and $SVX^D_I$) have significant economic value for investors.

5. Conclusion

This paper is novel in that it proposes a general equilibrium framework to price volatility in the same manner as is the case for all securities in the market, following Arrow and Debreu (1954). Using state prices estimated from S&P 500 index options, we illustrate how we can derive ex-ante volatility measures $SVX_I$ for industry portfolios, in which there are no traded options. The $SVX_I$ measures generate superior forecasting abilities for the future realized volatility and serve as qualified fear gauges. We show that our approach is flexible and general by extending it to downside risk and upside opportunity. Finally, we demonstrate that the superior forecasting ability of our general equilibrium volatility measure can create significant economic value through a simple volatility timing strategy. Our findings, together with the fact that the industry volatility indices can be easily constructed under the general equilibrium framework, offer practitioners an appealing alternative tool for managing volatility. Our general equilibrium framework is not limited to pricing volatility, but can be applied to price any moments of the return distribution.
References


Smith, T., Walsh, K., 2013. Why the CAPM is Half-Right and Everything Else is Wrong. Abacus 49, 73-78


This figure plots the Industry SVX\textsubscript{1} of 49 industry portfolios. The data are from 4 January 1996 to 29 April 2016.
TABLE 1

Summary Statistics of Industry SVX\textsubscript{i}

<table>
<thead>
<tr>
<th>Industry</th>
<th>Obs</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agric</td>
<td>5,099</td>
<td>0.150</td>
<td>0.103</td>
<td>0.039</td>
<td>0.950</td>
</tr>
<tr>
<td>Food</td>
<td>5,099</td>
<td>0.132</td>
<td>0.051</td>
<td>0.052</td>
<td>0.482</td>
</tr>
<tr>
<td>Soda</td>
<td>5,099</td>
<td>0.138</td>
<td>0.062</td>
<td>0.005</td>
<td>0.602</td>
</tr>
<tr>
<td>Beer</td>
<td>5,099</td>
<td>0.136</td>
<td>0.067</td>
<td>0.044</td>
<td>0.481</td>
</tr>
<tr>
<td>Smoke</td>
<td>5,099</td>
<td>0.143</td>
<td>0.066</td>
<td>0.049</td>
<td>0.645</td>
</tr>
<tr>
<td>Toys</td>
<td>5,099</td>
<td>0.180</td>
<td>0.066</td>
<td>0.080</td>
<td>0.603</td>
</tr>
<tr>
<td>Fun</td>
<td>5,099</td>
<td>0.216</td>
<td>0.098</td>
<td>0.088</td>
<td>0.893</td>
</tr>
<tr>
<td>Books</td>
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<td>0.165</td>
<td>0.079</td>
<td>0.060</td>
<td>0.772</td>
</tr>
<tr>
<td>Hshld</td>
<td>5,099</td>
<td>0.145</td>
<td>0.060</td>
<td>0.059</td>
<td>0.534</td>
</tr>
<tr>
<td>Clths</td>
<td>5,099</td>
<td>0.179</td>
<td>0.078</td>
<td>0.066</td>
<td>0.714</td>
</tr>
<tr>
<td>Hlth</td>
<td>5,099</td>
<td>0.155</td>
<td>0.066</td>
<td>0.055</td>
<td>0.553</td>
</tr>
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<td>MedEq</td>
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<td>0.160</td>
<td>0.063</td>
<td>0.059</td>
<td>0.581</td>
</tr>
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<td>Drugs</td>
<td>5,099</td>
<td>0.164</td>
<td>0.065</td>
<td>0.068</td>
<td>0.558</td>
</tr>
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<td>Chems</td>
<td>5,099</td>
<td>0.188</td>
<td>0.083</td>
<td>0.091</td>
<td>0.818</td>
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<tr>
<td>Rubbr</td>
<td>5,099</td>
<td>0.155</td>
<td>0.061</td>
<td>0.083</td>
<td>0.557</td>
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<tr>
<td>Txtls</td>
<td>5,099</td>
<td>0.166</td>
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<td>0.711</td>
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<td>BdMnt</td>
<td>5,099</td>
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<td>0.686</td>
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<td>0.224</td>
<td>0.095</td>
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<td>1.080</td>
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<td>Steel</td>
<td>5,099</td>
<td>0.243</td>
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<td>0.084</td>
<td>1.207</td>
</tr>
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<td>0.182</td>
<td>0.086</td>
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<td>0.780</td>
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<td>0.210</td>
<td>0.088</td>
<td>0.104</td>
<td>0.863</td>
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<td>ElecEq</td>
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<td>0.211</td>
<td>0.081</td>
<td>0.105</td>
<td>0.798</td>
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<td>Autos</td>
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<td>0.212</td>
<td>0.089</td>
<td>0.101</td>
<td>0.839</td>
</tr>
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<td>0.203</td>
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<td>0.089</td>
<td>0.756</td>
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<td>0.784</td>
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<td>0.554</td>
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<td>0.184</td>
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<td>0.208</td>
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<td>0.182</td>
<td>0.106</td>
<td>0.039</td>
<td>1.068</td>
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<td>Util</td>
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<td>0.129</td>
<td>0.076</td>
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<td>0.735</td>
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<td>Telnm</td>
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<td>0.184</td>
<td>0.083</td>
<td>0.075</td>
<td>0.836</td>
</tr>
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<td>PerSv</td>
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<td>0.171</td>
<td>0.063</td>
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<td>0.601</td>
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<td>BusSv</td>
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<td>0.084</td>
<td>0.615</td>
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<td>Softw</td>
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<td>0.678</td>
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<td>0.619</td>
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<tr>
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<td>0.078</td>
<td>0.605</td>
</tr>
<tr>
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<td>0.182</td>
<td>0.074</td>
<td>0.073</td>
<td>0.626</td>
</tr>
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<td>0.056</td>
<td>0.070</td>
<td>0.549</td>
</tr>
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<td>Banks</td>
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<td>0.229</td>
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<td>0.083</td>
<td>1.494</td>
</tr>
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<td>Insur</td>
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<td>0.191</td>
<td>0.096</td>
<td>0.077</td>
<td>1.005</td>
</tr>
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<td>RISt</td>
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<td>0.119</td>
<td>0.033</td>
<td>0.950</td>
</tr>
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<td>0.088</td>
<td>0.060</td>
<td>0.669</td>
</tr>
</tbody>
</table>

This table presents summary statistics of Industry SVX\textsubscript{i} of 49 industry portfolios. The data are from 4 January 1996 to 29 April 2016.
**TABLE 2**

Forecasting Realized Volatility with SVX$_i$, VIX$_i$, and HV$_i$

<table>
<thead>
<tr>
<th>Dep Variable: RV$_i$</th>
<th>Average of R$^2$</th>
<th>Average of Intercept</th>
<th>Average of $\beta$ for SVX$_i$</th>
<th>Average of $\beta$ for VIX$_i$</th>
<th>Average of $\beta$ for HV$_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>47.2%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coef</td>
<td>0.046**</td>
<td>1.010***</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p-value</td>
<td>0.041</td>
<td>1.15E-133</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Std Err</td>
<td>0.003</td>
<td>0.015</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>45.8%</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Coef</td>
<td>0.044*</td>
<td>0.918***</td>
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<td></td>
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<tr>
<td>p-value</td>
<td>0.057</td>
<td>3.6E-100</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Std Err</td>
<td>0.003</td>
<td>0.014</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>28.2%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coef</td>
<td>0.072***</td>
<td>0.677***</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p-value</td>
<td>1.3E-25</td>
<td>9.2E-137</td>
<td></td>
<td></td>
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<tr>
<td>Std Err</td>
<td>0.004</td>
<td>0.015</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This table presents the average OLS estimates of regressions in Equation (9). The regressions take the following general form:

$$Y = \alpha + \beta X + \varepsilon$$

All coefficients, p-values, and standard errors are an average of the corresponding measures from 49 regressions on all industry portfolios. Dependent variable RV$_i$ is annualized realized volatility in future 30 days of one of 49 industry portfolios, where $RV = 100 \times \left(\frac{365}{30} \times \sum_{t=1}^{22} R_i^2\right)^{1/2}$ and $R_i$ is the daily portfolio return. The data are from 4 January 1996 to 29 April 2016. To correct for autocorrelation and heteroskedasticity, we use the Newey-West estimator for covariance matric with automatically selected lags as in Newey and West (1994). ***, **, and * denote significance at the 0.01, 0.05, and 0.10 level, respectively.
TABLE 3

Regression Results of Rate Change of SVXI Against Returns of Industry Portfolios

<table>
<thead>
<tr>
<th>Dep Variable: ΔSVXI</th>
<th>Average of R2</th>
<th>Average of Intercept</th>
<th>Average of β₁ for RI,t</th>
<th>Average of β₂ for RI,t⁻¹</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coef</td>
<td>-0.001*</td>
<td>-0.517***</td>
<td>-0.221***</td>
<td></td>
</tr>
<tr>
<td>p-value</td>
<td>0.051</td>
<td>0.000</td>
<td>0.001</td>
<td></td>
</tr>
<tr>
<td>Std Err</td>
<td>0.000</td>
<td>0.019</td>
<td>0.032</td>
<td></td>
</tr>
</tbody>
</table>

This table presents the average OLS estimates of regressions in Equation (10). The regressions take the following general form:

\[ \Delta SVX_{t,t} = \alpha + \beta_1 R_{I,t} + \beta_2 R_{I,t}^- + \varepsilon \]

All coefficients, p-values, and standard errors are an average of the corresponding measures from 49 regressions on all industry portfolios. Independent variables include RI, daily return of the corresponding industry portfolio; and RI, daily return of the corresponding industry portfolio conditional on whether the return is below 0, i.e., RI = min(RI, 0). The dependent variable is the daily return of SVXI, where \( \Delta SVX_{t,t} = \ln(SVX_{t,t}/SVX_{t,t-1}) \). The return data are from 4 January 1996 to 29 April 2016. To correct for autocorrelation and heteroskedasticity, we use the Newey-West estimator for covariance matrix with automatically selected lags as in Newey and West (1994). ***, **, and * denote significance at the 0.01, 0.05, and 0.10 level, respectively.
### TABLE 4
Forecasting Realized Volatility and Downside Realized Volatility with SVX$^{D_1}$ and BEX$^I$

<table>
<thead>
<tr>
<th>Panel</th>
<th>Dep Variable</th>
<th>Average of R2</th>
<th>Average of Intercept</th>
<th>Average of $\beta$ for SVX$^{D_1}$</th>
<th>Average of $\beta$ for BEX$^I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>RV$^I$</td>
<td>47.3%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Coef</td>
<td>0.047**</td>
<td>1.003***</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>p-value</td>
<td>0.046</td>
<td>1.27E-133</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>Std Err</td>
<td>0.003</td>
<td>0.015</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>RV$^{D_1}$</td>
<td>34.7%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Coef</td>
<td>0.029*</td>
<td>0.917***</td>
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</tr>
<tr>
<td></td>
<td>p-value</td>
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<td>8.3E-166</td>
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<tr>
<td></td>
<td>Std Err</td>
<td>0.003</td>
<td>0.018</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This table presents the average OLS estimates of regressions in Sections III C and D. The regressions take the following general form:

$$ Y = \alpha + \beta X + \varepsilon $$

All coefficients, p-values, and standard errors are an average of the corresponding measures from 49 regressions on all industry portfolios. Dependent variables include; (1) RV is annualized realized volatility in future 30 days of one of 49 industry portfolios, where RV = $100 \times \left(365/30 \times \sum_{i=1}^{22} R_i^2\right)^{1/2}$ and $R_i$ is the daily portfolio return; and (2) RVD is the realized downside volatility in future 30 days of one of 49 industry portfolios, where RVD = $100 \times \left(365/30 \times \sum_{i=1}^{22} R_i^2 1_{R_i \leq 0}\right)^{1/2}$. The data are from 4 January 1996 to 29 April 2016. To correct for autocorrelation and heteroskedasticity, we use the Newey-West estimator for covariance matrix with automatically selected lags as in Newey and West (1994). ***, **, and * denote significance at the 0.01, 0.05, and 0.10 level, respectively.
### TABLE 5

Regression Results of Rate Change of SVX\(^D_t\) and ΔBEX\(_I\) against Returns of Industry Portfolios

<table>
<thead>
<tr>
<th>Panel</th>
<th>Dep Variable</th>
<th>Average of R2</th>
<th>Average of Intercept</th>
<th>Average of β(<em>1) for (R</em>{t,t})</th>
<th>Average of β(<em>2) for (R</em>{t,t}^)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>ΔSVX(^D_t)</td>
<td>37.9%</td>
<td>-0.001*</td>
<td>-0.514***</td>
<td>-0.222***</td>
</tr>
<tr>
<td></td>
<td>Coef</td>
<td></td>
<td>0.051</td>
<td>0.000</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>p-value</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Std Err</td>
<td></td>
<td>0.000</td>
<td>0.019</td>
<td>0.031</td>
</tr>
<tr>
<td>B</td>
<td>ΔBEX(_I)</td>
<td>37.6%</td>
<td>-0.001*</td>
<td>-0.397***</td>
<td>-0.171***</td>
</tr>
<tr>
<td></td>
<td>Coef</td>
<td></td>
<td>0.055</td>
<td>0.003</td>
<td>0.002</td>
</tr>
<tr>
<td></td>
<td>p-value</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Std Err</td>
<td></td>
<td>0.000</td>
<td>0.015</td>
<td>0.025</td>
</tr>
</tbody>
</table>

This table presents the average OLS estimates of regressions in Equation (22). The regressions take the following general form:

\[
ΔSVX_{t,t}^D = \alpha + \beta_1 R_{t,t} + \beta_2 R_{t,t}^\gamma + \varepsilon
\]

All coefficients, p-values, and standard errors are an average of the corresponding measures from 49 regressions on all industry portfolios. Independent variables include RI, daily return of the corresponding industry portfolio; and RI- daily return of the corresponding industry portfolio conditional on whether the return is below 0, i.e. \(R_{t,t} = \min(R_{t,t}, 0)\). The dependent variable is the daily return of SVX\(_I\), where \(ΔSVX_{t,t}^D = \ln(SVX_{t,t}^D/SVX_{t,t}^D)\). The return data are from 4 January 1996 to 29 April 2016. To correct for autocorrelation and heteroskedasticity, we use the Newey-West estimator for covariance matric with automatically selected lags as in Newey and West (1994). ***, **, and * denote significance at the 0.01, 0.05, and 0.10 level, respectively.
### TABLE 6 Out-of-Sample Volatility Forecasting Results

<table>
<thead>
<tr>
<th></th>
<th>Panel A: 1-year rolling window</th>
<th></th>
<th>Panel B: 2-year rolling window</th>
<th></th>
<th>Panel C: 3-year rolling window</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RMSE</td>
<td>MAE</td>
<td>MAPE</td>
<td>RMSE</td>
<td>MAE</td>
</tr>
<tr>
<td>HV$_{t}$</td>
<td>0.1005</td>
<td>0.0665</td>
<td>27.8261</td>
<td>HV$_{t}$</td>
<td>0.1086</td>
</tr>
<tr>
<td>VIX$_{t}$</td>
<td>0.0838</td>
<td>0.0555</td>
<td>23.2998</td>
<td>VIX$_{t}$</td>
<td>0.0879</td>
</tr>
<tr>
<td>SVX$_{t}$</td>
<td>0.0832</td>
<td>0.0554</td>
<td>23.3153</td>
<td>SVX$_{t}$</td>
<td>0.0868</td>
</tr>
<tr>
<td>SVX$^{D}_{t}$</td>
<td>0.0832</td>
<td>0.0554</td>
<td>23.3428</td>
<td>SVX$^{D}_{t}$</td>
<td>0.0867</td>
</tr>
</tbody>
</table>

This table reports the out-of-sample forecasting results from January 1996 to April 2016 with a fixed rolling window approach. We report average values of RMSE, MAE and MAPE for 49 industries in the 1-year window, 2-year window, and 3-year window.
TABLE 7
Out-of-Sample Portfolio Performance: Monthly Rebalancing

<table>
<thead>
<tr>
<th>Predictor</th>
<th>Panel A: Without Transaction Cost</th>
<th>Panel B: With Transaction Cost (0.25%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \gamma = 3 )</td>
<td>( \gamma = 4 )</td>
</tr>
<tr>
<td>BH</td>
<td>0.335</td>
<td>N/A</td>
</tr>
<tr>
<td>HV(_1)</td>
<td>0.326</td>
<td>249</td>
</tr>
<tr>
<td>VIX(_1)</td>
<td>0.389</td>
<td>326</td>
</tr>
<tr>
<td>SVX(_1)</td>
<td>0.393</td>
<td>330</td>
</tr>
<tr>
<td>SVX(_D)</td>
<td>0.396</td>
<td>331</td>
</tr>
</tbody>
</table>

This table reports the monthly out-of-sample portfolio allocation results from January 1998 to April 2016. We compare five strategies: buy-and-hold (BH), volatility timing based on HV\(_1\), volatility timing based on VIX\(_1\), volatility timing based on SVX\(_1\), and volatility timing based on SVX\(_D\). We report average annualized Sharpe ratio and certainty equivalent return (CER) gains for 49 industries under risk aversion coefficients of 3, 4, and 5. The CER gain (expressed in annualized basis points) is for a mean-variance investor who allocates between the industry portfolio and risk-free asset using the volatility timing strategy, relative to the naive buy-and-hold passive strategy (BH). Panel A presents the results without transaction cost and Panel B presents the results counting for transaction cost.
Appendix 1: Estimation of state prices

Breeden and Litzenberger (1978) and Banz and Miller (1978) show that state prices can be modelled as the second derivative of a European call option. This result comes from constructing a butterfly spread with a unit payoff. Assume that an investor longs one call option at a strike price of \( X = M - \Delta M \), one call option at \( X = M + \Delta M \), and shorts two calls at \( X = M \). The maturity of these calls is \( T \). If the market price of the underlying asset is \( M \), the payoff of this portfolio would be \( \Delta M \) and zero otherwise. Henceforth, when dividing this portfolio by \( \Delta M \), we can obtain a portfolio that produces a payment of $1 if the market price is \( M \), and zero otherwise.

The price of this portfolio is \( P(M, T; \Delta M) = \frac{[c(M + \Delta M) - c(M)] - [c(M) - c(M - \Delta M)]}{\Delta M} \). If we divide the price \( P(M, T; \Delta M) \) by \( \Delta M \) and take a limit of \( \Delta M \) to zero, we can have:\(^{11}\)

\[
\lim_{\Delta M \to 0} \frac{[c(M + \Delta M) - c(M)] - [c(M) - c(M - \Delta M)]}{(\Delta M)^2} = \frac{\partial^2 c(K, T)}{\partial K^2} \bigg|_{K=M}
\]

(A1.1)

There are two different approaches to estimate Eq A1.1: (i) the Black and Scholes (1973) risk-neutral framework, and (ii) the model-free approach using traded option prices. As suggested by Breeden and Litzenberger (1978), the market portfolio is sufficient to represent different states in the economy. In this paper, we use the S&P 500 index (SPX) as the market portfolio.

(i) The Black and Scholes (1973) approach

Under the assumptions of Black and Scholes (1973), the second derivative of a European call option equals to:

\[
\left. \frac{\partial^2 C}{\partial K^2} \right|_{K=F} = \frac{e^{-rT} \phi(d_2)}{K\sigma\sqrt{T}} ,
\]

(A1.2)

\(^{11}\) According to Martin (2018), the state pricing approach does not depend on whether the call option price is twice differentiable. Merton (1973) showed that call option prices is convex function of strike price, and by Alexandrov’s theorem, the second derivatives exist almost everywhere.
where \( d_2 = \frac{\ln((F_0 - PVD) / K) - 0.5 \sigma^2 T}{\sigma \sqrt{T}} \) is the familiar term in Black and Scholes (1973) pricing formula; \( \phi(.) \) refers to the standard normal probability density function evaluated at \( d_2 \); \( T \) is the maturity of the option, which is set to be 30/365 in the current study; \( r \) and \( \sigma \) refer to the annualized risk-free rate and volatility\(^{12}\) of the SPX options; \( PVD \) is the present value of the SPX dividends; and \( K \) is the strike price of the SPX options (i.e., the level of the SPX at which the state price is required).

The value of the contingent claim which generates the unit payoff if the price of the underlying asset is greater than or equal to some level \( F_s \) can be calculated as follows:

\[
G(F_s) = \int_{F_s}^{\infty} e^{-rT} \frac{e^{-rT} n(d_2)}{K \sigma \sqrt{T}} dK = e^{-rT} N[d_2(K = F_s)] \tag{A1.3}
\]

We can then define the state price as the cost of a security with a unit payoff if the level of the underlying asset is between levels \( F_s \) and \( F_{s+1} \), using the Eq. A1.4 below:

\[
\phi(F_s, F_{s+1}) = e^{-rT} \{ N[d_2^{Black}(K = F_s)] - N[d_2^{Black}(K = F_{s+1})] \} \tag{A1.4}
\]

where SPX index level \( F_{s+1} > F_s \).

In the calculation, the maximum and minimum values of state \( Y \) are bounded between 0.1 and 9,999. A state of 2,000 indicates the SPX will be at 2,000 after one month. State prices are assessed in increments of 0.1 between \( F_s \) and \( F_{s+1} \), representing 0.1 index point on the SPX.

\(^{12}\) Following Yan (2011), we estimate the volatility \( \sigma \) as the average of the implied volatilities of two at-the-money calls and two at-the-money put options, each with a maturity nearest to the 30-day period.
(ii) The model-free approach using traded option prices

State prices can also be estimated using the numerical derivative from traded options as:

\[
\frac{\partial^2 c(X,T)}{\partial X^2} \bigg|_{X=M} \approx \frac{c(M + \Delta M) - 2c(M) - c(M - \Delta M)}{(\Delta M)^2}
\]  

(A1.5)

Eq. A1.5 can be directly estimated using observed SPX option prices.\(^{13}\) However, Liu and O’Neill (2015) point out several concerns regarding this approach, including generating zero state prices due to same deep out of the money (OTM) option prices, and negative state prices due to irrational bids for deep OTM options.

We adopt the simple Black and Scholes (1973) analytical approach as it is simpler to use and less subject to numerical estimation issues. Liu and O’Neill (2015) also show that the state prices under this approach lead to pricing results which are almost perfectly correlated with the model-free approach.

\(^{13}\) To more precisely estimate Eq. A1.5 in practice, we can use the Taylor series expansion to include more options (see e.g., Eberly 2008).
Appendix 2: Equilibrium price and CAPM

The Arrow and Debreu (1954) approach states that the value of the market is:

\[ P_m = \sum_{s=1}^{S} \phi_{ms} F_{ms} \]  

(A2.1)

where \( F_{ms} \) is the payoff on the market in state \( s \) and \( P_m \) is the current value of the market.

Using the set of state prices on the market \( \phi_{ms} \), we can now find the value of any asset. We simply find the expected value of asset \( i \) conditional on the level of the market \( E(F_i \mid F_m) \), and multiply by the market state prices to obtain:

\[ P_i = \sum_{s=1}^{S} \phi_{ms} E[F_{is} \mid F_{ms}] \]  

(A2.2)

The common way to project this expectation is the linear projection of \( E(F_i \mid F_m) = \alpha_i + \beta_i F_m \), which yields:

\[ P_i = \sum_{s=1}^{S} \phi_{ms} [\alpha_i + \beta_i F_{ms}] \]  

(A2.3)

Now Eq. A2.3 looks very familiar and it closely resembles the CAPM in payoffs. However, it is different from CAPM because:

1. The state prices \( \phi_{ms} \) depend on the volatility of the market, the level of the market, the risk-free rate, and the usual determinants of option value. Therefore the state prices could change continuously yet the CAPM assumes constant coefficients.
2. The conditional value of asset $i$, $E(F_i | F_m)$ (Eq. A2.3) may not be linear in $F_m$. There are all sorts of non-linear ways of taking this conditional expectation (e.g., the excellent test of Friedman et al. (2001)), but the CAPM only implies a linear projection with constant intercept and slope coefficients.
Appendix 3: Construction method of industry volatility

We begin our construction by estimating a state-price volatility index for the whole market. This process is similar to the methodology described in Liu and O’Neill (2015). As the first step, we estimate the price of a security that pays off a dollar amount of $ \log \frac{M_s}{M_0} \frac{2}{\gamma} $ if the S&P 500 index moves to state $ M_s $ from an initial value of $ M_0 $ in 30 days’ time.\(^{14}\) When summing across all possible future states, we arrive at the price of this asset in the following form:

$$ SVX_M = \sqrt{\frac{365}{30}} \sum_{s=1}^{365} \log \frac{M_s}{M_0} \frac{2}{\gamma} $$

(A3.1)

where $ T $ is set to be exactly 30 calendar days (or equivalently, 22 trading days). $ SVX_M^2 $ can be viewed as a financial asset that pays off a dollar amount that is equivalent to the fair value of future variance. The factor 365/30 is to obtain an annualized volatility figure.

The second step involves estimating the relationship between each industry portfolio and the market portfolio, using a linear projection on the market model. That is, we are interested in the expected payoff for the industry conditional on the level of the index. The industry payoff here is the expected squared log returns $ E[R^2_i | R^2_M] $. There are many ways to estimate this relationship (see e.g., Friedman et al. 2001). Linear projection is the most common method in the extant literature (Smith & Walsh 2013), so we adopt this approach and use a simple linear least squares regression of daily squared industry returns on the squared S&P 500 market returns. In this setting, we use the market state prices and assume complete markets. The method captures the systematic

\(^{14}\) A common approach is to use price levels (returns) to represent the market states (Ross 2015).
components of industry volatility to the extent that the coefficient is allowed to vary over time. Specifically, we are interested in the alphas and betas in the following regression:

$$R_i^2 = \alpha_i + \beta_i R_M^2 + \varepsilon_i \quad (A3.2)$$

The return $R_i$ is computed using the close value at the end of day. We estimate each beta using a two-year fixed rolling window. That is, on 505th day, we use the past two years (504 trading days) of return squared to estimate the beta in the above regressions.

The third step is to work out the individual industry portfolio volatility index based on the estimated alphas and betas and their corresponding volatility asset. To illustrate this, we substitute Eq. A3.2 into Eq. A3.1:

$$SVX_i = \sqrt{\alpha_i \frac{365}{30} \sum_{s=1}^{S} \phi_{ms} + \beta_i SVX_M^2} \quad (A3.3)$$

The last step is to create an ad-hoc industry volatility index using the widely available CBOE volatility index VIX. VIX is a sum of weighted-average out-of-the-money S&P 500 put and call options. For a detailed discussion on VIX, we refer to Whaley (2009). As we do not have traded options for the industry portfolios, we cannot replicate the CBOE VIX methodology to reproduce the industry measures. Our ad-hoc estimation takes the following form by replacing $SVX_M^2$ with $VIX_M^2$:

$$VIX_i = \sqrt{\alpha_i \frac{365}{30} \sum_{s=1}^{S} \phi_{ms} + \beta_i VIX_M^2} \quad (A3.4)$$

where $VIX_M$ is the CBOE VIX.