A General Equilibrium Approach to Pricing Volatility Risk

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Abstract

This paper provides a general equilibrium approach to pricing volatility. Existing models (e.g., ARCH/GARCH, stochastic volatility) take a statistical approach to estimating volatility, volatility indices (e.g., CBOE VIX) use a weighted combination of options, and utility based models assume a specific type of preferences (e.g. Bansal et al., 2014; Tauchen, 2014). In contrast we treat volatility as any other asset and price it using the Arrow and Debreu (1954) general equilibrium state pricing framework. Our results show that the general equilibrium volatility method developed in this paper provides superior forecasting ability for realized volatility and serves as an effective fear gauge. We demonstrate the flexibility and generality of our approach by pricing downside risk and upside opportunity. Finally, we show that the superior forecasting ability of our approach generates significant economic value through volatility timing.

Keywords: General equilibrium, volatility, state pricing, fear gauge

Classification codes: G12, G13, G17

1. Introduction

Volatility modelling has proceeded as a field separate from asset pricing. Statistical models, such as ARCH (Engle 1982; Engle 2002), GARCH (Bollerslev 1986), stochastic volatility (Barndorff-Nielsen 2002), and option prices (Fleming *et al.* 1995) are commonly used to estimate volatility, without reference to modern asset pricing theory. In this paper, we propose to *price* volatility using a general equilibrium asset-pricing framework. The advantage of such an approach is that volatility can be priced and measured in the most general setting available. The approach also allows us to extend measurement and pricing (such as downside risk pricing and upside opportunity pricing), which cannot be achieved with current approaches to volatility modelling.

This paper assumes a complete market setting where state prices are available for each time and state. State prices are derived from the Arrow and Debreu (1954) general equilibrium state pricing framework. Following Breeden and Litzenberger (1978), state prices are obtained for each time and state of the market that can be used to price all assets. We apply this state pricing approach to market volatility risk and are able to derive prices that are almost perfectly correlated with the CBOE Volatility Index (VIX), but are the result of a general equilibrium model. There are several advantages to treating volatility as any other asset and pricing it using the Arrow and Debreu (1954) approach. First, we treat volatility pricing the same as any other pricing exercise. Second, we do not have to assume a specific form of utility preferences to implement this approach. Third, the equilibrium price of volatility provides not only a more general approach that is very flexible and can also be used for individual securities, but also serves as a better predictor of future volatility and as an investor fear gauge.

Empirically, we generate market state prices from the S&P 500 index options and use them to price the 49 Fama and French industry volatility. The ex-ante industry volatility measure constructed in this approach yields better forecasts of realized volatility than existing approaches and serves as a qualified investor fear gauge. We then demonstrate the flexibility and generality of our approach by pricing downside and upside opportunity, which complements the work of Bekaert and Engstrom (2017), Feunou *et al.* (2017), Kilic and Shaliastovich (2016) and Segal *et al.* (2015). We show that these new upside and downside volatility measures work well empirically. Finally, we analyze the economic value of volatility timing using the general equilibrium measures versus existing volatility measures. We show that the superior forecasting ability of our general equilibrium volatility measure has greater economic value for investors wishing to manage volatility.

Our paper is related to the recent literature on understanding volatility in a general equilibrium framework. As an extension of Bansal *et al.* (2004) and Bansal and Yaron (2004), Tauchen (2011) proposes a consumption-based general equilibrium model that assumes stochastic consumption. The model generates a two-factor structure for stock market volatility along with time-varying risk premiums on consumptions and volatility risk, and also the leverage effect. In the general equilibrium framework, Bansal *et al.* (2014) demonstrate that besides the cash flow risk and discount rate risk, volatility risk is an important and separate risk source that cannot be ignored. Different from this strand of literature, we impose no assumptions on consumption dynamics and rely only on state prices extracted from the options market. The focus of these existing studies is to explain the stylized facts (e.g., leverage effect, equity risk premium) in the market, while we aim to provide market participants with an easy and flexible tool to measure and manage the volatilities of asset portfolios or individual securities.

Our paper is also related to the prior literature on constructing volatility indices based on options. Britten-Jones and Neuberger (2000) use a replicating strategy to synthesize a variance swap using options contracts, assuming continuity in the underlying asset price. Jiang and Tian (2005) build on a similar concept by incorporating a jump-diffusion stochastic volatility model. We adopt the general equilibrium approach and are able to construct the volatility indices for assets without options written on. Our paper also relates to recent efforts to disentangle the effects of upside and downside uncertainty on asset prices (Segal *et al.* 2015; Bekaert & Engstrom 2017) by proposing a new approach for volatility decomposition. One common approach for decomposing volatility into upside and downside components is to use a threshold to compute risk-neutral expectations of semi-variances (see e.g., Andersen and Bondarenko (2007); Feunou *et al.* (2017); Kilic and Shaliastovich (2015)). However, this approach still depends on the existence of traded options, while our approach does not.

The paper is organized as follows. Section 2 outlines volatility pricing using a general equilibrium model. Section 3 extends our approach to price downside risk and upside opportunity. Section 4 analyzes the economic value of our volatility measures in a volatility timing framework. Section 5 concludes.

2. Pricing volatility in a general equilibrium model

This section outlines the general equilibrium approach for pricing market volatility and shows how it can be extended to pricing industry volatility. Industry volatility prices are compared with existing approaches to forecasting realized volatility and evaluated as a gauge of investor fear.

2.1 Method

Under a state pricing approach, the value of any asset is the sum of the state prices multiplied by the payoff in each state. If, for example, we were to price the market portfolio M which pays off F_{ms} in each of S states one period (set as 30 days in this paper) from now, the price is given by:[1]

$$P_m = \sum_{s=1}^{S} \phi_{ms} F_{ms} \qquad (1)$$

For an arbitrage asset i, whose payoff F_i depends on the level of the market, under the complete market setting, its price is given by:

$$P_i = \sum_{s=1}^{S} \phi_{ms} E[F_{is} \mid F_{ms}] \qquad (2)$$

If we were to take a linear projection[2] of F_i onto F_M, then we would obtain:

$$P_i = \sum_{s=1}^{S} \phi_{ms} [\alpha_i + \beta_i F_{ms}] \text{ or } P_i = \alpha_{rfi} + \beta_i P_m \qquad (3)$$

Since $\sum_{s=1}^{S} \phi_{ms}$ is the price of a risk-free asset with payoff of 1, α_{rfi} is the price of a riskless asset with payoff α_i .

This is a relation that closely resembles the Sharpe-Lintner Capital Asset Pricing Model (Sharpe 1964); however, the derivation contains obvious differences. First, the market price of risk

[1] Breeden and Litzenberger (1978) argue that the market portfolio, as a proxy for aggregate consumption, is sufficient to represent the different states in the economy. We show in Appendix 1 how we obtain the state prices using market options, where the market is represented by S&P 500 index (SPX).

[2] We could obtain this conditional expectation using more general non-linear methods as outlined in Friedman *et al.* (2001).

will vary over time as the state prices change. Second, the risk-free factor will be different for each asset *i* depending on the magnitude of α_i.[3]

We now consider pricing market volatility. Here, the payoff is the squared market return at each state. The price of market volatility under the Arrow and Debreu (1954) approach is given by:

$$SVX_M^2 = \sum_{s=1}^{S} \phi_{ms} R_{ms}^2 \tag{4}$$

where SVX_M is the state pricing volatility index for the market. It is the general equilibrium price of market volatility.

Compared to the calculation of the VIX, the SVX_M formula offers a more straightforward approach.[4] To price volatility on an arbitrary asset, for example industry *I*, the approach above yields:

$$SVX_I^2 = \sum_{s=1}^{S} \phi_{ms} E[R_{Is}^2 \mid R_{ms}^2] \tag{5}$$

An assumption of a linear relation between individual asset return and market return (as in Eq. 3) would lead to a linear relation between individual asset return squared and market return squared conditional on the given market return, R_m. Naturally, we have:

[3] Appendix 2 provides a more detailed discussion of the relation between state pricing theory and the CAPM.

[4] For a more detailed discussion on how to construct SVX_M and how it performs against other volatility measures in predicting future market volatility, see Liu and O'Neill (2015). The approach of using Arrow-Debreu securities to price squared returns has also been adopted by Brennan and Cao (1996) and Cao and Ou-Yang (2008).

$$SVX_I^2 = \sum_{s=1}^{S} \phi_{ms} [\alpha_I + \beta_I R_{ms}^2] \quad (6)$$

or

$$SVX_I^2 = \alpha_{rfI} + \beta_I SVX_M^2 \quad (7)$$

where α_{rfI} is the price of a riskless asset with payoff α_I, SVX_M^2 is as defined above in Eq. 4.

The details of the construction of SVX_I are presented in Appendix 3. We see that under the linear projection approach, the volatility price of any asset depends on the market price of volatility in a straightforward manner.

We compare two other volatility measures to our measure. The first is an ad-hoc industry volatility index using the widely available CBOE volatility index VIX. To achieve that, we simply replace SVX_M^2 by VIX_M^2:

$$VIX_I^2 = a_{rfI} + b_I VIX_M^2 \quad (8)$$

where VIX_M is the CBOE VIX.

The second measure is the historical volatility, HV_I , which is the realized volatility in the previous year.

2.2 Data

To estimate state prices in the complete market setting, we obtain prices and implied volatilities of S&P 500 index options and S&P 500 index dividend yields from the Ivy DB US OptionMetrics, available through Wharton Research Data Services. The options data are available on a daily basis from January 4, 1996 to April 29, 2016. Interest rates are taken from the Center for Research in Security Prices (CRSP) Zero Curve file. We apply a cubic spline to the interest rate term-structure data to match the length of the risk-free rate with the corresponding option maturity.

The 49 industry portfolios are obtained from Kenneth R. French's Data Library.[5] The daily return data are available from July 1927 to February 2017. However, our sample period is dictated by the availability of the S&P 500 options data in estimating the state prices, which are only available from January 4, 1996 to April 29, 2016. Given we need to estimate the betas in Eq. 7 using a fixed two-year rolling window, we examine daily industry returns since January 1994.

[Table 1 about here.]

Summary statistics of SVX_I are reported in Table 1. The mean of the annualized SVX_I varies across industries, ranging from 0.129 for the utility industry to 0.273 for the coal industry. Consistent with economic intuition, the "necessities" industries, such as Food, Soda, Beer, Smoke, and Utilities (as defined in Boudoukh *et al.* 1994) are insensitive to business cycles, and are the least volatile, on average.

[Figure 1 about here.]

SVX_I for the 49 industry portfolios is illustrated in Figure 1. It is apparent that there is a strong positive correlation among the volatility indices. Figure 1 also reveals that there is a significant time variation in volatility for all industries in the analysis, showing an upward spike

[5] We are grateful for Kenneth French for supplying this data: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

in the 1998 Asian Financial Crisis and during the technology bubble of the early 2000's. A peak occurs around the 2008 Global Financial Crisis.

2.3 Forecasting future 30-day realized volatility

The primary goal of a volatility index is to serve as a measure of the next 30-day expected volatility (CBOE 2009). In this section, we examine the information content of SVX_I in predicting subsequent 30-day realized return volatility from January 1996 to April 2016. We regress the future 30-day close-to-close realized volatility on different volatility measures in the following models:

$$RV_{I,t,t+30} = \alpha_I + \beta_I SVX_{I,t} + \varepsilon_{I,t,t+30} \tag{9}$$

where I stands for one of the 49 industry portfolios, $RV_{I,t,t+30}$ denotes the annualized realized volatility over the next 30 days and it is defined as $RV_{I,t,t+30} = 100(\frac{365}{30}\sum_{i=1}^{22} R_i^2)^{1/2}$, where R_i^2 is the square of the daily portfolio return. We also run this regression for VIX_I and HV_I, where HV_I is the annualized realized volatility in the previous year. These measures are highly correlated on average, so we cannot include both in the same regression to determine which index is more statistically significant. Instead, we use the Mincer and Zarnowitz (1969) regression to compare the prediction performances of these three volatility measures.

To be an unbiased volatility predictor, we expect alphas to be not significantly different from 0 and betas to be not significantly different from 1. If SVX_I is a better predictor than the other measures, the forecasting regression using it as the predictor is expected to generate the highest model explanatory power, as expressed in adjusted R-squared.

[Table 2 about here.]

We run the above regressions for each of 49 industry portfolios and report the arithmetic average of regression results in Table 2. The covariance matrix is computed according to Newey and West (1994) to correct for any potential serial correlations in the error terms.

The mean level of the β of SVX_I is 1.01, with an average standard error of 1.5%. As expected, β of SVX_I is significantly different from 0 at 1% level, and not significantly different from 1 at any levels. In comparison, β of VIX_I has a mean level of 0.918 with a standard error of 1.4%. It is significantly different from both 0 and 1 at 1% level. β of HV_I has the lowest mean level out of the three measures; namely, 0.677 with a standard error of 1.5%. On average, α of SVX_I is 0.046, which is marginally significantly different from 0 at 5% level. The mean for α of VIX_I is 0.04 and is not significantly different from 0 at 5% level. In comparison, the α of HV_I has an average value of 0.072 and is significantly different from 0. SVX_I provides higher adjusted R-squared than the regressions using VIX_I and HV_I . On average, the model with SVX_I as the predictor has a 1.4% higher adjusted R-squared than VIX_I and 19% higher than HV_I . Therefore we conclude that the state price volatility index SVX_I is a more efficient forecaster of future realized volatility than its counterparts. [6]This industry-level result reinforces Liu and O'Neill's (2015) finding that state price volatility outperforms *VIX* and other predictors at the market level.

2.4 Fear gauge

It is well-documented that there is a negative correlation between the rate of change in volatility (e.g., CBOE VIX) and daily market returns (see for example, Carr & Wu 2009). As expected market volatility increases, investors demand a higher rate of return on stocks and prices fall, which ultimately leads to a drop in the current market return. Therefore, we study the contemporaneous relation between rates of change in various industry volatility measures and daily industry portfolio returns. In particular, we investigate whether these indices contain any fear information from the market state prices. Generally, a fall in an industry portfolio usually implies a rally in investor fear in the segment. Therefore, we expect to see negative betas in all volatility measures. A fear gauge, such as CBOE VIX (Whaley 2009), should respond more to negative changes in portfolio returns than positive changes. We are interested in testing whether industry state price volatility measures can capture this asymmetric fear gauge effect. We regress the daily changes of various measures against industry portfolio returns in the following forms:

$$\Delta SVX_{I,t} = \alpha_I + \beta_{1,I} R_{I,t} + \beta_{2,I} R^-_{I,t} + \varepsilon_{I,t} \qquad (10)$$

where *I* stands for each of the 49 industry portfolios, Δ measures the daily changes, and $R_{I,t}$ is the daily industry portfolio return, $R^-_{I,t}$ is defined as $\min(R_{I,t}, 0)$.

[Table 3 about here.]

[6] In untabulated results, we also compare the forecasting abilities of ARCH and GARCH volatility measures with that of SVX_I. We find that SVX_I outperforms these two measures in terms of adjusted R-squared.

We run the above regressions for each of 49 industry portfolios and report the arithmetic average of regression results in Table 3. The covariance matrix is computed according to Newey and West (1994) to correct for any potential serial correlations in the error terms. On average, β_1 is significantly less than 0 at the 1% level, implying that there is an inverse relation between contemporaneous changes in volatility indices and changes in portfolio returns. β_2 is also significantly less than 0 at the 1% level. These results show that the response to different swings in portfolio returns is strongly asymmetric, and are consistent with the findings of Whaley (2009) and Liu and O'Neill (2015). Besides statistical significance, the coefficients are also economically meaningful: on average, an increase in the industry return by 100 basis would result in 51.7 basis decrease in SVX_I and an decrease in the industry return by 100 basis would lead to 73.8 basis increase in SVX_I .

3. Extensions

This section describes the flexibility and generality of the general equilibrium approach by modelling downside risk and upside opportunity. We show how these measures can be used to better forecast realized volatility, as well as downside risk and upside opportunity. We also consider how these measures can be used as an investor fear gauge.

3.1 Downside risk and upside opportunity in a general equilibrium model

In Section 2.1, we showed how to construct the industry state-price volatility index SVX_I in a general equilibrium setting. Here, we use the general pricing approach to price downside risk and upside risk. [7]

Downside market risk is given by:

$$BEX_M = \sqrt{\sum_{s=1}^{S} \phi_{ms} R_{ms}^2 I_{R_{ms}<0}} \quad (11)$$

where $I_{R_{ms}<0}$ is an indicator variable equal to 1 if R_{ms} is less than zero.

Similarly upside market risk is given by:

$$BUX_M = \sqrt{\sum_{s=1}^{S} \phi_{ms} R_{ms}^2 I_{R_{ms}>0}} \quad (12)$$

where $I_{R_{ms}>0}$ is an indicator variable equal to 1 if R_{ms} is greater than zero.

Industry measures of *BEX* and *BUX* are obtained in an analogous manner to industry volatility SVX_I (Eq. 6) and can be represented as:

$$BEX_I^2 = \alpha_{Down,I} + \beta_{Down,I} BEX_M^2 \quad (13)$$

$$BUX_I^2 = \alpha_{Up,I} + \beta_{Up,I} BUX_M^2 \quad (14)$$

[7] Recent literature highlights the importance of distinguishing downside and upside volatility risk. See Patton and Sheppard (2015), Bekaert and Engstrom (2017), and Segal *et al.* (2015).

To estimate the alphas and betas in Eq. 13 and Eq. 14, we use a linear least squares regression of squared daily industry returns on squared S&P 500 market returns. Specifically, we are interested in the alphas and betas in the following regressions:

$$R_I^2 = \alpha_{Down,I} + \beta_{Down,I} R_{M|Down}^2 + \varepsilon_I \tag{15}$$

$$R_I^2 = \alpha_{Up,I} + \beta_{Up,I} R_{M|Up}^2 + \varepsilon_I \tag{16}$$

where R_I^2 is the daily squared industry return, and $R_{M|Down}^2$ ($R_{M|Up}^2$) is the market return squared conditional on whether the market has gone down (up) from the previous day, regardless of movement in the industry portfolio.[8] The return is computed using the closing value at the end of day. We estimate each beta using a two-year fixed rolling window. That is, on the 505th day, we use the past two years (504 trading days) of return squared to estimate the alphas and betas in the above regressions.

3.2 Industry state-price volatility that incorporates downside market volatility

Prior studies have shown that returns become more correlated in a bear market (e.g., Ang & Chen 2002; Campbell *et al.* 2002). As a result, we extend the basic formula for SVX_I by using an alternative linear projection that incorporates market downside movement:

$$(SVX_I^D)^2 = \sum_{s=1}^{S} \phi_{ms} [\alpha_I + \beta_I R_{ms}^2 + \gamma_I R_{ms}^2 I_{R_{ms} \le 0}] \tag{17}$$

Which can be solved to yield:

[8] We also considered different definitions of conditional downturn return squared for the industry portfolio. Results are qualitatively the same when we measure the beta of the downturn industry return and the downturn market return. We selected the current definition because it is more meaningful for examining how the industry portfolio responds and contributes to a downturn in the whole market.

$$(SVX_I^D)^2 = \alpha_{rfI} + \beta_I SVX_M^2 + \gamma_I BEX_M^2 \qquad (18)$$

where α_{rfI} is the price of a riskless asset with payoff α_I, SVX_M^2 is as defined above in Eq. 4. and $I_{R_{ms}\leq 0}$ is an indicator variable equal to 1 if R_{ms} is less than zero. In this definition, the volatility price of any asset depends on its sensitivity to the price of market volatility and (in addition) to the price of market downside volatility.

To estimate the alphas, betas, and gammas in Eq. 18, we use a linear least square regression of squared daily industry returns on squared daily S&P 500 market returns.

$$R_I^2 = \alpha_{Down,I} + \beta_{Down,I} R_M^2 + \gamma_{Down,I} R_{M|Down}^2 + \varepsilon_I \qquad (19)$$

where R_I^2 is the daily squared industry return, and $R_{M|Down}^2$ is the market return squared conditional on whether the market has gone down from the previous day, regardless of movement in the industry portfolio.

3.3 Forecasting future 30-day realized volatility

To extend the analysis described in Section 2.3, we examine the information content of SVX_I^D in predicting the future 30-day realized volatility in each industry portfolio. We regress the future 30-day close-to-close realized volatility on different volatility measures in the following models:

$$RV_{I,t,t+30} = \alpha_I + \beta_I SVX_{I,t}^D + \varepsilon_{I,t,t+30} \qquad (20)$$

[Table 4 about here.]

Results are reported in Table 4 Panel A. The mean level of β of SVX_I^D is 1.003, with an average standard error of 1.5%. β of SVX_I^D is significantly different from 0 at 1% level, and not significantly different from 1 at any levels. Comparing to results in Table 2, we can see that SVX_I^D outperforms other volatility candidates in terms of adjusted R-squared.

3.4 Forecasting future 30-day realized downside volatility

A typical volatility measure does not describe the proportion of upside gain versus downside threat. In this paper, we solve this problem by introducing a downside (upside) volatility index BEX_I (BUX_I) for each industry portfolio as an *ex-ante* predictor of future downside (upside) volatility. It is important to note that, unlike the comparison with SVX_I and VIX_I in the previous section, we do not have a VIX benchmark per se because is not mathematically feasible to derive a downside VIX using market state prices. We regress the future 30-day close-to-close realized downside volatility in the following way:

$$RVD_{I,t,t+30} = \alpha_I + \beta_I BEX_{I,t}^2 + \varepsilon_{I,t,t+30} \quad (21)$$

where I stands for each of the 49 industry portfolios, $RVD_{I,t,t+30}$ denotes the realized downside volatility over the next 30 days and it is defined as $RVD_{I,t,t+30} = 100(\frac{365}{30}\sum_{i=1}^{22} R_{i=1}^{22} I_{R_i \le 0})^{1/2}$.

We expect alphas to be not significantly different from 0 and betas to be not significantly different from 1 if BEX_I is an unbiased forecaster, and betas to be significantly different from 0. We run the above regression for each of 49 industry portfolios and report the arithmetic average of regression results in Panel B in Table 4. The covariance matrix is computed according to Newey and West (1994) to correct for any potential serial correlations in the error terms. The mean level of β of BEX_I is 0.917, with an average standard error of 1.8%. β of BEX_I is significantly different from 0 and 1 at 1% level. On average, α of BEX_I is 0.029 and is not significantly different from 0 at 5% level. The average adjusted R-squared of Eq. 21 is 34.7%. We show that BEX_I is an efficient forecaster of future realized downside volatility.

3.5 Fear gauge

We further study the contemporaneous relation between rates of change in SVX_I^D and daily industry portfolio returns. We are interested in testing whether the modified industry volatility measure can better capture the fear gauge. We regress the daily changes of SVX_I^D against the industry portfolio returns in the following forms:

$$\Delta SVX_{I,t}^D = \alpha_I + \beta_{1,I} R_{I,t} + \beta_{2,I} R_{I,t}^- + \varepsilon_{I,t} \quad (22)$$

where *I* stands for each of the 49 industry portfolios, Δ measures the daily changes, and $R_{I,t}$ is the daily industry portfolio return, $R_{I,t}^-$ is defined as $\min(R_{I,t}, 0)$. We perform a similar analysis for $\Delta BEX_{I,t}^D$ to see whether the downside volatility measure can serve as a qualified fear gauge or not.

[Table 5 about here.]

We run the above regressions for each of 49 industry portfolios and report the arithmetic average of regression results in Table 5. The covariance matrix is computed according to Newey and West (1994) to correct for any potential serial correlations in the error terms. On average, for both $\Delta SVX_{I,t}^D$ and $\Delta BEX_{I,t}^D$, all β_1 are significantly less than 0 at 1% level, implying there is an inverse relation between the contemporaneous changes of volatility indices and those of portfolio returns. For both $\Delta SVX^{D}_{I,t}$ and $\Delta BEX^{D}_{I,t}$, β_2 are also significantly less than 0 at 1% level. These results shows that the response to different swings in portfolio returns is strongly asymmetric. This is consistent with findings reported in Section 2.4. By comparing the adjusted R-squared between Table 5 and Table 2, we can see that incorporating the downside risk into the volatility index can enhance monitoring effectiveness (37.9% and 37.6% for $\Delta SVX^{D}_{I,t}$ and $\Delta BEX^{D}_{I,t}$ in Table 5, and 36.7% for $\Delta SVX_{I,t}$ in Table 2). The results confirm that measures incorporating downside risk are better measures of fear gauge.

3.6 Volatility forecasting: out of sample evidence

Besides the in-sample forecasting evidence, here we further compare the volatility predictability by examining the out-of-sample tests for four volatility predictors: HV_I, VIX_I, SVX_I, and SVX^D_I. Specifically, we use a rolling fixed window approach. Every day, each forecasting model is estimated with a fixed rolling window to obtain the one-month-ahead volatility forecast. To ensure robustness, we use a one-year window, a two-year window and a three-year window. The out-of-sample forecasting accuracy is judged by three criteria: root-mean-square error (RMSE), mean-absolute error (MAE) and mean-absolute-percentage error (MAPE).

Table 6 reports the estimation results for the average values of RMSE, MAE and MAPE across 49 industries in three different rolling windows. First, HVI performs the worst among four volatility measures, regardless of criterion or rolling window. For instance, in the one-year rolling window case, the average RMSE values for VIX_I, SVX_I, and SVX^D_I are all around 0.083, while it is 0.100 for HVI. Second, in almost all the scenarios, SVX^D_I is the best predictor, with SVX_I and VIX_I being the second and third best predictors. Overall, the out-of-sample forecasting results reiterate the earlier in-sample test findings.

4. Economic value of volatility timing

This section investigates the economic value of using various predictors to forecast monthly industry volatility. First, consider an investor who allocates wealth between an industry portfolio and a risk-free asset using a volatility predictor to maximize utility gains. Similar to Fleming *et al.* (2001), Fleming *et al.* (2003), and Marquering and Verbeek (2004), we assume the investor maximizes the following utility based expression[9]:

$$\mathrm{U}[\mathrm{E_t}(R_{p,t+1}),\sigma^2_{p,t+1}] = E_t(R_{p,t+1}) - 0.5\gamma\sigma^2_{p,t+1}, \qquad (23)$$

where $E_t(R_{p,t+1})$ and $\sigma^2_{p,t+1}$ respectively are the conditional mean and variance of the portfolio returns, and γ refers to the coefficient of the investor's relative risk aversion. We set γ to a realistic estimate of 3, as suggested by Campbell and Thompson (2008) and Rapach *et al.* (2016). We also use values of γ of 4 and 5 for robustness and sensitivity analysis. The portfolio return is $E_t(R_{p,t+1}) = r_{f,t+1} + w_t(E_t(R_{I,t+1}) - r_{f,t+1})$, where w_t is the portfolio weight of industry portfolio *I*, $E_t(R_{I,t+1})$ is the conditional expected return of the industry portfolio, and $r_{f,t+1}$ denotes the risk-free rate, which is known *ex-ante*. The portfolio variance is $\sigma^2_{p,t+1} = w_t^2\sigma^2_{I,t+1}$, where $\sigma^2_{I,t+1}$ denotes the conditional variance of industry portfolio *I*. The optimal weight for industry *I* is given by:

$$w_t = \frac{E_t(R_{I,t+1}) - r_{f,t+1}}{\gamma\sigma^2_{I,t+1}}, \qquad (24)$$

The current study focuses on monthly realized volatility forecasting, so we assume the portfolio is rebalanced monthly. We set the month *t* expected return as the historical mean using

[9] Grossman (1976) and Admati (1985) show how negative exponential utility and normal distributions result in the above mean and variance maximization.

return data up to period t. The expected variance of portfolio I, $\sigma^2_{I,t+1}$, is based on the two-year rolling out-of-sample forecast using Eq. 11, with one of four different volatility measures (HV_I, VIX_I, SVX_I, and $SVX^D{}_I$) as the predictor. The accuracy of volatility forecasting determines the performance of this volatility timing strategy. Our benchmark strategy is the buy-and-hold strategy of the respective industry portfolios, I.

We adopt two criteria to compare the performance of different strategies. The first is the commonly used annualized Sharpe ratio. The second is the certainty equivalent return (CER) gain of a volatility timing strategy relative to that of a naïve buy-and-hold strategy:

$$\mathrm{CER} = \left(R_{\mathrm{p}} - 0.5\gamma\sigma_p^2\right) - \left(R_{\mathrm{naive}} - 0.5\gamma\sigma_{\mathrm{naive}}^2\right). \qquad (25)$$

Intuitively, the CER gains of Eq. 25 are the incremental management fees that the investor is willing to pay to invest in the volatility timing strategies based on the volatility forecasts over the buy-and-hold strategy.

[Table 6 about here.]

Table 7 presents the average performance of 49 industries from January 1998 to April 2016. In Panel A, we assume there is no transaction cost. We first examine the basic case of γ equal to 3. Here, in terms of Sharpe ratio, the volatility timing strategy based on historical volatility (HV_I) performs worse than the naïve buy-and-hold strategy (0.326 vs 0.335). However, the volatility timing strategies based on implied volatility series, that is, VIX_I, SVX_I, and SVX_I^D, generate higher Sharpe ratios than the buy-and-hold strategy. The highest Sharpe ratio is obtained by using SVX_I^D as the volatility predictor (0.396). The results on CER gains reveal that all the volatility timing strategies outperform the buy-and-hold strategy. SVX_I^D again performs best: the investor is prepared to pay a hefty incremental annual management fee of 311 basis points bps) to have access to predictive regression based on SVX_I^D, instead of the buy-and-hold strategy. In contrast, the investor is only willing to pay 249 bps for the strategy using HV_I. When the investor is more risk-averse, we find that the Sharpe ratios are similar to the basic case,[10] but the management fees that the investor is willing to pay to be involved with the volatility timing strategy using SVX_I^D increase from 331 bps ($\gamma = 3$) to 590 bps ($\gamma = 4$) and 866 bps ($\gamma = 5$). This result suggests that volatility timing is especially important for risk-averse investors.

The volatility timing strategy requires monthly rebalancing, so its performance might be sensitive to transaction costs. With this in mind, we analyze the impact of transaction costs on our results. Following Bandi *et al.* (2008) and Nolte and Xu (2015), we define the transaction cost adjusted portfolio return as follows:

$$\overline{R}_{p,t+1} = R_{p,t+1} - \rho(1 + R_{p,t+1}) \,|\, \Delta w_{t+1} \,|, \qquad (26)$$

where $\overline{R}_{p,t+1}$ is the transaction cost adjusted portfolio return, $R_{p,t+1}$ is the pre-adjusted portfolio return, ρ is the transaction cost parameter, and is set to be 0.0025, and Δw_{t+1} is the change of weight from month t to month t+1.

Panel B of Table 7 presents the results for transaction cost adjusted performance. It clearly shows that, even when we account for transaction cost, volatility timing strategies based on VIX_I, SVX_I, and SVX_I^D, still largely outperform the buy-and-hold strategy and the volatility timing

[10] Portfolio theory confirms that a change of weights has no impact on the Sharpe ratio when allocating assets between a portfolio and the risk-free rate.

strategy based on HV_I. Consistent with Panel A, SVX_I^D generates the highest Sharpe ratio (0.367) and CER gain (269 bps) for $\gamma = 3$, and again, economic values are larger when the investor is more risk-averse (CER gains of 544 bps for $\gamma = 4$ and 829 bps for $\gamma = 5$).

In summary, this section uses a volatility timing strategy to show that the strong forecasting abilities of the industry volatility indices (VIX_I, SVX_I, and SVX_I^D) have significant economic value for investors.

5. Conclusion

This paper is novel in that it proposes a general equilibrium framework to price volatility in the same manner as is the case for all securities in the market, following Arrow and Debreu (1954). Using state prices estimated from S&P 500 index options, we illustrate how we can derive *ex-ante* volatility measures SVX_I for industry portfolios, in which there are no traded options. The SVX_I measures generate superior forecasting abilities for the future realized volatility and serve as qualified fear gauges. We show that our approach is flexible and general by extending it to downside risk and upside opportunity. Finally, we demonstrate that the superior forecasting ability of our general equilibrium volatility measure can create significant economic value through a simple volatility timing strategy. Our findings, together with the fact that the industry volatility indices can be easily constructed under the general equilibrium framework, offer practitioners an appealing alternative tool for managing volatility. Our general equilibrium framework is not limited to pricing volatility, but can be applied to price any moments of the return distribution.

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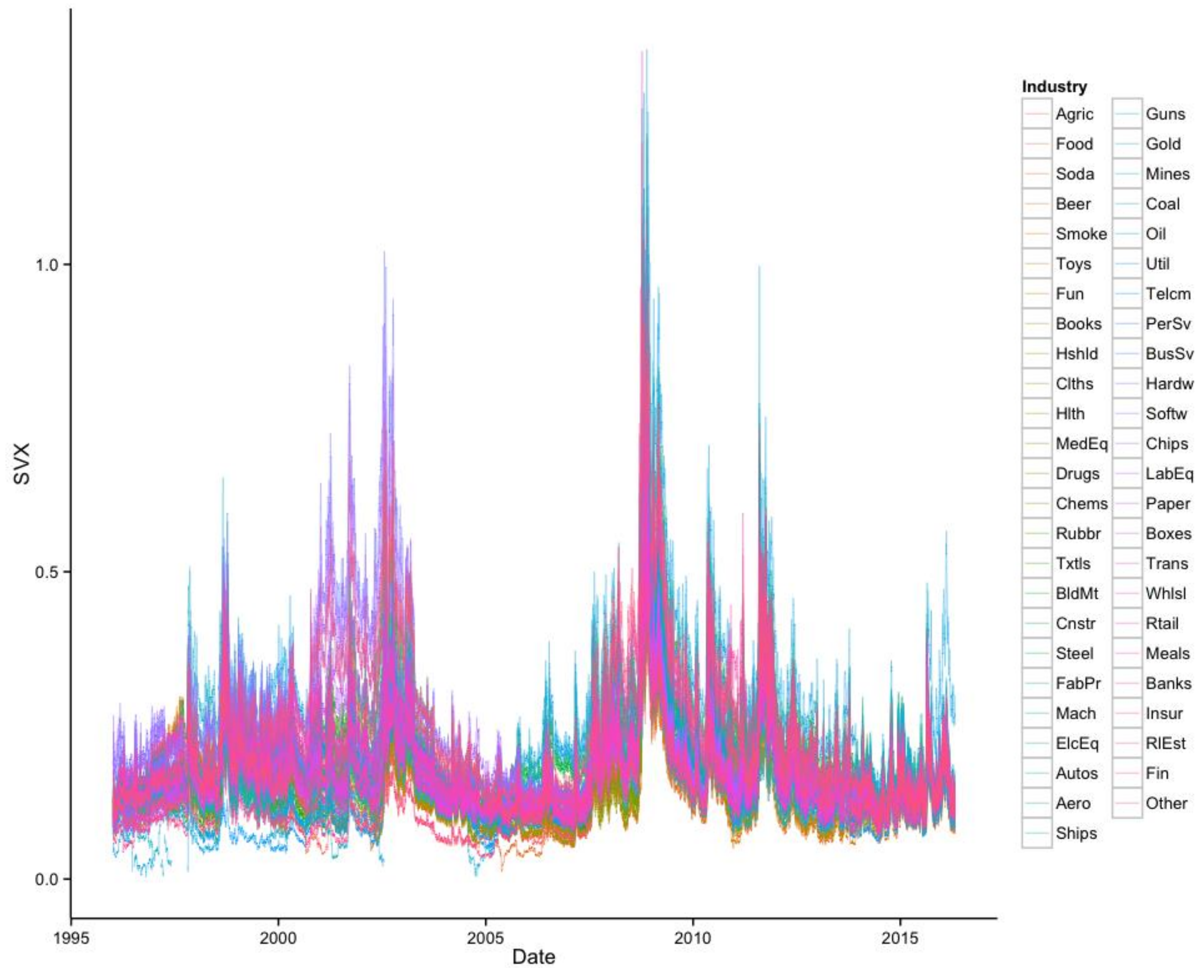
Figure 1

SVX_I for 49 Industries: 1996-2016



This figure plots the Industry SVX_I of 49 industry portfolios. The data are from 4 January 1996 to 29 April 2016.

TABLE 1

Summary Statistics of Industry SVX_I

Industry	Obs	Mean	Std. Dev.	Min	Max
Agric	5,099	0.150	0.103	0.039	0.950
Food	5,099	0.132	0.051	0.052	0.482
Soda	5,099	0.138	0.062	0.005	0.602
Beer	5,099	0.136	0.067	0.044	0.481
Smoke	5,099	0.143	0.066	0.049	0.645
Toys	5,099	0.180	0.066	0.080	0.603
Fun	5,099	0.216	0.098	0.088	0.893
Books	5,099	0.165	0.079	0.060	0.772
Hshld	5,099	0.145	0.060	0.059	0.534
Clths	5,099	0.179	0.078	0.066	0.714
Hlth	5,099	0.155	0.066	0.055	0.553
MedEq	5,099	0.160	0.063	0.059	0.581
Drugs	5,099	0.164	0.065	0.068	0.558
Chems	5,099	0.188	0.083	0.091	0.818
Rubbr	5,099	0.155	0.061	0.083	0.557
Txtls	5,099	0.166	0.076	0.066	0.711
BldMt	5,099	0.185	0.072	0.080	0.686
Cnstr	5,099	0.224	0.095	0.097	1.080
Steel	5,099	0.243	0.126	0.084	1.207
FabPr	5,099	0.182	0.086	0.056	0.780
Mach	5,099	0.210	0.088	0.104	0.863
ElcEq	5,099	0.211	0.081	0.105	0.798
Autos	5,099	0.212	0.089	0.101	0.839
Aero	5,099	0.203	0.098	0.089	0.756
Ships	5,099	0.168	0.073	0.073	0.784
Guns	5,099	0.146	0.067	0.028	0.554
Gold	5,043	0.184	0.102	0.005	0.925
Mines	5,099	0.208	0.133	0.049	1.199
Coal	4,735	0.273	0.154	0.007	1.363
Oil	5,099	0.182	0.106	0.039	1.068
Util	5,099	0.129	0.076	0.045	0.735
Telcm	5,099	0.184	0.083	0.075	0.836
PerSv	5,099	0.171	0.063	0.079	0.601
BusSv	5,099	0.177	0.066	0.084	0.615
Hardw	5,099	0.250	0.134	0.093	1.009
Softw	5,099	0.224	0.105	0.090	0.770
Chips	5,099	0.244	0.119	0.089	0.893
LabEq	5,099	0.204	0.081	0.095	0.667
Paper	5,099	0.162	0.061	0.085	0.631
Boxes	5,099	0.171	0.069	0.072	0.678
Trans	5,099	0.181	0.069	0.088	0.619
Whlsl	5,099	0.156	0.059	0.078	0.605
Rtail	5,099	0.182	0.074	0.073	0.626
Meals	5,099	0.156	0.056	0.070	0.549
Banks	5,099	0.229	0.114	0.083	1.494
Insur	5,099	0.191	0.096	0.077	1.005
RlEst	5,099	0.164	0.119	0.033	0.950
Fin	5,099	0.263	0.132	0.094	1.331
Other	5,099	0.179	0.088	0.060	0.669

This table presents summary statistics of Industry SVX_I of 49 industry portfolios. The data are from 4 January 1996 to 29 April 2016.

TABLE 2

Forecasting Realized Volatility with SVX_I, VIX_I, and HV_I

Dep Variable: RV_I	Average of R2	Average of Intercept	Average of β for SVX_I	Average of β for VIX_I	Average of β for HV_I
	47.2%				
Coef		0.046**	1.010***		
p-value		0.041	1.15E-133		
Std Err		0.003	0.015		
	45.8%				
Coef		0.044*		0.918***	
p-value		0.057		3.6E-100	
Std Err		0.003		0.014	
	28.2%				
Coef		0.072***			0.677***
p-value		1.3E-25			9.2E-137
Std Err		0.004			0.015

This table presents the average OLS estimates of regressions in Equation (9). The regressions take the following general form:

$$Y = \alpha + \beta X + \varepsilon$$

All coefficients, p-values, and standard errors are an average of the corresponding measures from 49 regressions on all industry portfolios. Dependent variable RVI is annualized realized volatility in future 30 days of one of 49 industry portfolios, where $RV = 100 \times (365/30 \times \sum_{i=1}^{22} R_i^2)^{1/2}$ and R_i is the daily portfolio return. The data are from 4 January 1996 to 29 April 2016. To correct for autocorrelation and heteroskedasiticity, we use the Newey-West estimator for covariance matric with automatically selected lags as in Newey and West (1994). ***, **, and * denote significance at the 0.01, 0.05, and 0.10 level, respectively.

TABLE 3

Regression Results of Rate Change of SVX_I Against Returns of Industry Portfolios

Dep Variable: ΔSVX_I	**Average of R2**	**Average of Intercept**	**Average of β_1 for $R_{I,t}$**	**Average of β_2 for $R_{I,t}^-$**
	36.7%			
Coef		-0.001*	-0.517***	-0.221***
p-value		0.051	0.000	0.001
Std Err		0.000	0.019	0.032

This table presents the average OLS estimates of regressions in Equation (10). The regressions take the following general form:

$$\Delta SVX_{I,t} = \alpha + \beta_1 R_{I,t} + \beta_2 R_{I,t}^- + \varepsilon$$

All coefficients, p-values, and standard errors are an average of the corresponding measures from 49 regressions on all industry portfolios. Independent variables include RI, daily return of the corresponding industry portfolio; and RI, daily return of the corresponding industry portfolio conditional on whether the return is below 0, i.e., RI^- = min(RI, 0). The dependent variable is the daily return of SVXI, where $\Delta SVX_{I,t} = \ln(SVX_{I,t}/SVX_{I,t-1})$. The return data are from 4 January 1996 to 29 April 2016. To correct for autocorrelation and heteroskedasiticity, we use the Newey-West estimator for covariance matric with automatically selected lags as in Newey and West (1994). ***, **, and * denote significance at the 0.01, 0.05, and 0.10 level, respectively.

TABLE 4

Forecasting Realized Volatility and Downside Realized Volatility with SVX^D_I and BEX_I

	Dep Variable	Average of R2	Average of Intercept	Average of β for SVX^D_I	Average of β for BEX_I
Panel A	**RV_I**	47.3%			
Coef			0.047**	1.003***	
p-value			0.046	1.27E-133	
Std Err			0.003	0.015	
Panel B	**RV^D_I**	34.7%			
Coef			0.029*		0.917***
p-value			0.058		8.3E-166
Std Err			0.003		0.018

This table presents the average OLS estimates of regressions in Sections III C and D. The regressions take the following general form:

$$Y = \alpha + \beta X + \varepsilon$$

All coefficients, p-values, and standard errors are an average of the corresponding measures from 49 regressions on all industry portfolios. Dependent variables include; (1) RV is annualized realized volatility in future 30 days of one of 49 industry portfolios, where $RV = 100 \times (365/30 \times \sum_{i=1}^{22} R_i^2)^{1/2}$ and R_i is the daily portfolio return; and (2) RVDI is the realized downside volatility in future 30 days of one of 49 industry portfolios, where $RVD = 100 \times (365/30 \times \sum_{i=1}^{22} R_i^2 I_{R_i \leq 0})^{1/2}$. The data are from 4 January 1996 to 29 April 2016. To correct for autocorrelation and heteroskedasiticity, we use the Newey-West estimator for covariance matric with automatically selected lags as in Newey and West (1994). ***, **, and * denote significance at the 0.01, 0.05, and 0.10 level, respectively.

TABLE 5

Regression Results of Rate Change of SVX^D_I and ΔBEX_I against Returns of Industry Portfolios

	Dep Variable	Average of R2	Average of Intercept	Average of β_1 for $R_{I,t}$	Average of β_2 for $R_{I,t}^-$
Panel A	**ΔSVX^D_I**	37.9%			
Coef			-0.001*	-0.514***	-0.222***
p-value			0.051	0.000	0.001
Std Err			0.000	0.019	0.031
Panel B	**ΔBEX_I**	37.6%			
Coef			-0.001*	-0.397***	-0.171***
p-value			0.055	0.003	0.002
Std Err			0.000	0.015	0.025

This table presents the average OLS estimates of regressions in Equation (22). The regressions take the following general form:

$$\Delta SVX^D_{I,t} = \alpha + \beta_1 R_{I,t} + \beta_2 R^-_{I,t} + \varepsilon$$

All coefficients, p-values, and standard errors are an average of the corresponding measures from 49 regressions on all industry portfolios. Independent variables include RI, daily return of the corresponding industry portfolio; and RI- daily return of the corresponding industry portfolio conditional on whether the return is below 0, i.e. $R_I = \min(R_I, 0)$. The dependent variable is the daily return of SVXDI, where $\Delta SVX^D_{I,t} = \ln(SVX^D_{I,t}/SVX^D_{I,t})$. The return data are from 4 January 1996 to 29 April 2016. To correct for autocorrelation and heteroskedasiticity, we use the Newey-West estimator for covariance matric with automatically selected lags as in Newey and West (1994). ***, **, and * denote significance at the 0.01, 0.05, and 0.10 level, respectively.

TABLE 6 Out-of-Sample Volatility Forecasting Results

	RMSE	MAE	MAPE
Panel A: 1-year rolling window			
HV_I	0.1005	0.0665	27.8261
VIX_I	0.0838	0.0555	23.2998
SVX_I	0.0832	0.0554	23.3153
SVX^D_I	0.0832	0.0554	23.3428
Panel B: 2-year rolling window			
	RMSE	MAE	MAPE
HV_I	0.1086	0.0753	32.7627
VIX_I	0.0879	0.0593	25.196
SVX_I	0.0868	0.0587	24.9739
SVX^D_I	0.0867	0.0586	24.961
Panel C: 3-year rolling window			
	RMSE	MAE	MAPE
HV_I	0.1102	0.0783	35.130
VIX_I	0.0906	0.0621	26.763
SVX_I	0.0894	0.0613	26.417
SVX^D_I	0.0891	0.0611	26.370

This table reports the out-of-sample forecasting results from January 1996 to April 2016 with a fixed rolling window approach. We report average values of RMSE, MAE and MAPE for 49 industries in the 1-year window, 2-year window, and 3-year window.

TABLE 7

Out-of-Sample Portfolio Performance: Monthly Rebalancing

Panel A: Without Transaction Cost						
	$\gamma = 3$		$\gamma = 4$		$\gamma = 5$	
Predictor	Sharp Ratio	CER(bps)	Sharp Ratio	CER(bps)	Sharp Ratio	CER(bps)
BH	0.335	**N/A**	0.335	**N/A**	0.335	**N/A**
HV_I	0.326	249	0.326	528	0.326	816
VIX_I	0.389	326	0.389	586	0.389	863
SVX_I	0.393	330	0.394	589	0.394	865
SVX^D_I	0.396	331	0.396	590	0.396	866
Panel B: With Transaction Cost (0.25%)						
	$\gamma = 3$		$\gamma = 4$		$\gamma = 5$	
Predictor	Sharp Ratio	CER(bps)	Sharp Ratio	CER(bps)	Sharp Ratio	CER(bps)
BH	0.335	**N/A**	0.335	**N/A**	0.335	**N/A**
HV_I	0.299	193	0.299	486	0.299	783
VIX_I	0.359	266	0.359	540	0.359	826
SVX_I	0.363	269	0.363	543	0.363	829
SVX^D_I	0.365	270	0.366	544	0.366	829

This table reports the monthly out-of-sample portfolio allocation results from January 1998 to April 2016. We compare five strategies: buy-and-hold (BH), volatility timing based on HV_I, volatility timing based on VIX_I, volatility timing based on SVX_I, and volatility timing based on SVX^D_I. We report average annualized Sharpe ratio and certainty equivalent return (CER) gains for 49 industries under risk aversion coefficients of 3, 4, and 5. The CER gain (expressed in annualized basis points) is for a mean-variance investor who allocates between the industry portfolio and risk-free asset using the volatility timing strategy, relative to the naïve buy-and-hold passive strategy (BH). Panel A presents the results without transaction cost and Panel B presents the results counting for transaction cost.

Appendix 1: Estimation of state prices

Breeden and Litzenberger (1978) and Banz and Miller (1978) show that state prices can be modelled as the second derivative of a European call option. This result comes from constructing a butterfly spread with a unit payoff. Assume that an investor longs one call option at a strike price of $X = M - \Delta M$, one call option at $X = M + \Delta M$, and shorts two calls at $X = M$. The maturity of these calls is *T*. If the market price of the underlying asset is M, the payoff of this portfolio would be ΔM and zero otherwise. Henceforth, when dividing this portfolio by ΔM, we can obtain a portfolio that produces a payment of $1 if the market price is M, and zero otherwise.

The price of this portfolio is $P(M,T;\Delta M) = \frac{[c(M+\Delta M) - c(M)] - [c(M) - c(M-\Delta M)]}{\Delta M}$. If we divide the price $P(M,T;\Delta M)$ by ΔM and take a limit of ΔM to zero, we can have:[11]

$$\lim_{\Delta M \to 0} \frac{[c(M + \Delta M) - c(M)] - [c(M) - c(M - \Delta M)]}{(\Delta M)^2} = \frac{\partial^2 c(K,T)}{\partial K^2}\Big|_{K=M} \tag{A1.1}$$

There are two different approaches to estimate Eq A1.1: (i) the Black and Scholes (1973) risk-neutral framework, and (ii) the model-free approach using traded option prices. As suggested by Breeden and Litzenberger (1978), the market portfolio is sufficient to represent different states in the economy. In this paper, we use the S&P 500 index (SPX) as the market portfolio.

(i) The Black and Scholes (1973) approach

Under the assumptions of Black and Scholes (1973), the second derivative of a European call option equals to:

$$\left.\frac{\partial^2 C}{\partial K^2}\right|_{K=F_s} = \frac{e^{-rT}\phi(d_2)}{K\sigma\sqrt{T}}, \tag{A1.2}$$

[11] According to Martin (2018), the state pricing approach does not depend on whether the call option price is twice differentiable. Merton (1973) showed that call option prices is convex function of strike price, and by Alexandrov's theorem, the second derivatives exist almost everywhere.

where $d_2 = \frac{\ln((F_0 - PVD)/K) - 0.5\sigma^2 T}{\sigma\sqrt{T}}$ is the familiar term in Black and Scholes (1973) pricing formula; $\phi(.)$ refers to the standard normal probability density function evaluated at d_2; T is the maturity of the option, which is set to be 30/365 in the current study; r and σ refer to the annualized risk-free rate and volatility[12] of the SPX options; *PVD* is the present value of the SPX dividends; and K is the strike price of the SPX options (i.e., the level of the SPX at which the state price is required).

The value of the contingent claim which generates the unit payoff if the price of the underlying asset is greater than or equal to some level F_s can be calculated as follows:

$$G(F_s) = \int_{F_S}^{\infty} \frac{e^{-rT} n(d_2)}{K\sigma\sqrt{T}} dK = e^{-rT} N[d_2(K = F_S)] \qquad \text{(A1.3)}$$

We can then define the state price as the cost of a security with a unit payoff if the level of the underlying asset is between levels F_s and F_{s+1}, using the Eq. A1.4 below:

$$\phi(F_s, F_{s+1}) = e^{-rT}\{N[d_2^{Black}(K = F_s)] - N[d_2^{Black}(K = F_{s+1})]\} \quad \text{(A1.4)}$$

where SPX index level $F_{s+1} > F_s$.

In the calculation, the maximum and minimum values of state Y are bounded between 0.1 and 9,999. A state of 2,000 indicates the SPX will be at 2,000 after one month. State prices are assessed in increments of 0.1 between F_s and F_{s+1}, representing 0.1 index point on the SPX.

[12] Following Yan (2011), we estimate the volatility σ as the average of the implied volatilities of two at-the-money calls and two at-the-money put options, each with a maturity nearest to the 30-day period.

(ii) The model-free approach using traded option prices

State prices can also be estimated using the numerical derivative from traded options as:

$$\frac{\partial^2 c(X,T)}{\partial X^2}\Big|_{X=M} \approx \frac{c(M+\Delta M)-2c(M)-c(M-\Delta M)}{(\Delta M)^2} \qquad \text{(A1.5)}$$

Eq. A1.5 can be directly estimated using observed SPX option prices.[13] However, Liu and O'Neill (2015) point out several concerns regarding this approach, including generating zero state prices due to same deep out of the money (OTM) option prices, and negative state prices due to irrational bids for deep OTM options.

We adopt the simple Black and Scholes (1973) analytical approach as it is simpler to use and less subject to numerical estimation issues. Liu and O'Neill (2015) also show that the state prices under this approach lead to pricing results which are almost perfectly correlated with the model-free approach.

[13] To more precisely estimate Eq. A1.5 in practice, we can use the Taylor series expansion to include more options (see e.g., Eberly 2008).

Appendix 2: Equilibrium price and CAPM

The Arrow and Debreu (1954) approach states that the value of the market is:

$$P_m = \sum_{s=1}^{S} \phi_{ms} F_{ms} \quad \text{(A2.1)}$$

where F_{tms} is the payoff on the market in state s and P_m is the current value of the market.

Using the set of state prices on the market ϕ_{ms}, we can now find the value of any asset. We simply find the expected value of asset i conditional on the level of the market $E(F_i \mid F_m)$, and multiply by the market state prices to obtain:

$$P_i = \sum_{s=1}^{S} \phi_{ms} E[F_{is} \mid F_{ms}] \quad \text{(A2.2)}$$

The common way to project this expectation is the linear projection of $E(F_i \mid F_m) = \alpha_i + \beta_i F_m$, which yields:

$$P_i = \sum_{s=1}^{S} \phi_{ms} [\alpha_i + \beta_i F_{ms}] \quad \text{(A2.3)}$$

Now Eq. A2.3 looks very familiar and it closely resembles the CAPM in payoffs. However, it is different from CAPM because:

1. The state prices ϕ_{ms} depend on the volatility of the market, the level of the market, the risk-free rate, and the usual determinants of option value. Therefore the state prices could change continuously yet the CAPM assumes constant coefficients.

2. The conditional value of asset i, $E(F_i \mid F_m)$ (Eq. A2.3) may not be linear in F_m. There are all sorts of non-linear ways of taking this conditional expectation (e.g., the excellent test of Friedman *et al.* (2001)), but the CAPM only implies a linear projection with constant intercept and slope coefficients.

Appendix 3: Construction method of industry volatility

We begin our construction by estimating a state-price volatility index for the whole market. This process is similar to the methodology described in Liu and O'Neill (2015). As the first step, we estimate the price of a security that pays off a dollar amount of $\log\left(\frac{M_s}{M_0}\right)^2$ if the S&P 500 index moves to state M_s from an initial value of M_0 in 30 days' time.[14] When summing across all possible future states, we arrive at the price of this asset in the following form:

$$SVX_M = \sqrt{\frac{365}{30}\sum_{s=1}^{S} \varphi_{ms} \log\left(\frac{M_s}{M_0}\right)^2} \quad \text{(A3.1)}$$

where T is set to be exactly 30 calendar days (or equivalently, 22 trading days). SVX_M^2 can be viewed as a financial asset that pays off a dollar amount that is equivalent to the fair value of future variance. The factor 365/30 is to obtain an annualized volatility figure.

The second step involves estimating the relationship between each industry portfolio and the market portfolio, using a linear projection on the market model. That is, we are interested in the expected payoff for the industry conditional on the level of the index. The industry payoff here is the expected squared log returns $E\left[R_I^2 \mid R_M^2\right]$. There are many ways to estimate this relationship (see e.g.,Friedman *et al.* 2001). Linear projection is the most common method in the extant literature (Smith & Walsh 2013), so we adopt this approach and use a simple linear least squares regression of daily squared industry returns on the squared S&P 500 market returns. In this setting, we use the market state prices and assume complete markets. The method captures the systematic

[14] A common approach is to use price levels (returns) to represent the market states (Ross 2015).

components of industry volatility to the extent that the coefficient is allowed to vary over time. Specifically, we are interested in the alphas and betas in the following regression:

$$R_I^2 = \alpha_I + \beta_I R_M^2 + \varepsilon_I \qquad \text{(A3.2)}$$

The return R_I is computed using the close value at the end of day. We estimate each beta using a two-year fixed rolling window. That is, on 505th day, we use the past two years (504 trading days) of return squared to estimate the beta in the above regressions.

The third step is to work out the individual industry portfolio volatility index based on the estimated alphas and betas and their corresponding volatility asset. To illustrate this, we substitute Eq. A3.2 into Eq. A3.1:

$$SVX_I = \sqrt{\alpha_I \frac{365}{30} \sum_{s=1}^{S} \phi_{ms} + \beta_I SVX_M^2} \qquad \text{(A3.3)}$$

The last step is to create an ad-hoc industry volatility index using the widely available CBOE volatility index *VIX*. *VIX* is a sum of weighted-average out-of-the-money S&P 500 put and call options. For a detailed discussion on *VIX*, we refer to Whaley (2009). As we do not have traded options for the industry portfolios, we cannot replicate the CBOE VIX methodology to reproduce the industry measures. Our ad-hoc estimation takes the following form by replacing SVX_M^2 with VIX_M^2:

$$VIX_I = \sqrt{\alpha_I \frac{365}{30} \sum_{s=1}^{S} \phi_{ms} + \beta_I VIX_M^2} \qquad \text{(A3.4)}$$

where VIX_M is the CBOE *VIX*.