A Consistent Stochastic Model of the Term Structure of Interest Rates for Multiple Tenors*

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Abstract

Explicitly taking into account the risk incurred when borrowing at a shorter tenor versus lending at a longer tenor ("roll-over risk"), we construct a stochastic model framework for the term structure of interest rates in which a frequency basis (i.e. a spread applied to one leg of a swap to exchange one floating interest rate for another of a different tenor in the same currency) arises endogenously. This roll-over risk consists of two components, a credit risk component due to the possibility of being downgraded and thus facing a higher credit spread when attempting to roll over short–term borrowing, and a component reflecting the (systemic) possibility of being unable to roll over short–term borrowing at the reference rate (e.g., LIBOR) due to an absence of liquidity in the market. The modelling framework is of "reduced form" in the sense that (similar to the credit risk literature) the *source* of credit risk is not modelled (nor is the source of liquidity risk). However, the framework has more structure than the literature seeking to simply model a different term structure of interest rates for each tenor frequency, since relationships between rates for all tenor frequencies are established based on the modelled roll-over risk. We proceed to consider a specific case within this framework, where the dynamics of interest rate and roll-over risk are driven by a multifactor Cox/Ingersoll/Ross–type process, show how such model can be calibrated to market data, and used for relative pricing of interest rate derivatives, including bespoke tenor frequencies not liquidly traded in the market.

Keywords: tenor swap, basis, frequency basis, liquidity risk, swap market
JEL Classfication: C6, C63, G1, G13

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1 Introduction

The phenomenon of the frequency basis (i.e. a spread applied to one leg of a swap to exchange one floating interest rate for another of a different tenor in the same currency) contradicts textbook no–arbitrage conditions and has become an important feature of interest rate markets since the beginning of the Global Financial Crisis (GFC) in 2008. As a consequence, stochastic interest rate term structure models for financial risk management and the pricing of derivative financial instruments in practice now reflect the existence of multiple term structures, i.e. possibly as many as there are tenor frequencies. While this pragmatic approach can be made mathematically consistent (see Grasselli and Miglietta (2016) and Grbac and Runggaldier (2015) for a recent treatise, as well as the literature cited therein), it does not seek to explain this proliferation of term structures, nor does it allow the extraction of information potentially relevant to risk management from the basis spreads observed in the market.

In the pre-GFC understanding of interest rate swaps (see explained in, e.g., Hull (2008)), the presence of a basis spread in a floating–for–floating interest rate swap would point to the existence of an arbitrage opportunity, unless this spread is too small to recover transaction costs. As documented by Chang and Schlögl (2015), post–GFC the basis spread cannot be explained by transaction costs alone, and therefore there must be a new perception by the market of risks involved in the execution of textbook "arbitrage" strategies. Since such textbook strategies to profit from the presence of basis spreads would involve lending at the longer tenor and borrowing at the shorter tenor, the prime candidate for this is "roll–over risk." This is the risk that in the future, once committed to the "arbitrage" strategy, one might not be able to refinance ("roll over") the borrowing at the prevailing market rate (i.e., the reference rate for the shorter tenor of the basis swap). This "roll–over risk," invalidating the "arbitrage" strategy, can be seen as a combination of "downgrade risk" (i.e., the risk faced by the potential arbitrageur that the credit spread demanded by its creditors will increase relative to the market average) and "funding liquidity risk" (i.e., the risk of a situation where funding in the market can only be accessed at an additional premium).

We propose to model this roll-over risk explicitly, which endogenously leads to the presence of basis spreads between interest rate term structures for different tenors. This is in essence the "reduced–form" or "spread–based" approach to multicurve modelling, similar to the approach taken in the credit risk literature, where the risk of loss due to default gives rise to credit spreads. The model allows us to extract the forward–looking "market's view" of roll-over risk from the observed basis spreads, to which the model is calibrated. Preliminary explorations using a simple model of deterministic basis spreads in Chang and Schlögl (2015) indicate an improving stability of the calibration, suggesting that the basis swap market has matured since the turmoil of the GFC and pointing toward the practicability of constructing and implementing a full stochastic model.

The bulk of the literature on modelling basis spreads is in a sense even more "reduced–form" than what we propose here, in the sense that basis spreads are recognised to exist, and are modelled to be either deterministic or stochastic in a mathematically consistent fashion, but there are no structural links between term structures of interest rates for different tenors (in a sense, the analogue of this approach applied to credit risk would be to model stochastic credit spreads directly, without any link to probabilities of default and losses in the event of default). This strand of the literature can be traced back to Boenkost and Schmidt (2004), who used this approach to construct a model for cross currency swap valuation in the presence of a basis spread. This was subsequently adapted by Kijima, Tanaka and Wong (2009) to modelling a single–currency basis spread. Henrard (2010) took an axiomatic approach to the problem, modelling a deterministic multiplicative spread between term structures associated with different tenors. Initially, these models were not reconciled with the requirement of the absence of arbitrage. Subsequent work, however, such as Fujii, Shimada and Takahashi (2009), gave explicit consideration to this requirement. This pragmatic way of modelling interest rates in the presence of spreads between term structures of interest rates for different tenors has been pursued further in a number of papers, including Mercurio (2009,2010) in a LIBOR Market Model setting, Kenyon (2010) in a short–rate modelling framework, stochastic additive basis spreads in Mercurio and Xie (2012), and Henrard (2013) for stochastic multiplicative basis spreads. Moreni and Pallavicini (2014) construct a model of two curves, riskfree instantaneous forward rates and forward LIBORs, which is Markovian in a common set of state variables.

Early work incorporating some of the potential causes of basis spreads into models of the single–currency "multicurve" environment post–GFC includes Morini (2009) and Bianchetti (2010), who focus on counterparty credit risk. The model of Crépey (2015) links funding cost and counterparty credit risk in a credit valuation adjustment (CVA) framework, but does not explicitly consider spreads between different tenor frequencies arising from roll–over risk.

Recently, there has been an emerging view that "roll–over risk" is what prevents pre–crisis textbook arbitrage strategies to exploit the basis spreads between tenor frequencies, and that modelling this risk can provide the link between overnight index swaps (OIS), the XIBOR (e.g. LIBOR, EURIBOR, etc.) style money market, the vanilla swap market, and the basis swap market. An important contribution in this vein is Filipović and Trolle (2013), who estimate the dynamics of interbank risk from time series data from these markets. They define "interbank risk" as "the risk of direct or indirect loss resulting from lending in the interbank money market." Decomposing the term structure of interbank risk into what they identify as default and non-default (liquidity) components, they study the associated risk premia. Filipović and Trolle interpret the "default" component in terms of the risk of a deterioration of creditworthiness of a LIBOR reference panel bank resulting in it dropping out of the LIBOR panel,[1] in which case this bank immediately would no longer be able to roll over debt at the overnight reference rate, while the rate on any LIBOR borrowing would remain fixed until the end of the accrual period (i.e., typically for several months). In their analysis, this differential

[1] This is also known as the "renewal effect," see Collin–Dufresne and Solnik (2001) and Grinblatt (2001).

impact of downgrade risk on rolling debt explains part of the LIBOR/OIS spread; the residual is labelled the "liquidity" component.[2] It is important to note that both components manifest themselves in the risk of additional cost when rolling over debt, i.e., "downgrade risk" and "funding liquidity risk" combining to form a total "roll–over risk."[3]

Based on similar considerations, Crépey and Douady (2013) model the spread between LIBOR and OIS as a combination of credit and liquidity risk premia, where in particular they focus on providing some model structure for the latter. They construct a stylised equilibrium model of credit risk and funding liquidity risk to explain the LIBOR/OIS spread, arguing (unlike Filipović and Trolle) that the overnight rate underlying OIS (e.g., the Fed Funds or EONIA rate) is riskfree (we will return to this point in our model setup below).

An alternative approach at the more fundamental end of the modelling spectrum is the recent work by Gallitschke, Müller and Seifried (2014), who propose a model for interbank cash transactions and the relevant credit and liquidity risk factors, which endogenously generates multiple term structures for different tenors. In particular, they explicitly model a mechanism by which XIBOR is determined by submissions of the member banks of a panel, which adds substantial complexity to the model.

Our aim is the construct a consistent stochastic model encompassing OIS, XIBOR, vanilla and basis swaps in a single currency.[4] Our "reduced–form" approach explicitly models both the credit and the funding component of roll–over risk to link multiple yield curves. In that, it is more parsimonious than the "pragmatic" way of modelling extant in the literature (reviewed above), where stochastic dynamics for basis spreads are specified directly without recourse to the underlying roll–over risk. In particular, this allows the relative pricing of bespoke tenors in a model calibrated to basis spreads between tenor frequencies for which liquid market data is available. It does not require the introduction of a new stochastic factor (or deterministic spread) for each new tenor frequency. However, the approach is "reduced–form" in the sense that it abstracts from structural causes of downgrade risk and funding liquidity risk — in this sense, our approach is closest in spirit to the "reduced–form" models of credit risk, doing for basis spreads what those models have done for credit spreads. The framework which we propose below departs from that of Chang and Schlögl (2015) in that, rather than focusing exclusively on

[2]Past empirical studies, in particular of the GFC, also indicate that credit risk alone is insufficient to explain the LIBOR/OIS spread; see e.g. Eisenschmidt and Tapking (2009).

[3]"Funding liquidity risk" has also been considered explicitly in a separate strand of the literature. For example, Acharya and Skeie (2011) model liquidity hoarding by participants in the interbank market. In their model, there is a positive feedback effect between roll–over risk and liquidity hoarding (via term premia on interbank lending rates), which in the extreme case can lead to a freeze of interbank lending. Brunnermeier and Pedersen (2009) model a similar adverse feedback effect between market liquidity and funding liquidity.

[4]Since the GFC a *cross currency basis* exceeding pre-crisis textbook arbitrage bounds has also emerged, see for example Chang and Schlögl (2012). The approach presented here could be extended to multiple currencies, but it is our view that across currencies there may be other factors than various forms of roll–over risk giving rise to a basis spread.

basis swaps, we treat OIS, XIBOR, vanilla and basis swaps in a unified framework, and model both the credit and funding liquidity components of roll–over risk in a "reduced–form" manner.

The remainder of the paper is organised as follows. Section 2 expresses the basic instruments in terms of the model variables, i.e., the overnight rate, credit spreads and a spread representing pure funding liquidity risk. Section 3 presents a concrete specification of the model in terms of multifactor Cox/Ingersoll/Ross–type dynamics. In Section 4, the model is calibrated to market data, demonstrating how instruments depending on bespoke tenors can be priced relative to the market. Section 5 concludes.

2 The model

2.1 Model variables

Denote by r_c the continuously compounded short rate abstraction of the interbank overnight rate (e.g., Fed funds rate or EONIA). This is equal to the riskless (default–free) continuously compounded short rate r plus a credit spread. In the simplest case, one could adopt a "fractional recovery in default," a.k.a. "recovery of market value," model[5] and denote by q the (assumed constant) loss fraction in default. Then

$$r_c(s) = r(s) + \Lambda(s)q \tag{2.1}$$

where $\Lambda(s)q$ is the average (market aggregated) credit spread across the panel and $\Lambda(s)$ is the corresponding default intensity. Note that although a significant part of the "multicurve" interest rate modelling literature mentioned in the introduction heuristically advances the argument that (mainly due to its short maturity) the interbank overnight rate is essentially free of default risk, in intensity–based models of default this is incorrect even in the instantaneous limit. In our modelling, we do not require this argument. Instead, it suffices that r_c is the appropriate rate at which to discount payoffs of any fully collateralised derivative transaction, because the standard ISDA Credit Support Annex (CSA) stipulates that posted collateral accrues interest at the interbank overnight rate.[6]

At the present stage of the modelling, r_c will be modelled directly. We are not attempting to disentagle the components $r(s)$ and $\Lambda(s)$, though an extension of the model to include credit default swaps may allow us to do so in the future. Roll–over risk is modelled via the introduction of a $\pi(s)$, denoting the spread over $r_c(s)$ which an arbitrary but fixed entity must pay when borrowing overnight. $\pi(s)$ has two components,

$$\pi(s) = \phi(s) + \lambda(s)q \tag{2.2}$$

where $\phi(s)$ is pure funding liquidity risk[7] (both idiosyncratic and systemic) and

[5]See Duffie and Singleton (1999).

[6]For a detailed discussion of this latter point, see Piterbarg (2010).

[7]In Section 3, $\phi(s)$ will be modelled as a diffusion as a "first–cut" concrete specification of our model. Modelling liquidity freezes properly may require permitting $\phi(s)$ to jump — the

$\lambda(s)q$ is the idiosyncratic credit spread over r_c (initially, e.g. at time 0, $\lambda(0) = 0$ by virtue of the fact that at time of calibration to basis spread data, we are considering market aggregated averages). The default intensity of any given (but representative) XIBOR panel member is $\Lambda(s) + \lambda(s)$. Thus $\lambda(s)q$ represents the "credit" (a.k.a. "renewal") risk component of roll–over risk, i.e. the risk that a particular borrower will be unable to roll over overnight (or instantaneously, in our mathematical abstraction) debt at r_c, instead having to pay an additional spread $\lambda(s)q$ because their credit quality is lower than that of the panel contributors determining r_c.

2.2 OIS with roll–over risk, one–period case

Borrowing overnight from t to T (setting $\delta = T - t$), rolling principal and interest forward until maturity, an arbitrary but fixed entity pays at time T:

$$- e^{\int_t^T r_c(s)ds} e^{\int_t^T \pi(s)ds} \tag{2.3}$$

Assuming symmetric treatment of credit risk when borrowing and lending,[8] lending to an arbitrary but fixed entity from t to T will incur credit risk with intensity $\Lambda(s) + \lambda(s)$. Discounting with

$$r(s) + (\Lambda(s) + \lambda(s))q = r_c(s) + \lambda(s)q$$

the present value of (2.3) is

$$- \mathbb{E}_t^{\mathbb{Q}} \left[e^{\int_t^T \phi(s)ds} \right] \tag{2.4}$$

Enter OIS to receive the overnight rate and pay the fixed rate $\mathrm{OIS}(t,T)$, discounting with r_c (due to collateralisation of OIS), the present value of the payments is

$$\mathbb{E}_t^{\mathbb{Q}} \left[1 - e^{-\int_t^T r_c(s)ds} - e^{-\int_t^T r_c(s)ds} \mathrm{OIS}(t,T)\delta \right] \tag{2.5}$$

Spot LIBOR observed at time t for the accrual period $[t,T]$ is denoted by $L(t,T)$. Lending at LIBOR $L(t,T)$, we receive (at time T) the credit risky payment

$$1 + \delta L(t,T)$$

Discounting as above with $r_c(s) + \lambda(s)q$, the present value of this is

$$\mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T (r_c(s)+\lambda(s)q)ds} (1 + \delta L(t,T)) \right] \tag{2.6}$$

framework laid out in the present section would allow for such an extension.

[8] We need to assume symmetric treatment of roll–over risk, both the "credit" and the "funding liquidity" component, in order to maintain additivity of basis spreads: Swapping a one–month tenor into a three–month tenor, and then swapping the three–month tenor into a 12–month tenor, is financially equivalent to swapping the one–month tenor into the 12–month tenor, and thus (ignoring transaction costs) the 1m/12m basis spread must equal the the sum of the 1m/3m and 3m/12m basis spreads.

We must have (because the initial investment in the strategy is zero) that the sum of the three terms ((2.4), (2.5) and (2.6)) is zero, i.e.

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T \phi(s)ds}\right] &= \\ &\mathbb{E}_t^{\mathbb{Q}}\left[1+e^{-\int_t^T(r_c(s)+\lambda(s)q)ds}(1+\delta L(t,T))-e^{-\int_t^T r_c(s)ds}(1+\delta\text{OIS}(t,T))\right] \quad (2.7)\end{aligned}$$

Consequently, if renewal risk is zero, i.e. $\lambda(s)\equiv 0$, then the LIBOR/OIS spread is solely due to funding liquidity risk:

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T \phi(s)ds}\right]=\mathbb{E}_t^{\mathbb{Q}}\left[1+e^{-\int_t^T r_c(s)ds}\delta(L(t,T)-\text{OIS}(t,T))\right] \quad (2.8)$$

Define the discount factor implied by the overnight rate as

$$D^{OIS}(t,T)=\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_c(s)ds}\right] \quad (2.9)$$

Since the mark–to–market value of the OIS at inception is zero, we have

$$\begin{aligned}\text{OIS}(t,T) &= \frac{1-D^{OIS}(t,T)}{\delta D^{OIS}(t,T)} && (2.10)\\ \Leftrightarrow D^{OIS}(t,T) &= \frac{1}{1+\delta\text{OIS}(t,T)} && (2.11)\end{aligned}$$

Thus the dynamics of r_c should be consistent with (2.11) (the term structure of the $D^{OIS}(t,T)$), and the dynamics of $\phi(s)$ and $\lambda(s)$ should be consistent with (2.7).

The following remarks are worth noting:

- (2.11) implies
$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_c(s)ds}(1+\delta\text{OIS}(t,T))\right]=1$$
Therefore (2.7) implies that we cannot have
$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T(r_c(s)+\lambda(s)q)ds}(1+\delta L(t,T))\right]=1 \quad (2.12)$$
unless $\phi(s)=0$, i.e. unless the LIBOR/OIS spread is solely due to renewal risk.

- It may seem counterintuitive that (2.12) doesn't hold, but this is due to the fact that the discounting in (2.12) only takes into account credit risk (including "renewal risk"), i.e. it does not take into account the premium a borrower of LIBOR (as opposed to rolling overnight borrowing) would pay for avoiding the roll–over risk inherent in $\phi(s)$.

2.3 OIS with roll–over risk, multiperiod case

OIS may pay more frequently than once at T for an accrual period $[t, T]$ (especially when the period covered by the OIS exceeds one year). In this case the strategy of the previous section needs to be modified as follows.

Borrowing overnight from $t = T_0$ to T_n (normalising $T_j - T_{j-1} = \delta, \forall 0 < j \leq n$), rolling principal until maturity and interest forward until each T_j, an arbitrary but fixed entity pays at time T_j ($\forall 0 < j < n$):

$$1 - e^{\int_{T_{j-1}}^{T_j} r_c(s)ds} e^{\int_{T_{j-1}}^{T_j} \pi(s)ds} \tag{2.13}$$

and at time T_n:

$$-e^{\int_{T_{n-1}}^{T_n} r_c(s)ds} e^{\int_{T_{n-1}}^{T_n} \pi(s)ds} \tag{2.14}$$

Discounting with $r_c(s) + q\lambda(s)$, the present value of this is

$$\sum_{j=1}^{n-1} \left(E_t^{\mathbb{Q}} \left[e^{-\int_t^{T_j} (r_c(s)+q\lambda(s))ds} \right] - \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^{T_{j-1}} (r_c(s)+q\lambda(s))ds} e^{\int_{T_{j-1}}^{T_i} \phi(s)ds} \right] \right) \\ - \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^{T_{n-1}} (r_c(s)+q\lambda(s))ds} e^{\int_{T_{n-1}}^{T_n} \phi(s)ds} \right] \tag{2.15}$$

Enter OIS to receive the overnight rate and pay the fixed rate $\text{OIS}(t, T_n)$ at each T_j ($\forall 0 < j \leq n$), discounting with r_c, the present value of the payments is

$$\sum_{j=1}^{n} (D^{OIS}(t, T_{j-1}) - (1 + \delta \text{OIS}(t, T_n)) D^{OIS}(t, T_j)) \tag{2.16}$$

If lending to time T_n at $L(t, T_n)$ is possible (i.e., a LIBOR $L(t, T_n)$ is quoted in the market), and supposing that LIBOR is quoted with annual compounding, with $T_n - t = m$, the present value of interest and repayment of principal is

$$\mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^{T_n} (r_c(s)+q\lambda(s))ds} (1 + L(t, T_n))^m \right] \tag{2.17}$$

We must have (because the initial investment in the strategy is zero) that the sum of the three terms ((2.15), (2.16) and (2.17)) is zero. Since the mark–to–market value of the OIS at inception is zero, we can drop (2.16) and write this directly as

$$\sum_{j=1}^{n} \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^{T_{j-1}} (r_c(s)+q\lambda(s))ds} e^{\int_{T_{j-1}}^{T_j} \phi(s)ds} \right] = \\ \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^{T_n} (r_c(s)+q\lambda(s))ds} (1 + L(t, T_n))^m \right] + \sum_{j=1}^{n-1} \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^{T_j} (r_c(s)+q\lambda(s))ds} \right] \tag{2.18}$$

Analogously to the single–period case, since the mark–to–market value of the OIS at inception is zero, we have

$$\text{OIS}(t, T_n) = \frac{1 - D^{OIS}(t, T_n)}{\delta \sum_{j=1}^{n} D^{OIS}(t, T_j)} \tag{2.19}$$

2.4 LIBOR dynamics

Substituting (2.11) into (2.7), we obtain the dynamics of the spot LIBORs $L(t,T)$ as conditional expectations over the dynamics of ϕ, r_c and λ:

$$L(t,T) = \frac{1}{\delta}\left(\frac{\mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T \phi(s)ds}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T (r_c(s)+\lambda(s)q)ds}\right]} - 1\right) \tag{2.20}$$

Rewriting this as a discount factor

$$D^L(t,T) = (1+\delta L(t,T))^{-1} = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T (r_c(s)+\lambda(s)q)ds}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T \phi(s)ds}\right]} \tag{2.21}$$

we see that if we assume independence of the dynamics of $r_c(s)+\lambda(s)q$ from $\phi(s)$, the "instantaneous spread" (admittedly a theoretical abstraction) over r_c inside the expectation becomes simply $\pi(s) = \phi(s) + \lambda(s)q$:

$$D^L(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T (r_c(s)+\phi(s)+\lambda(s)q)ds}\right] \tag{2.22}$$

2.5 Going beyond LIBOR maturities with interest rate swaps (IRS)

The application of roll–over risk must be symmetric, i.e. applied to roll–over of both borrowing and lending (with opposite sign for borrowing vs. lending), because otherwise contradictions are inescapable.[9]

Suppose we are swapping LIBOR with tenor structure $T^{(L)}$, from $T_0^{(L)} = T_0 = t$ to $T_{n_L}^{(L)}$ into a fixed rate $s^{(1)}(t, T_{n_1}^{(1)})$, paid based on a tenor structure $T^{(1)}$, with $T_{n_L}^{(L)} = T_{n_1}^{(1)}$, using an interest rate swap.[10] For a (fully collateralised) swap transaction, it must hold that

$$\begin{aligned} & \sum_{j=1}^{n_L} \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_j^{(L)}} r_c(s)ds}(T_j^{(L)} - T_{j-1}^{(L)})L(T_{j-1}^{(L)}, T_j^{(L)})\right] \\ = & \sum_{j=1}^{n_1} D^{OIS}(t, T_j^{(1)})(T_j^{(1)} - T_{j-1}^{(1)})s^{(1)}(t, T_{n_1}^{(1)}) \end{aligned} \tag{2.23}$$

[9] Such contradictions would arise in particular because asymmetry in the treatment of roll-over risk would prevent additivity of basis spreads: Combing a position of paying one–month LIBOR versus receiving three–month LIBOR with a position of paying three–month LIBOR versus receiving six–month LIBOR is equivalent to paying one–month LIBOR versus receiving six–month LIBOR; therefore (up to transaction costs), the one–month versus six–month basis spread should equal the sum of the one–month versus three–month and three–month versus six–month spreads.

[10] For example, a vanilla USD swap would exchange three–month LIBOR (paid every three months) against a stream of fixed payments paid every six months.

Typically, we have market data on (vanilla) fixed–for–floating swaps of different maturities, for a single tenor frequency. Combining this with market data on basis swaps, we get a matrix of market calibration conditions (2.23), i.e. one condition for each (maturity,tenor) combination. Note that the basis spread is typically added to the shorter tenor of a basis swap, so (2.23) needs to be modified accordingly. For example, if $T^{(L)}$ corresponds to a three–month frequency, $T^{(1)}$ corresponds to a six–month frequency, and $T^{(2)}$ corresponds to a one–month frequency of the same maturity ($T^{(L)}_{n_L} = T^{(2)}_{n_2}$), then combining (2.23) with a basis swap with spread $b(t, T^{(L)}_{n_L}, T^{(2)}_{n_2})$, we obtain

$$\begin{aligned}
&\sum_{j=1}^{n_2} \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_j^{(2)}} r_c(s)ds}(T_j^{(2)} - T_{j-1}^{(2)})L(T_{j-1}^{(2)}, T_j^{(2)})\right] \\
= &\sum_{j=1}^{n_1} D^{OIS}(t, T_j^{(1)})(T_j^{(1)} - T_{j-1}^{(1)})s^{(1)}(t, T_{n_1}^{(1)}) \\
&\qquad - \sum_{j=1}^{n_2} D^{OIS}(t, T_j^{(2)})(T_j^{(2)} - T_{j-1}^{(2)})b(t, T_{n_L}^{(L)}, T_{n_2}^{(2)}) \qquad (2.24)
\end{aligned}$$

Conversely, because the convention of basis swaps is that the basis swap spread is added to the shorter tenor, if $T^{(L)}$ corresponds to a three–month frequency, $T^{(1)}$ corresponds to a six–month frequency, and $T^{(3)}$ corresponds to a twelve–month frequency of the same maturity ($T^{(L)}_{n_L} = T^{(3)}_{n_3}$), then combining (2.23) with a basis swap with spread $b(t, T^{(L)}_{n_L}, T^{(3)}_{n_3})$, we obtain

$$\begin{aligned}
&\sum_{j=1}^{n_3} \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_j^{(3)}} r_c(s)ds}(T_j^{(3)} - T_{j-1}^{(3)})L(T_{j-1}^{(3)}, T_j^{(3)})\right] \\
= &\sum_{j=1}^{n_1} D^{OIS}(t, T_j^{(1)})(T_j^{(1)} - T_{j-1}^{(1)})s^{(1)}(t, T_{n_1}^{(1)}) \\
&\qquad + \sum_{j=1}^{n_3} D^{OIS}(t, T_j^{(3)})(T_j^{(3)} - T_{j-1}^{(3)})b(t, T_{n_L}^{(L)}, T_{n_3}^{(3)}) \qquad (2.25)
\end{aligned}$$

2.6 Options

Since (2.20) relates the dynamics of the spot LIBORs $L(t, T)$ to the dynamics of ϕ, r_c and λ, we can price caps and floors. Equation (2.23) relates the dynamics of swap rates to the spot LIBORs $L(t, T)$, and thereby to the dynamics of ϕ, r_c and λ, allowing us to price swaptions. Thus, this yields conditions by which the volatilities of ϕ, r_c and λ can be calibrated to market data (at the money, at least) of caps/floors and swaptions.

3 Stochastic modelling

We assume that the model is driven by d independent factors y_i, i.e.

$$r_c(t) = a_0(t) + \sum_{i=1}^{d} a_i y_i(t) \tag{3.1}$$

$$\lambda(t) = b_0(t) + \sum_{i=1}^{d} b_i y_i(t) \tag{3.2}$$

$$\phi(t) = c_0(t) + \sum_{i=1}^{d} c_i y_i(t) \tag{3.3}$$

where the y_i follow the Cox–Ingersoll–Ross (CIR) dynamics under the pricing measure, i.e.

$$dy_i(t) = \kappa_i(\theta_i - y_i(t))dt + \sigma_i \sqrt{y_i(t)} dW_i(t), \tag{3.4}$$

where $dW_i(t) \quad (i = 1, \cdots, d)$ are independent Wiener processes. The CIR–type model is chosen for its analytical tractability, as well as the guaranteed positivity of the modelled object, with the condition which ensures that the origin is inaccessible. In order to keep the model analytically tractable, we do not allow for time–dependent coefficients at this stage.[11] Since each of the factors follow independent CIR–type dynamics, the sufficient condition for each factor to remain positive is $2\kappa_i\theta_i \geq \sigma_i^2, \forall i$, as discussed for the one–factor case in Cox et al. (1985). Furthermore, by economic reasoning we might require that the model variable λ remains non-negative. A sufficient condition for this is to have the function b_0 and coefficients b_i all non-negative.

3.1 Some useful transforms

In this subsection we reproduce some standard results on the transforms of CIR processes that constitute the building blocks for the computations we have to perform in order to implement our formulae. Here we recall a slight generalisation of the celebrated Pitman and Yor (1982) formula giving the joint Laplace transform of the CIR process together with its integral.

Given a Brownian motion W defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, let $X^x = \{X^x(t), t \geq 0\}$ be the solution of the following (CIR) SDE

$$dX(t) = (a - bX(t))dt + \sigma\sqrt{X(t)}dW(t), \tag{3.5}$$

[11] For example, following Schlögl and Schlögl (2000), the coefficients could be made piecewise constant to facilitate calibration to at–the–money option price data, while retaining most analytical tractability.

with $X(0) = x > 0$, $a, b, \sigma > 0$ and $2a \geq \sigma^2$ (Feller condition). Consider $\mu, \alpha \in \mathbb{R}$ such that

$$\mu \geq -\frac{b^2}{2\sigma^2}, \tag{3.6}$$

$$\alpha \geq -\frac{\sqrt{A}+b}{\sigma^2}, \tag{3.7}$$

where $A = b^2 + 2\mu\sigma^2$.

Then the transform $\mathbb{E}\left[e^{-\mu\int_0^t X^x(s)ds-\alpha X^x(t)}\right]$ is well defined for all $t \geq 0$ and given by

$$\mathbb{E}\left[e^{-\mu\int_0^t X^x(s)ds-\alpha X^x(t)}\right] = \left(\frac{(\sigma^2\alpha+b)(e^{\sqrt{A}t}-1)+\sqrt{A}(e^{\sqrt{A}t}+1)}{2\sqrt{A}e^{\frac{\sqrt{A}+b}{2}t}}\right)^{-\frac{2a}{\sigma^2}} \tag{3.8}$$

$$\exp\left(x\frac{(\alpha b-2\mu)(e^{\sqrt{A}t}-1)-\alpha\sqrt{A}(e^{\sqrt{A}t}+1)}{(\sigma^2\alpha+b)(e^{\sqrt{A}t}-1)+\sqrt{A}(e^{\sqrt{A}t}+1)}\right). \tag{3.9}$$

If $\alpha < -\frac{\sqrt{A}+b}{\sigma^2}$ the transform is still valid, but up to a (maximal/explosion) time t_{max} given by

$$t_{max} = \frac{1}{\sqrt{A}}\ln\left(1-\frac{2\sqrt{A}}{b+\sigma^2\alpha+\sqrt{A}}\right). \tag{3.10}$$

The result is standard, for a proof see e.g. Theorem A.1 in Grasselli (2016), where it is shown that the result is valid also for $\mu = -\frac{b^2}{2\sigma^2}$ by taking the limit for $A \to 0$. We get immediately the following Lemma.

Lemma 3.1. *Let y be a $d-$dimensional Cox–Ingersoll–Ross (CIR) process, i.e.*

$$dy_i(t) = \kappa_i(\theta_i - y_i(t))dt + \sigma_i\sqrt{y_i(t)}dW_i(t), \tag{3.11}$$

where W_i $(i = 1, .., d)$ are independent Wiener processes under the (pricing) probability measure $\mathbb{Q}$. If for all $i = 1, .., d$

$$\mu \geq -\frac{\kappa_i^2}{2\sigma_i^2}, \tag{3.12}$$

$$\alpha \geq -\frac{\sqrt{A_i}+\kappa_i}{\sigma_i^2} \tag{3.13}$$

then for all $t \geq 0$ and $i = 1, .., d$

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\mu\int_0^t y_i(s)ds-\alpha y_i(t)}\right] = e^{\Phi_i(0,t,\mu,\alpha)+y_i(0)\Psi_i(0,t,\mu,\alpha)}, \tag{3.14}$$

where

$$\Phi_i(0,t,\mu,\alpha) = -\frac{2\theta_i\kappa_i}{\sigma_i^2}\ln\left(\frac{(\sigma_i^2\alpha+\kappa_i)(e^{\sqrt{A_i}t}-1)+\sqrt{A_i}(e^{\sqrt{A_i}t}+1)}{2\sqrt{A_i}e^{\frac{\sqrt{A_i}+\kappa_i}{2}t}}\right), \tag{3.15}$$

$$\Psi_i(0,t,\mu,\alpha) = \frac{(\alpha\kappa_i-2\mu)(e^{\sqrt{A_i}t}-1)-\alpha\sqrt{A_i}(e^{\sqrt{A_i}t}+1)}{(\sigma_i^2\alpha+\kappa_i)(e^{\sqrt{A_i}t}-1)+\sqrt{A_i}(e^{\sqrt{A_i}t}+1)}, \tag{3.16}$$

with $A_i = \kappa_i^2 + 2\mu\sigma_i^2$. *If for some* $i = 1, .., d$

$$\alpha < -\frac{\sqrt{A_i}+\kappa_i}{\sigma_i^2}, \tag{3.17}$$

then the transform (3.14) *is valid up to the maximal time* t_{max} *given by*

$$t_{max} = \min_{i\le d}\frac{1}{\sqrt{A_i}}\ln\left(1-\frac{2\sqrt{A_i}}{\kappa_i+\sigma_i^2\alpha+\sqrt{A_i}}\right). \tag{3.18}$$

3.2 Computation of the relevant expectations

We have now the ingredients to compute explicitly the expectations in (2.7) and (2.18).

Let us first consider the expectations involved in formula (2.7), i.e. $\mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T\phi(s)ds}\right]$, $\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T(r_c(s)+\lambda(s)q)ds}\right]$ and $D^{OIS}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_c(s)ds}\right]$.

Proposition 3.1. *Consider the processes* $y_i, i = 1, .., d$ *satisfying* (3.11) *and the processes* r_c, λ, ϕ *defined by resp.* (3.1), (3.2), (3.3). *Assume that the deterministic functions* $a_0(t), b_0(t), c_0(t)$ *are integrable and* $a_i, b_i, c_i, i = 1, .., d$ *are real constants.*
i) If for all $i = 1, .., d$

$$c_i \le \frac{\kappa_i^2}{2\sigma_i^2}, \tag{3.19}$$

then for all $t \ge 0$

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T\phi(s)ds}\right] = e^{\int_t^T c_0(s)ds}\prod_{i=1}^d\mathbb{E}_t^{\mathbb{Q}}\left[e^{c_i\int_t^T y_i(s)ds}\right] \tag{3.20}$$

$$= e^{\int_t^T c_0(s)ds+\sum_{i=1}^d(\Phi_i(t,T,-c_i,0)+y_i(t)\Psi_i(t,T,-c_i,0))}. \tag{3.21}$$

ii) If for all $i = 1, .., d$

$$a_i + b_i q \ge -\frac{\kappa_i^2}{2\sigma_i^2}, \tag{3.22}$$

then for all $t \ge 0$

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T(r_c(s)+\lambda(s)q)ds}\right] = e^{-\int_t^T(a_0(s)+b_0(s)q)ds}\prod_{i=1}^d\mathbb{E}_t^{\mathbb{Q}}\left[e^{-(a_i+b_iq)\int_t^T y_i(s)ds}\right] \tag{3.23}$$

$$= e^{-\int_t^T(a_0(s)+b_0(s)q)ds+\sum_{i=1}^d(\Phi_i(t,T,a_i+b_iq,0)+y_i(t)\Psi_i(t,T,a_i+b_iq,0))}. \tag{3.24}$$

iii) If for all $i = 1, .., d$

$$a_i \geq -\frac{\kappa_i^2}{2\sigma_i^2}, \tag{3.25}$$

then for all $t \geq 0$

$$D^{OIS}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_c(s)ds}\right] \tag{3.26}$$

$$= e^{-\int_t^T a_0(s)ds} \prod_{i=1}^d \mathbb{E}_t^{\mathbb{Q}}\left[e^{-a_i \int_t^T y_i(s)ds}\right] \tag{3.27}$$

$$= e^{-\int_t^T a_0(s)ds + \sum_{i=1}^d (\Phi_i(t,T,a_i,0) + y_i(t)\Psi_i(t,T,a_i,0))}. \tag{3.28}$$

Here the deterministic functions Φ_i, Ψ_i are defined by

$$\Phi_i(t,T,\mu,0) = -\frac{2\theta_i\kappa_i}{\sigma_i^2}\ln\left(\frac{\kappa_i(e^{\sqrt{A_i}(T-t)}-1)+\sqrt{A_i}(e^{\sqrt{A_i}(T-t)}+1)}{2\sqrt{A_i}e^{\frac{\sqrt{A_i}+\kappa_i}{2}(T-t)}}\right), \tag{3.29}$$

$$\Psi_i(t,T,\mu,0) = -\frac{2\mu(e^{\sqrt{A_i}(T-t)}-1)}{\kappa_i(e^{\sqrt{A_i}(T-t)}-1)+\sqrt{A_i}(e^{\sqrt{A_i}(T-t)}+1)}, \tag{3.30}$$

with $A_i = \kappa_i^2 + 2\mu\sigma_i^2$.

The proof follows immediately from the definition of r_c, λ, ϕ and the Lemma 3.1.

Let us now turn out attention to formula (2.18), which involves the following expectation:

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_{j-1}}(r_c(s)+\lambda(s)q)ds} e^{\int_{T_{j-1}}^{T_j}\phi(s)ds}\right] \tag{3.31}$$

$$= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_{j-1}}(r_c(s)+\lambda(s)q)ds}\mathbb{E}_{T_{j-1}}^{\mathbb{Q}}\left[e^{\int_{T_{j-1}}^{T_j}\phi(s)ds}\right]\right] \tag{3.32}$$

$$= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_{j-1}}(r_c(s)+\lambda(s)q)ds} e^{\int_{T_{j-1}}^{T_j}c_0(s)ds + \sum_{i=1}^d(\Phi_i(T_{j-1},T_j,-c_i,0)+y_i(T_{j-1})\Psi_i(T_{j-1},T_j,-c_i,0))}\right] \tag{3.33}$$

$$= e^{-\int_t^{T_{j-1}}(a_0(s)+b_0(s)q)ds + \int_{T_{j-1}}^{T_j}c_0(s)ds + \sum_{i=1}^d \Phi_i(T_{j-1},T_j,-c_i,0)} \tag{3.34}$$

$$\cdot\prod_{i=1}^d \mathbb{E}_t^{\mathbb{Q}}\left[e^{-(a_i+b_iq)\int_t^{T_{j-1}}y_i(s)ds + y_i(T_{j-1})\Psi_i(T_{j-1},T_j,-c_i,0)}\right] \tag{3.35}$$

$$= e^{-\int_t^{T_{j-1}}(a_0(s)+b_0(s)q)ds + \int_{T_{j-1}}^{T_j}c_0(s)ds + \sum_{i=1}^d \Phi_i(T_{j-1},T_j,-c_i,0)} \tag{3.36}$$

$$.e^{\sum_{i=1}^d \Phi_i(t,T_{j-1},a_i+b_iq,-\Psi_i(T_{j-1},T_j,-c_i,0)) + \sum_{i=1}^d y_i(t)\Psi_i(t,T,a_i+b_iq,-\Psi_i(T_{j-1},T_j,-c_i,0))} \tag{3.37}$$

We have then proved the following result.

Proposition 3.2. *Consider the processes $y_i, i = 1, .., d$ satisfying (3.11) and the processes r_c, λ, ϕ defined by resp. (3.1), (3.2), (3.3). Assume that the deterministic functions $a_0(t), b_0(t), c_0(t)$ are integrable and $a_i, b_i, c_i, i = 1, .., d$ are real constants. If for all $i = 1, .., d$*

$$c_i \leq \frac{\kappa_i^2}{2\sigma_i^2}, \tag{3.38}$$

$$a_i + b_i q \geq -\frac{\kappa_i^2}{2\sigma_i^2}, \tag{3.39}$$

$$\Psi_i(T_{j-1}, T_j, -c_i, 0) \leq \frac{\sqrt{A_i} + \kappa_i}{\sigma_i^2} \tag{3.40}$$

then for all $t \geq 0$

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_{j-1}}(r_c(s)+\lambda(s)q)ds} e^{\int_{T_{j-1}}^{T_j}\phi(s)ds}\right] = e^{\tilde{\Phi}(t)+\sum_{i=1}^d y_i(t)\tilde{\Psi}_i(t)}, \tag{3.41}$$

where

$$\tilde{\Phi}(t) = -\int_t^{T_{j-1}}(a_0(s) + b_0(s)q)ds + \int_{T_{j-1}}^{T_j} c_0(s)ds + \sum_{i=1}^d \Phi_i(T_{j-1}, T_j, -c_i, 0) \tag{3.42}$$

$$+ \sum_{i=1}^d \Phi_i(t, T_{j-1}, a_i + b_i q, -\Psi_i(T_{j-1}, T_j, -c_i, 0)), \tag{3.43}$$

$$\tilde{\Psi}_i(t) = \Psi_i(t, T, a_i + b_i q, -\Psi_i(T_{j-1}, T_j, -c_i, 0)). \tag{3.44}$$

If $\Psi_i(T_{j-1}, T_j, -c_i, 0) > \frac{\sqrt{A_i}+\kappa_i}{\sigma_i^2}$ for some $i = 1, .., d$ then the formula still holds true till the maximal time $t_{max} = T_{j-1} - t$ given by

$$t_{max} = \min_{i \leq d} \frac{1}{\sqrt{A_i}} \ln\left(1 - \frac{2\sqrt{A_i}}{\kappa_i - \sigma_i^2\Psi_i(T_{j-1}, T_j, -c_i, 0) + \sqrt{A_i}}\right). \tag{3.45}$$

In conclusion, all the expectations in formulae (2.7) and (2.18) can be explicitly computed.

3.3 The LIBOR and the pricing of swaps

In this subsection we compute the expectations appearing in swap transaction formula (2.23), which is crucial to the calibration procedure. The expression of D^{OIS} has already been computed in Proposition 3.1, so we focus now on the first expectation in (2.23) involving the forward LIBOR.

From (2.20) it follows immediately

$$\begin{aligned}\delta L(T_{j-1}, T_j) = &- 1 + e^{\int_{T_{j-1}}^{T_j}(c_0(s)+a_0(s)+qb_0(s))ds+\sum_{i=1}^d(\Phi_i(T_{j-1},T_j,-c_i,0)-\Phi_i(T_{j-1},T_j,a_i+qb_i,0))} \\ &. e^{\sum_{i=1}^d y_i(T_{j-1})(\Psi_i(T_{j-1},T_j,-c_i,0)-\Psi_i(T_{j-1},T_j,a_i+qb_i,0))}.\end{aligned}$$

We have

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_j} r_c(s)ds}L(T_{j-1},T_j)\right]$$
$$=\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_{j-1}} r_c(s)ds}L(T_{j-1},T_j)D^{OIS}(T_{j-1},T_j)\right]$$
$$=\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_{j-1}}(a_0(s)+\sum_{i=1}^d a_iy_i(s))ds-\int_{T_{j-1}}^{T_j}a_0(s)ds+\sum_{i=1}^d(\Phi_i(T_{j-1},T_j,a_i,0)+y_i(T_{j-1})\Psi_i(T_{j-1},T_j,a_i,0))}L(T_{j-1},T_j)\right]$$
$$=e^{-\int_t^{T_j}a_0(s)ds+\sum_{i=1}^d\Phi_i(T_{j-1},T_j,a_i,0)}\mathbb{E}_t^{\mathbb{Q}}\left[e^{\sum_{i=1}^d(-\int_t^{T_{j-1}}a_iy_i(s)ds+y_i(T_{j-1})\Psi_i(T_{j-1},T_j,a_i,0))}L(T_{j-1},T_j)\right].$$

Into this equation we insert the expression of the forward LIBOR in terms of the sum of the constant and the two exponentials and we apply Lemma 3.1:

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_j} r_c(s)ds}L(T_{j-1},T_j)\right]$$
$$=-\frac{1}{\delta}e^{-\int_t^{T_j}a_0(s)ds+\sum_{i=1}^d\Phi_i(T_{j-1},T_j,a_i,0)}\mathbb{E}_t^{\mathbb{Q}}\left[e^{\sum_{i=1}^d(-\int_t^{T_{j-1}}a_iy_i(s)ds+y_i(T_{j-1})\Psi_i(T_{j-1},T_j,a_i,0))}\right]$$
$$+\frac{1}{\delta}e^{-\int_t^{T_j}a_0(s)ds+\int_{T_{j-1}}^{T_j}(c_0(s)+a_0(s)+qb_0(s))ds+\sum_{i=1}^d(\Phi_i(T_{j-1},T_j,a_i,0)+\Phi_i(T_{j-1},T_j,-c_i,0)-\Phi_i(T_{j-1},T_j,a_i+qb_i,0))}$$
$$.\mathbb{E}_t^{\mathbb{Q}}\left[e^{\sum_{i=1}^d(-\int_t^{T_{j-1}}a_iy_i(s)ds+y_i(T_{j-1})(\Psi_i(T_{j-1},T_j,a_i,0)+\Psi_i(T_{j-1},T_j,-c_i,0)-\Psi_i(T_{j-1},T_j,a_i+qb_i,0)))}\right]$$
$$=-\frac{1}{\delta}e^{-\int_t^{T_j}a_0(s)ds+\sum_{i=1}^d(\Phi_i(T_{j-1},T_j,a_i,0)+\Phi_i(t,T_{j-1},a_i,-\Psi_i(T_{j-1},T_j,a_i,0))+y_i(t)\Psi_i(t,T_{j-1},a_i,-\Psi_i(T_{j-1},T_j,a_i,0)))}$$
$$+\frac{1}{\delta}e^{-\int_t^{T_j}a_0(s)ds+\int_{T_{j-1}}^{T_j}(c_0(s)+a_0(s)+qb_0(s))ds+\sum_{i=1}^d(\Phi_i(T_{j-1},T_j,a_i,0)+\Phi_i(T_{j-1},T_j,-c_i,0)-\Phi_i(T_{j-1},T_j,a_i+qb_i,0))}$$
$$.e^{\sum_{i=1}^d\Phi_i(t,T_{j-1},a_i,-(\Psi_i(T_{j-1},T_j,a_i,0)+\Psi_i(T_{j-1},T_j,-c_i,0)-\Psi_i(T_{j-1},T_j,a_i+qb_i,0)))}$$
$$.e^{\sum_{i=1}^d y_i(t)\Psi_i(t,T_{j-1},a_i,-(\Psi_i(T_{j-1},T_j,a_i,0)+\Psi_i(T_{j-1},T_j,-c_i,0)-\Psi_i(T_{j-1},T_j,a_i+qb_i,0)))}.$$

The previous formula is valid for all $t\le T_n$ provided that

$$a_i\ge-\frac{\kappa_i^2}{2\sigma_i^2},\qquad(3.46)$$
$$c_i\le\frac{\kappa_i^2}{2\sigma_i^2},\qquad(3.47)$$
$$\Psi_i(T_{j-1},T_j,a_i,0)\le\frac{\sqrt{A_i}+\kappa_i}{\sigma_i^2},\qquad(3.48)$$
$$\Psi_i(T_{j-1},T_j,a_i,0)+\Psi_i(T_{j-1},T_j,-c_i,0)-\Psi_i(T_{j-1},T_j,a_i+qb_i,0)\le\frac{\sqrt{A_i}+\kappa_i}{\sigma_i^2},\qquad(3.49)$$

where $A_i=\kappa_i^2+2\mu\sigma_i^2$.
If some of the inequalities (3.48)-(3.49) are not satisfied for some $i=1,..,d$, then the formula is still valid up to the maximal $t_{max}=T_{j-1}-t$ given by

$$t_{max}=\min_{i\le d}\frac{1}{\sqrt{A_i}}\ln\left(1-\frac{2\sqrt{A_i}}{\kappa_i+\sigma_i^2\alpha+\sqrt{A_i}}\right).\qquad(3.50)$$

with α equal to minus the LHS respectively of each of (3.48)-(3.49).

Therefore, also the expectations in (2.23) can be computed explicitly and the formula can be obtained in closed form.

3.4 The pricing of caps

Let us first consider a caplet on the LIBOR with maturity T_j, whose payoff is given by

$$K\delta\left(L(T_{j-1},T_j)-R\right)^+, \tag{3.51}$$

where K denotes the notional and R is the strike price of the caplet.

The price at time $t \le T_{j-1}$ is given by

$$\begin{aligned}
Caplet_t =& \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_j} r_c(s)ds} K\delta\left(L(T_{j-1},T_j)-R\right)^+\right] && (3.52)\\
=& D^{OIS}(t,T_j)\mathbb{E}_t^{\mathbb{Q}^{T_j}}\left[K\delta\left(L(T_{j-1},T_j)-R\right)^+\right] && (3.53)\\
=& D^{OIS}(t,T_j)K\delta\mathbb{E}_t^{\mathbb{Q}^{T_j}}\left[\frac{1}{\delta}\left(\frac{\mathbb{E}_{T_{j-1}}^{\mathbb{Q}}\left[e^{\int_{T_{j-1}}^{T_j}\phi(s)ds}\right]}{\mathbb{E}_{T_{j-1}}^{\mathbb{Q}}\left[e^{-\int_{T_{j-1}}^{T_j}(r_c(s)+\lambda(s)q)ds}\right]}-(1+\delta R)\right)^+\right] && (3.54)\\
=& D^{OIS}(t,T_j)K\mathbb{E}_t^{\mathbb{Q}^{T_j}}\left[\left(\frac{\mathbb{E}_{T_{j-1}}^{\mathbb{Q}}\left[e^{\int_{T_{j-1}}^{T_j}\phi(s)ds}\right]}{\mathbb{E}_{T_{j-1}}^{\mathbb{Q}}\left[e^{-\int_{T_{j-1}}^{T_j}(r_c(s)+\lambda(s)q)ds}\right]}-(1+\delta R)\right)^+\right] && (3.55)\\
=& D^{OIS}(t,T_j)K\left(1+\delta R\right)\mathbb{E}_t^{\mathbb{Q}^{T_j}}\left[\left(\frac{1}{1+\delta R}\frac{\mathbb{E}_{T_{j-1}}^{\mathbb{Q}}\left[e^{\int_{T_{j-1}}^{T_j}\phi(s)ds}\right]}{\mathbb{E}_{T_{j-1}}^{\mathbb{Q}}\left[e^{-\int_{T_{j-1}}^{T_j}(r_c(s)+\lambda(s)q)ds}\right]}-1\right)^+\right] && (3.56)\\
=& D^{OIS}(t,T_j)K\left(1+\delta R\right)\mathbb{E}_t^{\mathbb{Q}^{T_j}}\left[\left(e^{X(T_{j-1})}-1\right)^+\right], && (3.57)
\end{aligned}$$

where

$$\begin{aligned}
e^{X(T_{j-1})} =& \frac{1}{1+\delta R}\frac{\mathbb{E}_{T_{j-1}}^{\mathbb{Q}}\left[e^{\int_{T_{j-1}}^{T_j}\phi(s)ds}\right]}{\mathbb{E}_{T_{j-1}}^{\mathbb{Q}}\left[e^{-\int_{T_{j-1}}^{T_j}(r_c(s)+\lambda(s)q)ds}\right]}. && (3.58)\\
& && (3.59)
\end{aligned}$$

Using Proposition 3.1 we can now write

$$
\begin{aligned}
e^{X(T_{j-1})} =& \frac{1}{1+\delta R} \frac{e^{\int_{T_{j-1}}^{T_j} c_0(s)ds + \sum_{n=1}^{d}(\Phi_n(T_{j-1},T_j,-c_n,0)+y_n(T_{j-1})\Psi_n(T_{j-1},T_j,-c_n,0))}}{e^{-\int_{T_{j-1}}^{T_j}(a_0(s)+b_0(s)q)ds+\sum_{n=1}^{d}(\Phi_n(T_{j-1},T_j,a_n+b_nq,0)+y_n(T_{j-1})\Psi_n(T_{j-1},T_j,a_n+b_nq,0))}} && (3.60)\\
=& \frac{1}{1+\delta R} e^{\int_{T_{j-1}}^{T_j}(c_0(s)+a_0(s)+b_0(s)q)ds+\sum_{n=1}^{d}(\Phi_n(T_{j-1},T_j,-c_n,0)-\Phi_n(T_{j-1},T_j,a_n+b_nq,0))} && (3.61)\\
& .e^{\sum_{n=1}^{d} y_n(T_{j-1})(\Psi_n(T_{j-1},T_j,-c_n,0)-\Psi_n(T_{j-1},T_j,a_n+b_nq,0))} && (3.62)\\
=& e^{f(T_{j-1})+\sum_{n=1}^{d} y_n(T_{j-1})g(n,T_{j-1})}, && (3.63)
\end{aligned}
$$

where

$$
\begin{aligned}
f(T_{j-1}) =& -\ln(1+\delta R) + \int_{T_{j-1}}^{T_j}(c_0(s)+a_0(s)+b_0(s)q)ds && (3.64)\\
& + \sum_{n=1}^{d}(\Phi_n(T_{j-1},T_j,-c_n,0) - \Phi_n(T_{j-1},T_j,a_n+b_nq,0)), && (3.65)\\
g(n,T_{j-1}) =& \Psi_n(T_{j-1},T_j,-c_n,0) - \Psi_n(T_{j-1},T_j,a_n+b_nq,0). && (3.66)
\end{aligned}
$$

In order to compute Equation (3.57), we apply Fourier transform techniques well-explored by Eberlein, Glau and Papapantoleon (2010). Firstly, we derive the expression for the characteristic function or the moment generating function of X under the $\mathbb{Q}^{T_j}-$forward probability measure.

Proposition 3.3. *The conditional characteristic function under the $\mathbb{Q}^{T_j}-$forward probability measure of the random vector X defined in* (3.58) *is given by*

$$
\begin{aligned}
\varphi_X(u) = \mathbb{E}_t^{\mathbb{Q}^{T_j}}\left[e^{iuX(T_{j-1})}\right] &= e^{iuf(T_{j-1})}\mathbb{E}_t^{\mathbb{Q}^{T_j}}\left[e^{i\sum_{n=1}^{d} y_n(T_{j-1})ug(n,T_{j-1})}\right]\\
&= e^{iuf(T_{j-1})+\sum_{n=1}^{d}\Phi_n^{\kappa_n(T_{j-1},T_j),\theta_n(T_{j-1},T_j)}(t,T_{j-1},0,-iug(n,T_{j-1}))}\\
&\quad .e^{\sum_{n=1}^{d} y_n(t)\Psi_n^{\kappa_n(T_{j-1},T_j),\theta_n(T_{j-1},T_j)}(t,T_{j-1},0,-iug(n,T_{j-1})))},
\end{aligned}
$$

where the deterministic functions f,g are defined in (3.64), *while the functions $\Phi_n^{\kappa_n(T_{j-1},T_j),\theta_n(T_{j-1},T_j)}$, $\Psi_n^{\kappa_n(T_{j-1},T_j),\theta_n(T_{j-1},T_j)}$ are given by formulae* (3.15), (3.16) *by replacing κ_n,θ_n for $n=1,...,d$ with*

$$
\begin{aligned}
\kappa_n(T_{j-1},T_j) =& \kappa_n - \Psi_n(T_{j-1},T_j,a_n,0)\sigma_n^2 && (3.67)\\
\theta_n(T_{j-1},T_j) =& \frac{\kappa_n\theta_n}{\kappa_n(T_{j-1},T_j)}. && (3.68)
\end{aligned}
$$

The transform is well defined for $u\in\mathbb{C}$ and for all $n=1,...,d$.

Proof. From (3.28) it follows that

$$\frac{dD^{OIS}(t,T_j)}{D^{OIS}(t,T_j)} =(...)dt + \sum_{n=1}^{d} \Psi_n(t,T_j,a_n,0)\sigma_n\sqrt{y_n(t)}dW_n(t), \tag{3.69}$$

and the volatility term of the OIS-bond price involved in the change of measure from $\mathbb{Q}$ to the $\mathbb{Q}^{T_j}$−forward probability measure is given by the Girsanov shift

$$dW_n^{\mathbb{Q}^{T_j}}(t) =dW_n(t) - \Psi_n(t,T_j,a_n,0)\sigma_n\sqrt{y_n(t)}dt, \quad n=1,...,d. \tag{3.70}$$

Hence, under the $\mathbb{Q}^{T_j}$−forward probability measure, the dynamics (3.11) of the factor y_n become

$$dy_n(t) =\kappa_n(\theta_n - y_n(t))dt + \sigma_n\sqrt{y_n(t)}(dW_n^{\mathbb{Q}^{T_j}}(t) + \Psi_n(t,T_j,a_n,0)\sigma_n\sqrt{y_n(t)}dt) \tag{3.71}$$

$$=\kappa_n(t,T_j)(\theta_n(t,T_j) - y_n(t))dt + \sigma_n\sqrt{y_n(t)}dW_n^{\mathbb{Q}^{T_j}}(t), \tag{3.72}$$

where we define the deterministic parameters

$$\kappa_n(t,T_j) =\kappa_n - \Psi_n(t,T_j,a_n,0)\sigma_n^2 \tag{3.73}$$

$$\theta_n(t,T_j) =\frac{\kappa_n\theta_n}{\kappa_n(t,T_j)}. \tag{3.74}$$

Notice that under $\mathbb{Q}^{T_j}$ the Feller condition remains the same as under $\mathbb{Q}$. Now the result follows from a direct application of formulae (3.15), (3.16) to the process (3.72). □

Now we can apply Carr and Madan (1999) methodology in order to get the price of a caplet on LIBOR.

4 Calibration to market data

In this section, we proceed to calibrate the version of the model based on multi-factor Cox/Ingersoll/Ross (CIR) dynamics — as introduced above — to market data, step by step adding more instruments to the calibration. We begin with OIS, then include money market instruments such as LIBOR, vanilla swaps and basis swaps. Calibration to instruments with optionality (caps/floors and swaptions), and lastly credit default swaps, is work in progress and results will be added in a subsequent update of this paper. At each step calibration consists in minimising the sum of squared deviations from calibration conditions such as (2.9) and (2.23), or between market and model prices for products with optionality.

Since we are modelling several sources of risk (interest rate risk, downgrade risk and liquidity risk) simultaneously, the calibration problem necessarily becomes one of high-dimensional non-linear optimisation. To this end, we apply the "Adaptive Simulated Annealing" algorithm of Ingber (1993) and Differential Evolution of

Storn and Price (1997). The data set (of daily bid and ask values) for the instruments under consideration is sourced from Bloomberg, consisting of US dollar (USD) instruments commencing on 1 January 2013 and ending on 23 April 2016 (about 1395 quotes). Since we are calibrating the model to "cross–sectional" data, i.e. "snapshots" of the market on a single day, we present the results for an exemplary set of dates only, having verified that results for other days in our data period are qualitatively similar.

4.1 Market data processing

Our goal is to jointly calibrate the model to three different tenor frequencies, i.e., to 1-month, 3-month and 6-month tenors, as well as to the OIS–based discount factors. Table 1 and 2 show market data quoted on the 01/01/2013 and 18/06/2015 respectively.[12]

Table 1: Market data quote on 01/01/2013. Source: Bloomberg

	IRS		OIS		1m/3m		3m/6m	
T	bid	ask	bid	ask	bid	ask	bid	ask
0.5	0.50825	0.50825	0.13	0.17	9.6	9.6	19.32	21.32
1	0.311	0.331	0.125	0.165	8.29	9	16.25	18.25
2	0.37	0.395	0.125	0.165	7.8	9.8	13.56	15.56
3	0.5	0.5	0.12	0.16	7.21	9.21	11.7	13.7
4	0.657	0.667	0.12	0.16	7.7	7.7	10.64	12.64
5	0.83	0.87	0.109	0.16	7.3	7.3	9.74	11.74
6	1.0862	1.0978	0.13	0.17	6.8	6.8	10.2	10.2
8	1.5001	1.5149	0.226	0.276	6	6	9.7	9.7
9	1.6496	1.6684	0.368	0.418	5.7	5.7	9.7	9.7
10	1.836	1.837	0.563	0.613	5.3	5.3	8.67	10.67

[12] With maturities $T = \{0.5, 1, 2, 3, 4, 5, 6, 8, 9, 10\}$, as there are no quotes for maturity of 7 years in the basis swap market. IRS and OIS are given in % while basis swaps are quoted in basis points, i.e., one–hundredths of one percent.

Table 2: Market data quote on 18/06/2015. Source: Bloomberg

	IRS		OIS		1m/3m		3m/6m	
T	bid	ask	bid	ask	bid	ask	bid	ask
0.5	0.4434	0.4434	0.136	0.146	9.1	9.1	2.375	2.875
1	0.51	0.511	0.159	0.164	10	10.5	2.19675	2.19675
2	0.896	0.898	0.175	0.18	11.75	12.25	1.9375	2.05625
3	1.2354	1.2404	0.189	0.194	13	13.5	1.84375	1.96875
4	1.5103	1.5153	0.255	0.26	13.625	14.125	1.78125	1.90625
5	1.698	1.757	0.333	0.338	13.875	14.375	1.78125	1.90625
6	1.918	1.922	0.541	0.581	13.75	14.25	1.75	1.875
8	2.1877	2.1917	0.991	1.041	13.035	14.035	1.8125	1.9375
9	2.2857	2.288	1.255	1.305	13	13.5	1.84375	1.96875
10	2.3665	2.369	1.478	1.518	12.75	13.25	1.90625	2.03125

In the USD market, the benchmark interest rate swap (IRS) pays 3-month LIBOR floating vs. 6-month fixed, i.e. $\delta = 0.25$ and $\Delta = 0.5$. To get IRS indexed to LIBOR of other maturities, we can use combinations of basis swaps with the benchmark IRS. Calibration conditions for 1-month, 3-month and 6-month tenors are given in Equations (2.24), (2.23), and in (2.25), respectively. We compute the right–hand side of these equations using the market data. We shall call this "the market side."

We use the OIS equation (2.5) for interpolation. Our approach is to calibrate the model to OIS–based discount factors, then use the calibrated model to infer discount factors of any required intermediate maturities. In particular, we bootstrap the time-dependent parameter a_0 for both OIS bid and ask and then interpolate the bond prices using the OIS equation to get the corresponding bid and ask values for intermediate maturities. Table 3 and 4 show the thus calculated values for the market side of the respective equations (2.23)–(2.25).

Table 3: Processed basis swap data on 01/01/2013

Multiple Tenors						
	1–month		3–month		6–month	
T	bid	ask	bid	ask	bid	ask
0.5	0.00205927	0.00206	0.002539	0.00254	0.003504	0.003605
1	0.00220674	0.002479	0.003106	0.003307	0.004729	0.00513
2	0.00542728	0.00633	0.007385	0.007888	0.010092	0.010995
3	0.01219954	0.01281	0.014957	0.014968	0.018458	0.01907
4	0.02311226	0.023537	0.026185	0.026607	0.030426	0.03165
5	0.03767771	0.039724	0.041317	0.043359	0.046166	0.049211
6	0.06076268	0.061551	0.064828	0.065611	0.070917	0.071708
8	0.11423662	0.115646	0.119008	0.120409	0.126706	0.128121
9	0.14172589	0.143731	0.146814	0.148808	0.155451	0.157464
10	0.17552507	0.176075	0.180762	0.1813	0.189304	0.191838

Table 4: Processed basis swap data on 18/06/2015

Multiple Tenors						
	1–month		3–month		6–month	
T	bid	ask	bid	ask	bid	ask
0.5	0.001760563	0.001760687	0.00221538	0.002215493	0.002334068	0.00235917
1	0.004044806	0.004105048	0.00509396	0.005104206	0.005313423	0.005323675
2	0.015435797	0.015576907	0.01788161	0.017922752	0.018268365	0.018333239
3	0.032905173	0.033208001	0.03694427	0.037097184	0.037495769	0.037686126
4	0.054501841	0.054907621	0.06012901	0.060335029	0.060838404	0.061094287
5	0.077179624	0.080371194	0.08432434	0.087266504	0.085209295	0.088213692
6	0.105381587	0.105967608	0.11385359	0.114139546	0.11489315	0.115253782
8	0.159830424	0.161135776	0.17082487	0.171336062	0.172241977	0.172852572
9	0.187128495	0.188082654	0.19892604	0.19942727	0.200533269	0.201145956
10	0.213644683	0.21478477	0.22638097	0.227019875	0.228207946	0.228969985

4.2 Model calibration problem

Model calibration is a process of finding model parameters such that model prices match market prices as closely as possible, i.e., in Equation (2.23), we want the left hand side to be equal to the right hand side. In essence, we are implying model parameters from the market data. The most common approach is to minimise the deviation between model prices and market prices, i.e., in the "relative least–squares" sense of

$$\arg\min_{\theta\in\Theta} G(\theta) = \min_{\theta\in\Theta} \sum_{i=1}^{N} \left(\frac{P_{\text{model}}(\theta) - P_{\text{market}}}{P_{\text{market}}} \right)^2, \tag{4.1}$$

where N is the number of calibration instruments. θ represents a choice of parameters from Θ, the admissible set of model parameter values. P_{model} and P_{market} are the model and market prices, respectively. In our case, P_{model} is the left hand side of (2.23) and P_{market} is the right hand side of the same equation. Market instruments are usually quoted as bid and ask prices, i.e. resulting in the values given in Tables 3 and 4. Thus, market prices are only observed to the accuracy of the bid/ask spread. Therefore, we solely require the model prices to fall withing bid–ask domain rather than attempting to fit the mid price exactly. The problem in (4.1) can be reformulated as

$$\arg\min_{\theta\in\Theta} G(\theta) = \min_{\theta\in\Theta}\sum_{i=1}^{N}\left(\max\left\{\frac{P_{\text{model}}(\theta)-P^{\text{ask}}_{\text{market}}}{P^{\text{ask}}_{\text{market}}},0\right\}+\max\left\{\frac{P^{\text{bid}}_{\text{market}}-P_{\text{model}}(\theta)}{P^{\text{bid}}_{\text{market}}},0\right\}\right)^2. \tag{4.2}$$

Gradient-based techniques typically are inadequate to handle high–dimensional problems of this type. Instead, we utilise two methods for global optimisation, Differential Evolution (DE) (see Storn and Price (1995, 1997)) and Adaptive Simulated Annealing (ASA) (see Ingber (1996, 1993)).

Model parameters

Recall that the model is made up of the following quantities: r_c which is the short rate abstraction of the interbank overnight rate, ϕ the pure funding liquidity risk, and λ which represents the credit or renewal risk portion of roll-over risk.

In order to keep the dimension of the optimisation problem manageable, we split the calibration into 3 stages:

- Stage 1: Fit the model to OIS data. The OIS model parameters is the set

  $$\theta = \{a_i, y_i(0), \sigma_i, \kappa_i, \theta_i,\ \forall i = 1,\cdots,d, \text{ and } a_0(t),\ t\in[T_{k-1},T_k],\ k=1,2,\cdots,10\}.$$

  where d is the number of factors (one or three in the cases considered here).

- Stage 2: Using Stage 1 calibrated parameters, calibrate the model to basis swaps. From the results in Section 3.3, note that basis swaps depend on $c_0(t)$ and $b_0(t)$ only via $c_0(t) + qb_0(t)$, i.e. the two cannot be separated based on basis swap data alone — this will be done when calibration to credit default swaps is implemented. Define $d_0(t) = c_0(t) + qb_0(t)$ and in this stage assume constant d_0. Thus, the parameters which need to be determined in this step are

  $$\theta = \{b_i, c_i, q,\ \forall i = 1,\cdots,d, \text{ and } d_0\}.$$

- Stage 3: Now, keeping all other parameters fixed, we improve the fit to vanilla and basis swaps by allowing $d_0(t)$ to be a piecewise constant function of time, i.e. $d_0(t) = d_0^{(k)}$ for $t\in[T_{k-1},T_k)$. Since the shortest tenor is one month, the $[T_{k-1},T_k)$ are chosen to be one–month time intervals. The calibration problem is then underdetermined, so we aim for the $d_0^{(k)}$ to vary as little as possible by imposing an additional penalty in the calibration objective function based on the sum of squared deviations between consecutive values, i.e.

$$\sum_k (d_0^{(k+1)} - d_0^{(k)})^2.$$

This type of smoothness criterion on time–varying parameters is well established in the literature on the calibration of interest rate term structure models, with its use in this context dating back to the seminal paper by Pedersen (1998).

4.3 Fitting to OIS discount factors

We use overnight index swap (OIS) data to extract the OIS discount factors $D^{\text{OIS}}(t,T)$ as per equations (2.11) and (2.19). Note that this is model–independent.

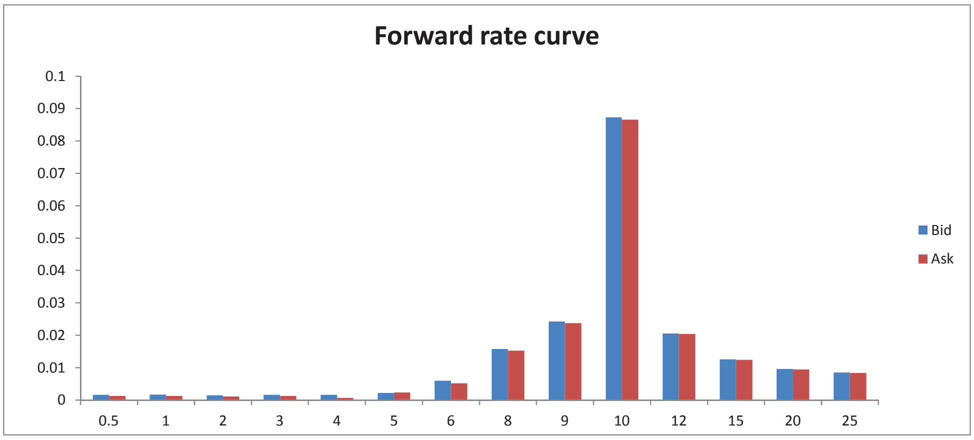
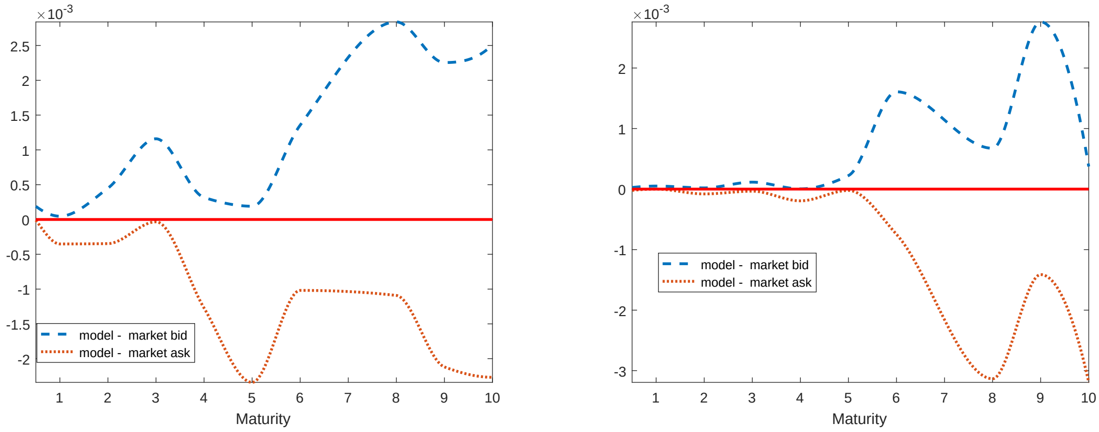


Figure 1: OIS–implied forward rates on 1 January 2013 (based on data from Bloomberg)

Bloomberg provides OIS data out to a maturity of 30 years. However, there seems to be a systematic difference in the treatment of OIS beyond 10 years compared to maturities of ten years or less. This becomes particularly evident when one looks at the forward rates implied by the OIS discount factors: As can be seen in Figure 1, there is a marked spike in the forward rate at the 10–year mark. Therefore, for the time being we will focus on maturities out to ten years only in our model calibration.



(a) 1 January 2013

(b) 18 June 2015

Figure 2: One–factor model fit to OIS discount factors (based on data from Bloomberg)

The OIS discount factors depend only on the dynamics of r_c, thus calibrating to OIS discount factors only involves the parameters in equation (3.1). As is standard practice in CIR–type term structure models,[13] we can ensure that the model fits the observed (initial) term structure by the time dependence in the (deterministic) function $a_0(t)$. In the one–factor case ($d = 1$), choosing $a_1 = 1$, we first fit the initial $y_1(0)$ and constant parameters a_0, κ_1, θ_1 and σ_1 in such a way as to match the observed OIS discount factors on a given day as closely as possible,[14] and then choose time–dependent (piecewise constant) $a_0(t)$ to achieve a perfect fit, as illustrated in Figure 2. In this figure, the difference between the discount factor based on OIS bid and the model discount factor is always negative, and the difference between the discount factor based on OIS ask and the model discount factor is always positive, this we have fitted the model to the market in the sense that the model price always lies between the market bid and ask.

Table 5: CIR model calibrated parameters on 01/01/2013

	$y_1(0)$	κ_1	θ_1	σ_1	a_1
Factor	0.726349	0.1972365	0.1525022	0.187127	0.001036
3	0.034559	0.9975629	0.01170301	0.008293	0.002325
	0.09391	0.682302	0.2431752	0.407768	0.002373
1	0.831418	0.2786581	0.7153432	0.22479	0.000517

[13]See e.g. Brigo and Mercurio (2006), Section 4.3.4.

[14]Once we include market instruments with option features in the calibration below, the volatility parameters will be calibrated primarily to those instruments, rather than the shape of the term structure of OIS discount factors.

Table 6: CIR model calibrated parameters on 18/06/2015

Factor	$y_1(0)$	κ_1	θ_1	σ_1	a_1
3	0.130307	0.9017996	0.06102752	0.092206	0.000657
	0.038801	0.0743267	0.05139999	0.081546	0.016585
	0.344269	0.04922985	0.7660736	0.073642	0.001173
1	0.483062	0.94945	0.326688	0.29234	0.001577

Table 7: Time-dependent parameter

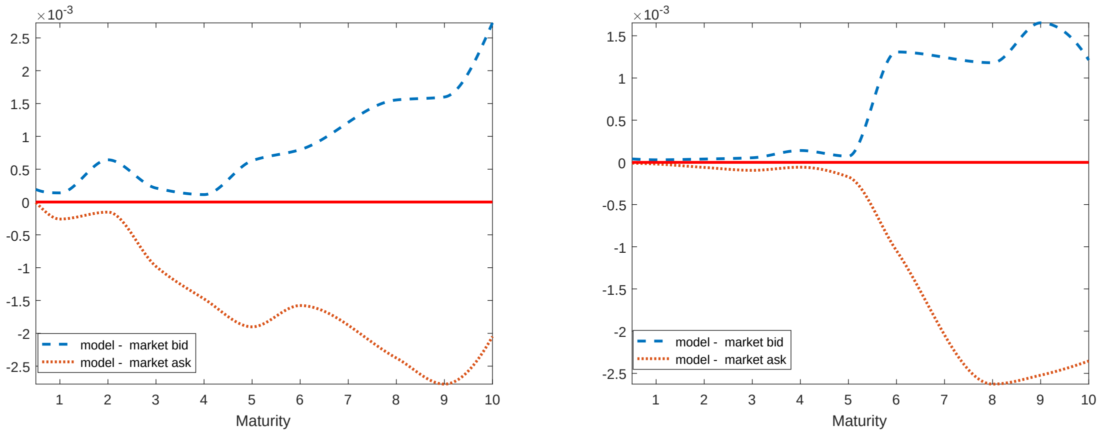
(a) 01 January 2013

T	$a_0^{(1)}$	$a_0^{(3)}$
0.5	0.000895	0.000244804
1	0.001464	0.000617087
2	0.000835	5.57785E-05
3	0.000385	0.000865304
4	0.002053	0.000668642
5	0.001338	8.77808E-05
6	0.000641	0.001074686
8	0.004826	0.004668318
9	0.015935	0.01480823
10	0.023585	0.02219484

(b) 18 June 2015

T	$a_0^{(1)}$	$a_0^{(3)}$
0.5	0.000694	0.000245494
1	0.001137	0.000699908
2	0.001411	0.000790489
3	0.001583	0.001018598
4	0.004171	0.003280662
5	0.005772	0.00533789
6	0.01618	0.01557728
8	0.024752	0.02352317
9	0.032731	0.03369083
10	0.038306	0.03521712

The fitted parameter are shown in Table 5, 6 and 7.

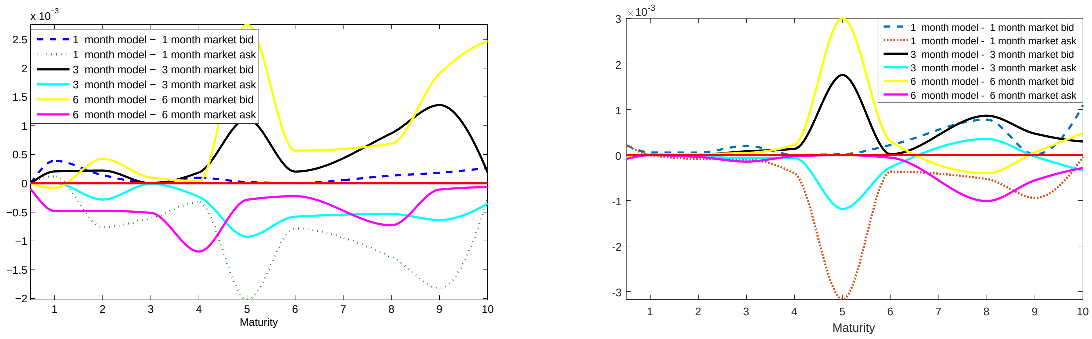


(a) 1 January 2013 (b) 18 June 2015

Figure 3: Three–factor model fit to OIS discount factors (based on data from Bloomberg)

4.4 Fitting to OIS, vanilla and basis swaps

The calibration condition for each vanilla swap is given by equation (2.23), and for each basis swap we obtain a condition as in (2.24) or (2.25). Specifically, in USD we have vanilla swaps exchanging a floating leg indexed to three–month LIBOR paid quarterly against a fixed leg paid semi-annually. We combine this with basis swaps for three months vs. six months and one months vs. three months, giving us three calibration conditions for each maturity (again, we restrict ourselves to maturities out to ten years for which data is available from Bloomberg, i.e. maturities of six months, 1, 2, 3, 4, 5, 6, 8, 9 and 10 years).



(a) 1 January 2013 (b) 18 June 2015

Figure 4: One–factor model fit to Basis Swaps (based on data from Bloomberg)

Figure 4 shows the fits of a one–factor model to vanilla and basis swap data obtained for 1 January 2013 and 18 June 2015, respectively. Tables 8 and 9 give the corresponding model parameters.

Table 8: Basis model parameters

Factor	b	c	q
3	0.051597	1.32E-05	0.000955017
	0.044587	0.000287	
	0.010496	0.000347	
1	0.000372	0.000268	0.5285594

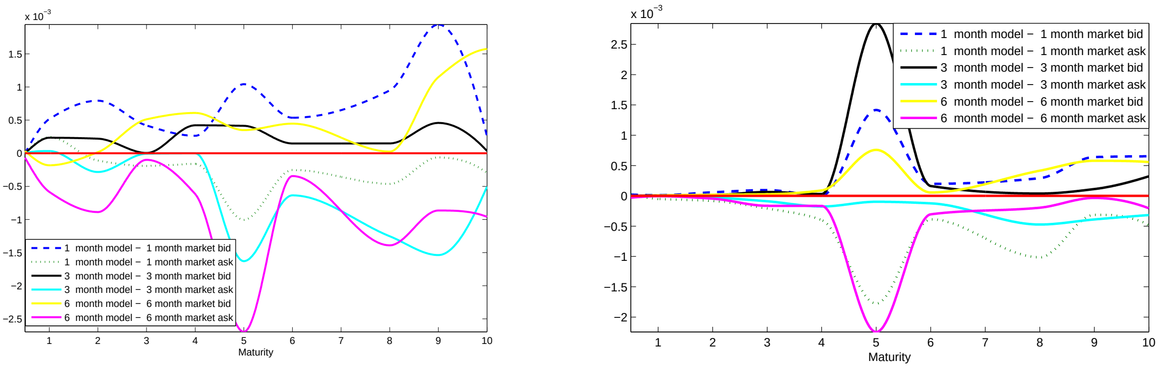
Table 9: Basis model parameters

(a) 01 January 2013

Factor	b	c	q
3	0.051597 0.044587 0.010496	1.32E-05 0.000287 0.000347	0.000955017
1	0.000372	0.000268	0.5285594

(b) 18 June 2015

Factor	b	c	q
3	0.631913 0.3881756 0.2936657	7.39E-05 0.000781247 0.006048256	0.00002286155
1	0.08377148	0.006292473	0.006214833



(a) 1 January 2013

(b) 18 June 2015

Figure 5: Three–factor model fit to Basis Swaps on 1 January 2013 (based on data from Bloomberg)

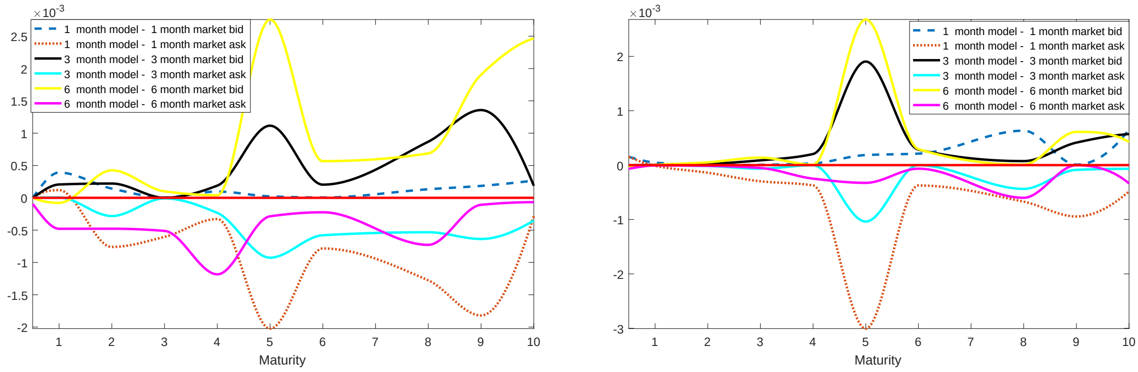
Expanding the model to three factors ($d = 3$), we modify the staged procedure to further facilitate the non-linear optimisation involved in the calibration: We first fit a one–factor model to OIS data as described in Section 4.3, and then keep those parameters fixed in the three-factor model, setting $a_2 = a_3 = 0$ to maintain the OIS calibration, and then fit the initial $y_2(0)$, $y_3(0)$ and constant parameters d_0, $b_1, b_2, b_3, c_1, c_2, c_3$, κ_2, κ_3, θ_2, θ_3 and σ_2, σ_3 in such a way as to match the swap calibration conditions on a given day as closely as possible.

Table 10: Parameters implied from basis swaps 1 January 2013

$y(0)$	a	b	c	q	σ	κ	θ
0.831418	0.000517	0.285696	0.000108		0.22479	0.278658	0.715343
0.094204	0	0.443564	0.000749	2.19E-05	0.220092	0.352275	0.383409
0.100625	0	0.587845	0.001809		0.343971	0.334376	0.797952

Table 11: Parameters implied from basis swaps 18 June 2015

$y(0)$	a	b	c	q	σ	κ	θ
0.483062	0.001577	0.969047	0.004607		0.29234	0.94945	0.326688
0.100874	0	0.027121	0.00285	0.00095	0.500709	0.780582	0.187569
0.326835	0	0.015832	0.000325		0.124841	0.898274	0.363523

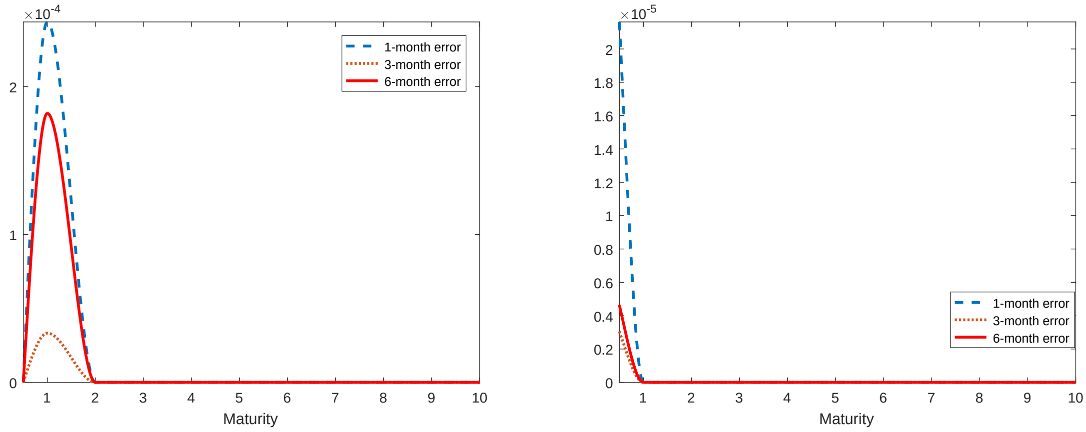


(a) 1 January 2013 (b) 18 June 2015

Figure 6: Tenors fit to market data (based on data from Bloomberg)

Lastly, in order to improve the fit further, we allow for time–dependent (piecewise constant) d_0. Figure 6 shows the resulting fits for 1 January 2013 and 18 June 2015, respectively, and Tables 10 and 11 give the corresponding model parameters.

From Figures 4, 5 and 6, we note that on each given day, the model can be simultaneously calibrated to all available OIS and vanilla/basis swap data, resulting in a fit between the bid and ask prices for all maturities, except for small discrepancies for maturities up to a year, as also illustrated in Figure 7.



(a) 1 January 2013 (b) 18 June 2015

Figure 7: Tenor calibration relative errors

5 Conclusion

Starting from the observation that in the context of pre-GFC models of the term structure of interest rates the presence of a frequency basis represents an arbitrage opportunity, we have presented a model framework which explicitly takes into account the roll-over risk which arises if one were to attempt such an "arbitrage." This allowed us to construct a consistent stochastic model, where the frequency basis between any (and all) arbitrary pairs of tenors arises endogenously from the two components of roll-over risk, i.e., "downgrade risk" and "funding liquidity risk." We have illustrated how such a model can be calibrated to observed market data at a given point in time, simultaneously to all available USD OIS, vanilla and basis swap data.

A model thus calibrated has a wide range of potential uses. One such use is the extraction of market–implied expectations of roll–over risk from current OIS, vanilla and basis swap market quotes. Furthermore, for any tenor pairs for which there is no frequency basis quoted in the market, the calibrated model will deliver as an output a value consistent (in a no-arbitrage sense) with the basis spreads of those tenor pairs for which basis swaps are traded in market. In economies where there is no liquid OIS market, this methodology could also be applied to infer OIS discount factors from other, traded tenors — this could be useful for the appropriate discounting of collateralised derivatives transactions in those economies. Also, since the roll-over risk we are modelling represents the risk to a bank's unsecured borrowing rate beyond the market interest rate risk, this approach opens an avenue for calibrating funding valuation adjustments (FVA) to the OIS, vanilla and basis swap markets. These applications will be explored further in follow-on papers.

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